Topological Criteria of Global Attraction with Applications in Population Dynamics

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Abstract

In this paper we derive a criterion of trivial dynamics based on the theory of translation arcs. This criterion extends and unifies some results in the literature. Applications in continuous and discrete models of population dynamics are given.

Key words: translation arcs, trivial dynamics, topological linear graphs, orientation preserving embeddings .

1 Introduction

Convergence of all solutions to equilibria is the simplest asymptotic behavior of a dynamical system. In planar flows, Poincaré-Bendixson's theory can be used to derive criteria ensuring this simple behavior. In particular, this dynamics occurs when there are no closed orbits or poly-cycles. In planar discrete-time dynamical systems, however, the situation is more delicate since chaotic behavior can appear. This fact has motivated a broad literature dealing with criteria of global attraction for planar systems using different tools such as the theory of monotone systems, the notion of translation arcs developed by Brouwer, Carathéodory's prime ends just to mention a few different approaches (see, for instance [1, 3, 4, 6, 11, 14, 16, 22, 26, 28, 37, 38, 39] for some significant examples and applications).

The purpose of the present paper is to establish a new criterion of trivial dynamics

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for planar discrete-time dynamical systems. In this framework, we will say that there is trivial dynamics *if the omega limit set of any bounded orbit (in the future) is a connected set contained in the fixed point set.* After this definition we properly explain our criterion. Indeed, consider

$$p_{n+1} = H(p_n), \tag{1.1}$$

where $H : M \longrightarrow M$ is an orientation preserving embedding (not necessarily onto) and the phase space M is a simply connected two dimensional manifold with boundary ∂M. Assume that every fixed point in the interior of M can be connected with a fixed point on ∂M through an invariant arc. Under these conditions, H has trivial dynamics.
The prototype of manifold M is the first quadrant of $\mathbb{R}^2$, that is

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}.$$

The figure below illustrates a hypothetical situation where our result can be applied.

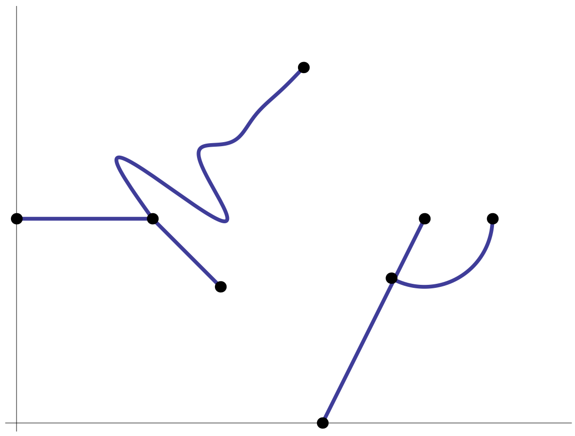

Figure 1: Example where we can use our criterion. The blue arcs are invariant under H and the black points represent the fixed points of H.

In connection with the related literature, in [6], Campos et al. considered Δ a topological disk and $H : \Delta \longrightarrow \Delta$ an orientation preserving homeomorphism (H must be onto) satisfying that the fixed point set is contained in the boundary of Δ. In this situation, they proved that H has trivial dynamics. Notice that we can recover this result in a more general setting. To be more precise, in our criterion, we do not need to suppose that H is onto or $\partial\Delta$ is invariant or Δ is compact. This extension is meaningful in applications, see Sections 3 and 4.
As a direct consequence of our arguments, we are able to deduce criteria of trivial dynamics when H is defined on whole $\mathbb{R}^2$. In this direction if we assume that $H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an orientation preserving embedding so that there exist disjoint sets $\gamma_1, ..., \gamma_n \subset \mathbb{R}^2$ satisfying that for all $i = 1, ..., n$,

- $H(\gamma_i) \subset \gamma_i$,
- $\gamma_i = \Phi_i([0,+\infty[)$ where $\Phi_i : [0,+\infty[\longrightarrow \Phi_i([0,+\infty[) \subset \mathbb{R}^2$ is a homeomorphism with $\Phi_i(0)$ a fixed point of H and $\lim_{t\to\infty} |\Phi_i(t)| = +\infty$,

then H has trivial dynamics.
This result is mainly motivated by Alarcón et al. in [1] and Ortega and Ruiz del Portal in [28]. In these interesting papers, the authors give a weaker variant. Specifically, they assume that there is a unique set γ_1 enjoying the previous conditions and a unique fixed point for H being locally asymptotically stable (see Theorem 11 in [28]).
The main motivation of our abstract results comes from the study of certain planar models of population dynamics in both discrete and continuous setting. Firstly we describe the applications in the discrete models. Mainly, our aim in this context is to relax some conditions of monotony considered in Liang-Jiang [22], Smith [35], [37] and Wang-Jiang [39]. Indeed, consider the system

$$\begin{cases} x_{n+1} = x_n g_1(x_n, y_n) \\ y_{n+1} = y_n g_2(x_n, y_n), \end{cases} \tag{1.2}$$

where g_i is a strictly positive function and the map $G : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ defined by

$$G(x, y) = (xg_1(x, y), yg_2(x, y))$$

is one-to-one and of class $\mathcal{C}^1$. For this system we derive the following:

- **Trichotomy in system (1.2) without assuming conditions of monotony.** Roughly speaking, this problem is the following. An usual situation in system (1.2) is that in each axis, there exists a positive equilibrium, namely $V_1^* > 0$ in the x-axis and $V_2^* > 0$ in the y-axis, attracting all non-zero orbits with initial condition in such an axis. If we also assume that each orbit is bounded in the future and the origin is a repeller then given $(x_0, y_0) \in Int(\mathbb{R}^2_+)$, one of the following condition holds:
  - there exists a fixed point of G in $Int(\mathbb{R}^2_+)$,
  - $G^N(x_0, y_0) \longrightarrow (V_1^*, 0)$,
  - $G^N(x_0, y_0) \longrightarrow (0, V_2^*)$.
- **Partially competitive maps.** In this part of the paper we obtain a criterion of global attraction for a fixed point $p = (p_1, p_2) \in Int(\mathbb{R}^2_+)$ assuming only conditions of monotony in a concrete region. More precisely, assume that p is the unique fixed point of G in $Int(\mathbb{R}^2_+)$, every orbit is bounded in the future, and

  $$G'(x, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

  is a competitive matrix (i.e. $a_{11}, a_{22} > 0$ and $a_{12}, a_{21} < 0$) for all $(x, y) \in \{(x, y) : 0 \le x \le p_1, 0 \le y \le p_2\} \cap Int(\mathbb{R}^2_+)$. Then p is a global attractor in

$Int(\mathbb{R}^2_+)$ if and only if $W^s(q) \cap Int(\mathbb{R}^2_+) = \emptyset$ for all q fixed point of G on the boundary of $\mathbb{R}^2_+$. Here, we have employed the notation

$$W^s(q) = \{z \in \mathbb{R}^2_+ : \lim_{N \longrightarrow \infty} G^{\sigma(N)}(z) = q \text{ with } \{\sigma(N)\} \subset \mathbb{N}\}.$$

For continuous models, we consider the system of differential equations

$$\begin{cases} x_1' = x_1 f_1(t, x_1, x_2) \\ x_2' = x_2 f_2(t, x_1, x_2), \end{cases} \tag{1.3}$$

where each $f_i : \mathbb{R} \times \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is a function of class $\mathcal{C}^1$ and T-periodic in time. This kind of systems has been extensively used to model the evolution of two species sharing the same environment. We recall that the periodicity in time is introduced to model day-night or seasonal forcing. Apart from this hypothesis, due to the limitations of the environment, it is natural to assume that system (1.3) is dissipative, i.e. there exists a constant $R > 0$ satisfying that for all $z_0 = (x_0, y_0) \in \mathbb{R}^2_+$, the solution with this initial condition is defined for all $t > 0$ and

$$\limsup_{t \to \infty} \|(x_1(t; z_0), x_2(t; z_0))\| < R.$$

In this setting, the Poincaré map associated with system (1.3),

$$P : \mathbb{R}^2_+ \longrightarrow P(\mathbb{R}^2_+) \subset \mathbb{R}^2_+$$

$$P(\xi) = (x_1(T; \xi), x_2(T; \xi)),$$

is well defined on $\mathbb{R}^2_+$ and an orientation preserving embedding. This map will be the key to link our abstract results with the dynamics of system (1.3). Namely we must take $M = \mathbb{R}^2_+$ and $H = P$. Observe that, by using the bounded behavior of the solutions, the concept of trivial dynamics implies that the omega limit set of any orbit is a connected set contained in the fixed point set of P.
After this discussion, we summarize some aims for system (1.3).

- **Global attraction for semi-coexistence states.** A fundamental issue in population dynamics is to give the minimal conditions to ensure the extinction of a concrete species. In this paper we show that there is a natural connection between this problem in case of two species and the notion of index on the convex set $\mathbb{R}^2_+$. More precisely, assume that there is intra-species competition in system (1.3), i.e.

$$\frac{\partial f_1}{\partial x_1}(t, x_1, 0) < 0$$

$$\frac{\partial f_2}{\partial x_2}(t, 0, x_2) < 0,$$

  and each species has logistic growth with semi-trivial coexistence states $(V_1(t), 0)$ and $(0, V_2(t))$ (see **P3** in Section 4.1). Then solution $(V_1(t), 0)$ is a global attractor in $Int(\mathbb{R}^2_+)$ if and only if system (1.3) has no T-periodic solutions in $Int(\mathbb{R}^2_+)$ and

$$index_{\mathbb{R}^2_+}(P, (V_1(0), 0)) = 1.$$

Notice that this result is somehow unusual because no conditions on $(0, V_2(t))$ are assumed.

- **Attraction of invariant curves.** A classical result of monotone systems says that if (1.3) is competitive together with certain additional conditions, then the Poincaré map admits an invariant curve γ, the so-called **carrying simplex**, joining all its fixed points. Moreover system (1.3) has trivial dynamics (see [15, 19, 20, 29, 32]). As a consequence of our results, we show that the presence of an invariant curve like γ implies, independently of any property of monotony, the previous simple dynamics for (1.3).

- **Coexistence states via permanence in predator prey systems**. From a biological point of view, we understand that system (1.3) is permanent if for every positive initial data, both species survive in the time. This definition is translated mathematically assuming that there is a compact set in the interior of $\mathbb{R}^2_+$ such that every solution with positive initial condition enters and remain in such a compact set. Apart from the biological consequences of this notion, it is well known that a permanent system always has a coexistence state. An end of this paper will be to prove that the presence of a coexistence state is also a sufficient condition in the class of predator prey systems.

The structure of the paper is as follows. In Section 2 we give some definitions and the main results. In Section 3 we apply our results in discrete models. In Section 4 we apply our results in continuous models. In the last section we prove the main results.
Notation:
The usual omega limit set is denoted by $\omega(z, H)$ and represents

$$\omega(z, H) = \{q : H^{\sigma(n)}(z) \longrightarrow q \text{ where } \{\sigma(n)\}_{n\in\mathbb{N}} \subset \mathbb{N} \text{ is strictly increasing}\}.$$

Given H a continuous map, we define $Fix(H)$ as the fixed point set of H.
$deg_{\mathbb{R}^2}(F, \Omega)$ denotes Brouwer's degree of F in the set Ω and $index_{\mathbb{R}^2}(F, p)$ (resp. $index_{\mathbb{R}^2_+}(F, p)$) denotes the usual index (resp. the usual index on the convex set $\mathbb{R}^2_+$) of F at the point p. See [10] for the definitions and elementary properties of these notions. In this reference,

$$deg_{\mathbb{R}^2}(F, \Omega) := deg_{\mathbb{R}^2}(F, 0, \Omega),$$

$$index_{\mathbb{R}^2}(F, p) := index_{\mathbb{R}^2}(id - F, p),$$

$$index_{\mathbb{R}^2_+}(F, p) := index_{\mathbb{R}^2_+}(id - F, p).$$

It is important to note that the fixed point index considered in this paper is a particular case of the index in ENR's considered by Dold [12], see also [18]. Finally, $Int(U)$, ∂U, $\overline{U}$ are used to denote the interior, boundary and closure of U respectively. $\{e_1, e_2\}$ refers to the usual basis of $\mathbb{R}^2$ and $\|\cdot\|$ denotes the Euclidean norm.

2 Embeddings and Topological Linear Graphs

In this section we present the main results of this paper together with some direct consequences. First of all, we need to introduce some concepts. Given two different points $p, q \in \mathbb{R}^2$, we can define the 1-simplex with vertices at p, q as

$$\{tp + (1-t)q : t \in [0,1]\}.$$

A point $\{p\}$ will be a 0-simplex. A **linear graph** is a finite collection K of 0 or 1 simplices in $\mathbb{R}^2$ with the following properties:

- K contains all vertices of all 1 simplices of K.
- If $\sigma, \tau \in K$ are two different 1 simplices with $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a vertex of both of them.

Given K a linear graph we will say that the dimension of K is 1 if K contains some 1-simplex and is 0 otherwise. A triplet $(\mathcal{A}, K, \phi)$ is a **topological linear graph** if $\mathcal{A} \subset \mathbb{R}^2$, K is a linear graph and $\phi : \mathcal{A} \longrightarrow |K|$ is a homeomorphism where $|K|$ denotes the union of the simplices of K. In such a case we define the topological 1 or 0 simplices of $(\mathcal{A}, K, \phi)$ in a natural way.

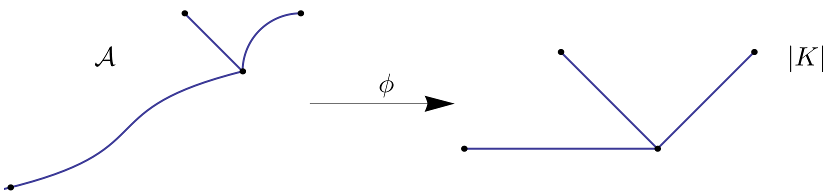

Figure 2: Example of topological linear graph.

The concept of topological linear graph allows us to introduce a natural notion of invariance for these sets. Specifically, given $g : \Xi \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a continuous map, we will say that a topological linear graph $(\mathcal{A}, K, \phi)$ is **graph invariant** under g if $\mathcal{A} \subset \Xi$ and every topological simplex of $(\mathcal{A}, K, \phi)$ is invariant under g. Clearly, if $(\mathcal{A}, K, \phi)$ is graph invariant then $\mathcal{A}$ is invariant. Next we give an example to illustrate that the converse is false. Indeed, consider $g : [0,1] \longrightarrow [0,1]$ a continuous map satisfying that $Fix(g) = \{0, \frac{1}{2}, 1\}$, $g(\frac{1}{4}) = \frac{3}{4}$ and $g(\frac{3}{4}) = \frac{1}{4}$. In this case $\mathcal{A} = [0,1]$ is invariant but $(\mathcal{A}, K, \phi)$ with $K = \{[0, \frac{1}{2}], [\frac{1}{2}, 1], \{0\}, \{1\}, \{\frac{1}{2}\}\}$ and $\phi(x) = x$ is not graph invariant since the topological 1-simplices are not invariant. In the following result we study the notion of graph invariance when g is one-to-one. Notice that this last property does not hold in the previous example.

Lemma 2.1 *Assume that* $g : \Xi \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *is continuous, one-to-one and* $(\mathcal{A}, K, \phi)$ *is a topological linear graph with* $\mathcal{A} \subset \Xi$*. Then* $(\mathcal{A}, K, \phi)$ *is graph invariant if and only if the set of vertices of* $(\mathcal{A}, K, \phi)$ *are fixed points of* g *and* $\mathcal{A}$ *is invariant.*

Proof. Firstly we observe that by definition of topological linear graph, $(\mathcal{A}, K, \phi)$ does not have two different 1-simplices with the same vertices. This fact enables us to conclude that if $\mathcal{A}$ is invariant and $V \subset Fix(g)$ then the connected components of $\mathcal{A}\backslash V$ are invariant under g where V denotes the set of vertices of $(\mathcal{A}, K, \phi)$. The proof follows from these comments. □

Let $M \subset \mathbb{R}^2$ be a simply connected two dimensional manifold with boundary, (see Pag. 224 in [24] for the precise definition of manifold with boundary). A map $H : M \longrightarrow M$ is an embedding if H is continuous and one-to-one. It is convenient to stress that H is not necessarily onto. If we also assume that

$$deg_{\mathbb{R}^2}(H - H(p_0), U) = 1 \tag{2.4}$$

for all U bounded open set with $p_0 \in U$ and $\overline{U} \subset Int(M)$, we will say that H is an **orientation preserving embedding**. Along the paper we use the notation $\mathcal{E}(\mathcal{D})$ for the class of embeddings and $\mathcal{E}_*(\mathcal{D})$ for the class of orientation preserving embeddings. Next we give our precise definitions of attractor in the system

$$p_{n+1} = H(p_n). \tag{2.5}$$

Indeed, an equilibrium $p = (p_1, p_2) \in \mathbb{R}^2$ of (2.5) is a global attractor in a set $D \subset \mathbb{R}^2$ if $\lim_{N\to\infty} H^N(z) = p$ for all $z \in D$.
Now we give the main result of this paper.

Theorem 2.1 *Let* $M \subset \mathbb{R}^2$ *be a simply connected two dimensional manifold with boundary and consider* $H : M \longrightarrow M$ *so that* $H \in \mathcal{E}_*(M)$*. Moreover we assume that there exists a family of connected and disjoint topological linear graphs* $(\mathcal{A}_1, K_1, \phi_1), \ldots, (\mathcal{A}_n, K_n, \phi_n)$ *with* $\mathcal{A}_1, ..., \mathcal{A}_n \subset M$ *and satisfying the following properties:*

- $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ *is connected and* $Fix(H) \cap Int(M) \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup ... \cup \mathcal{A}_n$.
- *For all* $i = 1, ..., n$, $\mathcal{A}_i \cap \partial M$ *is a non empty subset of* $Fix(H)$.
- *For all* $i = 1, ..., n$, $(\mathcal{A}_i, K_i, \phi_i)$ *is graph invariant under* H.

Then H *has trivial dynamics.*

Under the conditions of the previous theorem, clearly if the fixed point set is totally disconnected then for each $z \in M$ with $\{H^n(z) : n \in \mathbb{N}\}$ bounded,

$$\omega(z, H) = \{p\} \subset Fix(H). \tag{2.6}$$

Our following aim will be to guarantee (2.6) in a more general setting.

Theorem 2.2 *Let M, H and $(\mathcal{A}_1, K_1, \phi_1), ..., (\mathcal{A}_n, K_n, \phi_n)$ be as in Theorem 2.1. Assume that $H \in \mathcal{C}^1(M)$ and every non isolated fixed point p is partially hyperbolic i.e. $H'(p)$ has an eigenvalue with modulus different from 1. Then for each $z \in M$, $\omega(z, H)$ is a unique fixed point of H, (depending on z).*

There are some remarks to be made concerning the previous theorems. We say that H is of class $\mathcal{C}^1$ if there is an open set $U \supset M$ and an extension of H defined on U, namely $\widetilde{H}$, such that $\widetilde{H}$ is of class $\mathcal{C}^1$ in U. The previous results are not true in higher dimensions, (see Example 3 in [6]). The condition of manifold with boundary is essential for the validity of the theorems. For instance, if we replace M, $Int(M)$, ∂M in the previous theorems by $\overline{\Omega}$, Ω, $\partial\Omega$ with Ω an open and simply connected set, the previous results are false. Indeed, consider any continuous flow in the plane Ψ with the following dynamics, (Γ is a limit cycle and p is an equilibrium)

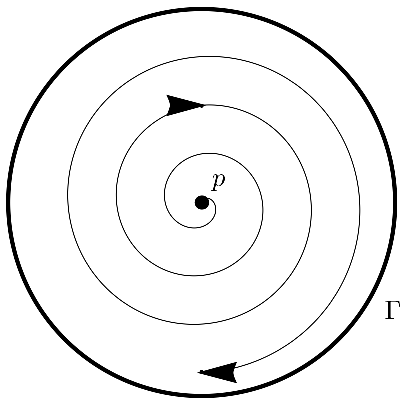

Figure 3: Dynamics of Ψ.

Define $\Omega = Int(\widetilde{D})\backslash(\{\Psi(t;q) : t \in \mathbb{R}\} \cup \{p\})$ where $\widetilde{D}$ is the topological disk limited by Γ and q is any point in $Int(\widetilde{D})\backslash\{p\}$. Observe that $\partial\Omega = \{\Psi(t;q) : t \in \mathbb{R}\} \cup \{p\} \cup \Gamma$ and $\overline{\Omega} = \widetilde{D}$. Next take two points r, s such that

- $r \in \Omega$,
- $s \in \Gamma$,
- $\Psi(\sigma(n), r) \longrightarrow s$ for some strictly increasing sequence $\{\sigma(n)\}_{\mathbb{N}} \subset \mathbb{N}$,
- $\Psi(\frac{1}{n_0}, s) \neq s$ for some $n_0 \in \mathbb{N}$.

Finally, we consider $H = \Psi(\frac{1}{n_0}, \cdot)$ and the topological graph $\mathcal{A}_1 = (\{p\}, \{p\}, id)$. Clearly, the conditions of Theorem 2.1 hold, and $\omega(r, H) \not\subset Fix(H)$.
After this example we study the condition used in Theorem 2.1. As mentioned in the introduction, a possible setting where we can apply the previous theorem is illustrated in Figure 1. Another interesting situation appears when $\emptyset \neq Fix(H) \subset \partial\mathcal{D}$. In this case we pick p a fixed point on the boundary of $\mathcal{D}$ and apply our results with the topological linear graph $(\mathcal{A}, K, \phi)$ where $\mathcal{A} = K = \{p\}$ and $\phi = id$. It is important to see that if $Fix(H) = \emptyset$ then we can directly deduce that there is trivial dynamics since in this case, all the orbits are unbounded (and so for all $z \in M$, $\omega(z, H) = \emptyset$). This fact for homeomorphisms can be found in [3] and for embeddings in [26, 27]. Next we collect these comments in the next result.

Corollary 2.1 *Let $M \subset \mathbb{R}^2$ be a simply connected two dimensional manifold with boundary and consider $H : M \longrightarrow M$ with $H \in \mathcal{E}_*(M)$ and $Fix(H) \subset \partial M$. Under these conditions, H has trivial dynamics.*

The previous result for homeomorphisms and M a topological disk (i.e. a simply connected and compact two dimensional manifold with boundary) was obtained in [6]. Notice that in [6], these two conditions are used in the proofs.
To finish this section we present the following result of trivial dynamics.

Theorem 2.3 *Suppose that $H \in \mathcal{E}_*(\mathbb{R}^2)$ and there exist disjoint sets $\gamma_1, ..., \gamma_n \subset \mathbb{R}^2$ with the following properties:*

- *for all $i = 1, ..., n$, $H(\gamma_i) \subset \gamma_i$,*
- *for all $i = 1, ..., n$, $\gamma_i = \Phi_i([0, +\infty[)$ where $\Phi_i : [0, +\infty[\longrightarrow \Phi_i([0, +\infty[) \subset \mathbb{R}^2$ is a homeomorphism with $\Phi_i(0) \in Fix(H)$ and $\lim_{t\to\infty} |\Phi_i(t)| = \infty$.*

Then, H has trivial dynamics.

3 Applications in discrete models

Throughout this section we apply the previous results to discrete equations. Specifically we study a trichotomy result for the class of orientation preserving embeddings and the notion of partially competitive maps. With this last concept, we refer to maps enjoying properties of monotony in a concrete region.

3.1 Trichotomy for orientation preserving embeddings

As a direct consequence of Theorem 2.1 we can obtain a version of Theorem 5.2 in [37] for the class of orientation-preserving embeddings. Specifically, we have.

Theorem 3.1 *Let $J = [0, a] \times [0, b] \subset \mathbb{R}^2$ with $0 < a, b$ or $J = \mathbb{R}^2_+$ be and let $P : J \longrightarrow J$ be a continuous map with the following properties:*

A1 *$P(0) = 0$ and 0 is a repeller, that is, there is $\delta > 0$ such that for all $\xi \in J \backslash \{0\}$, there exists $N := N(\xi) > 0$ so that $\|P^n(\xi)\| > \delta$ for all $n > N(\xi)$.*

A2 *$Fix(P) \cap \partial J = \{0, \widehat{u}e_1, \widehat{v}e_2\}$ with $0 < \widehat{u} < a, 0 < \widehat{v} < b$.*

A3 *$P \in \mathcal{E}_*(J)$ and for all $z \in J$, $\{P^n(z) : n \in \mathbb{N}\}$ is bounded.*

Then given $z_0 \in Int(J)$, one of the following holds:

1. *There exists a fixed point E_* of P in $Int(J)$.*
2. *$P^n(z_0) \longrightarrow \widehat{u}e_1$.*
3. *$P^n(z_0) \longrightarrow \widehat{v}e_2$.*

Proof. Assume that $Fix(P) \cap Int(J) = \emptyset$, otherwise the proof is complete. After that apply Corollary 2.1 with $M = J$ and $H = P$ in order to deduce that P has trivial dynamics. Observe that, by **A3**, the notion of trivial dynamics says that the omega limit set of any orbit is a connected set of $Fix(P)$. □

The previous theorem for competitive systems in the framework of Banach spaces can be deduced using [38] (also see [11] and [14]). The advantage of our result is that we do not need any property of monotony. However it must be noted that our proof only works in the Euclidean plane.

3.2 Partially competitive maps

Consider the system of difference equations

$$\begin{cases} x_{n+1} = x_n g_1(x_n, y_n) \\ y_{n+1} = y_n g_2(y_n, x_n) \end{cases} \tag{3.7}$$

where g_1, g_2 are strictly positive functions and the map $G : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ defined by the right-hand side of (3.7) is of class $\mathcal{C}^1$.

The aim of this section will be to derive a criterion of global attraction for a fixed point lying in $Int(\mathbb{R}^2_+)$. In this direction we can find interesting results developed by Smith in [37]. Namely if we assume that

i) $\det G'(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2_+$,

ii) $G'(x, y)$ is a competitive matrix for all $(x, y) \in Int(\mathbb{R}^2_+)$, (by a competitive matrix we understand a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{22} > 0$ and $a_{12}, a_{21} < 0$),

iii) for all $z \in \mathbb{R}^2_+$, $\{G^N(z) : N \in \mathbb{N}\}$ is bounded,

then system (3.7) has trivial dynamics, (see Proposition 2.1, Theorem 4.2 and Lemma 4.3 in [37]). Next we prove that the condition **ii)** can be refined. Specifically, it is enough to impose **ii)** in a smaller set.

Theorem 3.2 *Assume that G satisfies **i)**, **iii)**, and the following conditions:*

- $Fix(G) \cap Int(\mathbb{R}^2_+) = p$,
- $G'(x,y)$ *is a competitive matrix for all* $(x,y) \in C = \{(z_1,z_2) : z_1 \leq p_1, z_2 \leq p_2\} \cap Int(\mathbb{R}^2_+)$, *(* $p = (p_1,p_2)$ *).*

Then p is a global attractor in $Int(\mathbb{R}^2_+)$ if and only if $W^s(q) \cap Int(\mathbb{R}^2_+) = \emptyset$ for all $q \in Fix(G) \cap \partial\mathbb{R}^2_+$.

In the previous result $W^s(q)$ is defined as

$$W^s(q) = \{z \in \mathbb{R}^2_+ : \lim_{N \longrightarrow \infty} G^{\sigma(N)}(z) = q \text{ with } \{\sigma(N)\}_{N \in \mathbb{N}} \subset \mathbb{N}\}.$$

Proof. Firstly we notice that G is one-to-one in $\mathbb{R}^2_+$. For it, we use that $G^{-1}(\{0\}) = \{0\}$ together with the following elementary result.

Lemma 3.1 *(Lemma 2.3.4 in [8]) Assume that $K \subset \mathbb{R}^n$ is a compact set and*

$$f : K \longrightarrow f(K)$$

is a local homeomorphism. Then for all $y \in f(K)$, the cardinal of $f^{-1}(y)$ is finite. If $f(K)$ is also connected then there exists a constant r so that the cardinal of $f^{-1}(y)$ is exactly r for all $y \in f(K)$.

At this moment, using **i)** we know that $G \in \mathcal{E}_*(\mathbb{R}^2_+)$.

After that, we prove that $C \subset G(C)$. By using that $G'(x,y)$ is a competitive matrix in C, we deduce that

$$G_1(p_1,t) \geq p_1 = G_1(p_1,p_2) \quad 0 \leq t \leq p_2$$

$$G_2(t,p_2) \geq p_2 = G_2(p_1,p_2) \quad 0 \leq t \leq p_1.$$

These inequalities and $G_1(0,p_2) = 0 = G_2(p_1,0)$ imply that $C \subset G(C)$. Consequently

$$G(\overline{\mathbb{R}^2_+ \backslash C}) \subset \overline{\mathbb{R}^2_+ \backslash C}$$

and

$$Fix(G) \cap Int(\overline{\mathbb{R}^2_+ \backslash C}) = \emptyset.$$

Now we apply Corollary 2.1 to G, $\overline{\mathbb{R}^2_+ \backslash C}$, in order to obtain that for all $z \in \overline{\mathbb{R}^2_+ \backslash C}$, $\omega(z,G)$ is a connected set contained in $Fix(G)$. Notice that this behavior also holds if $z \in C$ and for some $j \in \mathbb{N}$, $G^j(z) \in \overline{\mathbb{R}^2_+ \backslash C}$. Finally we take $z \in C$ so that $G^j(z) \in C$ for all $j \in \mathbb{N}$. In such a case, $\omega(z,G)$ is a fixed point by applying Proposition 2.1, Theorem 4.2, Lemma 4.3 in [37]. The proof follows from the previous comments. $\square$

4 Applications in Continuous Models

The evolution of two species sharing the same environment can be modelled by a system of differential equations of the type

$$\begin{cases} x_1' = x_1 f_1(t, x_1, x_2) \\ x_2' = x_2 f_2(t, x_1, x_2), \end{cases} \tag{4.8}$$

where $f_i : \mathbb{R} \times \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^1$ and T-periodic in time. In our model, $x(t;\xi) = (x_1(t;\xi), x_2(t;\xi))$ denotes the maximal solution of (4.8) with $x(0;\xi) = \xi$ and represents the size of both populations at the instant t. As mentioned in the introduction, due to the limitations of the environment, the solutions of (4.8) are bounded in the future in an uniform way. More precisely, we assume that system (4.8) is dissipative, i.e. there exists a constant $R > 0$ so that for all $z_0 = (x_0, y_0) \in \mathbb{R}^2_+$, the solution with this initial condition is defined for all $t > 0$ and

$$\limsup_{t \to \infty} \|x(t; z_0))\| < R.$$

To link the dynamics of (4.8) with our abstract results, we will use the **Poincaré map**,

$$P : \mathbb{R}^2_+ \longrightarrow P(\mathbb{R}^2_+) \subset \mathbb{R}^2_+$$
$$P(\xi) = x(T;\xi).$$

It is well known that P is an orientation preserving embedding, the fixed points of P correspond to the periodic solutions of (4.8), and from the expression of system (4.8) and a straightforward computation, P satisfies that

$$P(\xi_1, \xi_2) = (\xi_1 e^{\int_0^T f_1(t, x(t;\xi))dt}, \xi_2 e^{\int_0^T f_2(t, x(t;\xi))dt}). \tag{4.9}$$

Notice that if P enjoys the conditions of Theorem 2.2 then every solution of (4.8) is asymptotic T-periodic, i.e., for all $\xi \in Int(\mathbb{R}^2_+)$, there is a T-periodic solution $\phi(t)$ such that $x(t;\xi) - \phi(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Moreover, observe that if $x(t,\xi) = (x_1(t,\xi), x_2(t,\xi))$ is a T-periodic solution of (4.8) and $x_i(t,\xi) \neq 0$ then

$$\int_0^T f_i(t, x(t,\xi))dt = 0. \tag{4.10}$$

4.1 Global attraction for semi-coexistence states

The aim of this subsection is to give the minima conditions ensuring the extinction of a concrete species in (4.8) for all initial condition lying in the interior of $\mathbb{R}^2_+$. For it, we assume the following properties for our system:

P1 System (4.8) is **dissipative**.

P2 There are no T-periodic solutions in $Int(\mathbb{R}^2_+)$.

P3 **Logistic growth** on the axes. For all $i \in \{1,2\}$, the scalar equation

$$\dot{x}_i = x_i f_i(t, x_i e_i), \tag{4.11}$$

has a unique positive T-periodic solution $V_i(t)$ and

$$\lim_{t\to\infty}[x_i(t) - V_i(t)] = 0$$

for $x_i(t) > 0$ any positive solution of (4.11).

P4 The origin is a **repeller**, (see **A1** in Theorem 3.1).

By Theorem 3.1, we can deduce that if system (4.8) satisfies **P1-P4** then for all $\xi \in Int(\mathbb{R}^2_+)$ either

$$\lim_{t\to\infty} \|x(t,\xi), -V_1(t)e_1\| = 0 \quad \text{or}$$
$$\lim_{t\to\infty} \|x(t,\xi) - V_2(t)e_2\| = 0.$$

In view of this fact, it arises the following question: under the conditions **P1-P4**, is there an index $i \in \{1,2\}$ so that for all $\xi \in Int(\mathbb{R}^2_+)$, we have that $\lim_{t\to\infty} \|x(t,\xi) - V_i(t)e_i\| = 0$? If we only assume **P1-P4**, the answer is negative. For instance, it is not hard to construct an autonomous system of type (4.8) with the following phase portrait.

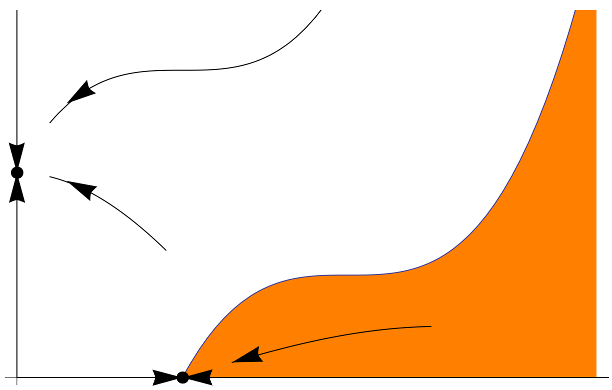

Figure 4: System without a global attractor.

However, the previous example is very pathological. In fact, if we assume that there is intra-species competition, i.e.

$$\frac{\partial f_1}{\partial x_1}(t, x_1, 0) < 0, \qquad \frac{\partial f_2}{\partial x_2}(t, 0, x_2) < 0, \tag{4.12}$$

then we will prove that our system has a global attractor in $Int(\mathbb{R}^2_+)$. Notice that this condition together with **P3** imply **P4**. This is a direct consequence of

$$\exp(\int_0^T f_1(t,0)dt), \exp(\int_0^T f_2(t,0)dt) > 1, \tag{4.13}$$

(see (4.10) and **P3**).

Theorem 4.1 *Assume that system (4.8) satisfies the properties **P1-P3** and (4.12). Then there is a global attractor in* $Int(\mathbb{R}^2_+)$*. Moreover,* $V_i(t)e_i$ *is a global attractor (resp. global repeller) in* $Int(\mathbb{R}^2_+)$ *if and only if* $index_{\mathbb{R}^2_+}(P,\, p_i^* e_i) = 1$ *(resp. 0), where* $p_i^* = V_i(0)$.

Proof. From expression (4.9), it is easy to prove that the Floquet multipliers at $p_1^* e_1$ (resp. $p_2^* e_2$) are

$$e^{\int_0^T f_2(t,V_1(t)e_1)dt} \quad (\text{resp. } e^{\int_0^T f_1(t,V_2(t)e_2)dt})$$

$$e^{\int_0^T \frac{\partial f_1}{\partial x_1}(t,V_1(t)e_1)dt} \quad (\text{resp. } e^{\int_0^T \frac{\partial f_2}{\partial x_2}(t,V_2(t)e_2)dt}).$$

Using (4.12), we have that the second Floquet multiplier is positive and strictly less than 1. In this situation, by Theorem 3.1 in [31], we can characterize the local stability via the index on $\mathbb{R}^2_+$. Namely, we know that

$$index_{\mathbb{R}^2_+}(P, p_i^* e_i) \in \{0, 1\}$$

and in addition, either $index_{\mathbb{R}^2_+}(P, p_i^* e_i) = 0$ if $p_i^* e_i$ is a local repeller in $Int(\mathbb{R}^2_+)$ or $index_{\mathbb{R}^2_+}(P, p_i^* e_i) = 1$ if $p_i^* e_i$ is a local attractor in $Int(\mathbb{R}^2_+)$. Next we introduce an auxiliar map $\widehat{P}$ defined as

$$\widehat{P}(\xi_1, \xi_2) = (\xi_1 e^{\int_0^T f_1(t,x(t,|\xi|))dt}, \xi_2 e^{\int_0^T f_2(t,x(t,|\xi|))dt}) \tag{4.14}$$

where $|\xi| = (|\xi_1|, |\xi_2|)$. Observe that $\widehat{P}$ corresponds with the Poincaré map of the system

$$\dot{x}_i = x_i\ f_i(t, |x_1|, |x_2|) \qquad i = 1, 2. \tag{4.15}$$

As a consequence of Remark 3.1 in [31] we have that, in this situation,

$$index_{\mathbb{R}^2_+}(P, p_i^* e_i) = 0 \Longleftrightarrow index_{\mathbb{R}^2}(\widehat{P}, p_i^* e_i) = -1$$

and

$$index_{\mathbb{R}^2_+}(P, p_1^* e_1) = 1 \Longleftrightarrow index_{\mathbb{R}^2}(\widehat{P}, p_i^* e_i) = 1.$$

Moreover, it is clear that $\widehat{P}$ commutes with symmetries with respect to the axes. More precisely

$$\widehat{P} \circ s_i = s_i \circ \widehat{P} \tag{4.16}$$

where $s_1(\xi_1, \xi_2) = (-\xi_1, \xi_2)$ and $s_2(\xi_1, \xi_2) = (\xi_1, -\xi_2)$. At this moment it is important to collect all the information. Specifically, we have that

$$index_{\mathbb{R}^2}(\widehat{P}, p_i^* e_i) \in \{-1, 1\}$$

and in addition, either $index_{\mathbb{R}^2}(\widehat{P}, p_i^* e_i) = 1$ if $p_i^* e_i$ is a local attractor in $Int(\mathbb{R}^2_+)$ for $\widehat{P}$ (and so, a local attractor for P in $Int(\mathbb{R}^2_+)$) or $index_{\mathbb{R}^2}(\widehat{P}, p_i^* e_i) = -1$ if $p_i^* e_i$ is a local repeller for $\widehat{P}$.

Once these properties have been shown, we continue with the proof. By Theorem 3.1, it is enough to prove that

$$index_{\mathbb{R}^2}(\widehat{P}, p_1^* e_1) \neq index_{\mathbb{R}^2}(\widehat{P}, p_2^* e_2). \tag{4.17}$$

Now we focus our attention on (4.17). Indeed, firstly we notice that, by (4.15) the fixed points of $\widehat{P}$ are $\pm p_1^* e_1$, $\pm p_2^* e_2$, 0 and they satisfy

$$index_{\mathbb{R}^2}(\widehat{P}, p_1^* e_1) = index_{\mathbb{R}^2}(\widehat{P}, -p_1^* e_1)$$

$$index_{\mathbb{R}^2}(\widehat{P}, p_2^* e_2) = index_{\mathbb{R}^2}(\widehat{P}, -p_2^* e_2).$$

This last property is a consequence of (4.16). After that, using that system (4.8) is dissipative and Browder's theorem (see [2]), we can prove that there exists $S > 0$ large enough such that every fixed point of $\widehat{P}$ is contained in $B(0, S)$, the ball with center at the origin and radius S, and

$$deg_{\mathbb{R}^2}(id - \widehat{P}, B(0, S)) = 1$$

(see step 1 in the proof of the main theorem in [30]). Moreover, using (4.13), we can check that

$$index_{\mathbb{R}^2}(\widehat{P}, (0, 0)) = 1$$

(this argument can be found in step 2 of the main theorem in [30]). Finally, an excision argument enables us to conclude that

$$deg_{\mathbb{R}^2}(id - \widehat{P}, B(0, S)) = index_{\mathbb{R}^2}(\widehat{P}, 0) + 2index_{\mathbb{R}^2}(\widehat{P}, p_1^* e_1) + 2index_{\mathbb{R}^2}(\widehat{P}, p_2^* e_2)$$

This equality clearly implies (4.17). □

Remark 4.1 *Notice that in the previous theorem solutions $V_1(t)e_1$ or $V_2(t)e_2$ can be partially hyperbolic.*

4.2 Attraction of invariant curves

Linking fixed points through invariant arcs is a typical behavior of the Poincaré map associated with (4.8) in the competitive case. This phenomenon can be deduced by applying the notion of carrying simplex in two dimensions, see [19], [20], [29], [32]. The aim of this section will be to give some dynamical consequences in (4.8) when the Poincaré map has an invariant curve joining all its fixed points. For it, we introduce the following definition. A curve γ is a **CS** for system (4.8) if it satisfies the next properties:

CS1 $\gamma : [0, 1] \longrightarrow \gamma([0, 1]) \subset \mathbb{R}^2_+ \backslash \{0\}$ is a homeomorphism with $\gamma(]0, 1[) \subset Int(\mathbb{R}^2_+)$,

CS2 $\gamma([0, 1])$ is invariant under P,

CS3 $Fix(P) \cap Int(\mathbb{R}^2_+) \subset \gamma([0, 1])$,

CS4 $Fix(P) \cap \{(x_1, 0) : x_1 > 0\} = \gamma(0)$ and $Fix(P) \cap \{(0, x_2) : x_2 > 0\} = \gamma(1)$.

Theorem 4.2 *Assume that system (4.8) is dissipative and every non trivial fixed point of P is partially hyperbolic. If there exists a CS for system (4.8) then for all $z \in \mathbb{R}^2_+$, $\omega(z, P)$ is a fixed point of P.*

Remark 4.2 *Notice that if*

$$\frac{\partial f_1}{\partial x_1}(t, x_1, x_2) + \frac{\partial f_2}{\partial x_2}(t, x_1, x_2) < 0$$

then every fixed point of P is partially hyperbolic. On the other hand, if we assume that $\frac{\partial f_i}{\partial x_j}(t, x_1, x_2) < 0$, $\int_0^T f_i(t, 0)dt > 0$ for all $i, j = 1, 2$, and system (4.8) is dissipative then (4.8) has a CS, (see [29]).

Proof. For the proof we distinguish two cases:

- *Case 1: For all $t \in [0, 1]$, $\gamma(t)$ is a fixed point of P.*
  In this case we can prove that $P(\mathcal{D}_1) = \mathcal{D}_1$ where $\mathcal{D}_1$ is the bounded connected component of $Int(\mathbb{R}^2_+)\backslash\gamma([0, 1])$. For it, we notice that the boundary of $\mathcal{D}_1$ is invariant under P. Now it is also clear that $P(Int(\mathbb{R}^2_+)\backslash(\mathcal{D}_1 \cup \gamma([0, 1]))) \subset Int(\mathbb{R}^2_+)\backslash(\mathcal{D}_1 \cup \gamma([0, 1]))$.

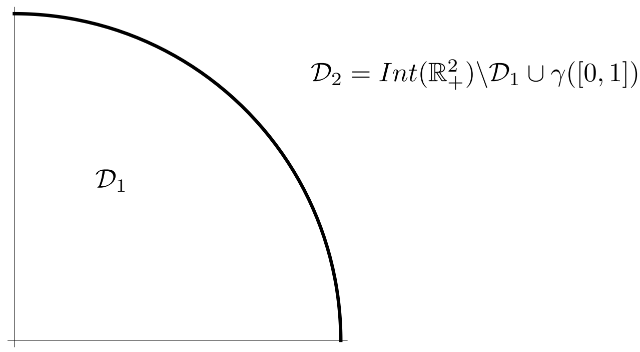

Figure 5: CS in case 1.

  Finally we separately apply Theorem 2.2 to $\overline{\mathcal{D}_1}$ and $\overline{\mathcal{D}_2}$ with topological linear graph any fixed point, for instance, $(\gamma(\frac{1}{2}), \gamma(\frac{1}{2}), id)$.

- *Case 2: There exists $]s_0, s_1[\subset [0, 1]$ such that $P(\gamma(t)) \neq \gamma(t)$ for all $t \in]s_0, s_1[$.*
  In this case we consider the topological linear graphs $(\mathcal{A}_1, K_1, \gamma)$ and $(\mathcal{A}_2, K_2, \gamma)$ with $\mathcal{A}_1 = \gamma([0, s_0])$ and $K_1 = \{\{s_0\}, \{s_1\}, [s_0, s_1]\}$, $\mathcal{A}_2 = \gamma([s_1, 1])$ and $K_2 = \{\{s_1\}, \{1\}, [s_1, 1]\}$.

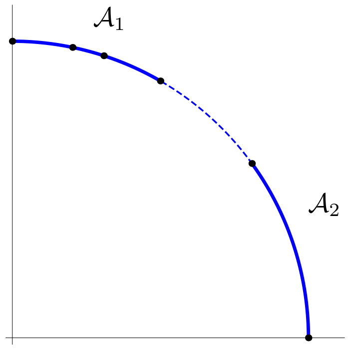

Figure 6: CS in case 2

Finally we apply Theorem 2.2.

□

4.3 Predator Prey Systems

The aim of this subsection is to characterize the presence of a T-periodic solution in $Int(\mathbb{R}^2_+)$ via permanence in predator prey systems. In fact, for this interaction, we are going to prove that if our system is not permanent then there is a global attractor on the boundary of $\mathbb{R}^2_+$. Next we fix the pertinent definitions.

Definition 4.1 *System (4.8) is said to be* ***permanent*** *if it is possible to find two constants* $0 < \underline{\sigma} < \overline{\sigma}$ *such that given initial conditions* $x_0 > 0, y_0 > 0$ *there exists* $T^* = T^*(x_0, y_0)$ *with*

$$\overline{\sigma} > x_1(t, (x_0, y_0)) > \underline{\sigma}$$

$$\overline{\sigma} > x_2(t, (x_0, y_0)) > \underline{\sigma}$$

for all $t > T^*$*. Notice that the numbers* $\underline{\sigma}$ *and* $\overline{\sigma}$ *are independent of the initial conditions.*

Observe that with the previous notion we guarantee that both species survive in the future.

In order to model the predator prey interaction we impose the following conditions in (4.8) (in our case, x_1 is predator and x_2 is prey).

1) $f_1(t, \cdot, x_2)$ and $f_2(t, x_1, \cdot)$ are strictly decreasing.

2) $f_1(t, x_1, \cdot)$ is strictly increasing.

3) $f_2(t, \cdot, x_2)$ is strictly decreasing.

4) For $i = 1, 2$, the scalar equation

$$x_i' = x_i f_i(t, x_i e_i) \tag{4.18}$$

has a strictly positive T periodic solution $V_i(t) > 0$ and attracts all positive solutions of (4.18).

5) For all $M > 0$,

$$x_1' = x_1 f_1(t, x_1, M) \tag{4.19}$$

has a nonnegative T periodic solution attracting all positive solutions of (4.19).

The next result collects the main aim of this section.

Theorem 4.3 *Assume that system (4.8) satisfies **1)-5)**. Then*
***i)** if $\int_0^T f_2(t, V_1(t), 0)dt > 0$, the system is permanent,*
***ii)** if $\int_0^T f_2(t, V_1(t), 0)dt \leq 0$, the solution $V_1(t)e_1$ is a global attractor in $Int(\mathbb{R}^2_+)$.*

Proof. Firstly, we prove that system (4.8) is dissipative. Indeed, consider $(\xi_1, \xi_2) \in Int(\mathbb{R}^2_+)$. Using **3)** we deduce that

$$x_2'(t; (\xi_1, \xi_2)) \leq x_2(t; (\xi_1, \xi_2)) f_2(t, 0, x_2(t; (\xi_1, \xi_2))).$$

By this inequality and **4)**, there exists $T_1 > 0$ large enough, so that for $t > T_1$, $x_2(t, (\xi_1, \xi_2)) \leq V_2(t) + 1 < \widetilde{M}$. After that, by **2)**,

$$x_1'(t; (\xi_1, \xi_2)) \leq x_1(t; (\xi_1, \xi_2)) f_1(t, x_1(t; (\xi_1, \xi_2)), \widetilde{M})$$

for all $t > T_1$. Finally, we use **5)** to obtain that there exists $T_2 > 0$ such that $x_1(t, (\xi_1, \xi_2)) \leq V_{1\,\widetilde{M}}(t) + 1$ for all $t > T_2$, where $V_{1\,\widetilde{M}}(t)$ is the solution considered in **5)** with constant $\widetilde{M}$. Now it is clear that our system is dissipative.

Proof of i). Clearly, using that $V_i(t)e_i$ is a T-periodic solution of (4.8),

$$\int_0^T f_i(t, V_i(t)e_i)dt = 0 \quad \text{for } i = 1, 2.$$

These equalities and **1)** allow us to conclude that

$$\int_0^T f_i(t, 0, 0)dt > 0 \ \text{ for } i = 1, 2. \tag{4.20}$$

Now, by (4.20) and **2)** we have that

$$\int_0^T f_1(t, 0, V_2(t))dt > 0. \tag{4.21}$$

At this point we know that the Poincaré map associated with system (4.8) has an expression of the type

$$P(x, y) = (x g_1(x, y), y g_2(x, y))$$

with $g_i > 0$, (see (4.9). On the boundary of $\mathbb{R}^2_+$, P has exactly three fixed points, namely $(0,0), (V_1(0),0)$ and $(0,V_2(0))$, and these fixed points satisfy that

$$g_1(0,0), g_2(0,0) > 1, (\text{see } (4.20))$$

$$g_1(0,V_2(0)) > 1, (\text{see } (4.21))$$

Also, by assumption,

$$g_2(V_1(0),0) > 1.$$

Moreover by **4)**, we have that $V_1(0)e_1$ and $V_2(0)e_2$ attract all positive solutions on the axes. On the other hand, as our system is dissipative, there exists a constant $R > 0$ such that

$$\limsup_n \|P^n(\xi)\| \leq R$$

for all $\xi \in \mathbb{R}^2_+$. Putting all the information together and using a standard argument we conclude that our system is permanent. This can be done by applying results in [17] with Lyapunov functions $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ or see [31].
Proof of ii). First of all we prove that system (4.8) has no T-periodic solutions in $Int(\mathbb{R}^2_+)$. Assume by contradiction that $(W_1(t), W_2(t))$ is a T-periodic solution of (4.8) in $Int(\mathbb{R}^2_+)$. In this case, by **2)**,

$$W_1'(t) = W_1(t)f_1(t, W_1(t), W_2(t)) > W_1(t)f_1(t, W_1(t), 0)$$

and so $W_1(t) > V_1(t)$. Next we apply condition **3)** and obtain that

$$0 = \int_0^T f_2(t, W_1(t), W_2(t))dt < \int_0^T f_2(t, V_1(t), 0)dt \leq 0.$$

(In the first equality we use that $(W_1(t), W_2(t))$ is T-periodic solution and in the second inequality we use **ii)**). This contradiction implies that there are no T-periodic solutions in $Int(\mathbb{R}^2_+)$. In the previous statement we have already proved that

$$\int_0^T f_1(t, 0, V_2(t))dt > 0.$$

Thus, the solution $V_2(t)e_2$ is always a local repeller. Observe that the inequalities (4.20) also hold and so the origin is a local repeller. Finally, by Theorem 3.1 we conclude that $V_1(t)e_1$ is a global attractor in $Int(\mathbb{R}^2_+)$. □

As a consequence of the previous theorem we obtain the following result.

Corollary 4.1 *Under conditions **1)-5)** for system (4.8) the following statements are equivalent:*

- *The system has a T-periodic solution in $Int(\mathbb{R}^2_+)$.*
- *The system is permanent.*

The previous results extend some of those in [5],[9],[13],[28].

5 Proofs

This section is devoted to prove the theorems of Section 2. Firstly we recall some known results.
Given $h \in \mathcal{E}(\Omega)$ for $\Omega \subset \mathbb{R}^2$ a simply connected and open set in the plane, we will say that an arc α with end points at p and q is a **translation arc** if

- $h(p) = q$.
- $h(\alpha \backslash \{q\}) \cap (\alpha \backslash \{q\}) = \emptyset$.

This notion is important in dynamical systems by the following result.

Lemma 5.1 *Assume that $h \in \mathcal{E}_*(\Omega)$ and there exists a translation arc α with $h^n(\alpha) \cap \alpha \neq \emptyset$ for some $n \geq 2$. Then there exists a Jordan curve $\Gamma \subset \Omega$ such that*

$$deg_{\mathbb{R}^2}(id - h, \mathcal{D}_\Gamma) = 1$$

where $\mathcal{D}_\Gamma$ is the interior of the domain limited by Γ.

In the proof of our results we will use the following criterion on construction of translation arcs.

Lemma 5.2 *Assume that $h \in \mathcal{E}(\Omega)$ and Δ is a topological disk in the plane such that Δ and $h(\Delta)$ lie in the same component of $\Omega \backslash Fix(h)$. In addition, assume that $h(\Delta) \cap \Delta = \emptyset$. Then, given $\xi_1, ..., \xi_n \in \Delta$, there exists a translation arc α contained in Ω and passing through these points.*

The previous results can be found in [27] and [4]. Next we proceed with the proofs.
Proof of Theorem 2.1. Take $z \in M$ so that $\{H^n(z) : n \in \mathbb{N}\}$ is bounded. Firstly we prove that $\omega(z, H)$ is contained in $Fix(H)$. We distinguish three different situations:

- $z \in Int(M) \backslash (\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$. Assume, by contradiction, that there is $p \in \omega(z, H)$ such that $p \notin Fix(H)$. Under this condition we can take a topological disk D_1 satisfying that

  - $p \in Int(D_1)$,
  - $D_1 \cap H(D_1 \cap M) = \emptyset$,
  - $D_1 \cap (M \backslash \mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ has a finite number of connected components.

  This fact is clear if $p \in \partial M$ by using the notion of manifold with boundary. In the case $p \in Int(M) \backslash \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$, the existence of D_1 is clear. Finally for the case $p \in \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$, firstly we observe that p belongs to the interior of a 1-simplex since the vertices of $(\mathcal{A}_1, K_i, \phi_i)$ are fixed points of H. Next we apply Theorem 8 in [25] in order to obtain a homeomorphism $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ so that $\Psi(\mathcal{A}_i) = |K_i|$. After that we define $\Psi^{-1}(B_1) = D_1$ where B_1 is a ball centered at $\Psi(p)$ satisfying that $B_1 \backslash \Psi(\mathcal{A}_i)$ has exactly two connected components. We illustrate the previous argument with the next figure.

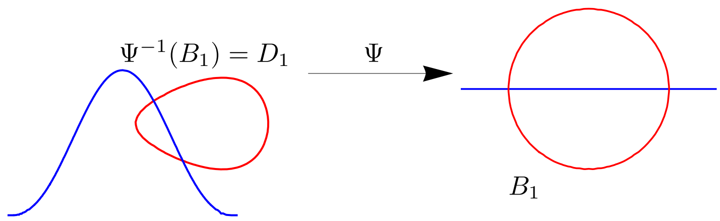

Figure 7: Illustration of Theorem 8 in [25].

Notice that this argument is genuinely two dimensional since in three dimensions we can have wild arcs, (see Section 4 in [25]).
After this discussion, by using that $p \in \omega(z, H)$, $z \in Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ and $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ is positively invariant, we can take a connected component K_1 of $D_1\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ so that $H^{n_1}(z), H^{n_2}(z) \in Int(K_1)$ with $n_2 > n_1$. Here we have used that the number of connected components is finite. At this moment we consider a closed topological disk $\widetilde{D_1} \subset K_1$ such that $H^{n_1}(z), H^{n_2}(z) \in \widetilde{D_1}$. The construction of $\widetilde{D_1}$ is as follows. Using that $Int(K_1)$ is arcwise connected we can take an arc β joining $H^{n_1}(z), H^{n_2}(z)$ such that $\beta \subset Int(K_1)$. Finally we inflate β without getting out from $Int(K_1)$. Once this reasoning has been done, we apply Lemma 5.2 to $\widetilde{D_1}$, $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ and H, in order to conclude that there exists $\alpha \subset Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ a translation arc passing through $H^{n_1}(z)$ and $H^{n_2}(z)$, (it is important to realize that, by standard topological arguments, $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ is simply connected). This fact is a contradiction. Indeed we know in advance that H does not have any fixed point in $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$. On the other hand, $H^{n_2 - n_1}(\alpha) \cap \alpha \neq \emptyset$ and by Lemma 5.1, H has a fixed point in $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$. This contradiction implies that for all $z \in Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$, $\omega(z, H) \subset Fix(H)$.

- $z \in \partial M$. Assume by contradiction that $\omega(z, H) \not\subset Fix(H)$. Under this condition, we see that $z \notin Fix(H)$ and so $z \notin \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$, (see second condition of the theorem). Consequently we realize that $H^N(z) \notin \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$ since $\mathcal{A}_1 \cup ... \cup \mathcal{A}_n$ is invariant. After this discussion, clearly, if for some $n \in \mathbb{N}$, $H^n(z) \in Int(M)$, the conclusion is clear by the previous reasoning. Therefore we have to study the case when $H^n(z) \in \partial M$ for all $n \in \mathbb{N}$. Indeed, take $p \in \omega(z, H)$ and assume that $H(p) \neq p$. In this setting, we can take a topological disk D_1 such that $D_1 \cap H(D_1 \cap M) = \emptyset$, $D_1 \cap Int(M)$ is simply connected and $p \in Int D_1$. Clearly, using that $p \in \omega(z, H)$, there exist $q \in D_1 \cap Int(M)$ and n_0 with $H^{n_0}(q) \in D_1 \cap Int(M)$. We reason as above in order to obtain a contradiction.

- Assume now that $z \in \mathcal{A}_i$. In this case we have that $\omega(z, H) \subset Fix(H)$ by using the definition of graph invariant together with some elementary notions of dynamics in $\mathbb{R}$.

To finish the proof of this theorem we use Proposition 8 Chapter 3 in [27] ensuring that the omega limit set is connected provided $\{H^n(z) : n \in \mathbb{N}\}$ is bounded. □

Proof of Theorem 2.2. The proof of this theorem is a direct consequence of the previous proof together with Proposition 3 in [7]. □

Proof of Theorem 2.3. Again, take a point $z \in \mathbb{R}^2$ such that $\{H^n(z) : n \in \mathbb{N}\}$ is bounded. Assume by contradiction that there is $q \in \mathbb{R}^2 \backslash Fix(H)$ so that $q \in \omega(z, H)$. In this situation we can take a topological disk D satisfying that $q \in D$ and $D \backslash (\gamma_1 \cup ... \cup \gamma_n)$ has at most two connected components. Indeed, if $q \notin \gamma_1 \cup ... \cup \gamma_n$ the construction is clear. Otherwise the construction of D is as follows. Assume that $q \in \gamma_j$. Consequently there exists $t_0 > 0$ with $\Phi_j(t_0) = q$. By Theorem 8 in [25] we can take $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ homeomorphism so that

$$\Psi(\{\Phi_j(t) : t \in [0, 2t_0]\}) = \{(x, 0) : x \in [0, T]\}.$$

It is clear that $\lim_{t\to\infty} |\Psi \circ \Phi_j(t)| = \infty$ (Ψ is a homeomorphism) and therefore, we can take $\delta > 0$ such that

$$B = B(\Psi(q), \delta) \not\ni \Psi(\Phi_j(t)) \quad \text{for all } t \geq 2t_0$$

and $B(\Psi(q), \delta) \backslash \Psi(\{\Phi_j(t) : t \in [0, 2t_0]\})$ has two connected components. Finally consider $D = \Psi^{-1}(B)$. The remainder of the proof is the same as the proof of Theorem 2.1. □

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