EMS Surv. Math. Sci. 2 (2015), 131–218
DOI 10.4171/EMSS/11

EMS Surveys in Mathematical Sciences

Flux-saturated porous media equations and applications

J. Calvo, J. Campos, V. Caselles, O. Sánchez and J. Soler*

To Vicent Caselles, in memoriam

Abstract. The aim of this paper is to review the main recent results about the dynamics of nonlinear partial differential equations describing flux-saturated transport mechanisms, eventually in combination with porous media flow and/or reactions terms. The result is a system characterized by the presence of wave fronts which move defining an interface. This can be used to model different process in applications in a variety of areas as developmental biology or astrophysics. The concept of solution and its properties (well-posedness in a bounded variation scenario, Rankine–Hugoniot and geometric conditions for jumps, regularity results, finite speed of propagation, ...), qualitative study of these fronts (traveling waves in particular) and application in morphogenesis cover the panorama of this review.

*J. Calvo, J. Campos, O. Sánchez and J. Soler were supported in part by MINECO (Spain), project MTM2014-53406-R, FEDER resources, and Junta de Andalucía Project P12-FQM-954. J. Calvo is also partially supported by a Juan de la Cierva grant of the Spanish MEC and La Caixa "Collaborative Mathematical Research". Juan Campos is also supported by MICINN Grant with FEDER funds MTM2011-23652. V. Caselles was supported in part by MICINN (Spain), project MTM2009-08171, and also acknowledges the partial support by GRC reference 2009 SGR 773, and by "ICREA Acadèmia" prize for excellence in research funded both by the Generalitat de Catalunya.

J. Calvo, Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C, 08193 Bellaterra, Barcelona, Spain
E-mail: jcalvo@crm.cat

J. Campos, Departamento de Matemática Aplicada, Facultad de Ciencias, Campus de Fuentenueva, Universidad de Granada, 18071 Granada, Spain
E-mail: campos@ugr.es

V. Caselles, Departamento de Tecnología, Universitat Pompeu-Fabra, Roc Boronat, 138, 08018 Barcelona, Spain
E-mail: vicent.caselles@upf.edu

O. Sánchez, Departamento de Matemática Aplicada, Facultad de Ciencias, Campus de Fuentenueva, Universidad de Granada, 18071 Granada, Spain
E-mail: ossanche@ugr.es

J. Soler, Departamento de Matemática Aplicada, Facultad de Ciencias, Campus de Fuentenueva, Universidad de Granada, 18071 Granada, Spain
E-mail: jsoler@ugr.es

Mathematics Subject Classification (*2010*). 35K57, 35B36, 35K67, 34Cxx, 70Kxx; 35B60, 37Dxx, 76B15, 35Q35, 37D50, 35Q99.

Keywords. Flux limitation, flux-saturation, porous media equations, waiting time, pattern formation,traveling waves, optimal mass transportation, entropy solutions, relativistic heat equation, complex systems.

Contents

1. Introduction

Flux saturated porous media equations is a shorthand that encompasses a class of degenerate parabolic equations combining two non-linear diffusion mechanisms: That of porous-media-type equations plus a flux saturation mechanism.

The archetypical porous media equation is

$$\frac{\partial u}{\partial t} = \operatorname{div}(u^{m-1}\nabla u), \tag{1.1}$$

while flux-limited equations (also known as flux saturated equations or tempered diffusion equations) are essentially equations in divergence form such that their flux saturates to a constant value whenever the size of the gradients is big enough. Probably the most representative example of such an equation, at least in the mathematical literature, is the so-called "relativistic heat equation"

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}}\right). \tag{1.2}$$

One of the easiest hybrid model we can come up with is the following (apparently first introduced in [61]):

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{u^m \nabla u}{\sqrt{u^2 + |\nabla u|^2}}\right). \tag{1.3}$$

This is quite representative of what we mean by a combination of two non-linear diffusion mechanisms, at least from the formal point of view. There are of course other ways of combining flux-saturated and porous media influences already present in [62, 67]. We will see lots of examples in the sequel. In fact the previous is just the tip of the iceberg, as both (1.1) and (1.2) lend themselves easily to generalization. The standard porous media equation is customarily generalized to the filtration equation

$$\frac{\partial u}{\partial t} = \Delta \Phi(u),$$

while flux-saturated equations come in a variety of ways, as long as they follow the vague requirement set above on the structure of the flux. This has led to a plethora of general forms or templates trying to embody large families of equations combining these two types of nonlinear diffusion. To name a few, we have

$$u_t = [u^n Q(g(u)_x)]_x \quad \text{and} \quad u_t = [\varphi(u) Q(u_x)]_x,$$

in [67, 68],

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\varphi(u)\psi(\nabla u/u)\right)$$

in [48],

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\Lambda(u)\nabla\Phi(u)}{\sqrt{1 + |\nabla\Phi(u)|^2}}\right)$$

in [62, 63] or even the very general formulation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, D\Phi(u)),$$

characterized by a Lagrangian having linear growth at infinity with respect to gradients, as introduced in [62].

All these models are interesting for a number of reasons. First, they appear from time to time in some areas of mathematical physics (they are even common currency in radiative transport theory and astrophysics, for instance) and they are starting to find interesting applications in some areas of mathematical biology as well (for instance in morphogenesis). Second, they allow us to compare both the porous media diffusion mechanism and the flux-saturated one in order to get a deeper understanding on each of them. It is not clear cut if these mechanisms will really interact in some way, and, if so, if they will cooperate and give rise to some emergent behavior. They might also be competing and as the outcome of such competition one of them could dominate the overall dynamics after a while. Which of these possibilities will show up in a given situation is not at all clear. Possibilities greatly increase if we also add reaction terms. Finally, all this is connected with the apparition of new mathematical phenomena which are interesting and quite challenging to analyze at the same time. All these aspects will be reflected in the present survey.

In fact, one of the main reasons favoring the use of these models in the aforementioned areas is the property of finite speed of propagation, which makes them stand in clear contrast with standard linear diffusion models like the heat equation. It is well known that if we start with a point source the corresponding solution of the linear diffusion equation is strictly positive at any point and for every positive time independently on the value of the diffusion coefficient. This property is known as infinite speed of propagation or instantaneous spreading and was already pointed out in the pioneering work of Einstein over a century ago [81]. He was able to deduce the linear diffusion equation from Brownian motion, but paying the price of an unexpected quality of the limit: The infinite speed of propagation, which Einstein remarked as impossible from the physical point of view. Einstein [81] points out that

> *... the mean velocity of change of the observable (...) becomes infinitely great for an infinitely small interval of time; which is evidently impossible ...*

Then, he concluded that depending on the context of applicability this property may actually pose some problems. One of the main reasons for the introduction of flux limited models was precisely to try to solve this shortcoming. This is one of the fundamental ideas that we will try to convey during the document. It will pop out as soon as we review the series of works that initiated the subject, it will manifest itself virtually in every section of this survey from there on (ranging from well-posedness to numerical simulation of those models, all along a journey through their various analytical properties), and it will be paramount in the closing section concerning applications in mathematical biology.

Indeed, the scope of the review is to give a detailed account of what is known nowadays for this class of hybrid models and to point out some of their applications. This includes in particular those flux-saturated models that do not have any porous media type term — like (1.2) — whose theory is more recent than that of the porous media equations, and not as complete. The document is essentially divided in five blocks. This introduction, together with the next section concerning examples will give a rough idea on the main features of the subject. The second block is devoted to well-posedness, which is a highly non-trivial issue from the mathematical point of view; it comprises Sections 4 to 7. Then the third main block analyzes from both qualitative and quantitative points of view various properties of these models. The finite propagation speed property, front propagation, smoothing effects and asymptotic regimes will be some of the topics to be treated in Sections 8–12. The coupling of these models with reaction terms deserves by itself a separate discussion. This will be our fourth block, constituted by Section 13. A final block presents some specific applications of the previous material.

This introduction itself is divided in various sections. We briefly review some of the properties of porous media equations and flux-limited equations before giving a first summary of the properties of hybrid models. An historical account on the subject will follow, together with some ideas concerning the applications to be discussed in the sequel.

1.1. A cavalier look into porous media equations. Let us recall that the standard porous media equation reads

$$\frac{\partial u}{\partial t} = \nu \operatorname{div}((u/\kappa)^{m-1}\nabla u), \quad \nu,\kappa > 0,\ m > 1 \quad x \in \mathbb{R}^d,\ t > 0. \tag{1.4}$$

The case $m = 1$ is just the standard diffusion equation. When $m < 1$ (1.4) is known as the fast diffusion equation. The mathematical theory for (1.4) and its generalization

$$\frac{\partial u}{\partial t} = \Delta\Phi(u) \quad (= \operatorname{div}(\Phi'(u)\nabla u)), \quad \Phi : \mathbb{R}^+ \to \mathbb{R}^+,\ \Phi' \geq 0$$

is presently quite satisfactory. We refer to [127, 128] and the very many references therein for details. Nevertheless, we want to point out a number of distinctive features of (1.4):

(1) Finite propagation speed. As stated in [127]:

> *Disturbances from the level $u = 0$ propagate in time with finite speed for solutions of the porous medium equation.*

Note that this spreading rate is not at all universal: It is data-dependent, in fact it depends on the associated initial pressure. Anyhow, every point of the domain is absorbed by the support in finite time.

(2) Presence of interfaces. Compactly supported initial data remain so during evolution, which makes the concept of free boundary (or interface) meaningful: It is the set of points separating the region in which the solution is strictly positive from that in which it is zero. Spatial sections of the support constitute a non-decreasing family (local positivity is maintained during evolution).

(3) Waiting times. Depending on the particular features on the initial datum (and more specifically of its behavior at the interface), the associated solution may not start to spread instantaneously but remain still for a while (meanwhile the solution redistributes itself inside the support) until certain conditions are met. Afterwards the solution will start to spread and will continue to do so indefinitely. This elapsed time is termed as the “waiting time” (strictly speaking, there is a waiting time for each point at the initial interface).

(4) Asymptotic behavior. Solutions to (1.4) resemble on the long time run the so-called Barenblat profiles:

$$U(t,x) = t^{-\alpha}\left(C - k|x|^2 t^{-2\beta}\right)_+^{\frac{1}{m-1}}, \quad C > 0$$

with

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \frac{\alpha}{d}, \quad k = \frac{\alpha(m-1)}{2md}.$$

These are solutions associated with a point source.

(5) Fast diffusion case. Equations (1.4) can be meaningfully considered in the range $0 < m < 1$. The behavior in this regime is more related to that of the standard heat equation. But there are additional phenomena related to loss of mass at infinity and finite time extinction, or even instantaneous extinction, see [128, 127] and references therein. The range $m < 0$ can be also considered, but properties can be even more pathological than in the previous regime (non-existence and non-uniqueness phenomena show up).

We would like to mention here the fact that another ingredient to be coupled with porous media equations (1.4) has been proposed recently in [46, 47]. It goes under the name of "fractional porous media equations", which read

$$u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u), \ 0 < s < 1.$$

The previous authors prove the existence of mass-preserving, nonnegative weak solutions satisfying energy estimates and finite propagation, C^α Hölder regularity, as well as the boundedness of nonnegative solutions with L^1 data. We won't consider this type of fractional porous media mechanism in the sequel.

1.2. What is to be expected of a flux-saturated mechanism? As opposed to the previous case, a mathematical theory for the relativistic heat equation

$$\frac{\partial u}{\partial t} = \nu \operatorname{div} \left(\frac{u \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2} |\nabla u|^2}} \right), \quad \nu, c > 0 \tag{1.5}$$

(here in dimensional form) and related models has not been available until very recently (see [11, 12] for well-posedness), and by no means our present understanding of these models is complete. Nevertheless, there are certain heuristics that we can easily grasp which are quite helpful in order to figure out what the behavior of solutions to (1.2) could be. Let us mention:

(1) Universal finite speed of propagation. As before, compactly suppoted initial data will launch solutions which are compactly supported as well. However, the spreading rate is given now by a finite, universal quantity which is usually found easily by direct inspection (it would be exactly c for the case of (1.5)).

(2) No waiting times. Solutions start to spread instantaneously, no matter their features at the interface.

(3) No regularization at the boundary. Initially discontinuous interfaces (meaning more precisely that the initial datum has a jump discontinuity across the interface) will remain so forever. Moreover, there may be some loss of regularity at the interface, specially if it is discontinuous (as the solution may develop infinite slopes at the jump site).

(4) Smoothing effects inside the support. This may not be an instantaneous effect, but we expect some kind of smoothing to take place for advanced times. In fact, (1.5) seems to behave more like a usual parabolic equation inside the support.

(5) Concavity properties. Solution profiles are expected to be concave inside the support on the long time run. Indeed, entropy conditions impose this sort of concavity properties right at the interfaces, if they are discontinuous (which triggers an instantaneous singularization of slopes there).

These features can be easily checked on numerical simulations, see those in Section 3 and in the references quoted therein. Later on we will see to what extent are they analytically demonstrated.

The asymptotic regime $\nu \to \infty$ is described by the transparent media equation. Namely, if $|\nabla u| \gg u$, then (1.5) formally converges to

$$\frac{\partial u}{\partial t} = c \operatorname{div}\left(u \frac{\nabla u}{|\nabla u|}\right). \tag{1.6}$$

Comparing this with the wave-like equation $u_t = \nu u_x$ gives a hint of the wave-like behavior of solutions in the large gradient regime. Quite the contrary, when $c \to \infty$ (1.2) formally converges to the standard heat equation.

1.3. Flux-saturated porous media equations in a nutshell. There are lots of hybrid models we may come up with. Hence their properties combine in a variety of ways those of standard porous media equations and those of flux-limited equations, also giving rise to some new properties. Let us briefly outline below what is know for this wide family of equations. Much of the content of this review is dedicated to explain in more detail the points below.

- Well-posedness can be shown generically in the class of so-called *entropy solutions*. This theory was introduced in the pioneering works [11, 12]. Just to give some hints about the subtleties of the theory, let us note that explicit solutions for (1.6) can be computed having as initial data the characteristic function of a ball [18]. These solutions[1] display front-like behavior and in fact their regularity is $u \in BV([\tau, T] \times \mathbb{R}^d)$, for any $0 < \tau < T$. Then this is in general the maximal regularity of solutions that we can expect for any kind of flux-saturated model (with or without porous media type terms). It is also seen that we cannot expect distributional solutions to be unique, as in the previous case the static solution given by the initial datum itself would qualify as distributional solution. These features lead to the necessity of introducing the concept of entropy solutions (being partially related with those introduced in [96]) for obtaining well-posedness. We will analyze these issues in detail in Sections 5, 6 and 7.
- Variable speed of propagation. Given a specific model, the spreading rate of its solutions may not be a universal feature, but depend instead on several constants of

[1]Specifically, given a ball $B_1 \in \mathbb{R}^d$, the entropy solution of (1.6) with initial datum $\alpha\chi_{B_1}$ is

$$u(t,x) = \alpha \frac{|B_1|}{|B_1 \oplus B(0,ct)|} \chi_{B_1 \oplus B(0,ct)}(x)$$

with $B(x,r)$ an open ball centered at x with radius r and $\oplus$ the Minkowsky sum. See Theorem 5.2 in [18] for details.

the model, on the initial datum and on the particular features at the interface. This property will be discussed in Section 9.

- Waiting times may show up in a number of cases. This may be related with losses of regularity (that may be either transitorial or perpetual) during evolution. This loss of regularity may show up either at the interior or at the interfaces. Part of the material in Section 10 is devoted to this topic.
- The previous phenomena may also be related with the formation and/or propagation of traveling fronts. The evolution of such fronts is controlled by a suitable form of Rankine–Hugoniot's law together with entropy conditions. We will give details on this in Section 8.
- In some cases singular interfaces are regularized during evolution. This is not known to be true in general and is connected with the previous remarks concerning waiting times and loss of regularity.
- As mentioned before, some loss of regularity at intermediate time scales is not at all to be discarded. Nevertheless, we expect smoothing effects to operate inside the support at longer time scales. To analyze this behavior is one of the goals of Section 10.
- Some asymptotic regimes of these models are well described by porous media equations. Due to the variety of cases, this has to be considered separately for each model of interest. We will discuss some results concerning porous media limits of various families of models in Section 11.

The addition of reaction terms to flux-limited models already incorporating porous media terms is meaningful in a number of situations from the modeling point of view as we explain below. Such extra terms can give rise to even more varied phenomenology. For instance, there are some cases in which solutions having eternal singularities inside the support are known to exist (see Section 13).

1.4. Some historical remarks about flux-limited models. To the best of our knowledge, flux-saturated equations — no porous media terms here yet — originated in radiative transfer theories in astrophysics, mainly due to the contributions of Levermore and coworkers [99, 101, 100] and an unpublished work by Wilson (see [106] for an account of it). The introduction of such theories is related to particular instances of closure problems and leads to formulations that are more tractable from the numerical point of view than the original set of equations for radiative transfer. Since then, flux-limited theories seem to have become part of the background in radiation transport (see [107, 109] and references therein). We will comment on such models in the sequel.

A great impetus for the use of such models was given later by Rosenau and co-workers in a series of papers [116, 114, 115, 67, 68, 97, 69]. Let us mention here:

- [116], in which they study to what extent an equation of the form

$$u_t = [G(u_x)]_x$$

(which they already term "a flux-limited diffusion process", as it is assumed that $G(\infty) < \infty$ and $G'(s) \geq 0$) is able to sustain an initially imposed sharp front –i.e. a jump discontinuity at the edge of the support. They found the following: Provided that

$$G(s) \sim G_0 - \frac{K}{s^\gamma}, \quad s \gg 1,$$

sharp fronts are not resolved inmediatly for $\gamma > 1$, but within a finite time, while such fronts are resolved immediately if $\gamma < 1$. Sharp fronts are not sustained for $\gamma = 1$, see [115].

- [115], in which the model (1.5) is introduced, as a way to produce "a sensible transport theory without trying to track the detailed hyperbolic aspects of the original process". This is obtained as a modification of the standard diffusion equation which takes into account the fact that the speed of sound gives an upper bound on the allowed propagation speed in a medium. Details on this will be given below.

  The telegraph equation (also called Cattaneo's model) [64, 85, 124]

$$u_t = \nu u_{xx} - \tau u_{tt}$$

  is also mentioned in [115] as a suitable model for predicting finite propagation speed and delayed resolution of large gradients. In 1992, Rubin [118] showed that Cattaneo's model of hyperbolic heat conduction [64] violates the second law of thermodynamics, which may make its use questionable in some particular application.

Another important and very influential contribution can be found in the work of Bertsch–dalPasso [35], who noticed that a certain class of degenerate parabolic equations (in one spatial dimension) may undergo an extreme regime in which they display hyperbolic behavior. This class of equations reads

$$u_t = (\varphi(u)\psi(u_x))_x, \quad \text{such that } \lim_{s\to\infty} \psi(s) = \psi_\infty < \infty$$

and arises in the theory of phase transitions when the free energy functional has a linear growth rate with respect to the gradient (compare with the rationale in [116, 114]). They construct solutions for the Cauchy problem which satisfy entropy-like conditions (in Oleinik's form). Constructed solutions need not be continuous and in fact when $u_x \to \infty$ they behave like solutions to $u_t = \psi_\infty(\varphi(u))_x$. They give a very detailed account of what kind of hyperbolic behavior shows up. More precise statements about finite time regularization are also given when $\varphi = 1$ (compare with [116]). There are also solutions of the Cauchy problem that do not satisfy the aforementioned entropy condition, hence such condition is necesary for uniqueness. It is also sufficient, as shown in [76]. We also mention here the approach by Blanc [38, 39], which coexisted in time with the previous set of ideas. He studied the homogeneous Neumann problem in $(0, T) \times (0, 1)$ for equations of the form $u_t = (\mathbf{a}(u, u_x))_x$, with $\mathbf{a} \in C^{1,\alpha}([0,\infty) \times \mathbb{R})$ such that $\mathbf{a}(u, 0) = 0$ and $\partial_\xi \mathbf{a}(u, \xi) < 0$ (together with some additional assumptions). He observed that there are no classical solutions of the previous problem in general, then he constructed mild solutions for it and showed their uniqueness (however he did not characterize such in distributional terms). This line of thought in [35, 76, 38, 39] is present in the general well-posedness theory that we review in Section 5 and motivated it to some extent.

As these ideas were starting to blossom, a different point of view on the subject was introduced by Brenier in [42], who connects (1.5) with optimal transport theory. Incidentally, the name "relativistic heat equation" was coined in [42], as the cost function allowing to deduce (1.5) from Jordan–Kinderlerher–Otto's framework [93] has an "obvious geometric and relativistic flavor", although the equation is not Lorentz invariant. This connection is stated at a formal level, but in can be made rigorous. Such is the idea behind [103], which puts on solid grounds this connection and extends it to a wider class of models. Then the optimal transport point of view is used as a way to construct actual solutions.

Yet there is another point of view that leads to another subclass of degenerate parabolic equations that can fairly be termed as "flux-saturated". This is given by [70] in an indirect way. Namely, they study the macroscopic limit of the one-dimensional kinetic model

$$\varepsilon \partial_t f_\varepsilon + v \partial_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon), \quad f_\varepsilon = f_\varepsilon(t,x,v) \tag{1.7}$$

as the mean free path ε is made very small. The collision kernel Q above is assumed to be linear and to satisfy a number of extra properties, see [70] for details. To solve a number of pitfalls when determining the asymptotic regime $\varepsilon \to 0$, it is suggested in [70] to replace the standard Hilbert expansion for f_ε with the following non-linear expansion

$$f_\varepsilon = \exp\left(\sum_{k\geq 0} \varepsilon^k \Phi_k\right).$$

Then it turns out that, if we truncate at first order in ε, the approximate density is well described (see Theorem 7.3 below for more details) in such asymptotic regime by solutions to an equation of the form

$$\partial_t u - \partial_x \left(\frac{u}{\varepsilon} \mathbb{G}\left(\frac{\varepsilon \partial_x u}{u} \right) \right) = 0, \tag{1.8}$$

where the specific form of the function $\mathbb{G}$ depends on the properties of the collision kernel Q. As a general rule $\mathbb{G}$ saturates to a constant value at infinity, thus (1.8) describes a flux-saturated process. For instance, if the collision kernel is given by

$$Q(f_\varepsilon) = \rho_\varepsilon - f_\varepsilon, \quad \rho_\varepsilon := \frac{1}{2}\int_{-1}^{1} f_\varepsilon(v)\, dv,$$

the asymptotic regime $\varepsilon \to 0$ can be described using the flux-saturated model proposed by Levermore and Pomraning in [101]. Lots of flux-saturated models arise in this way; we will comment on some of the models arising in this way below.

Since 2005 a series of results were made [11, 12, 19, 60] that put the foundations to treat well-posedness of virtually every flux limited model we just mentioned (not only in dimension one but in arbitrary dimension) and even strive to extend this framework to more general flux saturating porous media mixtures [62]. This provides maybe for the first time a unifying framework in order to deal with such proliferation of models from the mathematical point of view, which constitutes a very important achievement from the modeling point of view. As a matter of fact, this is groundbreaking material also from the purely mathematical point of view and will be explained in more detail below in Sections 4 and 5.

1.5. Some remarks concerning biological applications. One of the most important motivations for studying the previous models is concerned with the fact that generation and evolution of singularities (be it with or without reaction terms in the model) is involved in the dynamics of a large number of real problems. In the latter case, we have three different competing and/or collaborating mechanisms: The saturation of flux, the porous media effect and the reaction term. There are many applications in which this functional context takes on full meaning; for example, this combination of components is related to gradient formation in morphogenesis which develops propagation fronts. In this case, the problem is how to characterize the velocities and the jump structure of those fronts. In connection with this question, it is interesting to highlight an aspect that deserves a detailed study: The analysis of waiting times (namely, elapsed times prior to front propagation, during which the density of particles is reordered in order to form a propagating front that will afterwards move forward in the direction of propagation of the morphogen under study). The spectacular success of reaction-diffusion models (based on linear diffusion) motivated enormous attention from the scientific community thanks to their ability to generate patterns. Reaction–diffusion models are able to reproduce wave-like phenomena, which have been used to understand problems of competition (growth, invasion, . . .) of different individuals or substances. However, classical traveling wave solutions in linear diffusion systems coupled with reaction terms could not represent correctly the generation and evolution of fronts. In fact, the infinite tails of Gaussian-type classical traveling waves prevent the creation of fronts. To overtake this problem, cutting these tails by introducing an artificial threshold is not a solution because this "surgical" treatment modifies (as we should have expected) the dynamics of the system under consideration, as we will show in more detail in Section 14.

Still, the applicability of these models to specific biological situations is debatable. Then a fundamental issue here is to be able to connect those continuous models of interest with microscopic descriptions. See Section 2.3 on this account.

2. Towards a complete catalog of flux-saturated mechanisms and general formulations

This section is structured around a number of ways in which the "relativistic heat equation" (1.5) was historically introduced. These ideas can be generalized to produce many other flux-saturated and porous media flux-saturated models as we detail below. Specifically, we discuss first the derivation by Rosenau (amounting to an ad hoc truncation of the flux at large values). Then we move on to introduce optimal transport derivations, which have the double advantage of providing a method to construct actual solutions and to give a neat interpretation of the finite propagation speed property. The third strategy we detail is related with macroscopic limits of microscopic kinetic models. Note that this is also the idea in [70]. Nevertheless, while that line of reasoning is a way to introduce lots of models that were not present in the literature at that time, it must be pointed out that it only justifies their usage in the regime in which the mean free path ε is very small. This amounts to the fact that the maximum propagation speed would be way too large. That is, the arguments in [70] cannot be taken at all as a rigorous derivation of flux-saturated equations in the regime in which the maximum propagation speed is of order one (say), which is the one in which we are mostly interested for the sake of applications.

2.1. Rosenau's derivation. It is instructive to consider the argument in [115] leading to (1.5), as it can be generalized to justify many other flux-saturated equations (in fact the arguments to follow were already present in the rationale of those researchers in the field of radiation transport). The basis of his rationale is to write abstract diffusion models as

$$u_t = \operatorname{div}(\mathcal{F}), \tag{2.1}$$

being $\mathcal{F}$ the flux of the equation. For instance, the standard diffusion equation

$$u_t = \nu \Delta u \tag{2.2}$$

(also known as the heat equation or even as the Fokker-Planck equation) is replicated using Fourier/Fick's law

$$\mathcal{F} = -\nu \nabla u, \tag{2.3}$$

while porous media equations (1.4) are obtained using Darcy's law:

$$\mathcal{F} = -\frac{\nu}{m} \nabla u^m \quad (= -\nu u^{m-1} \nabla u). \tag{2.4}$$

As a way to fix some of the shortcomings of the standard diffusion equation (2.2), Rosenau proposed to replace the classical flux (2.3) with a flux that saturates as gradients become unbounded. In order to do so, he relates u and $\mathcal{F}$ through the velocity V defined by $\mathcal{F} = u\,V$. This amounts to write now (2.1) as a transport equation,

$$u_t = \operatorname{div}(u\,V). \tag{2.5}$$

In the case of (2.2) we have that

$$V = -\nu \frac{\nabla u}{u}, \tag{2.6}$$

which is clearly unbounded as $|\nabla u/u| \nearrow \infty$. However, V should be bounded on physical grounds by the sound speed or the light speed c. To take this into account, Rosenau modifies (2.6) as

$$\nu \frac{\nabla u}{u} = -\frac{V}{\sqrt{1 - |V|^2/c^2}}. \tag{2.7}$$

Hence

$$V = -\frac{\nu \nabla u/u}{\sqrt{1 + \left(\frac{\nu|\nabla u|}{cu}\right)^2}}, \quad \text{so that } \mathcal{F} = -\frac{\nu \nabla u}{\sqrt{1 + \left(\frac{\nu|\nabla u|}{cu}\right)^2}} \tag{2.8}$$

and we obtain (1.5) upon replacing this new flux into (2.1).

Clearly many different models can be obtained just by making variations of (2.7)–(2.8). The choice

$$\nu \frac{\nabla u}{u} = -\frac{V}{1 + V/c}, \quad \text{that is } \mathcal{F} = -\nu \frac{u \nabla u}{u + \nu|\nabla u|/c}$$

gives rise to Wilson's flux-saturated model (see [106]):

$$\frac{\partial u}{\partial t} = \nu \operatorname{div}\left(\frac{u \nabla u}{u + \frac{\nu}{c}|\nabla u|}\right).$$

Some porous media flux-saturated equations can be readily obtained if we modify Darcy's law in the same spirit of (2.7) and define V by (say)

$$\nu \nabla u^m = -\frac{V}{\sqrt{1 - |V|^2/c^2}}.$$

This gives the flux

$$\mathcal{F} = -\frac{u \nabla u^m}{\sqrt{1 + \frac{\nu^2}{c^2}|\nabla u^m|^2}}$$

and we obtain the following flux-limited porous media model ([62, 63]):

$$\frac{\partial u}{\partial t} = \nu \operatorname{div}\left(\frac{u \nabla u^m}{\sqrt{1 + \frac{\nu^2}{c^2}|\nabla u^m|^2}}\right). \tag{2.9}$$

Some models in the theory of radiation transport were indeed introduced by a similar reasoning. In fact the model proposed in [99, 101], which essentially boils down to

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|} u \left(\frac{u}{|\nabla u|} - \coth\left(\frac{|\nabla u|}{u}\right)\right)\right)$$

corresponds to a "flux-limited form of Fick's law of diffusion" [101]

$$\mathcal{F} = -\frac{\nabla u}{|\nabla u|} u \left(\frac{u}{|\nabla u|} - \coth\left(\frac{|\nabla u|}{u}\right)\right).$$

The one-dimensional version of this model is also recovered in [70] as already mentioned in the introduction.

Another example in the radiative transfer literature is the following family of models (which seems to have been first introduced by Larsen, see [109])

$$u_t = \operatorname{div}\left(\frac{|u|\nabla u}{\left(u^p + \frac{\nu^p}{c^p}|\nabla u|^p\right)^{1/p}}\right), \quad p > 1. \tag{2.10}$$

We clearly see that the flux $\mathcal{F}$ of (2.10) is bounded for large values of $|\nabla u|$.

In general, there have been many flux-limited models in the literature that have been introduced in an ad hoc fashion. We can always check that they fulfill the requirement of having a flux $\mathcal{F}$ that is bounded for large values of ∇u. Some additional models we would like to mention that were presented in such a way are

$$u_t = \left(\frac{u^n u_x}{\sqrt{1 + (u_x)^2}}\right)_x, \quad n > 0,$$

which is known as porous media curvature model, and

$$\frac{\partial u}{\partial t} = \nu \operatorname{div}\left(\frac{|u|^m \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla u|^2}}\right) \tag{2.11}$$

presented in [67], [115] and [61] respectively, being some of the first hybrid models to appear in the literature. We also have the so-called limited speed porous media equation [122]

$$u_t = \left(\frac{\nu(u - \log(1+u))u_x}{\sqrt{u^2 + \frac{\nu}{c}u_x^2}}\right)_x.$$

We would finally like to mention the so-called plasma equation [79]

$$\frac{\partial u}{\partial t} = \left(\frac{u^{5/2} u_x}{1 + u|u_x|}\right)_x,$$

which also follows the previous guidelines.

2.2. Optimal transport approach. The use of optimal mass transport problems to solve parabolic equations was pioneered by [93] and further developed by many authors, see [1, 3, 42] for instance. We give here a brief account on it. Let $k : \mathbb{R}^d \to [0,\infty]$ be a convex cost function and let us define the associated Wasserstein distance between two probability distributions ρ_0 and ρ_1 by

$$W_k^h(\rho_0, \rho_1) := \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} k\left(\frac{x-y}{h}\right) d\gamma(x,y) \Big/ \gamma \in \Gamma(\rho_0,\rho_1)\right\},$$

being $h > 0$. Here $\Gamma(\rho_0, \rho_1)$ stands for the set of probability measures in $\mathbb{R}^d \times \mathbb{R}^d$ whose marginals are ρ_0 and ρ_1.

Now let $F : [0, \infty) \to [0, \infty)$ be a convex function and let $\mathcal{P}(\mathbb{R}^d)$ be the set of probability density functions $\rho : \mathbb{R}^d \to [0, \infty)$. Starting from $\rho_0^h = \rho_0 \in \mathcal{P}(\mathbb{R}^d)$, we can solve iteratively

$$\inf_{\rho \in \mathcal{P}(\mathbb{R}^d)} h W_k^h(\rho_{n-1}^h, \rho) + \int_{\mathbb{R}^d} F(\rho(x))\, dx.$$

This is known as the Jordan–Kinderlehrer–Otto minimization scheme. Define $\rho^h(t) = \rho_n^h$ for $t \in [nh, (n+1)h)$. Then as $h \to 0^+$ the solution of this minimization scheme formally converges to a limit u which solves the following equation

$$u_t = \operatorname{div}(u \nabla k^*(\nabla F'(u))). \tag{2.12}$$

This convergence has been shown to be rigorous in certain cases [93, 1, 103].

Based on these ideas, [42] found a new way to arrive to (1.5). The idea is to regard (1.5) as the gradient flow of Boltzmann's entropy

$$F(r) = \nu(r \log r - r),$$

for the Wasserstein metric which corresponds to the following cost function

$$k(v) = \begin{cases} \left(1 - \sqrt{1 - \frac{|v|^2}{c^2}}\right) c^2 & \text{if } |v| \le c \\ +\infty & \text{if } |v| > c \end{cases}. \tag{2.13}$$

In particular, this gives some insight into the finite propagation speed property for (1.5). Let us stress that this connection, as introduced in [42], is just a *formal* argument. It was late made rigorous in [103] (we comment on this below).

Other flux-limited models can be recovered following this strategy (see [103, 48]). For instance,

- Wilson's equation can be recovered by means of the following cost function

$$k(v) = \begin{cases} -c|v| - c^2 \log\left(\frac{c-|v|}{c^2}\right) & \text{if } |v| < c \\ +\infty & \text{if } |v| \ge c \end{cases}.$$

- Similarly, the model

$$p_t = ((p/\varepsilon)\tanh(\varepsilon p_x/(\gamma p)))_x$$

(introduced in [70]) can be obtained if we use the following cost function

$$k(v) = \begin{cases} \frac{v}{\varepsilon}\operatorname{arctanh}(v\varepsilon) - \frac{1}{\varepsilon^2}\log(\cosh(\operatorname{arctanh}(v\varepsilon))) & \text{if } |v| < 1/\varepsilon \\ \log(2)/\varepsilon & \text{if } |v| = 1/\varepsilon \\ +\infty & \text{if } |v| > 1/\varepsilon \end{cases}.$$

This transport approach can be extended in order to generate some flux-limited porous media models, provided that we replace Boltzmann's entropy with a suitable functional.

Choosing

$$F(r) = \frac{r^{m+1}}{m}, \quad m > 0$$

(known in some contests as Tsallis q-entropy functional [125]) and keeping the same cost function as before, we formally arrive to the following variant of (2.9) [63]

$$\frac{\partial u}{\partial t} = \frac{m+1}{m} \operatorname{div} \left(\frac{u \nabla u^m}{\sqrt{1 + (\frac{m+1}{mc})^2 |\nabla u^m|^2}} \right). \tag{2.14}$$

This is in fact a rigorous statement when $m \geq 2$, thanks to the results in [103].

2.3. Kinetic derivations. Another aspect of interest in the study of these models is the connection between macroscopic equations and the dynamics at the microscopic scale, i.e. between individual-based models and the continuum flux-saturated approach. Can continuum models be derived from the underlying description at the microscale? Although this subject is yet to be developed, there are different approaches to the derivation of the flux saturated terms in connection with the classical kinetic theory: Through non-linear Hilbert expansions [70] as we have mentioned before, or from the kinetic theory of multicellular growing system [24], which is based on low (parabolic) and high (hyperbolic) field limits [112, 108, 86]. Models based on the kinetic theory of multicellular growing system includes the specific knowledge of interactions that modify the biological state without generating proliferation or destruction phenomena (originating cooperation, aggregation, reproduction, velocity-jump processes, gain and loss of individuals with specific biological properties due to conservative encounters, ...) as well as a complete description at the micro-scale level. As it has been suggested in [25, 26], flux-saturated mechanisms may have the capacity of reproducing some of the emerging behaviors that happen only at the level of collective phenomena (i.e. involving all the interacting individuals but not being directly related to the dynamics of a few entities). This is also a very interesting feature of flux saturated mechanism that we will develop in Section 14.

2.4. A word on general templates. Well-posedness theory for (porous media) flux-limited equations did not proceed by analyzing each model of interest separately. Quite the contrary, general classes of degenerate parabolic equations were proposed and their well-posedness analyzed in one single stroke. First, Andreu, Caselles and Mazón [12] dealt with well-posedness of equations

$$u_t = \operatorname{div} \mathbf{a}(u, Du) \tag{2.15}$$

such that certain degeneracy and growth conditions are satisfied. Afterwards in [62] that theory has been readapted in order to cover models of the form

$$u_t = \operatorname{div} \mathbf{a}(u, D\Phi(u)) \tag{2.16}$$

under very general assumptions on Φ. We will ellaborate on this in Sections 5 and 6.

It is important to point out that, while the previous very general classes of equations (2.15)–(2.16) provide arguably the largest framework in which those techniques by Andreu, Caselles and Mazón grant well-posedness, these classes are too general to be able to derive qualitative properties for solutions of such. Hence a number of sub-classes have been proposed in the literature with the aim of being able to provide more complete descriptions of the solutions (and completely on the other end, (1.5) alone has been the subject of some very specific mathematical investigations regarding its qualitative behavior, probably to a greater extent than any other model among those listed before). Let us enumerate some of them below.

a) Rosenau and his various collaborators have dealt with a number of families of flux-saturated equations. Let us mention here that in [67],

$$u_t = [u^n Q(g(u)_x)]_x, \quad n > 0, \tag{2.17}$$

being g a smooth function and Q a bounded function with $Q(0) = 0$ and $Q' > 0$. Previous studies [116] were conducted on

$$u_t = [Q(u_x)]_x$$

with $G(0) = 0$, $G(\infty) < \infty$, $G(-s) = -G(s)\ \forall s$ and $G' \geq 0$.

b) Bertsch and dal Passo introduced in [35] the following very general family of equations in dimension one:

$$u_t = [\varphi(u)\psi(u_x)]_x. \tag{2.18}$$

Here $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a smooth, strictly positive function and $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth, odd function which is strictly increasing and satisfies $\lim_{s\to\infty} \psi(s) = \psi_\infty < \infty$. Some extra technical assumptions are also needed; there is no specific attempt in [35] aimed at getting minimal assumptions.

c) [62] deals with various families of flux-saturated equations, among which we mention here

$$\frac{\partial u}{\partial t} = \alpha \operatorname{div}\left(\frac{\Lambda(u)\nabla\Phi(u)}{\sqrt{1+\beta|\nabla\Phi(u)|^2}}\right), \quad \alpha, \beta > 0, \tag{2.19}$$

which requires the following

Assumptions 2.1. *Let the functions* $\Phi, \Lambda : [0,\infty) \to [0,\infty)$ *be continuous and satisfy*

- $\Phi(0) = \Lambda(0) = 0$,
- Φ *is strictly increasing,*
- $\Phi, \Phi^{-1} \in W^{1,\infty}_{loc}([0,\infty))$,
- $\Lambda(z) > 0\ \forall z > 0$,
- $\Lambda(z) = \tilde{\Lambda}(z^m)$ *with* $m \geq 1$, $\tilde{\Lambda}(z) \geq c_0 z\ \forall z \geq 0$ *for some* $c_0 > 0$ *and* $\tilde{\Lambda} \in W^{1,\infty}_{loc}([0,\infty))$.

d) Another general family of models was presented in [48], which reads

$$\frac{\partial u}{\partial t} = \mathrm{div}\,(\varphi(u)\psi(\nabla u/u)) \tag{2.20}$$

under the following

Assumptions 2.2. *Let* $\psi = (\psi^{(1)}, \dots, \psi^{(d)}) : \mathbb{R}^d \to \mathbb{R}^d$ *enjoy the following properties*

(a) $\psi \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$.

(b) $\psi(0) = 0$.

(c) $\lim_{|r|\to\infty} |\psi(r) - r/|r|| = 0$. *Thus* $\lim_{|r|\to\infty} |\psi(r)| = 1$.

(d) *If* $d = 1$ *it is required that*

  (i) ψ *be odd, monotonically increasing and verifying* $|\psi'(r)| = O(1/|r|)$ *for* $|r| \gg 1$,

  while the following properties are required for dimension greater than one:

  (i) ψ *is a conservative vector field,*

  (ii) $\psi(-r) = -\psi(r) \quad \forall r \in \mathbb{R}^d$,

  (iii) *The Jacobian matrix of* ψ, $D\psi$, *is a non-negative definite (symmetric) matrix.*

  (iv) $\|D\psi\|_\infty(r) = O(1/|r|)$ *for* $|r| \gg 1$.

and

Assumptions 2.3. *Let* $\varphi : \mathbb{R} \mapsto \mathbb{R}_0^+$ *satisfy the following:*

(a) φ *is Lipschitz continuous.*

(b) $\varphi(0) = 0$ *and* $\lim_{z\to 0} \varphi(z)/|z| = \varphi'(0)$ *exists and is finite.*

(c) $\varphi(z) > 0$ *if* $z \neq 0$.

e) We will show in this document that the previous structure can be easily generalized to

$$\frac{\partial u}{\partial t} = \mathrm{div}\,(\varphi(u)\psi(\theta(u)\nabla u)) \tag{2.21}$$

while keeping many of its properties. Here ψ, φ satisfy the previous set of assumptions, while θ verifies

Assumptions 2.4. *Let* $\theta : \mathbb{R} \mapsto \mathbb{R}_0^+$ *be such that:*

(a) $\theta \in W^{1,\infty}_{loc}(\mathbb{R}\backslash\{0\})$,

(b) $\theta(z) > 0$ *for* $z \neq 0$.

(c) *The limit* $\lim_{z\to 0} \frac{\varphi(z)\theta'(z)}{\theta(z)}$ *exists and is finite.*

Note that this set of assumptions is authomatically satisfied for any power law. Clearly (2.20) is recovered for $\theta(u) = 1/u$. The idea here is the same as the rationale given in [48] to introduce the class (2.20): Here we declare gradients to be large in order to impose saturation effects if $\theta(u)|\nabla u| \gg 1$. The particular features of the problem of interest will determine what the appropriate scale (or gauge) θ would be; what we have in mind is essentially the case of power laws.

This list is by no means exhaustive (see for instance [122]). Some interconnections between these general families of models we have presented are easily found:

- Taking $g(u) = u$ in (2.17) we fall under the scope of (2.18).
- It is never possible to recast (2.20) as (2.18).
- If $g(u) = \log u$ then (2.17) stands as a sub-case of (2.20).
- (2.19) and (2.17) are connected by the identifications

$$g(u) = \Phi(u), \quad \Lambda(u) = u^n, \quad Q(s) = \frac{s}{\sqrt{1 + \beta s^2}}$$

  whenever they make sense.
- Again, it is never possible to recast (2.20) as (2.19). This would amount to have $\Phi(u) = \log u$, but this is clearly forbidden by Assumptions 2.1.
- The family (2.18) is contained in (2.21) for $\theta \equiv 1$.
- The class (2.17) is formally a subclass (and then (2.19) as well) of (2.21), as we see taking $\varphi(u) = u^n$, $\theta(u) = g'(u)$.

3. What kind of phenomena could be expected from flux-saturated mechanisms? Numerical approach to the analytic difficulties

As mentioned in the introduction, the variety and richness of phenomena that flux-saturated equations in combination with mechanisms associated with porous media originate (with or without reaction terms), is a source of mathematical problems, partially motivated by experimental evidence in some applications.

The key word in this type of systems is "singularities" or "interfaces", whose study concerns the following aspects: Their appearance or disappearance in the dynamics associated with these equations; their evolution in relation to their speed of propagation; properties of regularization or emergent fronts inside the support; waiting time for the evolution of the boundary of solutions support; existence of special solutions (steady states, traveling waves) or sub and super solutions that control the above properties; construction of singular traveling waves; asymptotic behavior and stability of solutions with respect to those of classical models.

Before beginning with the analysis of these problems, it is interesting that we acquire some intuition about the kind of phenomena with which we are confronted by some numerical examples to show the variety and difficulty of these phenomena. Then the purpose of the present section is to display some numerical simulations in order to support a number of statements in the text, as well as getting an additional insight into the nature of those phenomena under study. Note that the above mentioned characteristics and properties of the solutions of these equations show that they combine diverse parabolic and hyperbolic effects and this is reflected in the type of techniques needed for their study. In particular, the numerical approach requires to use a number of techniques in the field of conservation laws in order to preserve any singularities that solutions may display.

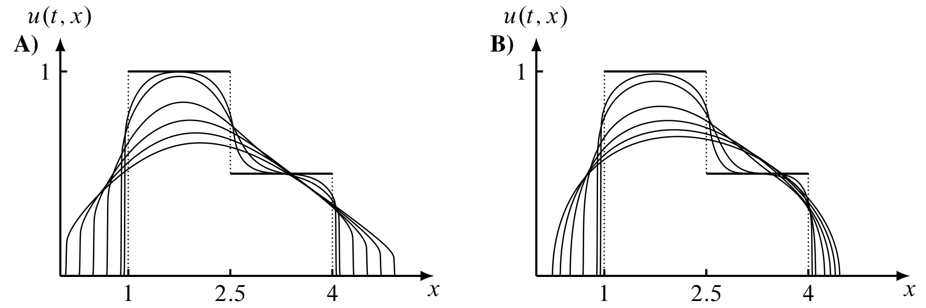

Figure 1. Numerical evolution of a stepwise initial condition by the relativistic equation (1.5) in A), and the flux-saturated porous media equation (2.9) with $m = 2$ in B). In both examples we have taken $\nu = c = 1$ and $t \in [0, 0.9]$ (the smaller the height, the more advanced the times). The time step between different profiles is 0.05 for the three first profiles and 0.2 for the rest in order to capture the different velocities. Note the instantaneous regularization at the interior jumps although this is not the case at the boundary. We remark the persistence of the discontinuous jumps at the boundary in A), which disappear in B). It also interesting to observe that velocity of support is constant in A) against the fact that it decreases in B).

In fact, we will display a number of simulations not just here but rather throughout the text. We focus in this section (Figures 1, 2 and 3) in both pure flux-saturated equations and some porous media variants; later on we will show some simulations where coupling with a reaction term or even with a system of ODE's is featured. Regardless of these characteristics, the set-up used for those numerical simulations that are presented in the document is always the same. The numerical solution considers a spatial discretization of the flux-saturated transport equation by using a fifth-order finite difference WENO (Weighted Essentially Non-Oscillatory) scheme [92] with Lax–Friedrichs flux splitting [84]. For the time evolution we use a fourth order Runge–Kutta method, which also allows to deal with possible delay phenomena. A spatial grid between 1000 and 2000 points with an appropriate CFL condition is considered.

The previous procedure is but one among a number of different possibilities. There are in the literature different numerical approaches to flux-saturated equations, among which we mention [20, 129, 52, 50, 57, 67, 68, 102, 122].

4. Mathematical preliminaries: The bounded variation scenario

In this section we introduce a number of tools that are needed to set up the well-posedness framework in [11, 12] and some of its extensions. As pointed out in the introduction and in the previous numerical examples, in general we may not expect solutions of flux-saturated equations to be more regular than $u \in BV((0, T] \times \mathbb{R}^d)$. This makes operations like integration by parts already involved. But in fact this may be even worse. First, there is no particular reason why u_t should even be a Radon measure, which creates a number of

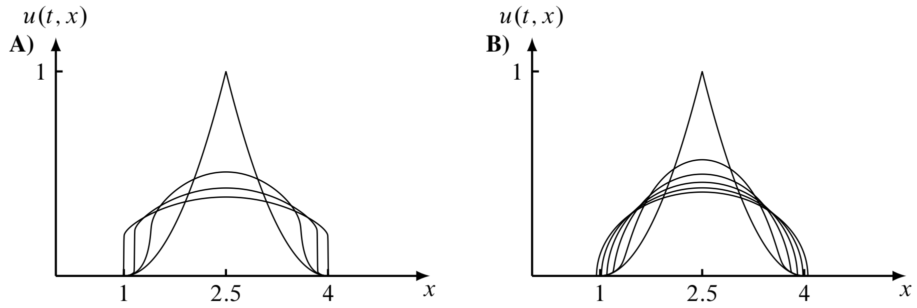

Figure 2. Numerical evolution of an initial condition given by a continuous polynomial spline. Plot A) depicts the evolution by (2.11), while plot B) shows the evolution by (2.9). In both cases $m = 2$ and $\nu = c = 1$. The time step between successive profiles is 0.48. In both cases we find a waiting time for support spreading which is longer for the case of (2.9). Note that the cusp is regularized in both cases but a discontinuity occurs in the derivative of the solution when sliding through the initial profile. The result is a continuous profile for (2.9), which reduces its velocity, and the emergence of a jump discontinuity in the case of (2.11), which is a moving front in which the velocity of propagation depend on the parameters of the system via a Rankine–Hugoniot-type condition. Therefore, in both cases there is a smoothing process but, simultaneously, another one of singularization emerges either in the derivative in B) or as a jump in A).

technical issues (we will treat this in more detail in Section 10.2). And second, the degeneracy of the equation may spoil the previous spatial regularity on the zeroth level set. Then it will be mandatory to avoid this set when dealing with certain delicate technical issues related with well-posedness. This motivates the introduction of a specific set of truncation functions that will be essential in order to construct the functional framework in which well-posedness can be proved. Another set of specific truncation functions will be required for the sole purpose of showing uniqueness, as the extremely low regularity of solutions requires to use Kruzkov's doubling variables methodology. This proof uses very complicated combinations of terms involving functions with extremely low regularity. A very specific functional calculus needs to be defined in order to make sense of the previous. In particular, lower semicontinuity results for energy functionals in this degenerate framework will be needed.

The required toolkit to cope with the above technical issues will be introduced in this section. Its contents are extracted from [11, 12, 17, 13, 56, 62].

Let us take the opportunity to state here some generic purpose notations. Throughout the document $B(x, r)$ denotes an open ball centered at x with radius r. Let us denote by $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$ the d-dimensional Lebesgue measure and the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$ respectively. Given an open set Ω in $\mathbb{R}^d$ we denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. The space of continuous functions in Ω will be denoted by $\mathcal{C}(\Omega)$ (resp. $\mathcal{C}_c(\Omega)$ for continuous functions with compact support in Ω). Likewise, $\mathcal{C}^{k,\alpha}$ denotes the class of k-times differentiable functions whose kth derivatives are Hölder-continuous with exponent α. Lebesgue and

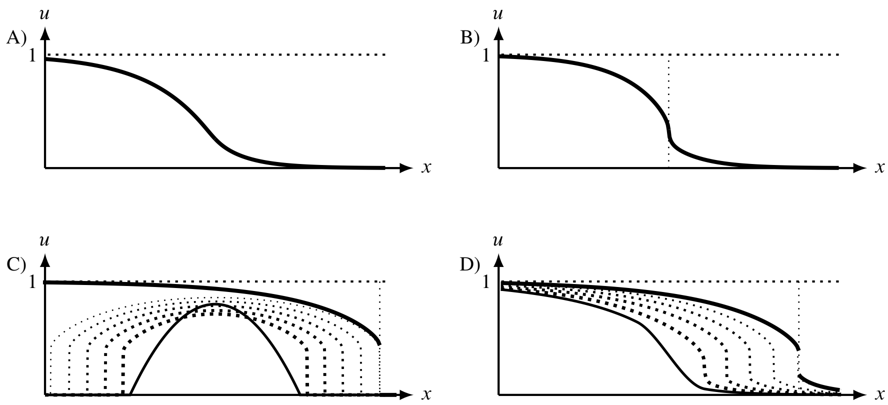

Figure 3. The above figures represent different traveling waves associated to (2.11) with a Fisher–Kolmogorov–Petrovski–Piskunov reaction term, i.e. a term of the form $F(u) = ku(1-u)$. Vertical dotted lines show points with infinite slope. The possible development of each of these traveling waves depends on the initial conditions and on the parameters associated with the system: The degree of porosity m, the characteristic velocity of the system c and the factor k of the reaction term, known as the intrinsic growth rate of the population. A) represent a classical regular traveling wave, while B) has a discontinuity in the derivative with infinite slope. In C) we give a discontinuous traveling wave together with the evolution of a parabola-type initial condition towards this traveling wave which is an attractor for this initial data. In D) we show a traveling wave with a jump discontinuity but instead of connecting with zero as in C) it has a Gaussian tail like in the classical case A). We also prove numerically that this solution is an attractor and how the continuous initial datum is singularized during evolution.

Sobolev spaces are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$ respectively. Sometimes we will use $D(B)$ to denote the domain of an operator B and $R(B)$ to denote its rank. Note that I stands for the identity operator. We will use from time to time the positive part function, defined as $u^+ = \max\{u, 0\}$. The Minkowsky sum of two sets is indicated by $\oplus$.

4.1. A short introduction to functions of bounded variation. Recall that if Ω is an open subset of $\mathbb{R}^d$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du in Ω is

$$\sup\left\{\int_\Omega u \,\mathrm{div}\sigma \, dx \big/ \sigma \in \mathcal{D}(\Omega),\ |\sigma(x)| \le 1\ \forall x \in \Omega\right\}$$

and is denoted by $|Du|(\Omega)$. We will say that $u \in BV_{loc}(\Omega)$ if the previous supremum is finite for $\sigma \in \mathcal{D}(O)$, for each open set $O \subset\subset \Omega$.

Given $u \in BV(\Omega)$, the vector measure Du decomposes into its absolutely continuous and singular parts $Du = D^{ac}u + D^s u$. Then $D^{ac}u = \nabla u\, \mathcal{L}^d$, where ∇u is the Radon–Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^d$. We also split $D^s u$ in two parts: The *jump* part $D^j u$ and the *Cantor* part $D^c u$.

We say that u is approximately continuous at $x \in \Omega$ if there exists $\tilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{\rho \searrow 0} \frac{1}{\mathcal{L}^d(B(x,\rho))} \int_{B(x,\rho)} |u(y) - \tilde{u}(x)|\, dy = 0,$$

the value $\tilde{u}(x)$ being the *approximate limit* of u at x. We denote by S_u the set of all $x \in \Omega$ such that u is not approximately continuous at x. We say that $x \in \Omega$ is an approximate jump point of u if there exist $u^+(x) \neq u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in \mathbb{S}^{d-1}$ such that

$$\lim_{\rho \searrow 0} \frac{1}{\mathcal{L}(B_\rho^+(x,\nu_u(x)))} \int_{B_\rho^+(x,\nu_u(x))} |u(y) - u^+(x)|\, dy = 0$$

and

$$\lim_{\rho \searrow 0} \frac{1}{\mathcal{L}(B_\rho^-(x,\nu_u(x)))} \int_{B_\rho^-(x,\nu_u(x))} |u(y) - u^-(x)|\, dy = 0,$$

where

$$B_\rho^+(x,\nu_u(x)) = \{y \in B(x,\rho) / (y-x)\cdot \nu_u(x) > 0\}$$

and

$$B_\rho^-(x,\nu_u(x)) = \{y \in B(x,\rho) / (y-x)\cdot \nu_u(x) < 0\}.$$

We denote by J_u the set of approximate jump points of u. It is a Borel subset of S_u such that $\mathcal{H}^{d-1}(S_u \backslash J_u) = 0$ and

$$D^j u = D^s u \llcorner J_u, \quad D^c u = D^s u \llcorner (\Omega \backslash S_u)$$

holds. It is well known (see for instance [2]) that

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{d-1} \llcorner J_u,$$

with $\nu_u(x) = \frac{Du}{|Du|}(x)$, being $\frac{Du}{|Du|}$ the Radon–Nikodym derivative of Du with respect to its total variation $|Du|$.

We also introduce some specific notations (see [61]) for functions of bounded variation of time and space that will be useful to discuss several properties of entropy solutions to flux-saturated equations:

(1) By $L^1_w(0,T,BV(\mathbb{R}^d))$ (resp. $L^1_{loc,w}(0,T,BV(\mathbb{R}^d))$) we denote the space of weakly-* measurable functions $w : [0,T] \to BV(\mathbb{R}^d)$ (that is $t \in [0,T] \to \langle w(t), \phi\rangle$ is measurable for every ϕ in the predual of $BV(\mathbb{R}^d)$) such that $\int_0^T \|w(t)\|\, dt < \infty$ (resp. $t \in [0,T] \to \|w(t)\|$ is in $L^1_{loc}(0,T)$). Observe that, since $BV(\mathbb{R}^d)$ has a separable predual (see [2]), it follows easily that the map $t \in [0,T] \to \|w(t)\|$ is measurable.

(2) The following notions are useful in connection with Rankine–Hugoniot relations in Section 8 below. Assume that $u \in BV_{loc}((0,T)\times\mathbb{R}^d)$. Let $\nu := \nu_u = (\nu_t, \nu_x)$ be the unit normal to the jump set of u and $\nu^{J_{u(t)}}$ the unit normal to the jump set of $u(t)$. We write $[u](t,x) := u^+(t,x) - u^-(t,x)$ for the jump of u at $(t,x) \in J_u$ and $[u(t)](x) := u(t)^+(x) - u(t)^-(x)$ for the jump of $u(t)$ at the point $x \in J_{u(t)}$.

For more details on the subject we refer the reader to [2, 82, 133].

4.2. Several classes of test functions. We will use in the sequel a number of different truncation functions. A first reason for so doing can be easily grasped from the following a priori estimate on solutions to (1.5):

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |\nabla u^2|\, dxdt \leq (T+1)\int_{\mathbb{R}^d} u_0^2\, dx.$$

This is obtained thanks to the crucial requirement (5.2) below and is essentially all the spatial regularity we may hope to get generically. Using the chain rule we can expect solutions to be of bounded variation as long as we stay above zero. This observation plagues in one way or another all the theory to follow.

For $a < b$ and $l \in \mathbb{R}$, let $T_{a,b}(r) := \max\{\min\{b, r\}, a\}$, $T^l_{a,b} = T_{a,b} - l$. We denote [11, 12, 17]

$$\begin{aligned} \mathcal{T}_r &:= \{T_{a,b} \ : \ 0 < a < b\}, \\ \mathcal{T}^+ &:= \{T^l_{a,b} \ : \ 0 < a < b,\, l \in \mathbb{R},\, T^l_{a,b} \geq 0\}, \\ \mathcal{T}^- &:= \{T^l_{a,b} \ : \ 0 < a < b,\, l \in \mathbb{R},\, T^l_{a,b} \leq 0\}. \end{aligned}$$

Given any function w and $a, b \in \mathbb{R}$ we shall use the notation $\{w \geq a\} = \{x \in \mathbb{R}^d : w(x) \geq a\}$, $\{a \leq w \leq b\} = \{x \in \mathbb{R}^d : a \leq w(x) \leq b\}$, and similarly for the sets $\{w > a\}$, $\{w \leq a\}$, $\{w < a\}$, etc.

We need to consider the following function space

$$TBV_{\mathrm{r}}^+(\mathbb{R}^d) := \left\{ w \in L^1(\mathbb{R}^d)^+ \ : \ T_{a,b}(w) - a \in BV(\mathbb{R}^d),\ \ \forall\, T_{a,b} \in \mathcal{T}_r \right\}.$$

It is closely related to the space $GBV(\mathbb{R}^d)$ of generalized functions of bounded variation introduced by Di Giorgi and Ambrosio (see [2] for instance). The main reason for introducing this space is the following: Using the chain rule for BV-functions (see for instance [2]), one can give a sense to ∇u for a function $u \in TBV^+(\mathbb{R}^d)$ as the unique function v which satisfies

$$\nabla T_{a,b}(u) = v \chi_{\{a<u<b\}} \quad \mathcal{L}^d - a.e.,\ \ \forall\, T_{a,b} \in \mathcal{T}_r.$$

We refer to [27] for details.

Let us denote by $\mathcal{P}$ the set of Lipschitz continuous functions $p : [0, +\infty) \to \mathbb{R}$ satisfying $p'(s) = 0$ for s large enough. We write $\mathcal{P}^+ := \{p \in \mathcal{P} \ : \ p \geq 0\}$.

Due to several reasons a slightly more general class of test functions needs to be considered. This class was first introduced in [60] (see also [61, 62]). The aim for this was essentially twofold: To clarify the relation of the notions of sub- and super-solutions with the concept of entropy solution and to clarify the meaning of entropy conditions on jump sets (which was one of the achievements in [61], see Section 8).

We define $\mathcal{TSUB}$ as the class of functions $S, T \in \mathcal{P}$ such that

$$S \geq 0, \quad S' \geq 0 \quad \text{and} \quad T \geq 0, \quad T' \geq 0$$

and $p(r) = \tilde{p}(T_{a,b}(r))$ for some $0 < a < b$, being $\tilde{p}$ differentiable in a neighborhood of $[a,b]$ and $p = S, T$. Similarly, we introduce $\mathcal{TSUPER}$ as the class of functions $S, T \in \mathcal{P}$ such that

$$S \le 0, \quad S' \ge 0 \quad \text{and} \quad T \ge 0, \quad T' \le 0$$

and $p(r) = \tilde{p}(T_{a,b}(r))$ for some $0 < a < b$, being $\tilde{p}$ differentiable in a neighborhood of $[a,b]$ and $p = S, T$.

4.3. Weak traces, Anzellotti pairings and an integration by parts formula. In order to give a meaning to integrals of bounded vector fields with integrable divergence with respect to the gradient of a function of bounded variation several results by Anzellotti will be introduced. Assume that Ω is an open bounded set of $\mathbb{R}^d$ with Lipschitz continuous boundary. Let $p \ge 1$ and p' its dual exponent. Following [22], let us denote

$$X_p(\mathbb{R}^d) = \{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^d) : \mathrm{div}(\mathbf{z}) \in L^p(\mathbb{R}^d)\}.$$

If $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, we define the functional $(\mathbf{z} \cdot Dw) : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z} \cdot Dw), \varphi \rangle := -\int_\Omega w\, \varphi\, \mathrm{div}(\mathbf{z})\, dx - \int_\Omega w\, \mathbf{z} \cdot \nabla\varphi\, dx.$$

Then $(\mathbf{z} \cdot Dw)$ is a Radon measure in Ω [22], and

$$\int_\Omega (\mathbf{z} \cdot Dw) = \int_\Omega \mathbf{z} \cdot \nabla w\, dx, \qquad \forall\, w \in W^{1,1}(\Omega) \cap L^\infty(\Omega).$$

Moreover, $(\mathbf{z} \cdot Dw)$ is absolutely continuous with respect to $|Dw|$ [22].

Let us denote by $(\mathbf{z} \cdot Dw)^{ac}$, $(\mathbf{z} \cdot Dw)^s$ the absolutely continuous and singular parts of $(\mathbf{z} \cdot Dw)$ with respect to $\mathcal{L}^d$. One has that $(\mathbf{z} \cdot Dw)^s$ is absolutely continuous with respect to $D^s w$ and $(\mathbf{z} \cdot Dw)^{ac} = \mathbf{z} \cdot \nabla w$.

The weak trace on $\partial\Omega$ of the normal component of $\mathbf{z} \in X_p(\Omega)$ is defined in [22]. More precisely, let us denote by $\nu^\Omega(x)$ the normal vector at x which points outwards. Then it is proved that there exists a linear operator $\gamma : X_p(\Omega) \to L^\infty(\partial\Omega)$ such that $\|\gamma(\mathbf{z})\|_\infty \le \|\mathbf{z}\|_\infty$ and $\gamma(\mathbf{z})(x) = \mathbf{z}(x) \cdot \nu^\Omega(x)$ for all $x \in \partial\Omega$, provided that $\mathbf{z} \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^d)$. We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z} \cdot \nu^\Omega](x)$. Moreover, the following Green's formula, relating the function $[\mathbf{z} \cdot \nu^\Omega]$ and the measure $(\mathbf{z} \cdot Dw)$, for $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, is proved

$$\int_\Omega w\, \mathrm{div}(\mathbf{z})\, dx + \int_\Omega (\mathbf{z} \cdot Dw) = \int_{\partial\Omega} [\mathbf{z} \cdot \nu^\Omega] w\, d\mathcal{H}^{d-1}.$$

Similar results hold true for $\Omega = \mathbb{R}^d$. In this case we have the following Green's formula

$$\int_{\mathbb{R}^d} w\, \mathrm{div}(\mathbf{z})\, dx + \int_{\mathbb{R}^d} (\mathbf{z} \cdot Dw) = 0 \quad \forall \mathbf{z} \in X_1(\mathbb{R}^d),\ \forall w \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).$$

There is also a related integration by parts formula for vector fields $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^d)$ such that $\mathrm{div}\, \mathbf{z} \in \mathcal{M}(\Omega)$. See [61], Sec. 5 for details; see also [66].

4.4. Lower semicontinuity results for energy functionals in BV. In order to define the notion of entropy solutions for flux-saturated equations and give suitable characterizations for them, a functional calculus defined on functions whose truncations are in BV is required. For that we need to introduce some functionals defined on functions of bounded variation [11, 12].

Let Ω be an open subset of $\mathbb{R}^d$. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0, \infty)$ be a Borel function such that

$$C(x)|\xi| - D(x) \leq g(x, z, \xi) \leq M'(x) + M|\xi| \tag{4.1}$$

for any $(x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^d$, $|z| \leq R$, and any $R > 0$, where M is a positive constant and $C, D, M' \geq 0$ are bounded Borel functions which may depend on R. Assume that $C, D, M' \in L^1(\Omega)$. Following Dal Maso [75] we consider the following functional:

$$\mathcal{R}_g(u) := \int_\Omega g(x, u(x), \nabla u(x))\, dx + \int_\Omega g^0\left(x, \tilde{u}(x), \frac{Du}{|Du|}(x)\right) d|D^c u| \\ + \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} g^0(x, s, \nu_u(x))\, ds \right) d\mathcal{H}^{d-1}(x),$$

for $u \in BV(\Omega) \cap L^\infty(\Omega)$. The recession function g^0 of g is defined by

$$g^0(x, z, \xi) = \lim_{t \to 0^+} t\, g\left(x, z, \frac{\xi}{t}\right). \tag{4.2}$$

It is convex and homogeneous of degree 1 in ξ.

In case that Ω is a bounded set, and under standard continuity and coercivity assumptions, Dal Maso proved in [75] that $\mathcal{R}_g(u)$ is L^1-lower semi-continuous for $u \in BV(\Omega)$. A very general result about the L^1-lower semi-continuity of $\mathcal{R}_g$ in $BV(\mathbb{R}^d)$ can be found on [77].

Assume now that $g : \mathbb{R} \times \mathbb{R}^d \to [0, \infty)$ is a Borel function such that

$$C|\xi| - D \leq g(z, \xi) \leq M(1 + |\xi|) \qquad \forall (z, \xi) \in \mathbb{R}^d,\ |z| \leq R, \tag{4.3}$$

for any $R > 0$ and for some constants $C, D, M \geq 0$ which may depend on R. Assume also that

$$\chi_{\{u \leq a\}} \left(g(u(x), 0) - g(a, 0)\right), \chi_{\{u \geq b\}} \left(g(u(x), 0) - g(b, 0)\right) \in L^1(\mathbb{R}^d),$$

for any $u \in L^1(\mathbb{R}^d)^+$. Let $u \in TBV_r^+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $T = T_{a,b} - l \in \mathcal{T}^+$. For each $\phi \in C_c(\mathbb{R}^d)$, $\phi \geq 0$, we define the Radon measure $g(u, DT(u))$ by

$$\langle g(u, DT(u)), \phi \rangle := \mathcal{R}_{\phi g}(T_{a,b}(u)) + \int_{\{u \leq a\}} \phi(x) \left(g(u(x), 0) - g(a, 0)\right) dx \\ + \int_{\{u \geq b\}} \phi(x) \left(g(u(x), 0) - g(b, 0)\right) dx. \tag{4.4}$$

If $\phi \in C_c(\mathbb{R}^d)$, we write $\phi = \phi^+ - \phi^-$ with $\phi^+ = \max\{\phi, 0\}$, $\phi^- = -\min\{\phi, 0\}$, and we define $\langle g(u, DT(u)), \phi \rangle := \langle g(u, DT(u)), \phi^+ \rangle - \langle g(u, DT(u)), \phi^- \rangle$.

Note that the following is shown in [77]: If $g(z,\xi)$ is continuous in (z,ξ), convex in ξ for any $z \in \mathbb{R}$, and $\phi \in \mathcal{C}^1(\mathbb{R}^d)^+$ has compact support, then $\langle g(u, DT(u)), \phi\rangle$ is lower semi-continuous in $TBV^+(\mathbb{R}^d)$ with respect to the $L^1(\mathbb{R}^d)$-convergence.

4.5. Functional calculus. Using the tools introduced in Sections 4.2 and 4.4 we can define the required functional calculus (see [11, 12, 61]). Let $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$. We assume that $u \in TBV_{\mathrm{r}}^+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and

$$\chi_{\{u\le a\}} S(u)\,(g(u(x),0) - g(a,0))\,, \chi_{\{u\ge b\}} S(u)\,(g(u(x),0) - g(b,0)) \in L^1(\mathbb{R}^d).$$

Then we define $g_S(u, DT(u))$ as the Radon measure given by (4.4) with $g_S(z,\xi) = S(z)g(z,\xi)$.

Most of the times the former definition is applied in connection with a couple of specific objets related with equations of the form $u_t = \operatorname{div}(\mathbf{a}(u, Du))$:

(1) Let us introduce $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^+$ such that $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$. We define $f_S(u, DT(u))$ as the Radon measure given by (4.4) with $f_S(z,\xi) = S(z)f(z,\xi)$.

(2) Let us introduce $h : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ defined by

$$h(z,\xi) := \mathbf{a}(z,\xi)\xi. \tag{4.5}$$

We define $h_S(u, DT(u))$ as the Radon measure given by (4.4) with $h_S(z,\xi) = S(z)h(z,\xi)$.

Suitable assumptions on $\mathbf{a}(z,\xi)$ for the previous to make sense will be given in Section 5.1 below.

5. Well-posedness: Entropic solutions

This section is committed to show the well-posedness of the Cauchy problem

$$u_t = \operatorname{div}(\mathbf{a}(u, Du)), \quad u(0,x) = u_0 \tag{5.1}$$

in the context of entropic solutions (to be defined in the sequel). This theory was stablished in [11, 12] and can be regarded as a continuation of previous works by the same authors [5, 6, 7, 8, 9, 10].

5.1. Assumptions on the flux. We present now what requirements on the flux vector field $\mathbf{a}(z,\xi)$ are needed to develop a well-posedness theory in the class of *entropy solutions*. The following list of assumptions is essentially the one stated in [12], with slight improvements by [62].

(1) There exists some continuous $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^+$, convex and differentiable in its last variable, such that

$$\nabla_\xi f \in \mathcal{C}(\mathbb{R}\times\mathbb{R}^d)$$

and

$$\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi).$$

Note that f can be chosen so that $f(z,0) = 0$.

(2) The following linear growth condition is satisfied:

$$C_0(z)|\xi| - D_0(z) \le f(z,\xi) \le M_0(z)(1+|\xi|) \qquad \forall (z,\xi) \in \mathbb{R}\times\mathbb{R}^d, \tag{5.2}$$

for some continuous functions $C_0, C_1, D_0 \ge 0$, such that $C_0(z) > 0$ for $z \neq 0$ and $D_0(z)$ is locally bounded in $\mathbb{R}\backslash\{0\}$.

(3) We assume $\mathbf{a}(z,0) = 0$ for all $z \in \mathbb{R}^d$.

(4) It is assumed that

$$\mathbf{a}(z,\xi) = z\mathbf{b}(z,\xi) \quad \text{with } |\mathbf{b}(z,\xi)| \le M_1(z), \forall \xi \in \mathbb{R}^d$$

for some continuous $M_1 \ge 0$.

(5) It is also assumed that

$$\frac{\partial \mathbf{a}}{\partial \xi}(z,\xi) \in \mathcal{C}(\mathbb{R}\times\mathbb{R}^d) \quad \text{for any } i = 1,\dots,d.$$

This condition is not always strictly necessary. For instance, it is not required if $\mathbf{a}(z,\xi)$ can be written in separate variables form. See [9] for more information on this point.

(6) We require that h defined by (4.5) satisfies

$$h(z,\xi) = h(z,-\xi) \quad \forall z \in \mathbb{R},\ \xi \in \mathbb{R}^d.$$

(7) We assume that h^0 exists, and that $f^0(z,\xi) = h^0(z,\xi)$ for every $(z,\xi) \in \mathbb{R}\times\mathbb{R}^d$.

(8) We also assume that

$$h^0(z,\xi) = \varphi(z)\psi^0(\xi), \tag{5.3}$$

with φ Lipschitz continuous, such that $\varphi(z) > 0$ for $z \neq 0$, and ψ^0 convex and homogeneous of degree 1.

(9) We require that

$$\mathbf{a}(z,\xi)\cdot\eta \le h^0(z,\eta) \quad \text{for all } \xi,\eta \in \mathbb{R}^d,\ z \in \mathbb{R}.$$

(10) Finally, let us assume that for any $R > 0$ there is a constant $C > 0$ such that

$$|(\mathbf{a}(z_1,\xi_1) - \mathbf{a}(z_2,\xi_1))\cdot(\xi_1 - \xi_2)| \le C|z_1 - z_2|\,|\xi_1 - \xi_2|$$

for any $(z_1,\xi_1), (z_2,\xi_2) \in \mathbb{R}\times\mathbb{R}^d$, $|z_1|, |z_2| \le R$.

As a consequence of this set of assumptions several useful identities can be derived (see for instance [12]). Here we point out that the following monotonicity condition is satisfied

$$(\mathbf{a}(z,\xi) - \mathbf{a}(z,\hat{\xi}))\cdot(\xi - \hat{\xi}) \ge 0 \tag{5.4}$$

for any $(z,\xi), (z,\hat{\xi}) \in \mathbb{R}\times\mathbb{R}^d$, $|z| \le M$. We also mention that those requirements in order that $f_S(u, DT(u))$, $h_S(u, DT(u))$ can be defined (that is the bounds (5.2)) are satisfied thanks to the previous assumptions and the fact that they imply

$$C_0(z)|\xi| - D_1(z) \le h(z,\xi) \le M|\xi| \quad \text{for all } z \in \mathbb{R},\ \xi \in \mathbb{R}^d,\ |z| \le R. \tag{5.5}$$

5.2. The class of entropy solutions. In this section we give the concept of entropy solution for the Cauchy problem (5.1) under the previous set of assumptions. For that purpose it is handy to let $J_q(r)$ denote the primitive of q for any function q; i.e.

$$J_q(r) := \int_0^r q(s)\, ds.$$

The definition we introduce here is a simpler version (as it appears for instance in [13] and in more recent works) of that originally in [12]. See for instance the comments in [13] for an explanation of this simplification.

Definition 5.1. A measurable function $u : (0, T) \times \mathbb{R}^d \to \mathbb{R}^+$ is an *entropy solution* of (5.1) in $Q_T = (0, T) \times \mathbb{R}^d$ if

- $u \in \mathcal{C}([0, T]; L^1(\mathbb{R}^d))$
- $u(0, x) = u_0(x), x \in \mathbb{R}^d$,
- $T(u(\cdot)) \in L^1_{loc,w}(0, T, BV(\mathbb{R}^d))$ for all $T \in \mathcal{T}_r$

and the equation is satisfied in the following sense:

(i) $u_t = \operatorname{div}(\mathbf{a}(u(t), \nabla u(t)))$ in $\mathcal{D}'((0, T) \times \mathbb{R}^d)$

(ii) Given any truncations $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and any $\eta \in \mathcal{D}(Q_T)$, the following entropy inequality is satisfied:

$$\int_{Q_T} \eta h_S(u, DT(u))\, dt + \int_{Q_T} \eta h_T(u, DS(u))\, dt \\ \leq \int_{Q_T} J_{TS}(u) \partial_t \eta \, dxdt - \int_{Q_T} \mathbf{a}(u, \nabla u) \nabla \eta \, T(u) S(u)\, dxdt. \quad (5.6)$$

A number of useful remarks in order to understand why this is a reasonable concept of solution for our setting here can be found in [14, 17]. Note that entropy solutions can be also defined analogously using the class $\mathcal{TSUB}$ instead of $\mathcal{P}^+, \mathcal{T}^+$, see Definition 5.1 in [60].

It is to be noted that entropy solutions so defined *have finite mass*. Up to now there is no completely satisfactory extension of this class to solutions which are just bounded.

5.3. The existence and uniqueness result. The following result is the main achievement of [12]. It allows us to ensure well-posedness for virtually every flux-saturated model we may come up with, toghether with many of their porous media variants.

Theorem 5.2. *Let the list of assumptions in Section 5.1 be satisfied. Then, for any initial datum $0 \leq u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique entropy solution u of (5.1) in $Q_T = (0, T) \times \mathbb{R}^d$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if $u(t)$, $\overline{u}(t)$ are the entropy solutions corresponding to initial data u_0, $\overline{u}_0 \in L^1(\mathbb{R}^d)^+$, respectively, then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all } t \geq 0. \quad (5.7)$$

The proof of the existence part proceeds by generating mild solutions via semigroup techniques [73], then checking that those are indeed entropy solutions. Uniqueness follows from a variant of Kruzkov's doubling variables method [96, 58]. Let us sketch this:

Step 1: The elliptic problem. The first step in order to generate mild solutions is to study the associated elliptic problem

$$u - \operatorname{div}(\mathbf{a}(u, Du)) = v \text{ in } \mathbb{R}^d. \tag{5.8}$$

This was the purpose of [11]. The meaning of the expression

$$v = -\operatorname{div}(\mathbf{a}(u, Du)) \text{ in } \mathbb{R}^d. \tag{5.9}$$

is the following.

Definition 5.3. ([11]) Given $v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we say that $u \geq 0$ is an *entropy solution* of (5.9) if $u \in TBV^+(\mathbb{R}^d)$ and $\mathbf{a}(u, \nabla u) \in X_1(\mathbb{R}^d)$ both satisfy the following conditions:

$$v = -\operatorname{div} \mathbf{a}(u, \nabla u) \quad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

$$h(u, DT(u)) \leq \mathbf{a}(u, \nabla u)\, DT(u) \quad \text{as measures } \forall\, T \in \mathcal{T}^+ \tag{5.10}$$

$$h_S(u, DT(u)) \leq \mathbf{a}(u, \nabla u)\, DJ_{T'S}(u) \quad \text{as measures } \forall\, S \in \mathcal{P}^+,\ T \in \mathcal{T}^+. \tag{5.11}$$

Then the main result in [11] is as follows.

Theorem 5.4. *Let the list of assumptions in Section 5.1 be satisfied. Then, for any* $0 \leq v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *there exists a unique entropy solution* $u \in TBV^+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *of problem (5.8). Moreover, let* $u, \overline{u}$ *be two bounded entropy solutions of (5.8) associated to* $v, \overline{v} \in L^\infty(\mathbb{R}^d)^+$*, respectively. Then,*

$$\int_{\mathbb{R}^d} (u - \overline{u})^+ \, dx \leq \int_{\mathbb{R}^d} (v - \overline{v})^+ \, dx.$$

The existence part of this theorem proceeds adding a viscosity term to (5.8). The trickiest point when letting the viscosity vanish is to identify the limit of the flux, which can be achieved usign a variant of Minty-Browder's method, thanks to (5.4). Entropy conditions (5.10)–(5.11) are proved using those semicontinuity results in Section 4.4. Uniqueness and the contraction property are obtained by a variant of Kruzkov's doubling variables method, for which the previous entropy inequalities (5.10)–(5.11) are crucial.

Step 2: Getting a mild solution. The idea here is to associate an accretive operator in $L^1(\mathbb{R}^d)$ to the formal expression $-\operatorname{div} \mathbf{a}(u, \nabla u)$. This is done in [11] as a consequence of the results in the previous step. Let us explain this in more detail. We start defining the associated differential operator.

Definition 5.5. ([11]) $(u, v) \in \mathcal{B}$ if and only if $0 \leq u \in TBV^+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $0 \leq v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and u is the entropy solution of problem (5.9).

Theorem 5.4 is a key ingredient to show the following:

Proposition 5.6. *([11]) The following assertions hold true:*

- $\mathcal{B}$ *is accretive in* $L^1(\mathbb{R}^d)$,

- $\left(L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\right)^+ \subset R(I+B)$,
- $D(B)$ *is dense in* $L^1(\mathbb{R}^d)^+$.

Let us denote by $\mathcal{B}$ the closure of B in $L^1(\mathbb{R}^d)$. It follows that the operator $\mathcal{B}$ is accretive in $L^1(\mathbb{R}^d)$, satisfies the comparison principle and verifies the range condition

$$\overline{D(\mathcal{B})}^{L^1(\mathbb{R}^d)} = L^1(\mathbb{R}^d)^+ \subset R(I+\lambda\mathcal{B}) \quad \text{for all } \lambda > 0.$$

Then Crandall–Liggett's Theorem [73] applies. Thus we get that for any $0 \le u_0 \in L^1(\mathbb{R}^d)$ there exists a unique mild solution $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}^d))$ of the abstract Cauchy problem

$$u'(t) + \mathcal{B}u(t) \ni 0, \quad u(0) = u_0.$$

Moreover, $u(t) = T(t)u_0$ for all $t \ge 0$, where $(T(t))_{t\ge 0}$ is the semigroup in $L^1(\mathbb{R}^d)^+$ generated by Crandall–Liggett's exponential formula, i.e.,

$$T(t)u_0 = \lim_{n\to\infty}\left(I + \frac{t}{n}\mathcal{B}\right)^{-n} u_0.$$

On the other hand, we have that the comparison principle also holds for $T(t)$. Meaning that, if $u_0, \overline{u}_0 \in L^1(\mathbb{R}^d)^+$, we have the estimate

$$\|(T(t)u_0 - T(t)\overline{u}_0)^+\|_1 \le \|(u_0 - \overline{u}_0)^+\|_1. \tag{5.12}$$

Step 3: Characterize mild solutions as entropy solutions. The final step in the existence proof is to show that mild solutions are actually entropy solutions. Crandall–Liggett's iteration scheme provides the required compactness in order to identify the limit equation. The identification of the flux proceeds again by a variant of Minty–Browder's procedure. The entropy inequality (5.6) is also obtained thanks to lower semicontinuity results. Finally to prove uniqueness and the contraction property this entropy inequality (which acts as a kind of renormalized formulation of the equation) is used in combination with a variant of Kruzkov's doubling variable technique; this time both spatial and temporal variables are doubled.

Remark 5.7. We observe that $u(t) \in BV(\mathbb{R}^d)$ for any $t > 0$ if $u_0 \in BV(\mathbb{R}^d)$. Indeed, let $\tau_h u_0(x) = u_0(x+h)$, $h \in \mathbb{R}^d$. Let $u_h(t)$ be the entropy solution corresponding to the initial datum $\tau_h u_0$. Then by the uniqueness result of Theorem 5.2 we have that $u_h(t) = \tau_h u(t)$ for any $t \ge 0$. By applying estimate (5.7) we have

$$\|u(t) - \tau_h u(t)\|_1 \le \|u_0 - \tau_h u_0\|_1 \qquad \forall t > 0.$$

Since $u_0 \in BV(\mathbb{R}^d)$ we deduce that $u(t) \in BV(\mathbb{R}^d)$ for all $t > 0$ and $\|u(t)\|_{BV} \le \|u_0\|_{BV}$. Clearly $u \in L^1_w(0,T; BV(\mathbb{R}^d))$.

While the theory presented in the previous paragraphs may be regarded as the standard reference result on well-posedness, some improvements have been made afterwards which we would like to mention here. Well-posedness for the Cauchy problem (5.1) was extended in [60] to hold for initial data in $BV(\mathbb{R}^d)$, assuming that the flux satisfies the following additional requirement:

$$|\mathbf{a}(z,\xi)| \le Cz, \quad \forall z \in [0,\infty).$$

In this setting, an initial datum $0 \le u_0 \in BV(\mathbb{R}^d)$ launches a unique entropy solution $u(t) \in L^1_w(0,T;BV(\mathbb{R}^d))$ such that $\|u(t)\|_{BV} \le \|u_0\|_{BV}$ (according to Remark 5.7). Note also that the conditions under which (5.2) is to hold were relaxed in [62] (compare with [11, 12]); these are the ones we quoted indeed in Section 5.1.

5.4. Comparison principles. Sub- and super-solutions for flux-saturated equations can be introduced as in [17]. These are very useful tools for qualitative analysis thanks to some comparison principles stated there. In order to use them a certain technical condition is required.

Assumptions 5.8. *Let the function h defined by (4.5) satisfy*

$$h(z,\xi) \le M(z)|\xi|$$

for some positive continuous function $M(z)$ and for any $(z,\xi) \in \mathbb{R}\times\mathbb{R}^d$.

Definition 5.9. ([17]) A measurable function $u : (0,T)\times\mathbb{R}^d \to \mathbb{R}^+_0$ is an entropy sub- (resp. super) solution of (5.1) if $u \in \mathcal{C}([0,T],L^1(\mathbb{R}^d))$, $T_{a,b}(u) \in L^1_{loc,w}(0,T,BV(\mathbb{R}^d))$ for every $0 < a < b$, $\mathbf{a}(u,\nabla u) \in L^\infty(Q_T)$ and the following inequality is satisfied:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \phi h_S(u,DT(u))\,dt &+ \int_0^T \int_{\mathbb{R}^d} \phi h_T(u,DS(u))\,dt \\ \ge \int_0^T \int_{\mathbb{R}^d} &\Big\{J_{TS}(u(t))\phi'(t) - \mathbf{a}(u(t),\nabla u(t))\cdot\nabla\phi\; T(u(t))S(u(t))\Big\}dxdt, \end{aligned} \tag{5.13}$$

(resp. with $\le$) for any $\phi \in \mathcal{D}(Q_T)^+$ and any truncations $T \in \mathcal{T}^+$, $S \in \mathcal{T}^-$.

This implies that

$$u_t \le \operatorname{div}\mathbf{a}(u,\nabla u) \quad \text{in } \mathcal{D}'(Q_T) \tag{5.14}$$

(resp. with $\ge$). Note also that sub- and super-solutions can be defined using truncations from the classes $\mathcal{TSUB}, \mathcal{TSUPER}$, see Definition 5.1 and Remark 8 in [60].

The following comparison principle was shown in [17]:

Theorem 5.10. *Let the list of assumptions in Section 5.1 and Assumptions 5.8 hold. Given an entropy solution u of (5.1) corresponding to an initial datum $0 \le u_0 \in (L^\infty \cap L^1)(\mathbb{R}^d)$, the following statements hold true:*

(1) *if $\overline{u}$ is a super-solution of (5.1) such that $\overline{u}(t) \in BV(\mathbb{R}^d)$ for a.e. $t \in (0,T)$, then*

$$\|(u(t)-\overline{u}(t))^+\|_1 \le \|(u_0-\overline{u}(0))^+\|_1, \quad \forall t \in [0,T],$$

(2) *if $\overline{u}$ is a sub-solution of (5.1) such that $\overline{u}(t) \in BV(\mathbb{R}^d)$ for a.e. $t \in (0,T)$, then*

$$\|(\overline{u}(t)-u(t))^+\|_1 \le \|(\overline{u}(0)-u_0)^+\|_1, \quad \forall t \in [0,T].$$

Some extensions of this result have been shown in [83] in order to be able to consider super-solutions neither bounded nor integrable.

5.5. An application: Well-posedness for (2.21). With very similar arguments to those in [48], the class given by (2.21) can be shown to provide models that are well-posed in the entropic sense. We just point out the differences and new ideas for this more general setting.

By definition $\mathbf{a}(z=0,\xi)=0$. This time the Lagrangian is given by

$$f(z,\xi)=\frac{\varphi(z)}{\theta(z)}\Phi(\theta(z)\xi), \quad f(z=0,\xi)=0,$$

with Φ constructed as in [48] using Poincare's Lemma. Its main properties follow as in [48], as they don't depend at all on those of θ. Note that in our case

$$\Phi(\theta(z)\xi)\sim\theta(z)|\xi| \quad \text{for} \quad \theta(z)|\xi|\gg 1.$$

Then it is easy to show that $\lim_{|z|\to 0} f(z,\xi)=0$ regardless of the behavior of θ at zero. This implies the continuity of f. We also compute

$$f^0(z,\xi)=\lim_{t\to 0} t\frac{\varphi(z)}{\theta(z)}\Phi(\xi\theta(z)/t)=|\xi|\varphi(z).$$

The bound $f(z,\xi)\le\varphi(z)(1+|\xi|)$ is readily obtained. In order to obtain a lower bound for f, following [48] we get

$$\Phi(\theta(z)\xi)\ge\theta(z)\xi\psi(\theta(z)\xi).$$

Apart from that, we claim that there exist constants $0<C_0<1$ and $D_0>0$ such that

$$r\psi(r)\ge C_0|r|-D_0 \quad \text{for any } r\in\mathbb{R}^d. \tag{5.15}$$

This is easily seen in dimension one. First we check that

$$\psi(\lambda)\lambda\ge C_0\lambda-D_0 \quad \forall\lambda\ge 0 \tag{5.16}$$

for appropriate positive constants $C_0<1$, and D_0. This is clear since $\lim_{\lambda\to 0}\psi(\lambda)\lambda-C_0\lambda=0$ and $\lim_{\lambda\to\infty}\psi(\lambda)\lambda-C_0\lambda=\infty$. Being ψ an odd function we have that (5.16) holds indeed for any $\lambda\in\mathbb{R}$, thereby proving our claim. This argument can be easily generalized to arbitrary dimensions.

We choose now $r=\theta(z)\xi$ in (5.15), so that

$$\theta(z)\xi\psi(\theta(z)\xi)\ge C_0\theta(z)|\xi|-D_0.$$

Multiplication by $\varphi(z)/\theta(z)$ finally leads to the desired lower bound:

$$f(z,\xi)\ge\varphi(z)\xi\psi(\theta(z)\xi)\ge C_0|\xi|\varphi(z)-D_0\frac{\varphi(z)}{\theta(z)}.$$

Next we introduce the function $h(z,\xi)=\xi\varphi(z)\psi(\theta(z)\xi)\ge 0$, easily seen to be even. Moreover,

$$h^0(z,\xi)=\lim_{t\to 0} t\frac{\xi}{t}\varphi(z)\psi(\theta(z)\xi/t)=\lim_{t\to 0}\xi\varphi(z)\frac{\theta(z)\xi/t}{|\theta(z)\xi/t|}=|\xi|\varphi(z).$$

We also point out that

$$\frac{\partial \mathbf{a}^{(i)}}{\partial z} = \varphi'(z)\psi(\theta(z)\xi) + \varphi(z)\theta'(z)\partial_i\psi^{(j)}(\theta(z)\xi)\xi_i$$

is bounded locally in z and globally in ξ, thanks to our assumptions on θ. All the properties discussed in this paragraph suffice to show that the class (2.21) satisfies those assumptions in Section 5.1.

6. A more general well-posedness theory

There are a number of situations in which flux-saturated equations of porous media type are such that the porous media type term is not regular. These situations are not covered by the theory we have reviewed in the previous section. In order to treat these cases, a suitable well-posedness theory encompassing the already existing one was developed in [62]. We explain below what are the new ideas.

The focus is now on flux limited or tempered diffusion equations having the form

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, D\Phi(u)) \tag{6.1}$$

characterized by a bounded flux $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$, where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^+$ is as explained in Section 5.1. Here Φ need not be smooth; if it is smooth then (6.1) can be treated within the previous framework in many cases. However, even in the smooth case the present theory allows to cover some situations not included in [11, 12] (see Remark 3.4 in [62]). Indeed, we state now a list of assumptions on Φ and the flux $\mathbf{a}$ under which the theory in [62] applies.

Assumptions 6.1. *Let $\Phi : [0,\infty) \to [0,\infty)$ be such that the following properties hold:*

- *It is a continuous, strictly increasing function*
- $\Phi(0) = 0$
- $\Phi, \Phi^{-1} \in W^{1,\infty}([a,b])$ *for any* $0 < a < b$.

Let also $\mathbf{a}(z,\xi)$ be such that those assumptions in Section 5.1 hold, except 4 being replaced by the following: It is assumed that there exists a constant $M > 0$ and some $m \geq 1$ such that

$$\mathbf{a}(z,\xi) = z^m \mathbf{b}(z,\xi) \quad \textit{with } |\mathbf{b}(z,\xi)| \leq M, \forall (z,\xi) \in \mathbb{R}^+ \times \mathbb{R}^d.$$

This new assumption on the structure of the flux was initially considered in [60].

6.1. The class of entropy solutions: A suitable functional calculus. To define entropy solutions in this new context requires to reformulate (and actually extend) the functional calculus of previous paragraphs. The meaning of this new functional calculus we now summarize is explained in detail in [63].

Let $S \in \mathcal{C}([0,\infty))$ and $p \in \mathcal{P} \cap \mathcal{C}^1([0,\infty))$. We denote

$$f^{\Phi}(z,\xi)) = f(\Phi^{-1}(z,\xi)), \quad h^{\Phi}(z,\xi)) = h(\Phi^{-1}(z,\xi)),$$

$$f^{\Phi}_{S:p}(z,\xi)) = S(z)p'(z)f^{\Phi}(z,\xi), \quad h^{\Phi}_{S:p}(z,\xi)) = S(z)p'(z)h^{\Phi}(z,\xi).$$

Assume that $p(z) = p(T_{a,b}(z))$, $0 < a < b$ and that $w = \Phi(u) \in TBV_r^+(\mathbb{R}^d)$. We denote by

$$f^{\Phi}_{S:p}(w, DT_{a,b}(w)), \quad h^{\Phi}_{S:p}(w, DT_{a,b}(w)),$$

or simply by

$$f_{S\circ\Phi:p\circ\Phi}(u, DT_{a,b}(w)), \quad h_{S\circ\Phi:p\circ\Phi}(u, DT_{a,b}(w))$$

the Radon measures defined by (4.4) with $g(z,\xi) = f^{\Phi}_{S:p}(z,\xi)$ and $g(z,\xi) = h^{\Phi}_{S:p}(z,\xi)$ applied to w respectively.

The above definitions can be extended to any $p \in \mathcal{P}$ such that $p(z) = \tilde{p}(T_{a,b}(z))$ and $\tilde{p}$ is differentiable in a neighborhood of $[a,b]$, by writting

$$f^{\Phi}_{S:p}(z,\xi) = S(z)\tilde{p}'(z)f^{\Phi}(z,\xi), \quad h^{\Phi}_{S:p}(z,\xi) = S(z)\tilde{p}'(z)h^{\Phi}(z,\xi).$$

Notice that when $S \in \mathcal{T}^+$ and $\Phi(z) = z$ then $f_S(u, DT(u))$ and $h_S(u, DT(u))$ coincide with the definitions in [11, 12].

6.2. The existence and uniqueness result. We are now in a position to introduce the concept of entropy solution for the following Cauchy problem:

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, D\Phi(u)), \quad u(0,x) = u_0. \tag{6.2}$$

Definition 6.2. ([62]) A measurable function $u\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^+$ is an *entropy solution* of (6.2) in $Q_T = (0,T)\times\mathbb{R}^d$ if

- $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}^d))$
- $u(0,x) = u_0(x), x \in \mathbb{R}^d$,
- $T_{a,b}(\Phi(u(\cdot))) - a \in L^1_{loc,w}(0,T, BV(\mathbb{R}^d))$ for all $0 < a < b \le \infty$
- $\mathbf{a}(u(t), \nabla\Phi(u(t))) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for a.e. $t \in (0,T)$

and the equation is satisfied in the following sense:

(i) $u_t = \operatorname{div}(\mathbf{a}(u(t), \nabla\Phi(u(t))))$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$

(ii) Given any truncations $(S,T) \in \mathcal{TSUB}$ with $T = \tilde{T}\circ T_{a,b}$, $S = \tilde{S}\circ S_{c,d}$ and any $\phi \in \mathcal{D}(Q_T)$, the following entropy inequality is satisfied:

$$\begin{aligned}\int_{Q_T} \phi h^{\Phi}_{S:T}(\Phi(u), DT_{a,b}(\Phi(u)))\,dt &+ \int_{Q_T} \phi h^{\Phi}_{T:S}(\Phi(u), DS_{c,d}(\Phi(u)))\,dt \\ \le \int_{Q_T} & J_{T\circ\Phi\, S\circ\Phi}(u)\partial_t\phi\, dxdt \\ & - \int_{Q_T} \mathbf{a}(u(t), \nabla\Phi(u(t)))\nabla\phi\; T(\Phi(u(t)))S(\Phi(u(t)))\,dxdt. \end{aligned} \tag{6.3}$$

Then the following well-posedness result was proved in [62]:

Theorem 6.3. *Let Assumptions 6.1 be satisfied. Then, for any initial datum* $0 \le u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *there exists a unique entropy solution* u *of (6.2) in* $Q_T = (0,T)\times\mathbb{R}^d$ *for every* $T > 0$ *such that* $u(0) = u_0$*. Moreover, if* $u(t)$*,* $\overline{u}(t)$ *are the entropy solutions corresponding to initial data* u_0*,* $\overline{u}_0 \in L^1(\mathbb{R}^d)^+$*, respectively, then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \le \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all } \ t \ge 0. \tag{6.4}$$

As in the previous section, the proof for the existence part proceeds by generating mild solutions via semigroup techniques, then checking that those are indeed entropy solutions. Let us sketch this in order to make clear what are the differences:

Step 1: The elliptic problem. The first step in order to generate mild solutions is to study the associated elliptic problem, which this time reads

$$u - \operatorname{div} \mathbf{a}(u, D\Phi(u)) = v \quad \text{ in } \quad \mathbb{R}^d. \tag{6.5}$$

Definition 6.4. ([62]) Given $0 \le v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we say that $u \ge 0$ is an *entropy solution* of (6.5) if $u \in TBV_r^+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\mathbf{a}(u, \nabla\Phi(u)) \in X_1(\mathbb{R}^d, \mathbb{R}^d)$ both satisfy the following conditions:

$$u - \operatorname{div} \mathbf{a}(u, \nabla\Phi(u)) = v \qquad \text{in } \ \mathcal{D}'(\mathbb{R}^d),$$

$$h^\Phi_{S:T}(\Phi(u), DT_{a,b}(\Phi(u))) \le (\mathbf{a}(u, \nabla\Phi(u)),\ DJ_{T'S}(\Phi(u))) \\ \text{as measures } \ \forall\,(S,T) \in \mathcal{TSUB}.$$

Then the following result was proved in [62].

Theorem 6.5. *Let Assumptions 6.1 be satisfied. Then, for any* $0 \le v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *there exists a unique entropy solution* $u \in TBV^+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *of the problem* (6.5)*. Moreover, let* $u, \overline{u}$ *be two bounded entropy solutions of (6.5) associated to* $v, \overline{v} \in L^\infty(\mathbb{R}^d)^+$*, respectively. Then,*

$$\int_{\mathbb{R}^d} (u - \overline{u})^+ \, dx \le \int_{\mathbb{R}^d} (v - \overline{v})^+ \, dx. \tag{6.6}$$

The proof of this result follows the same lines of that for Theorem 5.4, making extensive use of the specifically adapted functional calculus that was introduced before.

Step 2: Getting a mild solution. Now the previous result is used to construct a semigroup.

Definition 6.6. ([62]) $(u, v) \in B$ if and only if $0 \le u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\Phi(u) \in TBV_r^+(\mathbb{R}^d)$, $0 \le v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\mathbf{a}(u, \nabla\Phi(u)) \in X_1(\mathbb{R}^d)$ satisfy:

(1) $v = -\operatorname{div} \mathbf{a}(u, \nabla\Phi(u))$ in $\mathcal{D}'(\mathbb{R}^d)$

(2) $h^\Phi_{S:T}(\Phi(u), DT_{a,b}(\Phi(u))) \le (\mathbf{a}(u, \nabla\Phi(u)),\ DJ_{T'S}(\Phi(u)))$ as measures $\forall\,(S,T) \in \mathcal{TSUB}$.

Then B is shown to satisfy the assumptions of Crandall–Liggett's theorem. This provides us with mild solutions of (6.2), a representation of them by means of an exponential formula and a contraction principle.

Step 3: Characterize mild solutions as entropy solutions. As in the proof of Theorem 5.2, the final step in the existence proof is to show that mild solutions are actually entropy solutions. This is done in the same vein as in the previous case, using again the specific functional calculus that was devised to deal with this framework. Uniqueness is shown likewise by a doubling variables argument.

7. Variants on well-posedness theory

7.1. Alternative approaches. *The optimal transport approach.* Solutions to the Neumann problem in bounded, convex domains for (2.12) were constructed in [103] as limits of an approximating scheme involving optimal mass transport problems as explained in Section 2.2. They can show the convergence of such approximating scheme when initial data are bounded from above and below, for a very wide range of cost functions k and entropy functionals F (including the model (1.5)).

Bertsch–dalPasso's theory [35]. They study the Cauchy problem for (2.18) with strictly increasing initial data (and hence with infinite mass, so the theory on Section 5 does not apply). Existence is shown by standard parabolic regularization, which enables to construct solutions in $L^\infty(Q_T) \cap BV_{loc}(Q_T)$ satisfying entropy-like conditions (in Oleinik's form). Uniqueness in the constructed class of solutions is proved in [76].

7.2. Construction of regular solutions. We will present in forthcoming Section 10.4 a number of regularity results that can actually be regarded as providing existence of solutions to (porous media) flux-limited models for certain sub-classes of initial data (uniqueness being a consequence of those uniqueness results for entropy solutions). For instance, Bernstein-type estimates on (2.17) yield the following statement.

Theorem 7.1. *([67]) Let $n > 0$ and consider the equation*

$$u_t = (u^n Q(u_x))_x\,, \quad u(0,x) = u_0. \tag{7.1}$$

The following statements hold true:

(1) Assume that $0 < n \leq 1$. If $0 \leq u_0 \in \mathcal{C}^3$ then (7.1) has a unique global classical solution, such that $u(t,x)$ is of class $\mathcal{C}^2$ with respect to x and of class $\mathcal{C}^1$ with respect to t.

(2) Assume that $n > 1$. If $u_0 \in \mathcal{C}^3$ is such that

$$0 < m := \min u_0(x) \leq \max u_0(x) = M$$

and

$$\left(\frac{M}{m}\right)^n \|Q((u_0)_x)\|_\infty < \min\{-a, b\}, \quad \text{being} \quad Q : \mathbb{R} \to (a,b).$$

Then (7.1) has a unique global classical solution, such that $u(t,x)$ is of class $\mathcal{C}^2$ with respect to x and of class $\mathcal{C}^1$ with respect to t.

The situation covered in [51] comprises a large number of sub-cases, which we will defer from analyzing here. Suffices to say that the arguments there lead to existence results like the following.

Theorem 7.2. *Let $m \geq 2$. Let $0 \leq u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, there exists some $t^* < \infty$ such that (2.9) has a unique solution $u \in L^\infty(0, t^*, W^{1,\infty}(\mathbb{R}^d))$ with u_0 as initial datum. The same statement holds for (2.11).*

From another point of view, local in time regularity properties (inside its support) on the entropy solutions of the relativistic heat equation (1.5) have been analyzed in [20] for radially symmetric smooth initial conditions. The smoothness argument is inspired on Angenent's discussion on a class of degenerate parabolic equations [21]. Again, these considerations provide a local existence result for certain classes of radially symmetric initial data.

As in Theorem 7.1, some existence results for (1.8) can be proved if we give away the idea of finite mass solutions [70]. Note that this is out of the scope of the theory for entropy solutions in Section 5.

Theorem 7.3. *([70]) Assume that the collision kernel Q in (1.8) satisfies conditions (C1)–(C4) in [70]. Let $\bar{\rho} > 0$. There exist three positive constants δ, C, T_* such that, for any $\varepsilon \in (0, 1]$ and for any ρ_0 with $\|\rho_0 - \bar{\rho}\|_{H^4(\mathbb{R})} \leq \delta$,*

(1) *There exists a unique solution $\rho_\varepsilon \in \mathcal{C}(0, T_*, \bar{\rho} + H^4(\mathbb{R}))$ to (1.8) with initial datum ρ_0.*

(2) *Moreover, the solution satisfies the estimates*

$$\sup_{[0,T_*]} \|\rho_\varepsilon(t) - \bar{\rho}\|_{H^4(\mathbb{R})} \leq C, \qquad \sup_{[0,T_*]} \|\rho_\varepsilon(t) - \bar{\rho}\|_{W^{1,\infty}(\mathbb{R})} \leq \bar{\rho}/2.$$

(3) *Let r be the solution to the heat equation $\rho_t = D\rho_{xx}$, with initial data ρ_0, being D computed explicitly from the collision kernel Q –using formula (6) in [70]. Then, we also have*

$$\sup_{[0,T_*]} \|\rho_\varepsilon(t) - r(t)\|_{L^2(\mathbb{R})} \leq C\varepsilon.$$

(4) *In fact, if $\bar{f}_\varepsilon$ is the solution to (1.7), we may set*

$$\tilde{f}_\varepsilon = \rho_\varepsilon \exp\left(-\varepsilon \frac{v\partial_x \rho_\varepsilon}{\rho_\varepsilon}\right), \qquad (t, x, v) \in (0, T*) \times \mathbb{R} \times V \subset (0, T*) \times \mathbb{R} \times \mathbb{R}$$

and there holds that

$$\|\tilde{f}_\varepsilon - \bar{f}_\varepsilon\|_{L^2((0,T*)\times\mathbb{R}\times V)} = O(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

7.3. Other types of initial value problems. The homogeneous Neumann problem in a bounded domain is already considered in [9, 10] when the initial condition is bounded and strictly positive, or when the Lagrangian is coercive. Similar techniques to those in [11, 12] may be used to address problems in bounded domains with zero Neumann boundary conditions, as stated in those references, as long as the Lagrangian verifies the previous set of hypotheses. The domain needs to be bounded and with $\mathcal{C}^1$-boundary. Then well-posedness for non-negative and bounded initial data can be shown. See [16] for a detailed exposition about the elliptic problem with homogenoeus Neumann boundary conditions. See also earlier accounts on the one-dimensional Neumann problem by Blanc [38, 39].

The Dirichlet problem on a bounded domain was undertaken in [19]. The extension of those arguments in [11, 12] is highly non-trivial: Boundary conditions need not be attained and they have to be weakend to an obstacle condition. Despite this fact, they show well-posedness of (1.5) for non-negative and bounded initial conditions when the domain is a bounded set with Lipschitz boundary and the boundary data is non-negative and bounded. Contractivity estimates also hold.

A mixed one-dimensional Dirichlet–Neumann problem with non-homogeneous Neumann boundary condition was studied in [4, 52]. Again, Dirichlet boundary condition does not hold in classical form.

To the best of our knowledge, the study of problems with periodic boundary conditions has not been touched upon. See however [51].

For all these initial value problems the concept of entropy solution has to be suitably re-adapted. We refer to the above references for details.

7.4. Well-posedness for reaction-diffusion equations. Here we consider flux-saturated equations (with or without porous media terms) with reaction terms in the following form.

$$\frac{\partial u}{\partial t} = \mathrm{div}(\mathbf{a}(u, Du)) + F(u) \quad \text{in } \mathbb{R}^d, \quad u(0,x) = u_0(x). \tag{7.2}$$

The extension of the theory of entropy solutions to the case of (7.2) was developed in [15]. Here we just quote the main results. For that we need to (re)formulate properly the concept of entropy solutions to (7.2).

Definition 7.4. ([15]) A measurable function $u\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^+$ is an *entropy solution* of (7.2) in $Q_T = (0,T)\times\mathbb{R}^d$ if

- $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}^d))$
- $F(u(t)) \in L^1_{loc}(\mathbb{R}^d)$ for a.e. $0 \le t \le T$,
- $u(0,x) = u_0(x), x \in \mathbb{R}^d$,
- $T(u(\cdot)) \in L^1_{loc,w}(0, T, BV(\mathbb{R}^d))$ for all $T \in \mathcal{T}_r$

and the equation is satisfied in the following sense:

(i) $u_t = \mathrm{div}(\mathbf{a}(u(t), \nabla u(t))) + F(u(t))$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$

(ii) Given any truncations $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and any $\eta \in \mathcal{D}(Q_T)$, the following entropy inequality is satisfied:

$$\begin{aligned}\int_{Q_T} \eta h_S(u, DT(u))\, dt \ &+ \int_{Q_T} \eta h_T(u, DS(u))\, dt \\ &\le \int_{Q_T} J_{TS}(u)\partial_t \eta \, dxdt \ - \int_{Q_T} \mathbf{a}(u, \nabla u)\nabla \eta \, T(u)S(u)\, dxdt \\ &\qquad + \int_{Q_T} \eta T(u(t))S(u(t))F(u(t))\, dxdt. \qquad (7.3)\end{aligned}$$

The notions of sub- and super-solutions can be reformulated accordingly to accomodate them to this framework. The main well-posedness result is the following.

Theorem 7.5. *([15]) Let the list of assumptions in Section 5.1 be satisfied and let F be Lipschitz continuous with $F(0) = 0$. Then, for any initial datum $0 \le u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ there exists a unique entropy solution u of (7.2) in Q_T for every $T > 0$ such that $u(0) = u_0$, satisfying $u \in \mathcal{C}([0, T]; L^1(\mathbb{R}^d))$ and $F(u(t)) \in L^1(\mathbb{R}^d)$ for almost all $0 \le t \le T$. Moreover, if $u(t)$, $\overline{u}(t)$ are entropy solutions corresponding to initial data u_0, $\overline{u}_0 \in \left(L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\right)^+$, respectively, then*

$$\|u(t) - \overline{u}(t)\|_1 \le e^{t\|F\|_{Lip}} \|u_0 - \overline{u}_0\|_1, \qquad \text{for all } \ t \ge 0.$$

While we cannot guarantee existence for initial data which are just bounded, we have uniqueness of entropy solutions for initial data in $L^\infty(\mathbb{R}^d)$ when they have null flux at infinity. This is particularly useful in order to deal with traveling wave solutions.

Definition 7.6. ([15]) Let u be a sub- or a super-solution of (7.2) We say that u has a null flux at infinity if

$$\lim_{R\to+\infty} \int_0^T \int_{\mathbb{R}^d} |\mathbf{a}(u(t), \nabla u(t))|\, |\nabla \psi_R(x)|\, dxdt = 0,$$

for all $\psi_R \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \le \psi_R \le 1$, $\psi_R \equiv 1$ on $B(0, R)$, $\text{supp}(\psi_R) \subset B(0, R + 2)$ and $\|\nabla \psi_R\|_\infty \le 1$.

Theorem 7.7. *([15]) Let the list of assumptions in Section 5.1 be satisfied and let F be Lipschitz continuous with $F(0) = 0$.*

(i) *Let $u(t)$, $\overline{u}(t)$ be two entropy solutions of (7.2) with initial data $u_0, \overline{u}_0 \in L^\infty(\mathbb{R}^d)^+$, respectively. Assume that $u(t)$ and $\overline{u}(t)$ have null flux at infinity. Then*

$$\|u(t) - \overline{u}(t)\|_1 \le e^{t\|F\|_{Lip}} \|u_0 - \overline{u}_0\|_1, \qquad \text{for all } \ t \ge 0.$$

(ii) *Let Assumptions 5.8 hold. Assume that $u_0 \in (L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))^+$, $\overline{u}_0 \in L^\infty(\mathbb{R}^d)^+$. Let $u(t)$ be the entropy solution of (7.2) with initial datum u_0. Let $\overline{u}(t)$ be an entropy super-solution of (7.2) with initial datum $\overline{u}_0 \in L^\infty(\mathbb{R}^d)^+$ having a null flux at infinity. Assume in addition that $\overline{u}(t) \in BV_{\text{loc}}(\mathbb{R}^d)$ for almost every $0 < t < T$. Then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \le e^{t\|F\|_{Lip}} \|(u_0 - \overline{u}_0)^+\|_1, \qquad \text{for all } \ t \ge 0.$$

A similar result holds for $\bar{u}$ a solution and u a sub-solution.

8. Geometric interpretation of entropy conditions. Rankine–Hugoniot relations

Rankine–Hugoniot conditions are quite well-known in combustion theory and in the theory of scalar conservation laws. They give some useful information on various parameters describing the time evolution of traveling fronts (commonly known as shocks and rarefaction waves elsewhere), which, together with some complementary information may allow to determine their laws of motion completely. On a technical level, they impose a condition on the structure of jump discontinuities of a given solution to a scalar conservation law, by the mere fact of satisfying a distributional formulation. Being sloppy for now, let u^+, u^- be the values at both sides of a given jump discontinuity and let $\mathcal{F}(u^+)$, $\mathcal{F}(u^-)$ be the fluxes at both sides of such a discontinuity. If v is the velocity by which such jump discontinuity is transversing the medium, then the Rankine–Hugoniot condition roughly says that

$$v = \frac{\mathcal{F}(u^+) - \mathcal{F}(u^-)}{u^+ - u^-}. \tag{8.1}$$

Of course this sole condition is not enough to determine the structure of the fluxes at both sides of the jump discontinuity; this extra amount of information is usually provided by additional conditions (entropy-like conditions).

We will see soon that flux-saturated equation (including those of porous media type) are not much different from scalar conservation laws in this regard. Namely, in the regime of large gradients they behave essentially like scalar conservation laws (as already noticed in [35] and commented above) and we expect some form of the Rankine–Hugoniot condition to hold. Let us mention that these ideas were already present in [67], where accurate formulas for front evolution of solutions to (2.17) with $g(u) = u$ were given, and also in [35], where similar formulas and a detailed analysis of shock discontinuities for solutions to (2.18) are shown.

Analytical studies of these issues were carried in [61, 62] which include equations (2.9) and (2.11). The point of view given there is slightly different: All in all, it is shown in [61, 62] that "the fact that

$$u \in BV([0,\tau] \times \mathbb{R}^d) \quad \text{for any} \quad 0 < \tau < T \tag{8.2}$$

permits to identify the Rankine–Hugoniot condition, to give a more concrete characterization of the entropy conditions on the jump set of u and to compute the speed of the moving discontinuity fronts". This has also been extended to the case of (2.20), see [48].

Therefore we take the fundamental assumption (8.2) for granted (we comment on this later in Section 10.2) and give a brief account on how do the aforementioned analysis proceed. For this we use the specific notations introduced back in Section 4.1. We assume that $u^+ > u^-$ in what follows (this determines if ν_x points inwards or outwards according to the conventions on Section 4.1); we also assume $u^- \geq 0$.

The following result was proved in [61].

Lemma 8.1. *Let $u \in BV_{loc}(Q_T)$ and let $\mathbf{z} \in L^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ be such that $u_t = \operatorname{div} \mathbf{z}$ in $\mathcal{D}'(Q_T)$. Then*

$$\mathcal{H}^d\left(\{(t,x) \in J_u / \nu_x(t,x) = 0\}\right) = 0.$$

Thanks to the previous result the following definition is meaningful.

Definition 8.2. ([61]) Let $u \in BV_{loc}(Q_T)$ and let $\mathbf{z} \in L^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ be such that $u_t = \operatorname{div} \mathbf{z}$ in $\mathcal{D}'(Q_T)$. We define the speed of the discontinuity set of u as $\mathbf{v}(t,x) = \frac{\nu_t(t,x)}{|\nu_x(t,x)|}$ $\mathcal{H}^d$-a.e. on J_u.

The following is a result encoding the Rankine–Hugoniot conditions that can be found in [61] too.

Proposition 8.3. *Let $u \in BV_{loc}(Q_T)$ and let $\mathbf{z} \in L^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ be such that $u_t = \operatorname{div} \mathbf{z}$. For a.e. $t \in (0,T)$ we have*

$$[u(t)](x)\mathbf{v}(t,x) = [[\mathbf{z} \cdot \nu^{J_{u(t)}}]]_{+-} \quad \mathcal{H}^{d-1} - a.e.\ in\ J_{u(t)},$$

where $[[\mathbf{z} \cdot \nu^{J_{u(t)}}]]_{+-}$ denotes the difference of traces from both sides of $J_{u(t)}$.

Next we state the following characterization of entropy conditions, which are now rephrased as two separate conditions on the jump and the Cantor parts of the original entropy inequality. This is quite helpful in order to describe solution behavior at jump discontinuities as we will show below.

Proposition 8.4. *Let $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}^d)) \cap BV_{loc}(Q_T)$. Assume that $u_t = \operatorname{div} \mathbf{z}$ in $\mathcal{D}'(Q_T)$, where $\mathbf{z} = \mathbf{a}(u, \nabla u)$. Assume also that $u_t(t)$ is a Radon measure for a.e. $t > 0$. Let φ defined by (5.3) be a locally Lipschitz continuous function such that $\varphi(0) = 0$. Then u is an entropy solution of (2.11) (resp. (2.21)) if and only if for any $(T,S) \in \mathcal{TSUB}$ (for any $(T,S) \in \mathcal{TSUB} \cup \mathcal{TSUPER}$) we have*

$$h_S(u, DT(u))^c + h_T(u, DS(u))^c \leq (\mathbf{z}(t,x) \cdot D(T(u)S(u)))^c$$

and for almost any $t > 0$ the inequality

$$\begin{aligned} &[ST\varphi(u(t))]_{+-} - [J_{TS\varphi'}(u(t))]_{+-} \\ &\qquad \leq -\mathbf{v}[J_{TS}(u(t))]_{+-} + [[\mathbf{z}(t) \cdot \nu^{J_{u(t)}}]T(u(t))S(u(t))]_{+-} \end{aligned} \tag{8.3}$$

holds $\mathcal{H}^{d-1}$-a.e. on $J_{u(t)}$.

This is just Proposition 8.1 in [61] plus an easy generalization of Proposition 7.8 in [48]. All in all, it is a particular case ($\Phi(u) = u$) of Proposition 6.8 in [62], which we now state.

Proposition 8.5. *Let $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}^d)) \cap BV_{loc}(Q_T)$. Assume that $u_t = \operatorname{div} \mathbf{z}$ in $\mathcal{D}'(Q_T)$, where $\mathbf{z} = \mathbf{a}(u, \nabla\Phi(u))$.* Assume also that $u_t(t)$ is a Radon measure for a.e. $t > 0$. *Then u is an entropy solution of (2.19) if and only if for any $(T,S) \in \mathcal{TSUB}$ (for any $(T,S) \in \mathcal{TSUB} \cup \mathcal{TSUPER}$) we have*

$$h^\Phi_{S:T}(\Phi(u), DT(\Phi(u)))^c + h^\Phi_{T:S}(\Phi(u), DS(\Phi(u)))^c \leq (\mathbf{z}(t,x) \cdot D(T(\Phi(u))S(\Phi(u))))^c$$

and for almost any $t > 0$ the inequality

$$\begin{aligned} &[ST\varphi^\Phi(\Phi(u(t)))]_{+-} - [J_{TS\varphi^{\Phi'}}(\Phi(u(t)))]_{+-} \\ &\qquad \leq -\mathbf{v}[J_{T\circ\Phi\, S\circ\Phi}(u(t))]_{+-} + [[\mathbf{z}(t) \cdot \nu^{J_{u(t)}}]T(\Phi(u(t)))S(\Phi(u(t)))]_{+-} \end{aligned} \tag{8.4}$$

holds $\mathcal{H}^{d-1}$-a.e. on $J_{u(t)}$.

We are now able to present the main result regarding Rankine–Hugoniot relations, which applies to equations of the form (2.11), (2.19) and (2.21) — see [62, 63, 48]. Note that any distributional solution satisfies a Rankine–Hugoniot law (8.1) at jump points — this is Proposition 8.3 already. The point here is that entropy conditions characterize completely the structure of the fluxes at both sides of a given jump discontinuity, as we now state.

Proposition 8.6. *Let $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}^d))$ be the entropy solution of (2.11), (2.19) or (2.20) with $0 \le u(0) = u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. Assume that $u \in BV_{loc}(Q_T)$. Then the following statements hold true:*

(1) *Case of (2.11): The entropy conditions (8.3) hold if and only if for almost any $t \in (0,T)$*

$$[\mathbf{z} \cdot \nu^{J_{u(t)}}]_+ = (u^+(t))^m \quad \textit{and} \quad [\mathbf{z} \cdot \nu^{J_{u(t)}}]_- = (u^-(t))^m \tag{8.5}$$

hold $\mathcal{H}^{d-1}$-a.e. on $J_{u(t)}$. Moreover the speed of any discontinuity front is

$$\mathbf{v} = \frac{(u^+(t))^m - (u^-(t))^m}{u^+(t) - u^-(t)}. \tag{8.6}$$

(2) *Case of (2.19): Assume further that Φ, Λ satisfy Assumptions 2.1. Then the entropy conditions (8.4) hold if and only if for almost any $t \in (0,T)$*

$$[\mathbf{z} \cdot \nu^{J_{u(t)}}]_+ = \Lambda(u^+(t)) \quad \textit{and} \quad [\mathbf{z} \cdot \nu^{J_{u(t)}}]_- = \Lambda(u^-(t)) \tag{8.7}$$

hold $\mathcal{H}^{d-1}$-a.e. on $J_{u(t)}$. Moreover the speed of any discontinuity front is

$$\mathbf{v} = \frac{\Lambda(u^+(t)) - \Lambda(u^-(t))}{u^+(t) - u^-(t)}. \tag{8.8}$$

(3) *Case of (2.21): Assume further that φ defined by (5.3) is a convex function. Then the entropy conditions (8.3) hold if and only if for almost any $t \in (0,T)$*

$$[\mathbf{z} \cdot \nu^{J_{u(t)}}]_+ = \varphi(u^+(t)) \quad \textit{and} \quad [\mathbf{z} \cdot \nu^{J_{u(t)}}]_- = \varphi(u^-(t)) \tag{8.9}$$

hold $\mathcal{H}^{d-1}$-a.e. on $J_{u(t)}$. Moreover the speed of any discontinuity front is

$$\mathbf{v} = \frac{\varphi(u^+(t)) - \varphi(u^-(t))}{u^+(t) - u^-(t)}. \tag{8.10}$$

Remark 8.7. Under some additional assumptions we may derive from either (8.6), (8.8) or (8.10) a vertical contact angle condition, as first pointed out in [61]. For that we assume that for $\mathcal{H}^d$-almost $x \in J_u$ there is a ball B_x centered at x such that either (a) $u_{|B_x} \ge \alpha > 0$ or (b) $J_u \cap B_x$ is the graph of a Lipschitz function with $B_x \backslash J_u = B_x^1 \cup B_x^2$, where

B_x^1, B_x^2 are open and connected and $u \geq \alpha > 0$ in B_x^1, while the trace of u on $J_u \cap \partial B_x^2$ computed from B_x^2 is zero. Then, the following statements (encoding in a weak form the fact that solutions display vertical contact angles at jump points) hold:

(1) For equation (2.11), in both cases $\left[\frac{\nabla u}{\sqrt{u^2+\frac{\nu^2}{c^2}|\nabla u|^2}} \cdot \nu^{J_{u(t)}}\right]_+ = 1$ on $J_u \cap B_x$. If (a) holds, we also have $\left[\frac{\nabla u}{\sqrt{u^2+\frac{\nu^2}{c^2}|\nabla u|^2}} \cdot \nu^{J_{u(t)}}\right]_- = 1$ on $J_u \cap B_x$.

(2) For equation (2.19), in both cases $\left[\frac{\nabla \Phi(u)}{\sqrt{1+\beta|\nabla \Phi(u)|^2}} \cdot \nu^{J_{u(t)}}\right]_+ = 1$ on $J_u \cap B_x$. If (a) holds, we also have $\left[\frac{\nabla \Phi(u)}{\sqrt{1+\beta|\nabla \Phi(u)|^2}} \cdot \nu^{J_{u(t)}}\right]_- = 1$ on $J_u \cap B_x$.

(3) For equation (2.21), in both cases $\left[\psi(\theta(u)\nabla u) \cdot \nu^{J_{u(t)}}\right]_+ = 1$ on $J_u \cap B_x$. If (a) holds, we also have $\left[\psi(\theta(u)\nabla u) \cdot \nu^{J_{u(t)}}\right]_- = 1$ on $J_u \cap B_x$.

9. Finite propagation speed in a general setting

Probably the most distinctive feature of flux-saturated models like (1.5) is the finite propagation speed property for support growth. It is not only the fact that the spreading rate of a compactly supported solution is finite, but the fact that it can be bounded above by a universal quantity (the speed of sound/light depending on the applications) and that generically this universal bound is indeed the actual spreading rate of the support.

This has been justified heuristically in a number of ways, as we have seen above, but the first rigorous installment of such properties was not given until 2006 in [17]. The main idea of that paper is that for compactly supported solutions to (1.5), the support at time t is generically the Minkowsky sum of the initial support and a ball of radius ct. Stated in another way, the spreading rate of the support is precisely given by the constant c.

Let us discuss in some detail those results in [17] leading to this conclusion. First, a general comparison principle was given, see Theorem 5.10 in Section 5. It has been improved in [83] to cover more general situations. This allows to use sub- and super-solutions to control the spreading rate of compactly supported solutions, which is precisely the next step taken in [17]. To bound from above the speed of propagation, they show that

$$\|u_0\|_\infty \chi_{\Omega(t)}, \quad \Omega(t) := \operatorname{supp} u_0 \oplus B(0, ct),$$

being $\oplus$ the Minkosky sum of two set, is a super-solution dominating the entropy solution $u(t)$ with initial datum u_0. This fact is easilly carried onto many other flux-limited models, as shown in [83] (which includes in particular the classes (2.19) [62], (2.20) [48] and easily generalizes without change to the case of (2.21)). The key idea is the fact that the fastest spreading rate for the support will be attained when the slopes at the interface are very high, that is, the case of a traveling front. Then the object that codifies the behavior of the equation at that regime is the recession function of the Lagrangian f (see section 5.1 and specially (5.3)).

It turns out that

$$U(t,x) = \|u_0\|_\infty \chi_{\Omega(t)}, \quad \text{being } \Omega(t) := \overline{(\operatorname{supp} u_0 \oplus B(0, \varphi'(\|u_0\|_\infty)t))}$$

is a super-solution. A number of immediate consequences follow:

(1) The speed of propagation of pure flux-saturated models — no porous media terms — is always bounded above by a universal constant, as claimed in the introduction. In particular, the speed of propagation for (1.5) and generally any model of the form (2.10) is bounded above by c.

(2) The speed of propagation for (2.9) is also bounded above by c.

(3) The speed of propagation for (2.11) is bounded above by $cm(\|u_0\|_\infty)^{m-1}$. Note that the bound is no longer universal but depends on both some of the constants of the model and on the particular initial datum.

To find suitable sub-solutions is a subtler task. One of the major contributions of [17] was to construct a family of compactly supported sub-solutions to (1.5) which spread with a rate given by c. These sub-solutions have the following form: Given $R_0, \alpha_0 > 0$ and $\gamma_0 \geq 0$, there are values $\beta_1, \beta_2 > 0$ large enough such that

$$u(t,x) = \begin{cases} \exp\{-\beta_1 t - \beta_2 t^2\} \left(\alpha_0 \frac{c}{\nu} \sqrt{(R_0+ct)^2 - |x|^2} + \gamma_0\right), & \text{if } |x| < R_0 + ct \\ 0, & \text{if } |x| \geq R_0 + ct \end{cases}$$

is an entropy sub-solution of the relativistic heat equation.

Hence we have that, generically, the spreading rate of compactly supported solutions to (1.5) is exactly c. But we also get some extra bits of information which are interesting. First, these sub-solutions ensure that local positivity is maintained during evolution. And second, they also show that discontinuous interfaces stay put forever in a number of cases (sandwiching with the previous super-solutions). Thus, there cannot be regularizing effects at the tip of the support for solutions of (1.5) and the maximum regularity which is to be expected for u_t is, generically, that it is a Radon measure. Let us also mention the fact that a similar family of subsolutions was constructed in [48] for some of the models having the form (2.20), thus extending the previous results to a broader class of equations.

Note that some of these results could have been partially derived using Rankine–Hugoniot's relation. The main point here is that Rankine–Hugoniot's law is valid as long as interfaces remain discontinuous, but it provides no estimate on the life span of such a discontinuous interface. This is of course one of the major contributions in [17, 48], as the families of sub-solutions so constructed allow to ensure that discontinuous interfaces are eternal in a varied number of cases.

Finally, let us mention that the finite propagation speed property can be also connected in some cases to optimal transport approaches. Following the ideas back in Section 2.2, if the model under study has the property of universal finite speed of propagation (let's say c), then the associated cost function should reflect this fact. Namely, if we want to displace something a distance greater than c per unit time as we follow the Jordan–Kinderlehrer–Otto optimal transport scheme, we should pay infinite to make that movement (thus banning such a possibility). This was shown for (1.5) in [103], providing a different justification of the finite propagation speed property. Pursuing this line further, note that [48] shows that any flux-saturated model whose structure agrees with that of (2.20) is connected with a cost function whose domain is contained in a ball of finite radius (being that radius the maximum speed allowed). This result has a suitable generalization to the case of those flux-saturated models having the form (2.21) when $\varphi(z) = z$. Equating (2.12) and (2.21), we readily get that $\psi(r) = \nabla k^*(r)$ (as in [48]) and $\nabla(F'(u)) = \theta(u)\nabla u$, which amounts to

$$F''(z) = \theta(z). \tag{9.1}$$

Provided that (9.1) holds, a statement like the one in [48] can be proved. For instance, if $\theta(z) = z^\alpha, \alpha > -1$ then the model (2.21) can be formally obtained when $\varphi(z) = z$ from a minimization scheme associated to a convex entropy of the form

$$F(z) = \frac{z^{\alpha+2}}{(\alpha+1)(\alpha+2)}$$

and a cost function which is finite only on a ball of finite radius. This is coherent with the results concerning (2.14) that were mentioned back in Section 2.2.

10. Regularity results

This section deals with further regularity properties of solutions to flux-saturated equations (with porous-media type terms eventually) than those barely predicted by well-posedness results in the class of entropy solutions. We are talking here mainly about time regularity, propagation of Lipschitz regularity and smoothing effects. To the best of our knowledge, most of the studies so far deal with these issues only for the Cauchy problem on the whole space. This will be the case below unless we explicitly state it otherwise. It is to be expected that the addition of boundary conditions may pose additional substantial challenges.

10.1. Earlier regularity results. First regularity results we know of focus in particular classes of solutions (with the exception of those in [67], already discussed in Section 7.2). Regularity results for the class of log-concave solutions to (1.5) (*i.e.* $x \mapsto \log(u(t,x))$ is concave) were provided in [13]. More precisely, consider the class of log-concave initial data which are compactly supported on a smooth convex domain, and such that they are smooth inside their support, bounded away from zero and with vertical contact angles at the boundary. Then the concept of entropy solution for this family of initial data boils down to check that certain degree of smoothness is fulfilled inside the support, that the contact angles at the boundary are vertical and that the equation is verified inside the support. Put it in another words, such class of initial data provides entropy solutions wich are log-concave, smooth inside their support and with vertical contact angles at the boundary. These results were also extended to treat with the case of a reaction term, see [15].

Spherically symmetric solutions are studied in [20]: Initial data which are compactly supported and with discontinuous interfaces are considered. Interior regularity for entropy solutions to (1.5) launched by these initial data is analyzed both from analytical and numerical points of view. Local-in-time existence of radially symmetric smooth solutions (inside the support) for smooth initial conditions whose only discontinuities are at the boundary of its support are obtained for (1.5), as explained in Section 7.2. These results are expected to be global in time (note that there is no waiting time phenomena for (1.5)), but no proof for this seems to be available up to date.

10.2. Partial regularity results on the time derivative. We have seen previously that flux-limited equations (and hence also those of porous media type) have a certain tendency to support discontinuous traveling fronts. Therefore, we cannot expect the time derivative of a solution to be better than a Radon measure. Is it possible to ensure that it won't be worse than that? To get some understanding of this problem is a paramount issue, as the Rankine–Hugoniot theory previously presented in Section 8 rests on the fact that entropy solutions $u(t)$ are assumed to verify

$$u \in BV_{loc}((0,T) \times \mathbb{R}^d). \tag{10.1}$$

Concerning spatial regularity, we have already seen that entropy solutions are of bounded variation if the initial condition is, cf Remark 5.7. Clearly this is not enough to fulfill (10.1); we need some extra knowledge about the time derivative. Namely, we need u_t to be a Radon measure. Therefore, solutions not complying with this regularity condition are pathological enough so that currently we do not know how to describe the evolution of their jump discontinuities. On top of that, boundary traces of the flux $\mathbf{a}(u, \nabla u)$ may fail to be uniquely defined now, see Section 5 in [61]. Thus, we want to be able to ensure that under suitable conditions a given initial datum launches an entropy solution such that u_t is a Radon measure. The main references analyzing this subject are [13, 61, 62].

Let us mention that the desired regularity for u_t is easily obtained when the operator is homogeneous of degree $m > 1$, as the homogeneity estimate of Benilan and Crandall in [28] implies that u_t is a finite Radon measure in $(0,T) \times \mathbb{R}^d$ for any $0 \le u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. This is the case with solutions to (2.11) if $m > 1$ and also the case of (2.20) when $\varphi(u) = u^m$ with $m > 1$ [61, 48].

The situation is much more delicate when the operator is homogeneous of degree one, which is for instance the case of (1.5) and (2.9). It is conjectured in [61, 62] that solutions to (1.5), (2.19) verify that $u \in BV([\tau, T] \times \mathbb{R}^d)$ for any $0 < \tau < T$ as soon as $u_0 \in BV(\mathbb{R}^d)$, but no proof for this is available yet. Some partial results have been obtained instead, showing that $u_t(t)$ is a Radon measure for any $t > 0$ under various sets of conditions. The rationale behind such is the same in all results up to date: Quoting [61, 62], what is used is the basic result that if the initial condition u_0 is in the domain of $\mathcal{B}$, then $\|u_t(t)\|_1 \le \|\mathcal{B}u_0\|_1$ [73]. To use this result, a set of conditions on u_0 are given depending on the case that guarantee that it can be approximated by $u_{0n} \in (L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))^+$ in the domain of $\mathcal{B}$ with $\|\mathcal{B}u_{0n}\|_1$ bounded. As a consequence, it is derived that if $u_n(t)$ is the entropy solution of the pertinent equation with $u_n(0) = u_{0n}$, then $\|u_{nt}(t)\|_1 \leq \|\mathcal{B}u_{0n}\|_1$ for any $n \geq 1$ and any $t > 0$. Letting $n \to \infty$ it is deduced that $u_t(t)$ is a Radon measure for any $t > 0$. As a consequence $S(u) \in BV([\tau, T] \times \mathbb{R}^d)$ for any $0 < \tau < T$ and any truncation $S \in \mathcal{T}_r$. If, in addition, $u_0 \in BV(\mathbb{R}^d)$, then finally $u \in BV([\tau, T] \times \mathbb{R}^d)$ for any $0 < \tau < T$.

The first such result concerns (1.5) and was proved in [13]. It specifies that the previous approximation procedure can be performed if the initial datum $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is supported in a domain Ω of class $\mathcal{C}^{2,1}$, it is bounded away from zero in its support and satisfies $u_0 \in W^{1,\infty}(\Omega) \cap W^{2,1}(\Omega)$; hence for such set of initial conditions the time derivative is a Radon measure. This result was later extended in [61] for the case of initial conditions having a jump set composed of smooth $\mathcal{C}^{2,1}$ hypersurfaces, such that u_0 is in $W^{1,\infty} \cap W^{2,1}$ away from such jump set (plus some other technical conditions, we refer the reader to the precise statement of Lemma 4.1 in [61]). Afterwards the result in [61] was generalized, under the same conditions on the initial data than in [61], to cover a subclass of those equations of the form (2.19) which includes in particular

$$u_t = \alpha \operatorname{div}\left(\frac{u^r \nabla u^m}{\sqrt{1+\beta|\nabla u^m|^2}}\right), \quad \alpha, \beta > 0,\ r \geq 1$$

for the cases $m \geq 2$ and $r \geq m$. See [62] for more details.

10.3. Waiting time phenomena. We know that initial conditions with discontinuous interfaces will start to spread instantaneously as per Rankine–Hugoniot relations. This may not be the case if an initial condition displays continuous interfaces. Some times the support will start to spread immediatly, as we have seen already for (1.5) and related models in Section 9, but this is not the case for some of the porous media counterparts of these models, in which case we say that there is a (finite) waiting time for the support to spread. Some numerical evidences of this phenomenon can be found in [67, 20, 56, 122, 49] and in Fig. 2. There are some analytical results in [83, 49] (see also [67], Theorem 2.2).

Now, we show the existence of waiting times for some one-dimensional models of the form (2.20). Specifically, we will deal with

$$\frac{\partial u}{\partial t} = \left(\varphi(u)\, \psi\left(\frac{u_x}{u}\right)\right)_x, \tag{10.2}$$

verifying that

$$\varphi(s) \leq c s^\delta \text{ and } \varphi'(s) \leq c\delta s^{\delta-1}, \text{ with } \delta > 1\,, \tag{10.3}$$

for some $c > 0$, together with the required well-possednes Assumptions 2.2 and 2.3. Following the arguments developed in [50] (see also [83]) based on the existence of supersolutions, we can deduce a finite waiting time for solutions to (10.2) with initial data compactly supported in $[a, b]$ and verifying some decay estimates at the boundary of the support.

Theorem 10.1. *Let $\delta > 1$ given by equation (10.2)–(10.3). Then, any solution u to (10.2) with initial condition $u_0 \in \mathrm{L}^\infty(\mathbb{R})$ such that*

- *u_0 is compactly supported on $[a, b]$,*
- *u_0 verifies strong enough decay estimates at the boundaries, i.e.*

$$\frac{u_0(x)}{(x-a)^\gamma}, \ \frac{u_0(x)}{(b-x)^\gamma} \quad \textit{belong to } \mathrm{L}^\infty(\mathbb{R}), \ \textit{with } \frac{2}{\gamma} < \delta - 1,$$

can be majorized by a super-solution supported on $[a, b]$ for $t \in [0, T]$, being $T > 0$ some time instant depending on u_0. Therefore, the support of u does not increase during that time interval.

Proof. Let us prove the local in time existence of a super solution of the form $\tilde{u}(t, x) = \alpha(t)\beta(x)$, being $\beta(x) = ((x-a)(b-x))_+^\gamma$, where $(\cdot)_+$ denotes the positive part. It is a simple mater to get the following estimates:

$$|\beta'(x)| \le \gamma(b-a)\beta^{\frac{\gamma-1}{\gamma}}(x)\,, \quad \left|\left(\frac{\beta'(x)}{\beta(x)}\right)'\right| \le \gamma \frac{(b-a)^2}{\beta^{\frac{2}{\gamma}}(x)}, \qquad \text{for } x \in [a, b]\,.$$

Combining the above estimates together with hypothesis (10.3) and the uniform bounds on ψ and ψ' (Assumptions 2.2), we find

$$\left(\varphi(\tilde{u})\,\psi\left(\frac{\tilde{u}_x}{\tilde{u}}\right)\right)_x \le Kc\gamma(b-a)\alpha^\delta(t)\left(\beta^{\delta-\frac{1}{\gamma}} + (b-a)\beta^{\delta-\frac{2}{\gamma}}\right)\,.$$

From this estimate we can conclude

$$\frac{\partial \tilde{u}}{\partial t} = \alpha'(t)\beta(x) \ge \left(\varphi(\tilde{u})\,\psi\left(\frac{\tilde{u}_x}{\tilde{u}}\right)\right)_x$$

provided that

$$\frac{\alpha'(t)}{\alpha^\delta(t)} \le Kc\gamma(b-a)\left(\beta^{\delta-\frac{1}{\gamma}-1}(x) + (b-a)\beta^{\delta-\frac{2}{\gamma}-1}(x)\right)$$

holds. Since $\frac{2}{\gamma} < \delta-1$ we get a global estimate of the right hand side allowing to establish the local-in-time existence of α. Note that $\alpha(0)$ should to be chosen big enough in order to have $u_0 \le \tilde{u}$, which is always possible thanks to the hypothesis on u_0. The comparison results quoted in Section 5.4 give the bound $u(t, x) \le \tilde{u}(t, x)$ justifying the waiting time during which $\alpha(t)$, and as a consequence $\tilde{u}$, exists. □

10.4. Partial regularity results based on Bernstein's approach. The idea behind Bernstein's approach is to get bounds on the spatial gradient by means of a maximum principle applied to an equation satisfied by $|\nabla u|^2$, which we derive from the equation satisfied by u itself. As long as there is some chance of getting waiting time phenomena, only local-in-time bounds may be furnished by this procedure. Namely, if waiting times are present, singularities in the spatial derivative will show up eventually, no matter the initial degree of smoothness. This explains why global-in-time regularity results are not usually obtained in this setting, which is reflected by the fact that those estimates provided by

Bernstein's method may blow up in finite time. Note that this provides a lower bound for the waiting time itself, were it present.

These ideas were applied to some extent in [67], where partial regularity results for (2.17) were stated when $g(u) = u$. Namely, when $0 < n \leq 1$ they show that $\|u_x\|_\infty \leq \|(u_0)_x\|_\infty$, while when $n > 1$ they show that $\|u_x\|_\infty$ is bounded for every finite time if certain requirements on the range on the initial datum are met (involving strong separation from zero, see Theorem 7.1 for details).

A more sistematic treatment was given in [51]. Here the following question is addressed for both (2.9) and (2.11): If we start with a Lipschitz continuous initial datum, will it remain so during evolution? The answer is affirmative at least for the one dimensional versions of (2.9)–(2.11) in the range $0 < m < 1$ (this goes in the same line of the results in [67]). Otherwise the question cannot be addressed by means of Bernstein type estimates, as those obtained in [51] are local in time. A number of remarks are in order, though:

- there are a number of sub-cases depending on the spatial dimension and the exponent m, which are summarized in table *1* in [51]. The large number of sub-cases suggests that those results are probably non-optimal.
- as we mentioned above, if we track the proofs in [51] we can produce explicit lower bounds on the waiting time (whenever applicable).

The results in [51] imply in particular that for the one-dimensional relativistic heat equation, Lipschitz initial conditions launch spatially Lipschitz solutions, altough not explicitly stated there (see their Proposition 4.5, compare with [13, 56]). This rationale can be partially extended to mode general models. For instance, the one-dimensional version of (2.20) is such that Lipschitz initial data launch spatially Lipschitz solutions. This can be obtained arguing as in [51] and using the following Bernstein-type estimates.

Lemma 10.2. *Let $d = 1$. Assume that $\psi' > 0$ and $\psi \in \mathcal{C}^2(\mathbb{R})$. Let $u(t,x)$ be a solution of (2.20) with $\mathcal{C}^1$-regularity with respect to t and $\mathcal{C}^3$-regularity with respect to x. Assume that there is some $\alpha > 0$ such that $u(t,x) \geq \alpha > 0$ for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Then there exists some finite $C \geq \|(u_0)_x\|_{L^\infty}$ such that $\|u_x(t)\|_{L^\infty} \leq C$, $\forall t \geq 0$.*

Proof. Let $w = (u_x)^2/2$. We have

$$w_x = u_x u_{xx}, \quad w_{xx} = (u_{xx})^2 + u_x u_{xxx}, \quad w_t = u_x u_{tx}. \tag{10.4}$$

and

$$u_t = u_x \psi(u_x/u) + u\psi'(u_x/u)[u_x/u]_x$$

so that

$$u_{tx} = u_{xx}\psi(u_x/u) + 2u_x\psi'(u_x/u)[u_x/u]_x \\ + u\psi''(u_x/u)([u_x/u]_x)^2 + u\psi'(u_x/u)[u_x/u]_{xx}.$$

Hence

$$
\begin{aligned}
w_t &= w_x \psi(u_x/u) + 4w\psi'(u_x/u)\frac{uu_{xx}-2w}{u^2} + u\psi''(u_x/u)\frac{u_x(uu_{xx}-2w)^2}{u^4} \\
&\qquad + uu_x\psi'(u_x/u)\left\{\frac{u_{xxx}}{u} - 3\frac{w_x}{u^2} + 4\frac{u_x w}{u^3}\right\} \\
&\le \psi'(u_x/u)w_{xx} + w_x\left\{\psi(u_x/u) - \frac{u_x w_x}{u}\psi'(u_x/u)\right\} + 4\frac{u_x}{u^3}\psi''(u_x/u)w^2.
\end{aligned}
$$

Then we have that $\psi''(u_x/u) \le 0$ for $u_x/u \gg 1$ and $\psi''(u_x/u) \ge 0$ for $-u_x/u \gg 1$. Thus, we can ensure that there exists some $C > 0$ (depending only on ψ) such that either $w \le C$ or

$$
w_t - \psi'(u_x/u)w_{xx} - w_x\left\{\psi(u_x/u) - \frac{u_x w_x}{u}\psi'(u_x/u)\right\} \le 0.
$$

The expression on the left hand side of the previous inequality is a strictly parabolic operator whose coefficients are bounded thanks to our hypothesis. Using a comparison principle we can derive the estimate in the statement. □

Then we would get the following regularity result for the one-dimensional case.

Proposition 10.3. *Let $d = 1$. Assume that $\psi' > 0$ and $\psi \in \mathcal{C}^2(\mathbb{R})$. Let $u_0 \in W^{1,\infty} \cap L^1$. Then the entropy solution u of (2.20) launched by u_0 verifies $u \in L^\infty(0,\infty, W^{1,\infty})$.*

Arguably the extra restrictions on ψ can be bypassed. We will not investigate this point here.

Remark 10.4. The same results hold for (2.21) provided that $2\theta'(z) + z\theta''(z) \le 0$ for every $z \ge 0$. This includes in particular the case of $\theta(z) = z^\alpha$ for $\alpha \in [-1, 0]$.

10.5. Partial regularity results: The one-dimensional relativistic heat equation. Bernstein-type regularity results for (1.5) can be carried much further in the one-dimensional case. Namely, it can be shown that solutions become smooth inside of their support after a finite time. The first major step taken in this direction was presented in [56]. They show that not only Lipschitz initial data remain so during evolution, but that if they are strongly separated from zero inside the support then the solution becomes inmediatly smooth inside the support (furthermore if $u_0 \in W^{1,2}$ then u_t is a Radon measure, as per the discussion back in Section 10.2). A similar result holds for the case of discontinuous interfaces and initial Lipschitz regularity inside the support. The main idea to show these is to pass to a dual problem stated in terms of the inverse distribution function by using a transformation called "the mass coordinate" of Lagrange. This dual problem, that transforms the support into a known domain, has some regularity properties that are typical of uniformly elliptic operators of second order, see [105] for references. Namely, let $u(t)$ an entropy solution to (1.5) in dimension one, let M be its mass and consider

$$
(a(t), b(t)) := (\min \operatorname{supp} u(t), \max \operatorname{supp} u(t)).
$$

Assume for simplicity that the support is connected. Then the inverse distribution function $\varphi(t,\cdot) : (0, M) \to (a(t), b(t))$ is defined by

$$\int_{a(t)}^{\varphi(t,m)} u(t,x)\,dx = m, \quad m \in (0, M). \tag{10.5}$$

Now we perform one further change of variables, letting $v(t,m) := \frac{\partial\varphi}{\partial m}(t,m)$, which relates to $u(t,x)$ by means of

$$v(t,m) = \frac{1}{u(t,\varphi(t,m))}. \tag{10.6}$$

This function v satisfies the following equation:

$$\frac{\partial v}{\partial t} = \left(\frac{\nu v_m}{\sqrt{v^4 + \frac{\nu^2}{c^2}(v_m)^2}} \right)_m, \quad t > 0,\ m \in (0, M). \tag{10.7}$$

Then (10.7) provides a dual formulation in which some features are analyzed in an easier way than in the original formulation (1.5). In particular, Bernstein-type estimates for (10.7) with suitable boundary conditions can be shown and then be translated for the original solution.

The second major step was taken in [49], which extends the finite time smoothing effect to the case of a finite number of non-Lipschitz continuity points (like Hölder cusps) and jump discontinuities. These ideas are able to handle also isolated zeros but require the a-priori knowledge of the fact that u_t be a Radon measure (because it relies heavily on the material in Section 8; as mentioned in Section 10.2 this fact is not yet completely characterized in terms of the initial datum, although there are some sufficient conditions for it to hold). The main observation here is that, if we think of u as a temperature, jump discontinuities determine (moving) adiabatic walls while they stand, thereby allowing to apply the techniques put forward in [56] to each isolated sub-system.

These ideas have also been partially extended to other one-dimensional models. Concerning (2.9), it is shown in [49] that compactly supported initial data supporting a finite number of singularities (non-Lipschitz continuity points and /or jump discontinuities) which are bounded away from zero do not develop further singularities *prior to contact time with zero* (and in fact anything but jump discontinuities is instantaneously smoothed out). This is a real issue for (2.9) in dimension one, as it was also shown in [49] that jump discontinuities in general and discontinuous interfaces for (2.9) in particular are dissolved within a finite time, at least in the one-dimensional case (which is in clear contrast to the case of (1.5) and other models already mentioned in Section 9, for which discontinuous interfaces are eternal). This was shown by comparison with a special family of traveling waves which serve as suitable super-solutions. Then interfaces become eventually continuous and what happens afterwards, from the point of view of inner regularity, is essentially open.

11. Asymptotic regimes from FLPME to classical counterparts: Heat and porous media equations

We saw already that when $c \to \infty$, solutions of (1.5) formally converge to (2.2), while when $\nu \to \infty$ we formally arrive to (1.6). These asymptotic regimes for the relativistic heat equation were rigorously stated in [59, 18]. More precisely, for the regime $c \to \infty$ we have that given an entropy solution u_c to the Cauchy problem for (1.5) such that the initial datum satisfies $0 \le u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then as $c \to \infty$, u_c converges in $\mathcal{C}([0,T], L^1(\mathbb{R}^d))$ to the solution of the Cauchy problem for (2.2) with initial datum u_0. This was shown in [59] with a proof that rests on an a priori estimate on $\|\nabla u/u\|_\infty$. This estimate is obtained by means of Bernstein's method, by considering v such that $u = e^v$, which verifies (after normalization)

$$\frac{\partial v}{\partial t} = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) + \frac{|\nabla v|^2}{\sqrt{1+|\nabla v|^2}}.$$

As regards the regime $\nu \to \infty$, analytical results were given in [18]. Given an entropy solution u_ν to the Cauchy problem for (1.5) such that the initial datum satisfies $0 \le u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then, as $\nu \to \infty$, u_ν converges in $\mathcal{C}([0,T], L^1(\mathbb{R}^d))$ to the entropy solution of the Cauchy problem for (1.6) with initial datum u_0. This follows as a consequence of the convergence of the associated resolvent operators. In fact, these results show indeed the well-posedness of (1.6). Note that (1.6) does not fit into the framework of Section 5, then a separate framework has to be formulated for it. This is precisely done passing to the limit those notions that operate nicely for (1.5). The associated elliptic problem for (1.6) is actually solved in this fashion; then the associated semigroup is constructed, which allows to conclude the existence for the parabolic problem by the convergence of the associated resolvent operators as $\nu \to \infty$. Let us finally mention that this convergence has been also treated from a numerical point of view in [56].

These results are expected to hold for a wider spectrum of models. As a matter of fact, [63] shows that solutions of the Cauchy problem for (2.9) converge when $m > 0$ to those of

$$u_t = \nu \operatorname{div}(u \nabla u^m),$$

while solutions of (2.11) converge to those of

$$u_t = \nu \operatorname{div}(u^{m-1} \nabla u)$$

for $m \ge 1$. Note that $0 < m < 1$ would amount to consider the porous media equation in the fast diffusion range, see the introduction and next section; since these models are troublesome from the point of view of well-posedness, we expect the analysis of this particular asymptotic regime to be more involved.

In fact, [63] proves a much more general statement. Namely, consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{\Lambda(u)\nabla\Phi(u)}{\sqrt{1+|\nabla\Phi(u)|^2/c^2}}\right), & \text{in} \quad Q_T = (0,T)\times\mathbb{R}^d \\ u(0,x) = u_0(x), & \text{in} \quad \mathbb{R}^d \end{cases}.$$

Assume that Λ, Φ satisfy Assumptions 2.1 together with $\Phi \in C^1(0,\infty)$ and Λ increasing. Letting u_c be the entropy solution of the previous Cauchy problem with $0 \le u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then as $c \to \infty$ the sequence u_c converges in $\mathcal{C}([0,T], L^1(\mathbb{R}^d))$ to the solution of the generalized porous media equation

$$u_t = \mathrm{div}(\Lambda(u)\nabla\Phi(u)), \quad u(0,x) = u_0.$$

Note that this includes some models of the form (2.17) [67].

The proof of this result is based on the convergence of the resolvents for both problems, an approach that also works for the case of (1.5) and (2.2). To show that the spatial derivative of the limit has no singular part a convex duality method by [41] is used (see also [65, 111]). This requires to construct a specific family of sub-solutions at the resolvent level, for which some extra conditions on Φ are needed, see [63] for details. The proof of a similar statement for (2.11) is essentially the same and is also contained in [63].

We also mention the work in [70], see Theorem 7.3, which can be regarded as a statement concerning the limit of those flux-saturated models found in [70] when the maximum propagation speed tends to infinity. Given that there are no porous media terms in their models, this limit is described by the standard heat equation.

12. The fast diffusion range

As we have seen in the previous section, solutions to (2.11) for $0 < m < 1$ formally converge to the porous media equation in the fast diffusion range. It is known that if the spatial dimension $d \ge 3$, then (1.4) in the range $0 < m < (d-2)/d$ features properties wich are radically different from those of the standard porous media equations [128]. In a nutshell, diffusion may be indeed too fast (hende the name “fast diffusion”), causing losses of mass to infinity and lack of uniqueness for the Cauchy problem. This poses a different scenario from the modeling point of view, as new phenomena show up from the same formal structure, but raises a number of problems in order to analyze these models in a completely rigorous way.

Then, we may expect to have some of these problems also when considering (2.11) in the “fast diffusion range” above specified. A quick glance shows that (2.11) is not included in the previous set of well-posedness theories when $0 < m < 1$. Presently this is not completely well understood and there are a number of interesting open problems pertaining the behavior of (2.11) (and related models) in the fast diffusion range. Though, several advances were made in [51]. A concept of entropy solution was proposed there, which is roughly the same as in Section 5 with the extra assumption of null flux at infinity. The idea is that preventing mass leakage through infinity will help avoid non-uniqueness phenomena. In fact, it was shown that, under such concept of solution, well-posedness for the Cauchy problem holds in dimension one for any initial condition in $(L^1 \cap L^\infty)^+$, while the same is true in higher dimension if the initial datum $u_0 \in (L^1 \cap L^\infty)^+$ satisfies the additional decay condition

$$u_0(x) = o(|x|^{\frac{1-d}{m}}) \quad \text{as} \quad |x| \to \infty \tag{12.1}$$

(and in fact this decay rate is preseved during evolution). Existence is proved as in Section 5: First the associated elliptic problem is studied, then Crandall–Liggett's theorem is used to construct mild solutions, which can be characterized afterwards in more operative terms as entropy solutions in the previous sense. Uniqueness is obtained by a suitable modification of the doubling variables strategy in which the null flux condition in Definition 7.6 — i.e. no leakage of mass — plays a paramount role: It is not known how to prove uniqueness if this condition is not satisfied and in fact we conjecture that uniqueness will break down if there is some loss of mass to infinity. The null flux condition can be shown to hold if the aforementioned decay condition is met, by means of a modified comparison principle for the elliptic problem in a bounded domain and the use of certain super-solutions decaying as power laws with exponent less or equal than $(1-d)/m$.

Apart from that, currently we only have heuristics and numerical evidences (see [20] for instance) in order to ascertain the behavior of such models. Although not proved analytically yet, it is pretty clear that in this case solutions will propagate with infinite speed; it is also quite plausible that smoothing effects will take place instantaneously (besides of those global Lipschitz regularity estimates that we already mentioned in Section 10.4).

13. Traveling waves for flux-saturated-reaction models

Since the apparition of the groundbreaking work by Turing [126], the study of reaction-diffusion equations ad their traveling wave solutions has grown to become a full research area in its own right. There are many situations for which such models can be applied, and the number of different problems in this area that deserve to be studied are virtually countless (see for instance [94]). Our humble contribution here to this body of research focuses on a study of traveling wave solutions to one dimensional flux-saturated equations coupled with a reaction term of Fisher–Kolmogorov type. Several applications in biology (morphogenesis) [4, 32, 52, 126], Social Sciences [25] or traffic flow [34] motivate the study of non-smooth densities such as singular traveling waves [45] . Do they really exist as solutions to flux-saturated models with reaction terms? To answer this question is one of the objectives of the analysis to be carried in this section.

Some of the results of this section are a recreation of the ideas previously introduced in [49, 54], but here we provide a vision and an original development based on the study of first order graph associated with the traveling wave equation.

We look for traveling wave solutions of reaction-diffusion equations having the following general form:

$$\partial_t u = \left(\varphi(u)\,\psi\left(\frac{u_x}{|u|}\right)\right)_x + uK(u), \quad u(t=0,x) = u_0(x),\ x \in \mathbb{R},\ t > 0. \tag{13.1}$$

Here we wrote the reaction term as $F(u) = uK(u)$; the properties of K are stated below.

The profiles we are seeking as solutions to (13.1) are non-negative functions $u(\tau) = u(x-\sigma t)$ defined on $\mathbb{R}$ for some $\sigma \geq 0$, such that

$$\lim_{\tau\to-\infty} u(\tau) = 1 \quad \text{and} \quad \lim_{\tau\to\infty} u(\tau) = 0\,. \tag{13.2}$$

From a different perspective, the existence of other kind of traveling waves such as those with pulses or soliton-type shapes constitutes an interesting problem to be explored which requires at least the choice of different reaction terms, since in the context studied in this section this alternative traveling waves do not appear in our analysis (see for example [113] or [80, 120] in another context). It can be also of interest to consider heterogenous media, where the heterogeneity property can come from the heterogeneous character of the equation and/or from the underlying domain (we refer to [31, 32, 33]), but again this is not the objective of this review.

In the first two subsections we introduce the results, while the proofs are postponed to the last part of this section for a better panoramic view of the theory.

13.1. Statement of the problem and main result. We will restrict ourselves to look for traveling wave solutions in the class of entropy solutions for (13.1). Then we will use those results in Section 7.4 to tackle this issue. As a brief summary, let us note that:

(1) There is no existence theory for solutions to (13.1) with infinite mass. This is not a problem for our particular purpose here as we will explicitly construct entropic solutions of traveling wave type; it will later turn out that none of these has finite mass.

(2) Nevertheless, uniqueness in the class of entropy solutions can be ensured if these traveling profiles happen to have null flux at infinity.

(3) Those results on Section 8 about geometric characterizations of entropy conditions are generalized without much effort to the case in which there is an additional reaction term in the equation. Such extended statements will be crucial for the developments below.

To be completely sure about the validity of the previous statements, functions ψ, φ and K have to be such that those results in Section 5 apply indeed. Let us detail what are our assumptions on these structure functions.

Assumptions 13.1. *Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be a function satisfying Assumptions 2.2 such that* $\psi'(x) > 0$, $\forall x > 0$.

In the case of the functions φ and K, the following properties are considered:

Assumptions 13.2. *Let the functions* $\varphi, K : [0,1] \to [0,\infty)$ *be such that*

(1) $\varphi \in \mathcal{C}^1([0,1]) \cap \mathcal{C}^2(]0,1])$

(2) $\varphi(0) = 0$ *and* $\varphi > 0$ *in* $]0,1]$.

(3) $\varphi'' \geq 0$ *in* $]0,1]$.

(4) $K \in \mathcal{C}([0,1]) \cap \mathcal{C}^1(]0,1])$

(5) $K(1) = 0$ *and* $K > 0$ *in* $]0,1[$

Altogether, Assumptions 13.1 and 13.2 enable to use the entropic well-posedness framework to deal with (13.1) (see Section 5 and [48]).

In order to construct traveling wave profiles we substitute the traveling wave ansatz $u(x-\sigma t)$ into (13.1). This leads to the study of the following equation:

$$\Big(\varphi(u)\psi(u'/u)\Big)' + \sigma u' + uK(u) = 0\,, \tag{13.3}$$

We can use (13.3) to construct piecewise smooth entropy solutions of (13.1). For that, it suffices to join together smooth solutions of (13.3) defined on intervals of $\mathbb{R}$ fulfilling the following rules:

(i) If solutions corresponding to two consecutive intervals match in a continuous way, then the first derivative cannot have a jump discontinuity. If the first derivative is $+\infty$ (resp. $-\infty$) on one side then it must be also $+\infty$ (resp. $-\infty$) on the other side. This can be justified as in [49].

(ii) If solutions corresponding to two consecutive intervals match forming a jump discontinuity, then the speed of the moving front should obey the Rankine–Hugoniot condition (8.10) and the slopes of the profile at both sides of the discontinuity must agree and be either plus or minus infinity (see the geometric interpretation of entropy conditions in Section 8), except when one of the solutions we are matching with is the zero solution. In that case, when looking for decreasing profiles, we only have to worry about the infinite slope condition on the left side of the discontinuity. If φ is not convex then we have no guarantee that the fulfillment of (8.10) plus the conditions in *i)* above ensure that the traveling profile so constructed qualifies as an entropy solution; in that case our construction will be at least formally correct (we produce distributional solutions).

Therefore, we focus on describing smooth solutions of (13.3) defined on $]\tau_{-\infty},\tau_\infty[$ for $\tau_{-\infty},\ \tau_\infty \in [-\infty,+\infty]$, which we will eventually extend afterwards to the whole real line by suitable matching procedures based on the previous observations. In order to accomplish the first stage of this prospect we will write (13.3) as an autonomous planar system. We will explain how to do this in what follows. Throughout the following construction we will assume the following:

$$u'(\tau) < 0 \quad \forall \tau \in]\tau_{-\infty},\tau_\infty[. \tag{13.4}$$

This is done only for technical convenience. Proposition 13.22 below shows that this assumption can be safely removed. Note in passing that the constant solutions $u=1$ and $u=0$ do not satisfy it; nevertheless they will play a very important role in the sequel.

Definition 13.3. We will say that u is a smooth solution of (13.3) if both u and $\varphi(u)\psi(u'/u)$ are $\mathcal{C}^1(]\tau_{-\infty},\tau_\infty[)$ functions verifying (13.3) pointwise.

Remark 13.4. Indeed, we will justify that if $0<u<1$ this will be equivalent to get a classical solution, that is $u\in\mathcal{C}^2(]\tau_{-\infty},\tau_\infty[)$ verifying (13.3).

Definition 13.5. We will say that a traveling wave profile defined in $\mathbb{R}$ saturates the flux if its slope at some point is not finite.

In order to state what are the traveling wave solutions of (13.1), let us first sort them into various convenient sub-classes.

Definition 13.6. Let u be a traveling wave profile defined in $\mathbb{R}$ which is an entropic solution of (13.1). We will say that it is a profile of type *a, b, c, d* or *e* respectively if it matches the corresponding item from the list below:

(a) it is a non-saturated smooth traveling wave solution (that is a $\mathcal{C}^1$ solution supported on the whole real line with finite slopes at any point).

(b) it is a continuous traveling wave supported on a half-line composed of the following two branches: A $\mathcal{C}^2$ solution in a interval $]-\infty, \tau_0[$ verifying $\lim_{\tau\to\tau_0^-} u(\tau) = 0$ and $\lim_{\tau\to\tau_0^-} u'(\tau) = -\infty$, which is extended continuously by zero.

(c) it is a discontinuous traveling wave supported on a half-line composed of the following two branches: A $\mathcal{C}^2$ solution in a interval $]-\infty, \tau_0[$ verifying $\lim_{\tau\to\tau_0^-} u(\tau) > 0$ and $\lim_{\tau\to\tau_0^-} u'(\tau) = -\infty$, which is extended by zero to the right (hence there is a saturated jump discontinuity at the matching point).

(d) it is a continuous traveling wave supported on the whole real line and constituted by two $\mathcal{C}^2$ branches u^-, u^+, defined respectively in intervals $]-\infty, \tau_0[$ and $]\tau_0, \infty[$, such that $\lim_{\tau\to\tau_0^-} u^-(\tau) = \lim_{\tau\to\tau_0^+} u^+(\tau)$ and $\lim_{\tau\to\tau_0^-} (u^-)'(\tau) = \lim_{\tau\to\tau_0^+} (u^+)'(\tau) = -\infty$ (the traveling profile saturates at the matching point).

(e) it is a discontinuous traveling wave supported on the whole real line and constituted by two $\mathcal{C}^2$ branches u^-, u^+, defined respectively in intervals $]-\infty, \tau_0[$ and $]\tau_0, \infty[$, such that $\lim_{\tau\to\tau_0^-} u^-(\tau) > \lim_{\tau\to\tau_0^+} u^+(\tau)$ and $\lim_{\tau\to\tau_0^-} (u^-)'(\tau) = \lim_{\tau\to\tau_0^+} (u^+)'(\tau) = -\infty$ (the traveling profile saturates at the matching point, in which a jump discontinuity is present).

We are now ready to present our main statement regarding the description of traveling wave solutions to (13.1).

Theorem 13.7. *Let Assumptions 13.1 and 13.2 be satisfied. Then, there exist two values $0 < \sigma_{ent} \le \sigma_{smooth}$ depending on φ, ψ and K, such that:*

(1) *If $\sigma > \sigma_{smooth}$ there exists a non-saturated smooth traveling wave solution to* (13.1) *(hence a type* a *profile),*

(2) *$\sigma_{ent} \neq \sigma_{smooth}$ if and only if $\varphi'(0) < \sigma_{smooth}$. In that case*

   (a) *$\varphi'(0) < \sigma_{ent} < \sigma_{smooth}$,*

   (b) *if $\sigma = \sigma_{ent}$ there exist a discontinuous traveling wave solution to* (13.1) *supported on a half-line (hence a type* c *profile),*

   (c) *if $\sigma_{ent} < \sigma < \sigma_{smooth}$ there exist a traveling wave solution to* (13.1) *supported on the whole real line which is discontinuous (hence a type* e *profile),*

(3) *If $\sigma = \sigma_{smooth}$ the following holds:*

   (a) *if $\sigma_{smooth} < \varphi'(0)$, there exists a non-saturated smooth traveling wave solution to* (13.1) *(hence a type* a *profile),*

   (b) *if $\varphi'(0) < \sigma_{smooth}$, then we have two sub-cases. If $(\varphi')^{-1}(\sigma_{smooth})$ is reduced to a single value, then there exists a continuous (non-smooth) traveling wave* *solution to* (13.1) *supported on the whole real line (that would be a type* d *profile). If* $\#(\varphi')^{-1}(\sigma_{smooth})$ *is greater than one, then there exists a traveling wave solution to* (13.1) *supported on the whole real line which is either continuous (non-smooth) or discontinuous (that would be a type* d *or* e *profile).*

(c) *if* $\varphi'(0) = \sigma_{smooth}$ *there exists a traveling wave solution to* (13.1) *that can be of any of the types in Definition 13.6.*

Furthermore:

- *All the previous traveling wave profiles have null flux at infinity and hence they are unique as entropy solutions.*
- *For a given value of* $\sigma \geq 0$ *there does not exist any other entropic traveling wave solution of* (13.1) *in the class of piecewise smooth functions*[2] *but those we enumerated above. In particular, there are no such solutions for* $\sigma < \sigma_{ent}$.

Remark 13.8. If $\psi'(0) > 0$ the non-saturated smooth traveling waves become classical traveling wave solution, that are $\mathcal{C}^2$ solution supported on the whole real line with finite slopes at any point.

Remark 13.9. To make the most out of the previous result we would need a generic recipe to compute the first bifurcation value σ_{smooth}. Unfortunately this seems not to be available in general and such computation has to be performed for each case of interest separately.

The proof of this result is outlined in the following sections. Some technical auxiliary results required for this will be addressed in Subsection 13.3.

13.2. Getting the constituents of the traveling wave profiles: The graph formulation. Hereafter Assumptions 13.1 and 13.2 will be taken for granted with no further mention. To analyze smooth solutions of (13.3) we reformulate it as a planar dynamical system. For that we consider the range of u to be restricted to $]0,1[$. We define for the given range of u an auxiliary variable:

$$r := -\psi(u'/u) \in]0,1[. \tag{13.5}$$

Note that the given range for r is due to Assumptions 13.1 and the fact that $u' < 0$ by (13.4). On this basis, we can invert for u' in the previous relation. Thus (13.3) can be recast as

$$-\varphi(u)r' + \big(\sigma - r\varphi'(u)\big)\, u' + uK(u) = 0$$

and then

$$\begin{cases} u' = & -ug(r), \\ r' = & \dfrac{u}{\varphi(u)}\Big(g(r)\big(r\varphi'(u) - \sigma\big) + K(u)\Big), \end{cases} \tag{13.6}$$

where $g(r) := \psi^{-1} :]0,1[\to]0,\infty[$ is a $\mathcal{C}^1$, strictly increasing function verifying $\lim_{r\to 0} g(r) = 0$, $\lim_{r\to 1} g(r) = \infty$ and $g(r) > 0\ \forall r \in]0,1[$ thanks to Assumptions 13.1. Let us notice that thanks to Assumptions 13.2 the flux of (13.6) is regular in $]0,1]\times]0,1[$.

[2]Meaning that there is a finite set $S = \{p_1, \ldots, p_r\} \subset \mathbb{R}$ such that u is smooth in $\mathbb{R}\backslash S$.

Remark 13.10. If, in addition ψ would verify

$$\psi'(0) > 0 \text{ then } \lim_{r\to 0} g'(r) = \frac{1}{\psi'(0)} < \infty \tag{13.7}$$

which would provide some extra regularity properties for the planar flow of (13.6) in the set $]0,1[\times\{0\}$.

Remark 13.11. If we look for decreasing profiles, we observe that $r(\xi) \in [0,1[$ for all $\xi \in \mathbb{R}$ (while $r(\xi) \in]-1,1[$ for all $\xi \in \mathbb{R}$ if no monotonicity assumption is made, where g has to be suitably extended to $]-1,1[$). Moreover, if $u(-\infty) = 1$, $u(+\infty) = 0$ and u is smooth, then $u(\xi) \in [0,1]$ for all $\xi \in \mathbb{R}$. Then, for smooth solutions, (13.3) is equivalent to the previous first order planar dynamical system. In what follows we shall focus on the study of decreasing traveling profiles, for these are the only reasonable traveling waves that can be obtained, as we show in forthcoming Proposition 13.22. Thus, we will restrict the study of (13.6) to the set $]0,1]\times]0,1[$; this will be implicitly assumed in every statement referring to (13.6).

To deal with the solutions of the planar system (13.6) in $]0,1[\times]0,1[$ we will analyze the equivalent first order equation:

$$R'(u) = \frac{1}{\varphi(u)}\left(\sigma - R(u)\varphi'(u) - \frac{K(u)}{g(R(u))}\right) \tag{13.8}$$

where R is such that $R(u(\tau)) = r(\tau)$ for any given solution (u,r) to (13.6). We will refer to (13.8) as the *graph formulation*. Sometimes we will find useful to stress the dependence of R on σ, for which we will use the notation R_σ.

Let us discuss now how to reflect boundary conditions (13.2) in this new formulation. In terms of the planar orbit (13.6), first condition in (13.2) implies that we seek solutions $(u(\tau), r(\tau))$ defined in $]\tau_{-\infty}, \tau_\infty[$ verifying

$$\lim_{\tau\to\tau_{-\infty}} u(\tau) = 1.$$

If this is verified we can prove that $\lim_{\tau\to\tau_{-\infty}} u'(\tau) = 0$ or equivalently

$$\lim_{\tau\to\tau_{-\infty}} (u(\tau), r(\tau)) = (1,0)\,. \tag{13.9}$$

When $\tau_{-\infty} = -\infty$ this can be rigorously justified arguing as in Lemma 2.1 in [54]. If $\tau_{-\infty} > -\infty$ this solution can be only matched with the constant solution 1 in the interval $]-\infty, \tau_{-\infty}[$ since we are looking for decreasing functions. Then $\lim_{\tau\to\tau_{-\infty}} u'(\tau) = 0$ according to the rules stated after (13.3). Thus, in any case, first condition in (13.2) is replicated in the graph formulation as

$$\lim_{u\to 1} R(u) = 0\,. \tag{13.10}$$

However, there is nothing meaningful that can be said about second condition in (13.2) at this stage; it will require a particular analysis later on.

Proposition 13.12. *For any $\sigma \geq 0$ there exists a unique maximal solution to (13.8) $R_\sigma :]u_\sigma, 1[\to]0, 1[$ verifying (13.10) and $u_\sigma \geq 0$. Furthermore:*

(1) *If $u_\sigma > 0$, then*

$$\lim_{u \to u_\sigma} R_\sigma(u) = 1\,. \tag{13.11}$$

(2) *If $\sigma_1 < \sigma_2$ then*

$$u_{\sigma_1} \geq u_{\sigma_2} \quad \textit{and} \quad R_{\sigma_1}(u) \geq R_{\sigma_2}(u)$$

in their common interval of definition $]u_{\sigma_1}, 1[$.

This is proved in Subsection 13.3. This result is an inestable manifold type theorem for system (13.6) giving us the existence of ordered regular branches of solutions $(\tilde{u}_{\sigma(\tau)}, \tilde{r}_\sigma(\tau))$ verifying (13.9) and such that $R_\sigma(\tilde{u}_\sigma(\tau)) = \tilde{r}_\sigma(\tau)$. Note that this ordering affects both the functions R_σ and their definition domains.

At this point we may formally define the first bifurcation value.

Definition 13.13. Let

$$\sigma_{smooth} = \inf\{\sigma > 0\,; u_\sigma = 0\}\,. \tag{13.12}$$

As a direct consequence of the previous Proposition we deduce that $u_\sigma > 0$ for any $\sigma < \sigma_{smooth}$ and $u_\sigma = 0$ for any $\sigma > \sigma_{smooth}$.

Our next goal is to describe the behavior of the corresponding solutions $(\tilde{u}_\sigma(\tau), \tilde{r}_\sigma(\tau))$. This traveling wave profile $\tilde{u}_\sigma(\tau)$ may or may not saturate the flux depending on R_σ hitting $r = 1$ or not. Our next result describes more precisely when does each of these possibilities take place.

Proposition 13.14. *The following statements hold true:*

(1) $0 < \sigma_{smooth} < \infty$.

(2) *If $\sigma < \sigma_{smooth}$, then (13.11) holds and $\sigma < \varphi'(u_\sigma)$.*

(3) *If $\sigma > \sigma_{smooth}$ then $\sup\left\{R_\sigma(u)\,; u \in]0, 1[\right\} < 1$.*

(4) *The mapping $\sigma \mapsto u_\sigma$ is continuous for any $\sigma \neq \sigma_{smooth}$.*

(5) *If $\sigma_{smooth} < \varphi'(0)$ then $u_{\sigma_{smooth}} = 0$ and $\sup\left\{R_{\sigma_{smooth}}(u)\,; u \in]0, 1[\right\} < 1$.*

(6) *If $\sigma_{smooth} > \varphi'(0)$ then $u_{\sigma_{smooth}} > 0$, (13.11) holds for $\sigma = \sigma_{smooth}$ and $\lim_{\sigma \to \sigma_{smooth}^-} u_\sigma = u_{\sigma_{smooth}} = \varphi'(u_{\sigma_{smooth}})$.*

Remark 13.15. The case $\sigma = \sigma_{smooth} = \varphi'(0)$ requires a more detailed analysis. There are examples in [54] where $u_{\sigma_{smooth}} = 0$ with R_σ not hitting $r = 1$, and also examples for which $u_{\sigma_{smooth}} > 0$ with R_σ actually converging towards 1. The possibility of having $u_{\sigma_{smooth}} = 0$ and $\sup\left\{R_{\sigma_{smooth}}(u)\,; u \in]0, 1[\right\} = 1$ at the same time cannot be discarded.

If

$$\sup\left\{R_\sigma(u)\,; u \in]0, 1[\right\} < 1 \tag{13.13}$$

it can be easily proved that the corresponding traveling wave profile $\tilde{u}_\sigma$ is defined in $]\tau_{-\infty}, \infty[$. Moreover, (13.13) implies that $g(\tilde{r}_\sigma(]\tau_{-\infty}, \tau_\infty[))$ is bounded from above by a certain positive contant C. Upon using (13.6) we can deduce that

$$u(t_2) \geq \exp(-Ct_2)\exp(Ct_1)u(t_1) \quad \text{for any} \quad \tau_{-\infty} \leq t_1 \leq t_2 \leq \tau_\infty. \tag{13.14}$$

Then τ_∞ cannot be finite because of the regularity of the flux of the system (13.6) in $]0,1]\times]0,1[$ (recall that $\lim_{\tau\to\infty}\tilde{u}_\sigma(\tau)=u_\sigma=0$). On the other hand, $\tau_{-\infty}$ can assume any finite value or the value $-\infty$. Regardless of which possibility shows up, we have that $\lim_{\tau\to\tau_{-\infty}}\tilde{u}_\sigma(\tau)=1$ and $\lim_{\tau\to\tau_{-\infty}}\tilde{u}'_\sigma(\tau)=0$ (by definition of R_σ).

In those cases when $u_\sigma>0$ (only possible if $\sigma<\sigma_{smooth}$ or $\sigma=\sigma_{smooth}\geq\varphi'(0)$)

$$\lim_{\tau\to\tau_\infty}\tilde{u}_\sigma(\tau)=u_\sigma \quad \text{and} \quad \lim_{\tau\to\tau_\infty}\tilde{u}'_\sigma(\tau)=-\infty \tag{13.15}$$

hold. Note that, as $g(r)$ is not bounded, (13.15) rules out the possibility of having $\tau_\infty=-\infty$.

In order to obtain a globally defined profile, we may match $\tilde{u}$ with another profile $\bar{u}_\sigma:]\tau_\infty,\bar{\tau}_\infty[\to]0,u_\sigma[$. If this matching is to be continuous then

$$\lim_{\tau\to\tau_\infty^+}\bar{u}_\sigma(\tau)=u_\sigma \quad \text{and} \quad \lim_{\tau\to\tau_\infty^+}\bar{u}'_\sigma(\tau)=-\infty\,.$$

If it is discontinuous then

$$\lim_{\tau\to\tau_\infty^+}\bar{u}'_\sigma(\tau)=-\infty \quad \text{and} \quad \sigma=\frac{\varphi(u^+)-\varphi(u^-)}{u^+-u^-}$$

where u^+ and u^- should assume the values u_σ and $\lim_{\tau\to\tau_\infty^+}\bar{u}_\sigma(\tau)$ respectively. Let us analyze these cases in turn.

In the continuous case we can easily ensure that $\bar{u}_\sigma$ can only exists for $\sigma=\sigma_{smooth}$ and in consequence $u^-=u_{\sigma_{smooth}}$. The existence of $\bar{u}_\sigma$ is equivalent to the existence of a solution R_{u_σ} to (13.8) defined on an interval $]u_\sigma-\delta,u_\sigma[$ such that $\lim_{u\to(u_\sigma)^-}R_{u_\sigma}(u)=1$. Then, there exists an increasing sequence $\{u_n\}_{n\in\mathbb{N}}\to u_\sigma$ such that $\lim_{n\to\infty}R_{u_\sigma}(u_n)=1$ and $R'_{u_\sigma}(u_n)>0$. By evaluating (13.8) in $\{u_n\}_{n\in\mathbb{N}}$ and making n large enough we can get $\sigma>\varphi'(u_\sigma)$. By a similar argument applied to $\tilde{u}$ and R_σ we deduce that $\sigma<\varphi'(u_\sigma)$, and as a by-product $\sigma=\varphi'(u_\sigma)$. This allow us to conclude that $\sigma=\sigma_{smooth}$ thanks to Proposition 13.14.

In the discontinuous case, we have to find those values of σ such that the equation

$$\sigma=\frac{\varphi(u_\sigma)-\varphi(u)}{u_\sigma-u} \tag{13.16}$$

has positive solutions $u\in]0,u_\sigma[$. Note that solutions u^- to (13.16) can only exist when $\sigma_{smooth}\geq\varphi'(0)$, as $\sigma\leq\sigma_{smooth}$ and $\frac{\varphi(u_\sigma)-\varphi(u)}{u_\sigma-u}\geq\varphi'(0)$ — thanks to Assumption 13.2.3. Now we have two different situations. If $\sigma_{smooth}=\varphi'(0)$ then clearly (13.16) can only hold for $\sigma=\sigma_{smooth}$. In fact, in this case φ is a linear function in the interval $[0,u_{\sigma_{smooth}}]$ with slope σ_{smooth}, as per

$$\sigma_{smooth}=\frac{\varphi(u_{\sigma_{smooth}})-\varphi(u)}{u_{\sigma_{smooth}}-u}=\varphi'(\xi)=\varphi'(0), \quad \xi\in]u,u_{\sigma_{smooth}}[.$$

In this case (13.16) would be achieved for any $u\in(\varphi)^{-1}(\sigma_{smooth})$.

In the case $\sigma_{smooth}>\varphi'(0)$ the study of solutions to (13.15) can be performed in terms of a second bifurcation value that we now introduce.

Proposition 13.16. *Assume that $\sigma_{smooth} > \varphi'(0)$. Then (13.17) has a unique solution, which belongs to the interval $]\varphi'(0), \sigma_{smooth}[$.*

Definition 13.17. Let σ_{ent} be the value of σ such that $u^- = 0$, that is, the unique solution to the following equation:

$$\sigma = \frac{\varphi(u_\sigma)}{u_\sigma} \quad \text{for } \sigma \leq \sigma_{smooth}. \tag{13.17}$$

In all those cases where $\sigma_{smooth} \leq \varphi'(0)$ we will define $\sigma_{ent} = \sigma_{smooth}$.

This enables us to state the following.

Proposition 13.18. *Assume that $\sigma_{smooth} > \varphi'(0)$. Then, for any $\sigma < \sigma_{smooth}$ equation (13.16) has a unique solution u^- if and only if $\sigma > \sigma_{ent}$.*

Once we have analyzed the existence of the value u^- depending on σ, we will be able to provide the existence of the function $\bar{u}$ enabling for a matching with the desired properties. In order to do this we need to introduce the following quantity:

$$u^* = u^*(\sigma) := \min\{u; \varphi'(u) = \sigma\}.$$

By Assumption 13.2.3., for any $\sigma \in (\varphi')^{-1}([0,1])$ the sets $(\varphi')^{-1}(\sigma)$ are non-empty intervals. If $\sigma > \varphi'(0)$ then zero is not contained in those. The equation $\varphi'(u) = \sigma$ formally corresponds to the limit case of the Rankine–Hugoniot conditions when $u^+,\ u^- \to u$.

Proposition 13.19. *Assume that $\sigma \geq \varphi'(0)$. Then, for any $u^- \in]0, u^*]$ there exists a unique solution $R_{u^-} :]0, u^-[\to]0, 1[$ to the first order equation (13.8) such that*

$$\lim_{u \to u^-} R_{u^-}(u) = 1.$$

Furthermore, this solution verifies

$$\limsup_{u \to 0} R_{u^-}(u) < 1.$$

If $u^- > u^*$ there does not exist any solution of the first order equation (13.8) such that $\lim_{u \to u^-} R_{u^-}(u) = 1$.

Remark 13.20. In the particular case $\sigma = \varphi'(0)$ last Proposition reduces to a non-existence result since $u^* = 0$, which is induced by the linearity of φ in $[0, \sigma_{smooth}]$.

Sumarizing, if we are given $\sigma \in]\sigma_{ent}, \sigma_{smooth}[$ we can ensure the existence and uniqueness of a function $\bar{u}_\sigma :]\tau_\infty, \infty[\to]0, u_\sigma[$ verifying

$$\lim_{\tau \to \tau_\infty^+} \bar{u}_\sigma(\tau) = u^-, \quad \lim_{\tau \to \tau_\infty^+} \bar{u}'_\sigma(\tau) = -\infty \quad \text{and} \quad \sigma = \frac{\varphi(u_\sigma) - \varphi(u^-)}{u_\sigma - u^-}$$

by using an estimate similar to (13.14). These will be called *continuation solutions*.

13.2.1. Assembling traveling wave profiles: Proof of Theorem 13.7. In this paragraph we will prove Theorem 13.7 collecting all the information of the previous section. From the definitions of σ_{smooth} and σ_{ent} we clearly have that $0 < \sigma_{ent} \leq \sigma_{smooth}$, with equality if and only if $\varphi'(0) \geq \sigma_{smooth}$.

In the cases $\sigma > \sigma_{smooth}$ and $\sigma = \sigma_{smooth} < \varphi'(0)$ those functions $\tilde{u}_\sigma$ defined in $]\tau_{-\infty}, \infty[$ are the unique solutions to (13.3) verifying $\lim_{\tau \to \tau_{-\infty}} \tilde{u}_\sigma(\tau) = 1$ and $\lim_{\tau \to \infty} \tilde{u}_\sigma(\tau) = 0$. If $\tau_{-\infty} = -\infty$ the profile $\tilde{u}$ constitutes by itself a classical traveling wave profile supported on the whole real line since $0 < \tilde{u} < 1$. If $\tau_{-\infty} > -\infty$ the profile $\tilde{u}$ shoud be matched continuously with the constant solution 1 in the interval $]-\infty, \tau_{-\infty}[$ to give rise to a traveling wave profile in $\mathcal{C}^1(]-\infty, \infty[)$. However, this is only possible if $\psi'(0) = 0$, due to the extra regularity of the planar flow (13.6). This justifies points (1) and (3)-(a) of Theorem 13.7.

When $\varphi'(0) < \sigma_{smooth}$, the profile $\tilde{u}_\sigma$ is defined on $]\tau_{-\infty}, \tau_\infty[$ with τ_∞ finite. To extend it to the whole real line we do as follows:

- If $\tau_{-\infty} > -\infty$ then $\tilde{u}$ can be extended continuously to the half-line $]-\infty, \tau_\infty[$ (up to a maching with the constant solution 1).
- If $\sigma = \sigma_{ent}$ then by definition 13.17 $\tilde{u}_{\sigma_{ent}}$ is extended to $]\tau_\infty, \infty[$ by zero, giving rise to a traveling wave suported on a half line.
- Finally, if $\sigma > \sigma_{ent}$ then a discontinuous traveling wave profile is obtained by joining $\tilde{u}$ with the corresponding continuation solution $\bar{u}$.

Those profiles above are the ones presented in Theorem 13.7-(2). The case $\sigma = \sigma_{smooth} > \varphi'(0)$ corresponds to Theorem 13.7-(3)-(b). Here $\tilde{u}$ gives a profile defined in $]-\infty, \tau_\infty[$ (up to a maching with 1), such that $\lim_{\tau \to \tau_\infty^+} \tilde{u}(\tau) = u_{\sigma_{smooth}} > 0$. By Proposion 13.14-(4), we have that u_{smooth} belongs to $\{u; \varphi'(u) = \sigma_{smooth}\}$. However, by Proposition 13.19 there exists only one compatible continuation solution verifying $\lim_{\tau \to \tau_\infty^+}(\tau) = u^*$. If $u^* = u_{\sigma_{smooth}}$ then the corresponding traveling wave is a continuous profile supported on the whole real line and saturating the flux at the maching point. This is always the case if $(\varphi')^{-1}(u_{\sigma_{smooth}})$ is a single point. If $u^* < u_{\sigma_{smooth}}$ the profile will give a discontinuous traveling wave supported on the whole real line.

In the critical case $\sigma = \sigma_{smooth} = \varphi'(0)$, which corresponds to Theorem 13.7-(3)-(c), Remark 13.15 shows that a number of different situations can occur:

- If $u_\sigma = 0$ and $\tau_\infty = \infty$ then the profile $\tilde{u}$ (up to a matching with 1) constitutes a non-saturated traveling wave.
- If $u_\sigma = 0$ and $\tau_\infty < \infty$ then $\tilde{u}$ (up to a matching with 1) is continuated continuously by zero in the interval $]\tau_\infty, \infty[$. This gives rise to a continuous traveling wave profile supported on a half-line.
- If $u_\sigma > 0$ and φ is linear in $]0, u_\sigma[$ then $\tilde{u}$ (up to a matching with 1) can be only extended by zero obtaining a discontinuous traveling wave supported on a half-line.

The uniqueness of all these profiles is consequence of the uniqueness of σ_{smooth}, σ_{ent}, $\tilde{u}_\sigma$, $\bar{u}_\sigma$ and the neccesary Rankine–Hugoniot conditions. Let us observe that even if u^- is not unique, in the case $\sigma = \sigma_{smooth}$ the continuation solution is, see results in Proposition 13.19 and Remark 13.20.

Remark 13.21. We have proved uniqueness for traveling wave profiles which are decreasing monotone functions and have at most two singular points; one of them is the connection with $u = 1$ that could be regular or singular. Our analysis also implies that there is at most one interior singular point. Other possible concept of solutions (that could include Cantor set of singularities) are not considered in this paper.

13.2.2. Uniqueness with respect to σ. Let us address now how many traveling solutions can be constructed with a given speed σ. For that we recall that when we say that a function is piecewise smooth, up to a finite number of points, we understand that at those singular points there is a jump either of the function or of its first derivative. A suitable adaptation of the techniques introduced in [49] shows that there are no other traveling profiles in this class of piecewise smooth profiles than those we already constructed.

Proposition 13.22. *Let φ be linear or strictly convex. Then the following statements hold true:*

(1) *Given $\sigma \in [\sigma_{ent}, +\infty[$, the only nontrivial entropy solution of* (13.1) *with the form $u(x-\sigma t)$, having its range in* $[0, 1]$ *and being piecewise smooth –up to a finite number of points– is (up to spatial shifts) the one provided by the previous development.*

(2) *Given $\sigma \in]-\infty, -\sigma_{ent}]$, the only nontrivial entropy solution of* (13.1) *with the form $u(x-\sigma t)$, having its range in* $[0, 1]$ *and being piecewise smooth –up to a finite number of points– is (up to spatial shifts) the mirror image of the one provided by the previous development for wave speed $-\sigma$.*

(3) *Given $\sigma \in]-\sigma_{ent}, \sigma_{ent}[$, there is no entropy solution of* (13.1) *with the form $u(x - \sigma t)$, having its range in* $[0, 1]$ *and being piecewise smooth –up to a finite number of points–.*

Note that we just spoke about entropy solutions of (13.1) in the previous statement. This does not assume any monotonicity property for the traveling profiles (see Remark 13.11), nor does it fix any particular behavior at $\pm\infty$. Hence this is a very strong uniqueness statement ruling out the apparition of any type of non-monotonic profile (solitons in particular). It also allows to settle some open questions that were raised in [54]. Entropy traveling waves there constructed are unique in the class of entropy piecewise smooth profiles, and there are no other such profiles in that class but the ones constructed in [54] (in particular there are certain regimes in which there are no admissible profiles).

13.3. Proof of the main results of Section 13. Here we collect the proofs of the results established in the first part of this Section 13, where the traveling wave construction was presented in a concise way for the sake of simplicity. Due to their technical character we present them separately. Some complex requirements of the proofs lead us to re-elaborate some of the statements in a more complete form including mathematical properties which were not taken into account in the previous subsection. This is the case of our first result that contains in particular the statements of Proposition 13.12.

Proposition 13.23. *There exist a relative open subset* $\Omega \subset [0,\infty[\times]0,1]$ *and a continuous function* $R : \Omega \to [0,1[$ *such that:*

(1) *For any* $\sigma \geq 0$ *the set* $\{u\,;\,(\sigma,u) \in \Omega\}$ *is an interval* $]u_\sigma, 1]$ *and* $R(\sigma,\cdot)$ *is the unique maximal solution to (13.8) in* $]u_\sigma, 1]$ *such that* $\lim_{u\to 1} R_\sigma(u) = 0$.

(2) *The map* $\sigma \mapsto u_\sigma$ *is lower-semicontinuous, that is,*

$$\limsup_{n\to\infty} u_{\sigma_n} \leq u_\sigma \text{ for any } \sigma_n \to \sigma.$$

(3) *If* $\sigma_1 < \sigma_2$*, then* $u_{\sigma_1} \geq u_{\sigma_2}$ *and*

$$R_{\sigma_1}(u) \geq R_{\sigma_2}(u)$$

in the common interval of definition $]u_{\sigma_1}, 1[$.

(4) $u_0 > 0$ *holds.*

(5) *If* $u_\sigma > 0$*, then* $\lim_{u\to u_\sigma} R_\sigma(u) = 1$ *and* $\sigma \leq \varphi'(u_\sigma)$.

(6) *If there exists some value* σ_0 *such that* $u_{\sigma_0} > 0$ *and* $\sigma_0 < \varphi'(u_{\sigma_0})$*, then the function* $\sigma \to u_\sigma$ *is continuous in* σ_0.

(7) *If* $\sigma \neq \varphi'(0)$ *and* $u_\sigma = 0$*, then* $\sup\left\{R_\sigma(u)\,;\,u \in]0,1[\right\} < 1$.

Furthermore, there exists $\varepsilon > 0$ *such that* $u_\sigma < 1-\varepsilon$ *for any* $\sigma \geq 0$.

The proof of this Proposition requires the study of some additional properties of the solutions to (13.8) that we introduce in some preliminary Lemmas.

Our analysis of the solutions to (13.8) is based on a general continuation process. The properties $K(1) = 0$ and $\lim_{r\to 1} g(r) = \infty$ allow to build the following continuous function

$$F(u,r) = \begin{cases} \frac{1}{\varphi(u)}\left(\sigma - r\varphi'(u) - \frac{K(u)}{g(r)}\right), & \text{for} \quad (u,r) \in]0,1[\times]0,1[, \\ \frac{1}{\varphi(u)}\left(\sigma - \varphi'(u)\right), & \text{for} \quad (u,r) \in]0,1[\times[1,\infty[, \\ \frac{1}{\varphi(1)}\left(\sigma - r\varphi'(1)\right), & \text{for} \quad (u,r) \in [1,\infty[\times]0,1[, \\ \frac{1}{\varphi(1)}\left(\sigma - \varphi'(1)\right), & \text{for} \quad (u,r) \in [1,\infty[\times[1,\infty[. \end{cases}$$

Then, we consider a continuous extension of the differential equation (13.8):

$$R'(u) = F\big(u, R(u)\big) \quad u \in]0,\infty[. \tag{13.18}$$

Peano's Theorem allows to deduce the local existence and uniqueness of a solution $R :]u_0-\varepsilon, u_0+\varepsilon] \to]0,\infty[$ to (13.18) such that $R(u_0) = r_0$, for any $(u_0, r_0) \in]0,\infty[\times]0,\infty[$. We are interested in solutions to (13.18) because these will provide solutions to (13.8) after restricting them to $]0,1[\times]0,1[$. The above extension will be specially useful when dealing with the behavior of solutions at the boundaries $\{1\}\times]0,1[$ and $]0,1[\times\{1\}$.

Lemma 13.24. *Let* $R :]u_{min}, u_{max}[\to]0,1[$ *be a maximal solution to (13.8) in* $]0,1[\times]0,1[$ *such that* $u_{min} > 0$. *Then* $\lim_{u\to u_{min}} R(u) = 1$ *and* $\lim_{u\to u_{min}} R'(u) = \frac{1}{\varphi(u_{min})}\left(\sigma - \varphi'(u_{min})\right) < 0$.

Proof. The maximal character of the solution assures that $R(u)$ tends to the boundary of the domain as $u \to u_{min}$. This is not the point $(u_{min}, 0)$ because in that case $\lim_{u\to u_{min}} R(u) = 0$ and $\lim_{u\to u_{min}} R'(u) = -\infty$. Thus, the solution is forced to verify $\lim_{u\to u_{min}} R(u) = 1$. The assertion about the sign and the convergence of $R'(u)$ can be deduced from the regularity of the extended differential equation (13.18). □

In the case where the boundary is $]0, 1[\times\{0\}$ the flux cannot be extended continuously because it diverges to $-\infty$. However, we have the next estimate for solutions close to this boundary.

Lemma 13.25. *Let $u_0 \in]0, 1[$. Consider the maximal solution R of (13.8) in $]0, 1[\times]0, 1[$ with $R(u_0) = r_0$. For any given $\delta > 0$, there exists some $\gamma > 0$ such that if $r_0 < \gamma$, then R is not defined for $u > u_0 + \delta$. As a consequence, we have $u_{max} < u_0 + \delta$ and* $\lim_{u\to u_{max}} R(u) = 0$.

Proof. Let M be a positive fixed quantity, then there exists $0 < \gamma < \delta M$ such that for any $(u, r) \in]0, 1[\times]0, 1[\cap B((u_0, 0), \delta)$ the inequality $F(u, r) \le -M$ holds. Thus, since $(u, R(u)) \in B((u_0, 0), \delta)$ the solution to (13.8) with initial data $R(u_0) = r_0$ and $r_0 \le \gamma$ verifies $R(u) \le r_0 - M(u - u_0)$, for any $u \in [u_0, u_{max}[$. In particular, this estimate implies that $u_{max} < u_0 + \frac{r_0}{M} \le u_0 + \delta$. □

The regularity of the equation in $]0, 1[\times]0, 1[$ allows to assure that solutions to (13.8), for a fixed value σ, cannot cross each other. It can be also proved that there exists an order with respect to σ between all these solutions in the following sense:

Lemma 13.26. *Let $\sigma_1 < \sigma_2$ and α_1, α_2 such that the maximal solutions in $]0, 1]\times]0, 1[$ $\tilde{R}_{\sigma_1}$ and $\tilde{R}_{\sigma_2}$ have a common intersection, i.e. there exists some point $u_0 \in]0, 1]$ such that $\tilde{R}_{\sigma_1}(u_0) = \tilde{R}_{\sigma_2}(u_0)$. Then*

$$\tilde{R}_{\sigma_1}(u) > \tilde{R}_{\sigma_2}(u), \quad \textit{for any } u \in]u^{\sigma_1}_{min}, u_0[,$$

and

$$\tilde{R}_{\sigma_1}(u) < \tilde{R}_{\sigma_2}(u), \quad \textit{for any } u \in]u_0, \min\{u^{\sigma_1}_{max}, u^{\sigma_2}_{max}\}[,$$

being $u^{\sigma_1}_{min}$, $u^{\sigma_1}_{max}$, $u^{\sigma_2}_{max}$ the lower and upper limits for the respective maximal existence interval of $\tilde{R}_{\sigma_1}$ and $\tilde{R}_{\sigma_2}$ in $]0, 1[\times]0, 1[$.

Proof. This property can be easily proved since at any crossing point u_0

$$\tilde{R}'_{\sigma_1}(u_0) - \tilde{R}'_{\sigma_2}(u_0) = \frac{\sigma_1 - \sigma_2}{\varphi(u_0)} < 0$$

holds, which justifies the result locally around u_0. Standart arguments show that it is not possible to have two or more points verifying this inequality. Thus, there is at most one crossing point. □

Proof. Proposition 13.23. The process of construction for the solutions stablished in the first statement of Proposition 13.23 will also provide the proof of assertions (2) and (3). The order relation given in Lemma 13.26 and the regularity of the extended system assure that any solution $\tilde{R}_\sigma$ to (13.18), for any $\sigma \geq 0$ and initial condition $\tilde{R}_\sigma(1) \leq \alpha$, is bounded from above by the solution $\tilde{R}_0 :]u^0_{min}, 1] \rightarrow]0, 1[$ with initial condition $\tilde{R}_0(1) = \alpha$, where $\alpha > 0$. This implies that those solutions are defined at least in $]u^0_{min}, 1]$, where we have used Lemma (13.24). Let u_{min} be such that $u^0_{min} < 1 - \varepsilon$, for any $\sigma \geq 0$ fixed but arbitrary and $0 < \varepsilon$. Then, the set

$$I_\sigma = \left\{r \in]0, 1[\mid \text{s.t. } \tilde{R}_\sigma \text{ solution to (13.18)}, \tilde{R}(1-\varepsilon) = r \text{ is defined in }]1-\varepsilon, 1]\right\}$$

is an open interval due to regularity of (13.18) and the usual continuous dependence results. Here, we have to take into consideration the concept of solution at $u = 1$. Now, we can easily determine that $r_\sigma := \inf I_\sigma$ is strictly positive because otherwise this would contradict Lemma 13.25 at $u_0 = 1 - \varepsilon$ with $\delta = \varepsilon/2$. Now, we will focus on the solution to (13.18) verifying $\tilde{R}_\sigma(1-\varepsilon) = r_\sigma$. In particular, we are going to prove that it is defined in $[1-\varepsilon, 1[$ and verifies $\lim_{u\to 1} \tilde{R}_\sigma(u) = 0$. Since $(1-\varepsilon, r_\sigma) \in]0, 1[\times]0, 1[$ the maximal solution R_σ passing through this point is defined in the interval $[1-\varepsilon, u_{max}[$. If $u_{max} < 1$, then $\lim_{u\to u_{max}} R_\sigma(u) = 0$. We focus on solutions passing through (u_{max}, γ) with $\gamma > 0$. The order relation allows to assure that these solutions have to be defined for $u = 1 - \varepsilon$ and, as a consequence, also for $u = 1$ due to the definition of r_σ. This contradicts Lemma 13.25 since γ can be chosen small enough. Hence, R_σ is defined in $]1-\varepsilon, 1[$. Using the regularity of the extended system and the continuation argument for (13.18) we also get $\lim_{u\to 1} \tilde{R}_\sigma(u) = 0$. Otherwise, the solution would be extended for $u \geq 1$ and consequently $r_\sigma \in I_\sigma$ which is a contradiction.

The uniqueness of R_σ for (13.8) with $\lim_{u\to 1} \tilde{R}_\sigma(u) = 0$ can be justified by realizing that the right hand side of this equation is just an addition of $\frac{1}{\varphi(u)}(\sigma - r\varphi'(u))$, a regular term at $(1, 0)$, to $-\frac{K(u)}{g(r)}$, an increasing term in r.

Once we have proved the existence and uniqueness we deal with the order property stated in (3). The construction of R_σ together with Lemma 13.26 provide that $r_{\sigma_1} \geq r_{\sigma_2}$, for $\sigma_1 < \sigma_2$ since $I_{\sigma_1} \subset I_{\sigma_2}$. Due to the uniqueness of R_σ and the fact that the previous bound does not depend on ε, we conclude that $R_{\sigma_1}(u) \geq R_{\sigma_2}(u)$ for $u \in [1-\varepsilon, 1[$. In fact, the inequality is strict thanks again to Lemma 13.26 since a common value implies $R_{\sigma_1}(u) < R_{\sigma_2}(u)$ for u smaller than the crossing point.

These solutions R_σ have to be extended by maximal solutions to intervals $]u_\sigma, 1[$. The order property is still valid in this maximal existence interval, otherwise the application of Lemma 13.26 in the upper crossing point would contradict $R_{\sigma_1}(u) \geq R_{\sigma_2}(u)$ for $u \in [1-\varepsilon, 1[$.

Now we can prove the continuity of R with respect to both σ and u, in the open set Ω . We have the following uniform convergence property: Consider a monotone sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to σ. Then, $R_{\sigma_n} \to R_\sigma$ uniformly in $[1-\varepsilon, 1]$, where ε was obtained previously as the minimal length of any existence interval for the solutions R_σ. For instance, if σ_n is increasing the order relation gives $R_{\sigma_0}(u) \geq R_{\sigma_n}(u) \geq R_\sigma(u)$, for any $u \in [1-\varepsilon, 1]$. These estimates allow to deduce that the pointwise limit of R_{σ_n}, $\tilde{R}$, is a continuous function in $[1-\varepsilon, 1[$, which is a solution of the Volterra integral equation for σ associated to (13.8). Furthermore, $\tilde{R}$ is continuous in $u = 1$ due to the bounds $R_{\sigma_0}(u) \geq \tilde{R}(u) \geq R_\sigma(u)$. Dini's Theorem assures the uniform convergence in $[1-\varepsilon, 1]$. The uniqueness of solution to (13.8) provides the identity $\tilde{R} = R_\sigma$. From this uniform convergence and the continuous dependence of the solutions in $\{1-\varepsilon\}\times]0,1[$ we can deduce the continuity of R and also that Ω is an open set. The lower-semicontinuity of $\sigma \to u_\sigma$ is a consequence of the open character of Ω.

The proof of point (4) is analogous to those exhibited in [49]. In the case $\sigma = 0$ it can be justified by using (13.8) that:

$$\big(r(u)\varphi(u)\big)' = -\frac{K(u)}{g(r)} \leq 0.$$

The case of $u_0 = 0$ implies that $r(u)\varphi(u)$ is a decreasing function such that $\lim_{u\to 0} r(u)\varphi(u) = 0$ and therefore identically zero. This is a contradiction with our construction, in particular with the fact that $r_\sigma > 0$.

Assertion (5) in Proposition 13.23 is a direct consequence of Lemma 13.24.

Now we focus on (6), for which continuity will be deduced by considering monotone sequences. Let $\{\sigma_n\}_n$ be an increasing sequence such that $\sigma_n \to \sigma_0$. Then $u_{\sigma_n} \geq u_{\sigma_0}$, which together with the lower semicontinuity of (2) and

$$u_{\sigma_0} \leq \liminf u_{\sigma_n} \leq \limsup u_{\sigma_n} \leq u_{\sigma_0}$$

justify the convergence of u_{σ_n} towards u_{σ_0}. If $\sigma_n \to \sigma_0$ is a decreasing sequence, then u_{σ_n} is increasing and bounded by u_{σ_0}. Thus $u_{\sigma_n} \to \alpha \leq u_{\sigma_0}$. Let us show that $\alpha = u_{\sigma_0}$. We argue by contradiction, assuming that $\alpha < u_{\sigma_0}$. Then, we can define the pointwise limit of R_{σ_n}, i.e. $R_p(u) = \lim_{n\to\infty} R_{\sigma_n}(u)$, for any $u \in]\alpha, 1[$. Using Proposition 13.23-(1) we deduce $R_{\sigma_0}(u) = R_p(u)$, for any $u \in]u_{\sigma_0}, 1[$. On the other hand, since the sequences R_{σ_n} and R'_{σ_n} are uniformly bounded in any interval $[a,b] \subset]\alpha, 1[$ the Áscoli-Árzela Theorem provides the uniform convergence of R_{σ_n} towards R_p. The uniform convergence and the continuous extension (13.18) also provide the uniform convergence of R'_{σ_n}. Thus R_p is a derivable function in $]\alpha, 1[$. To conclude, we have that $R_p(u_\sigma) = \lim_{u\to u_\sigma} R_{\sigma_0}(u) = 1$ and since $u = u_\sigma$ is a critical point we have that $0 = \lim_{u\to u_\sigma} R'_p(u) = \lim_{u\to u_\sigma} R'_{\sigma_0}(u)$. However, this contradicts the hypothesis $\sigma_0 < \varphi'(u_{\sigma_0})$ and Lemma 13.24, which implies $\alpha = u_{\sigma_0}$.

The proof of the last claim of Proposition 13.23 can be split in two cases. If $\sigma > \varphi'(0)$ there exists $\delta > 0$ such that

$$\sigma - r\varphi'(u) - \frac{K(u)}{g(r)} > 0 \qquad u \in [0,\delta]\,,\ r \in [1-\delta, 1].$$

That is, the flux in $[0,\delta]\times[1-\delta,1]$ is monotonically increasing. If the solution R_σ crosses this rectangular region, then its increasing character implies that the maximum value is achieved exactly at $u = \delta$. This together with the fact that $u_\sigma = 0$ justify that R_σ reaches an absolute maximum in $]0,1[$. In the opposite case $\sigma < \varphi'(0)$, a rectangular region with monotone decreasing flux $[0,\delta]\times[1-\delta,1]$, can be defined in a similar way. If the solution passes through this region, the maximal value of R should be reached when u converges to $u_\sigma = 0$. If $\lim_{u\to 0} R_\sigma(u) < 1$, then the result holds. Otherwise the fact that $\lim_{u\to 0} R(u) = 1$ provides a contradiction with the following argument: Consider

$v_\sigma = R_\sigma^{-1}$ in the rectangular region where monotonicity holds. Then, $v = v_\sigma$ is the solution to the IVP

$$v'(r) = Z(r)\varphi(v(r)), \quad v(1) = 0 \quad r \in [1-\delta, 1], \tag{13.19}$$

where $Z(r)$ is a continuous function such that

$$Z(r) = \frac{1}{\left(\sigma - r\varphi'(v_\sigma(r)) - \frac{K(v_\sigma(r))}{g(r)}\right)}, \quad \text{if } r < 1 \text{ and } Z(1) = \frac{1}{\sigma - \varphi'(0)} < 0.$$

Because of the regularity of φ, the solution to (13.19) is unique. The contradiction is then obtained from the fact that (13.19) also admits $v = 0$ as a solution. □

The proof of Proposition 13.14 is founded on the results of Proposition 13.23.

Proof. Proposition 13.14. First, we prove that σ_{smooth} is positive. Proposition 13.23-(5) implies $u_\sigma = 0$, for any $\sigma > \max\{\varphi'(u);\ u \in [0,1]\}$, and as a consequence $\sigma_{smooth} < \infty$. Furthermore, σ_{smooth} is nonnegative by definition. However, if $\sigma_{smooth} = 0$, then $u_\sigma = 0$, for any $\sigma > \sigma_{smooth} = 0$, while $u_{\sigma_{smooth}} = u_0 > 0$ by Proposition 13.23-(4). Hence, the function $\sigma \to u_\sigma$ is not continuous in σ_{smooth}. Using (5)-(6) in Proposition 13.23 we deduce that $0 = \sigma_{smooth} = \varphi'(u_{\sigma_{smooth}})$, which contradicts Assumption 13.2-(2), concluding the proof of the first statement.

Concerning the proof of the second claim, using the definition of σ_{smooth} and the order relation of Proposition 13.23-(3), we obtain $u_\sigma > 0$, for any $\sigma < \sigma_{smooth}$. Then, by Proposition 13.23-(5) we only have to prove that the equality $\varphi'(u_\sigma) = \sigma$ cannot be verified. The reason of this incompatibility is that $u_\sigma > 0$ together with $\varphi'(u_\sigma) = \sigma$ is only admissible if $\sigma = \sigma_{smooth}$. If this were not the case, there would exists $\tilde{\sigma} > \sigma$ such that $u_{\tilde{\sigma}} \leq u_\sigma$ and as a consequence:

$$\tilde{\sigma} > \sigma = \varphi'(u_\sigma) \geq \varphi'(u_{\tilde{\sigma}}),$$

which is in contradiction with Proposition 13.23-(5).

The third assertion establishes that for $\sigma > \sigma_{smooth}$ the solutions R_σ are always away from 1. When $\sigma \neq \varphi'(0)$ this is a direct consequence 13.23.7 since $u_\sigma = 0$. In the case $\sigma = \varphi'(0)$ we have $\varphi'(0) > \sigma_{smooth}$ and we can choose $\tilde{\sigma} \in (\sigma_{smooth}, \varphi'(0))$. Then, the boundedness of $R_{\tilde{\sigma}}$ also implies that of R_σ because of the order relation between them.

The last assertion deals with the continuity and jump discontinuities of the function $\sigma \to u_\sigma$. When $\sigma < \sigma_{smooth}$, the second statement of Proposition 13.14 provides $\sigma < \varphi'(u_\sigma)$ and thanks to Proposition 13.23-(6) continuity holds. On the other hand, if $\sigma > \sigma_{smooth}$ continuity is satisfied since $u_\sigma = 0$, for any $\sigma > \sigma_{smooth}$. At the critical value $\sigma = \sigma_{smooth}$ the function $\sigma \to u_\sigma$ may have different behaviors. If $\sigma_{smooth} < \varphi'(0)$, then u_σ is left-continuous since $u_{\sigma_{smooth}} \leq u_\sigma = 0$ by Proposition 13.23-(5), for any $\sigma \in]\sigma_{smooth}, \varphi'(0)[$. In the opposite case $\sigma_{smooth} > \varphi'(0)$ we have that u_σ is right-continuous. Firstly, note that $u_{\sigma_{smooth}}$ cannot be zero since in that case there would exists an increasing sequence $\sigma_n \to \sigma_{smooth}$ such that $u_{\sigma_n} > 0$ and $\varphi'(u_{\sigma_n}) \geq \sigma_n$. The lower-semicontinuity property established in Proposition 13.23-(2) also implies that $u_{\sigma_n} \to 0$ since

$$0 \leq \liminf u_{\sigma_n} \leq \limsup u_{\sigma_n} \leq u_{\sigma_{smooth}} = 0,$$

and $\varphi'(0) \geq \sigma_{smooth}$, which contradicts the hypothesis $\sigma_{smooth} > \varphi'(0)$. Using the fact that $u_{\sigma_{smooth}} > 0$ and the lack of continuity of $\sigma \to u_\sigma$ at $\sigma = \sigma_{smooth}$, Proposition 13.23-(6) leads to $\sigma_{smooth} = \varphi'(u_{\sigma_{smooth}})$. Finally, we check that $\lim_{\sigma \to \sigma_{smooth}^-} u_\sigma = u_{\sigma_{smooth}}$. Consider $\sigma_n \to \sigma_{smooth}$ such that $\sigma_n \leq \sigma_{smooth}$. Then, (3) and (2) in Proposition 13.23 give

$$u_{\sigma_{smooth}} \leq \liminf u_{\sigma_n} \leq \limsup u_{\sigma_n} \leq u_{\sigma_{smooth}},$$

which completes this proof. □

The proof of Proposition 13.18 is based on some general properties of convex functions provided by the next result.

Lemma 13.27. *Let $\varphi \in \mathcal{C}^1(I)$, being I an interval. Then $H(u,v) := \frac{\varphi(u)-\varphi(v)}{u-v}$ is an increasing function in both variables verifying*

$$\varphi'(u) \leq H(u,v) \leq \varphi'(v), \tag{13.20}$$

for any $u,\ v \in I$ such that $u < v$. Furthermore, if any of these inequalities becomes an equality, then the same would happen with the other one and φ would be linear in $[u,v]$.

Proof. Convexity properties provide

$$H(x_1,x_2) \leq H(x_1,x_3) \leq H(x_2,x_3),$$

for any values $x_1 < x_2 < x_3$ such that $x_2 = (1-\alpha)x_1 + \alpha x_3$ with $\alpha = \frac{x_2-x_1}{x_3-x_1}$. From this it can be proved that for any different $y,\ u,\ v \in I$ such that $u < v$

$$H(y,u) \leq H(y,v)$$

holds by taking into account all the relative positions of $y,\ u$, and v. Inequalities (13.20) are a result of the Mean Value Theorem and the increasing character of φ'.

Assume that $\varphi'(u) = H(u,v)$ holds. This implies that the continuous function $g(v) := H(u,v)$, for $v \neq u$, is an increasing function such that $g(u) = g(v)$ for $u > v$. This can be only possible if g is constant, i.e. if φ is linear in the interval $[u,v]$ and therefore $\varphi'(v) = H(u,v)$. We can argue in a similar way if $\varphi'(v) = H(u,v)$ holds. □

Proof of Proposition 13.18. Hypothesis $\sigma_{smooth} > \varphi'(0)$ implies that φ cannot be linear. This will follow as a consequence of Proposition 13.14-(2); note that the convexity of φ provides

$$\varphi'(0) < \varphi'(u_{\varphi'(0)}) \leq \varphi'(u_{\sigma_{smooth}}).$$

Then, equalities in (13.20) with $u = 0,\ v \in]0, u_{\sigma_{smooth}}[$ cannot be achieved, and hence φ cannot be linear in any subinterval of $[0, u_{\sigma_{smooth}}]$, proving our claim.

Furthermore, by using Proposition 13.14-(4), we also find

$$\sigma_{smooth} = \varphi'(u_{\sigma_{smooth}}) > H(u_{\sigma_{smooth}}, 0).$$

On the other hand, for $\sigma = \varphi'(0)$ we get

$$\sigma = \varphi'(0) < H(u_{\varphi'(0)}, 0)$$

since as in the previous argument φ cannot be linear in $[0, u_{\varphi'(0)}]$. The last two estimates together with the decreasing character of $H(u_\sigma, 0)$ with respect to σ prove the existence and uniqueness of σ_{ent} solution of (13.17). The same strategy also gives

$$\sigma > H(u_\sigma, 0) \iff \sigma > \sigma_{ent}. \tag{13.21}$$

Now, the aim is to deal with solutions to (13.16). If there exists a value $u^- \in]0, u_\sigma[$ solution to (13.16), then the monotone character of H gives $\sigma = H(u_\sigma, u^-) \geq H(u_\sigma, 0)$. Equality $H(u_\sigma, u^-) = H(u_\sigma, 0)$ is not possible since this would imply the linear character of φ in the interval $[0, u^-]$. Thus we have that $\sigma > H(u_\sigma, 0)$, which is only possible if $\sigma < \sigma_{ent}$, as we have mentioned previously. Consider $\sigma \in]\sigma_{ent}, \sigma_{smooth}[$. Then, using (13.21) and Proposition 13.14-(4), we deduce that $H(u_{,\sigma}, \cdot)$ is a continuous increasing function defined for $u \in [0, u_\sigma[$ such that $\lim_{u\to 0} H(u_\sigma, u) = H(u_\sigma, 0) < \sigma$ and $\lim_{u\to u_\sigma} H(u_\sigma, u) = \varphi'(u_\sigma) > \sigma$. This justifies the existence of at least one positive solution u^- to (13.16) in $]0, u_\sigma[$. The uniqueness of this solution is due to the fact that φ is non-linear in $[0, u_\sigma]$. □

We conclude this section by proving the last result of Section 13, Proposition 13.19.

Proof. Proposition 13.19. Let $u^* > 0$, which is possible only if $\sigma > \varphi'(0)$. Consider (13.18) the continuous extension of the differential equation (13.8). Then, Peano's Theorem implies the local existence of a solution $R_{u^-} :]u^- - \varepsilon, u^-] \to]0, \infty[$ to (13.18) such that $R_{u^-}(u^-) = 1$, for any $u^- \in]0, u^*[$. Since $F(u, r) > 0$ in $]0, u^-] \times [1, \infty[$, R_{u^-} cannot reach 1 for any $u \in]0, u^-[$, which justifies the extension of R_{u^-} to $]0, 1[$. By continuous dependence with respect to the initial condition u^-, we can deduce the existence of R_{u^*}, solution to (13.18) such that $R_{u^*} \leq 1$, for any $u \in]0, u^*[$ and $R_{u^*}(u^*) = 1$. In fact, due to the increasing character of the flux in $]0, u^*[\times 1$, R_{u^*} verifies the strict inequality $R_{u^*} < 1$, for any $u \in]0, u^*[$. Backward uniqueness comes from the fact that the flux F can be written as the sum of a locally Lipschitz function plus an increasing function of R.

If $u^- > u^*$ we have to consider two different cases. When $\varphi'(u^-) > \sigma$ any solution to (13.18) such that $R_{u^-}(u^-) = 1$ is bigger than 1, for $u < u^-$, because $F(u^-, 1) < 0$. When $\varphi'(u^-) = \sigma$ holds, the constant function $R = 1$ is the unique solution to (13.18) such that $\lim_{u\to u^-} R(u) = 1$, for $u \in (u^*, u^-)$. Since this is not a solution to (13.8) the existence of an extension R_{u^-} is then not possible. □

14. Applications to developmental biology

Embryonic morphogenesis is concerned with spatial organization and cell number allocation, giving rise to the proper shape and size of tissues and organs of the developing and adult organism. Morphogens are signaling molecules (proteins) secreted by localized sources that regulate tissue patterning by triggering distinct cellular fates in responding cells at different concentration ranges. In this section, we deal with the dynamics of the Sonic Hedgehog morphogenetic action and the consequences of the activation of the target gene *gli*. Progenitor or stem cells, mostly cycling, enter distinct differentiation programs at defined positions from the source, responding to the morphogen in a concentration–dependent manner. When this process is deregulated, it is usually associated with tumor growth.

14.1. Biological and mathematical basis on morphogenesis. A central question in biology is how secreted morphogens act to induce distinct cell fates in a concentration-dependent manner. This is highly relevant as it has a number of consequences on cell growth and tumor dynamics, for instance. In the sequel, we will focus on one of the most important proteins in development and stem cell biology: Hedgehog (Hh). In a variety of tissues, Hh acts as a morphogen to regulate growth and cell fate specification.

Mathematical models in developmental biology need to replicate in as much detail as possible, to the extent that technology allows, the events observed in experiments. Moreover, it is desirable that the predictive capability of such models goes further than just accurately mimicking known experimental results, aquiring in consequence more biological relevance.

Several hypotheses have been proposed to explain morphogen movement. The mathematics of morphogenesis has been classically based on reaction–diffusion models since the pioneering works of Turing, Crick and Meinhardt [74, 94, 98, 104, 126]. From that point on, the dominant paradigm when modeling morphogenetic processes has largely focused on parameter tuning. That is, adjusting parameters within reaction–diffusion systems in order to get quantitative fittings that reproduce previously established patterns. Quantitative accuracy is important, but there are no less important qualitative aspects of evolution that may shed light on morphogenesis processes and validate the mathematical models as a driving force for new biological predictions. On large time scales, qualitative and quantitative properties must converge. Note however that what happens at small scales has often a critical effect on pattern formation, as we shall see in the example of the signaling pathway of the morphogen Sonic Hedgehog (Shh) and its impact on the activation of target gene *gli*.

A large part of this section will focus on the dynamical properties of signaling pathway Shh–Gli. This pathway is involved in both normal embryonic development and tissue, when Gli is regulated, as well as in tumor progression when Gli is deregulated [78, 123]. In developmental biology, the vertebrate neural tube and the insect wing imaginal disc (where the morphogen Hh plays the same role than Shh in vertebrates) are prime examples of morphogenetic patterning. In the former, embryonic pseudostratified ventral epithelial cells are instructed to acquire specific fates in response to Shh morphogenetic signals derived from the ventral floor plate, and earlier the notochord [91]. In the latter, Hh signal from the posterior compartment establishes patterns in the anterior compartment of the developing wing epithelium. Cells closest to the source receive higher doses of a morphogen and for longer times than those further apart from it. But, importantly, distal–most cells do not respond.

In the above examples related to development, reaction–diffusion systems do not replicate the real biological events. Note that by default diffusion in such models is described by a linear differential operator of second order (namely, the Laplacian). In fact, a number of biological observations are incompatible with linear diffusion. First, Hh morphogens do not behave as very small particles in large spaces, thus invalidating one of the assumptions of Brownian motion implied by linear diffusion: Hh has been detected in *Drosophila* as visible aggregates of $\sim$ 20–300 nm. in diameter [89, 131] . Such particles have been shown to be multimers [132] , membrane vesicles [88], oligomers [131], and/or lipoprotein particles [53, 110]. Second, the dynamics of morphogen–induced pattern formation are inconsistent with the action of linear diffusion. Indeed, the time of exposure (and not only the large time asymptotics) of morphogen concentration is critical to specify cell fate. Modeling morphogen gradients with linear diffusion necessarily implies that every cell in the responding field receives a low level of morphogen instantaneously and, hence, predicts the same time of exposure for every cell in the whole tissue [78]. This is at odds with the observation that the development of morphogenetic responses requires time. Indeed, low level Hh signaling has cumulative effects in the neural tube and limb buds. The following sentence from [78] makes this clear:

> *In the vertebrate central nervous system and limbs, the pattern of cellular differentiation is controlled by both the amount and the time of Shh exposure.*

Similarly, while current linear diffusion models may reproduce natural final patterns they cannot account for how these are formed [90], and therefore the dynamics of the formation of genetic patterns is not very representative qualitatively. Indeed, in such models an arbitrary threshold of noninstructive signaling or noise had to be introduced in order to make the concentration curves fit the biological reality of a discontinuity at the front of the gradient (see Fig. 8 in [119]). Removing the tails of a Gaussian-type solution is a usual procedure in reaction–diffusion systems, specifically when dealing with traveling wave solutions. However, in the present context this produces artificial fronts out of no biological inputs. It is important to highlight that both the existence of a front and the velocity of morphogen transport (which is obviously not infinite, as opposed to what linear diffusion would predict) are critical elements in order to accurately describe pattern formation in morphogenetic processes. Without a threshold mechanism, the mathematical model in [119] predicts that the chemical signal arrives too fast to distant areas, thus triggering the chemical cascade too soon. Nonetheless, with a threshold mechanism the chemical signal would never be able to arrive at distant areas according to the model in [119]. As a matter of fact the chemical signal would be receding, having as a result that large sections of the neural tube would never be exposed to the action of the morphogen. In this way, previous work on linear diffusion simply approached known data but had to impose nonbiological thresholds to make some sense of it, as far as traveling fronts are concerned [119]. These models, then, might not predict the novelty introduced by biological processes such as negative feedback [78].

Moreover, such linear diffusion models do not appropriately account for the anatomy where the macroparticles (vesicles) are been transported. In fact, Hh proteins do not travel alone but in aggregates that are composed of different types of molecules grouped in the form of a vesicle. Note that there are continuous interactions between vesicles and the activation of target genes. Also, linear diffusion models do not account for the topography of its signaling landscape. This means that the precise physical relationships between the cells that produce signaling proteins and the fields of cells that receive these signals have not been properly described in the literature so far.

A satisfactory explanation for the previous couple of points (the cytoneme mechanism for protein transport) has been proposed recently [87, 95, 117, 121, 129]; it also provides a good experimental agreement. Cytonemes are actin-based cell protrusions that span several cell diameters. Various Hh signaling components localize to cytonemes, as well as to form structures (macroparticles) moving along cytonemes and are probably exovesicles. Cytonemes are dynamical structures: Hh gradients are established and correlated with cytoneme formation in space and time. This cytoneme–based model challenges previous linear diffusion–based models; we feel that these features should be incorporated in those mathematical models describing morphogen transport. In general, models that cannot accommodate all of the different settings have limited biological relevance.

The model in [129] features a flux saturation mechanism and indeed predicts nontrivial behavior, as it incorporates such interaction mechanisms between flux and target genes. As regards Shh signaling, the model takes into account the action of Ptc1 by introducing a negative feedback component. This model can be regarded as an extension of Fick's law and of the pioneering ideas of Turing, that tries to overcome the qualitative weaknesses associated with linear diffusion. Here, a general model for Shh-Gli signaling in the vertebrate neural tube, based on a continuous feedback between mathematical modeling, numerical analysis and data collection from experiments on Hh particle movement in the insect wing imaginal disc epithelium and in mammalian cells is developed. Our thesis is that the transport of active macroparticles (vesicles) throughout cytonemes is the biological counterpart of the flux-saturated mechanism.

To understand the implications of such a change of viewpoint, let us elaborate a bit more on the biological setting. As we have pointed out previously, in the Shh-Gli pathway the number of aggregates is infinitely large from a scale point of view, trafficking is large, and a molecular description is difficult to use since several molecular and cellular interactions are involved. These molecules positive or negatively regulate the Shh pathway. For instance, the cell adhesion molecule Cdon forms a heteromeric complex with the Hh receptor Patched 1. Cdon–mediated interference with Hh ligand dispersion is a mechanism by which Hh signaling information can be regulated in vertebrates. This receptor complex binds Hh and enhances signaling activation, indicating that Cdon positively regulates the pathway. In the case of pattern formation of the vertebrate optic vesicle into proximal/optic stalk and distal/neural retina, this Ptc-independent function protects the retinal primordium from Hh activity, defines the stalk/retina boundary and thus the correct proximo-distal patterning of the eye [55]. Then, there is evidence that in the absence of Ptc interaction with Hh, Cdon (but also other molecules as Boi or Ihog) has an evolutionary preserved function as Hh decoy receptors. This mechanism of limiting Hh activity acts in parallel to other, more intensively studied mechanisms. We have for example negative feedback regulation of the Ptc receptor, which inhibits pathway activation. Therefore, Shh molecules do not travel alone, but in complex aggregates with other molecules, forming vesicles, in which there are, as we have seen, emerging relations of cooperation and interactions towards a common goal beyond a single behavior.

The way in which these macromolecules are transported is not, of course, random. As we pointed out before, morphogens travel from producing to receiving cells, and the physical medium in which such displacements take place has been the subject of intense study, both from mathematical and biological points of view. Transport of information by cellular extensions (cytonemes) that point towards the morphogen source, is at present the mechanism supported by most of biological evidence [87, 95, 117, 121, 129]. Vesicles are transported through cytonemes, which form an extensive network that regulates traffic. In Drosophila an initial mechanism of synopsis has been described, so that epithelial cells behave as neural cells; emitting cells interact in the signaling process as neurons. This would strengthen the argument that morphogenetic signals do not diffuse freely to form an extracellular concentration gradient, but the transport and receipt of morphogens would only be mediated by cellular extensions that would be prolonged to cell contact. The formation, evolution and transfer of information by cytonemes and vesicles and the connection between cytonemes and vesicles is thus a complex process which would involve ligand-receptor interaction, which in turn may also determine the directionality of cell extensions and information exchange. This exchange could be carried across synapses or as simple transport paths forming a global distribution network. We stress again that our thesis claims that the biological mechanisms for the saturation of the flux are precisely the cooperation processes between cytonemes and vesicles.

14.2. Results. The main issue when describing the vertebrate neural tube is trying to understand how morphogen gradients are formed and interpreted (transduction by cells). The models proposed so far study DV patterning in the chick embryo spinal cord, beginning when Shh is first secreted by the floor plate. They do not focus on the whole neural tube, but only on the ventral-most binary cell fate (V3 interneurons). The most relevant proteins involved in the transduction process are PtcShh_{mem}, PtcShh_{cyt}, Ptc_{mem}, Ptc_{cyt}, $\mathrm{Gli1}^{Act}$, $\mathrm{Gli3}^{Act}$ and $\mathrm{Gli3}^{Rerp}$. They follow the interaction scheme of Fig. 4. The interaction scheme is similar but a bit simpler for *Drosophila*; there, the target genes are *ptc* and *ci (cubits interruptus)* instead of *ptc1* and *gli*.

Mathematical models for Shh transport must address the problem of infinite speed of propagation that plagues reaction-diffusion descriptions, due to the reasons we mentioned previously. Recall that linear diffusion models in mathematical biology arise essentially from Fick's theory, which is based on a microscopic linear relation between the concentration flux and the gradient $\nabla u(t,x)$ of the concentration function. The subsequent macroscopic equation associated with Fick's law yields $\partial_t u = k\operatorname{div}(\nabla u)$, and predicts an infinite speed of propagation for the concentration flux. It is mandatory to amend this unrealistic principle if we want to get a reasonable description in our particular setting, as previously argued. One way out is to resort to flux-limitation ideas. Flux-limitation mechanisms propose to modify the microscopic law defining the flow to make it saturate when concentration gradients become unbounded. The result is a non-linear spreading equation of the type

$$\frac{\partial u}{\partial t} = \nu \operatorname{div}\left(\frac{|u|^m \nabla_x u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla_x u|^2}}\right) + F(u)\,. \tag{14.1}$$

In our setting the reaction term $F(u)$ would take into account the chemical reactions taking place inside the cells. Provided that $m = 1$, the constant c is the maximum speed of propagation allowed in the medium, while ν reduces to a diffusion coefficient in the limit of the velocity $c \to \infty$, in which the usual Fick equation is recovered [59].

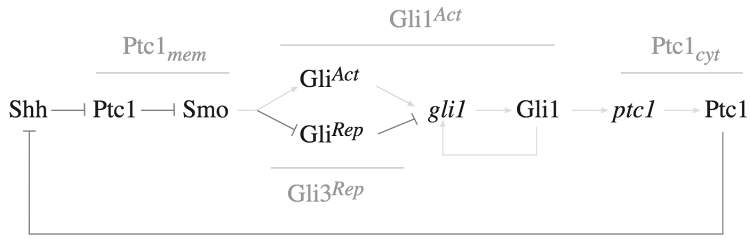

Figure 4. Schematic diagram of the Shh-Gli pathway. The temporal time line follows the flow of positive arrows (promotion) and negative T bars (repression) from the action of the secreted Shh morphogen on the left. Note that the Gli code includes the function of Gli2 and Gli3 activators as well as Gli2 repressors. For simplicity we incorporate all repressor function in the term $\mathrm{Gli3}^{Rep}$. For activators, we include $\mathrm{Gli3}^{Act}$ but consider $\mathrm{Gli1}^{Act}$ as the major Gli activator.

Using this kind of ideas, the following flux-limited spreading (FLS) equation for Shh concentration shows up once the mass law action is applied:

$$\frac{\partial[\mathrm{Shh}]}{\partial t} = \nu\, \partial_x \left(\frac{[\mathrm{Shh}]^{\mathrm{m}} \partial_{\mathrm{x}} [\mathrm{Shh}]}{\sqrt{[\mathrm{Shh}]^2 + \frac{2}{\mathrm{c}^2} (\partial_{\mathrm{x}}[\mathrm{Shh}])^2}} \right) + k_{off} [\mathrm{Ptc1Shh}_{mem}] - \mathrm{k_{on}} [\mathrm{Shh}][\mathrm{Ptc1}_{mem}].$$

Here, square brackets denote concentations. We will adopt this description and assume that $m = 1$ for now. Later on we will comment on experimental results concerning velocities of discontinuity fronts; if such speeds are not constant, then we may require to consider a different value of m in order to fit experimental data in a better way.

We propose to describe the chemical cascade of reactions taking place inside the cells with a different set of ordinary differential equations than those that have been traditionally used in the context of linear diffusion models [119]. Our aim is to take into account a number of facts that we believe quite important and become clearer when a linear description is disregarded. First, the chemical signal does not arrive instantaneously at the surface receptors, which makes such a big change in the intracellular dynamics compared to what linear models predict. Second, synthesis and transport to cell membrane of Ptc1 molecules is also not instantaneous, and in fact the elapsed time is not that short to be disregarded at once. This feature seems to have been overlooked in previous models and it entails an extra delay for the system of differential equations describing the chemical cascade inside those cells composing the neural tube. Below we represent this delay by a parameter τ (depending on the individuals, the value of τ is about 2 or 3 hours), so that the aforementioned set of differential equations describing the chemical cascade reads now as follows:

$$
\begin{aligned}
\frac{\partial[\mathrm{Ptc1Shh}_{mem}]}{\partial t} &= -(\mathrm{k_{off}} + \mathrm{k_{Cin}})[\mathrm{Ptc1Shh}_{mem}] + \mathrm{k_{on}}[\mathrm{Shh}][\mathrm{Ptc1}_{mem}] \\
&\qquad + k_{Cout}[\mathrm{Ptc1Shh}_{cyt}], \\
\frac{\partial[\mathrm{Ptc1Shh}_{cyt}]}{\partial t} &= k_{\mathrm{Cin}}[\mathrm{Ptc1Shh}_{mem}] - \mathrm{k_{Cout}}[\mathrm{Ptc1Shh}_{cyt}] - \mathrm{k_{Cdeg}}[\mathrm{Ptc1Shh}_{cyt}], \\
\frac{\partial[\mathrm{Ptc1}_{mem}]}{\partial t} &= k_{\mathrm{off}}[\mathrm{Ptc1Shh}_{mem}] - \mathrm{k_{on}}[\mathrm{Shh}][\mathrm{Ptc1}_{mem}] + \mathrm{k_{Pint}}[\mathrm{Ptc1}_{cyt}], \\
\frac{\partial[\mathrm{Ptc1}_{cyt}]}{\partial t} &= k_{\mathrm{P}}\mathrm{P_{tr}}\left\{[\mathrm{Gli1}^{Act}](\mathrm{t}-\mathrm{fi}), [\mathrm{Gli3}^{Act}](\mathrm{t}), [\mathrm{Gli3}^{Rep}](\mathrm{t})\right\}\Phi_{\mathrm{Ptc}} - \mathrm{k_{Pint}}[\mathrm{Ptc1}_{cyt}], \\
\frac{\partial[\mathrm{Gli1}^{Act}]}{\partial t} &= k_{\mathrm{G}}\mathrm{P_{tr}}\left\{[\mathrm{Gli1}^{Act}](\mathrm{t}-\mathrm{fi}), [\mathrm{Gli3}^{Act}](\mathrm{t}), [\mathrm{Gli3}^{Rep}](\mathrm{t})\right\}\Phi_{\mathrm{Ptc}} - \mathrm{k_{deg}}[\mathrm{Gli1}^{Act}], \\
\frac{\partial[\mathrm{Gli3}^{Rep}]}{\partial t} &= [\mathrm{Gli3}^{Act}]\frac{\mathrm{k_{g3r}}}{1+\mathrm{R_{Ptc}}} - \mathrm{k_{deg}}[\mathrm{Gli3}^{Rep}], \\
\frac{\partial}{\partial t}[\mathrm{Gli3}^{Act}] &= \frac{\gamma_{\mathrm{g3}}}{1+\mathrm{R_{Ptc}}} - [\mathrm{Gli3}^{Act}]\frac{\mathrm{k_{g3r}}}{1+\mathrm{R_{Ptc}}} - \mathrm{k_{deg}}[\mathrm{Gli3}^{Act}]\ ,
\end{aligned}
$$

being

$$
\Phi_{\mathrm{Ptc}} = \frac{[\mathrm{Ptc1}_{0}]}{[\mathrm{Ptc1}_{0}] + [\mathrm{Ptc1}_{mem}]}, \qquad R_{Ptc} = \frac{[\mathrm{Ptc1Shh}_{mem}]}{[\mathrm{Ptc1}_{mem}]},
$$

where $[\mathrm{Ptc1}_0]$ is the initial value of $[\mathrm{Ptc1}_{mem}]$. The values for the many parameters above are either taken from the literature or obtained experimentally [129]. From now on, we will refer to the coupling of the FLS equation with the ODEs system as the Gli-FLS model.

The mixed Dirichlet–Neumann problem for the FLS equation has been analyzed in [4] (wellposedness) and [52] (asymptotic behavior of solutions). In these studies the velocity at which the incoming chemical signal (vesicles) travels through the cytonemes in the extracellular matrix, as described by FLS, is taken constant (and agrees with the value of c, a mean value). This is precisely the behavior that we wanted to describe with a mathematical model, which cannot be replicated using a model as that in [119]. The value of c can be measured experimentally in different systems [129]. The mean value in the experiments [129] for c was found to be 0.07 $\mu m/s$ in the first few seconds, followed by an average value of 0.02 $\mu m/s$. Other measures in wing discs (8 hrs after Hh-GFP induction) and in early (embryonic day 8.5) mouse neural tubes, however, suggest a value of 0.0013 $\mu m/s$ (see [129] and references therein). The value of c thus appears to vary in different contexts (*Drosophila*, culture cells or vertebrates), at different times or stages of development, and in areas of equivalent distance from the source. These variations on speed at different stages of morphogen transport allows us to elucidate that a flux-saturated model taking into account the porous nature ($m \neq 1$) of the extracellular medium could improve the quantitative and qualitative aspects of the description.

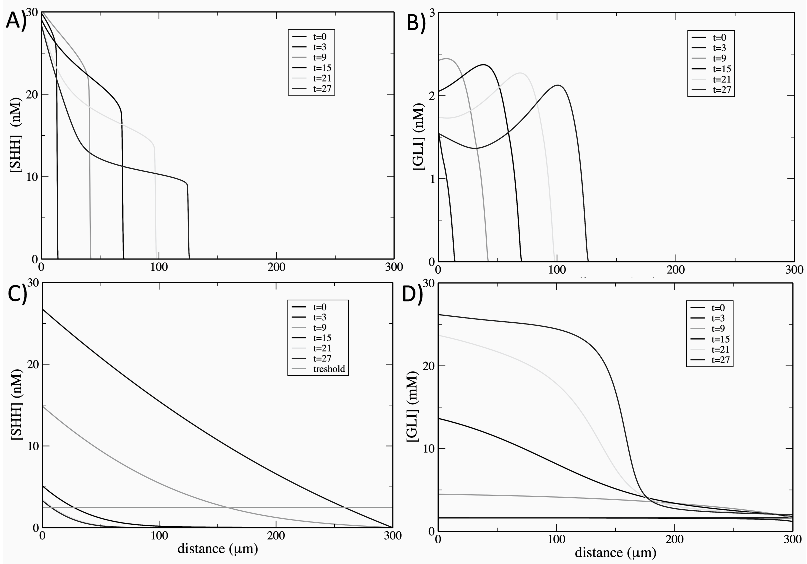

Figure 5. Plots of Shh and Gli1^{Act} concentrations versus distance from the floor plate at various times. The plots have been obtained numerically solving the flux-saturated model A) and B) and the linear diffusion model C) and D). Note that in linear diffusion modeling C) and D) there are no natural fronts and that an artificial and not dynamic threshold had to be imposed in C) at 2.5nM in order to achieve them (see Fig. 8 in [119]), which is independent of any biological reality and contradicts the results in [78]. Note also the correct pattern in the activation of Gli1^{Act} in B) while there are not concrete patterns in D).

14.3. Discussion. Previous studies in [24, 26] concerning macro and microscales and their connection via parabolic–hyperbolic asymptotics [112, 108, 86], point out that flux-saturated models of biological materials are related to the dynamical properties of living matter. This can be given an extended meaning as aggregates of living beings in the way of swarms, flocks, schools, bacteria colonies, cells, social entitites... [23, 26, 71]. Then, flux-saturated mechanisms may have the capacity of reproducing some of the emerging behavior that shows up at a collective level; here we term emerging behavior as that involving the interacting individuals but not being directly related to the dynamics of a few entities. This would relate to vesicles and their relationship with cytonemes in our context.

Understanding how these molecules gain information from each other, transfer it, can be synchronized [40], cooperate and make decisions is a fascinating issue. To the best of our knowledge, the applicability of flux-limitation ideas to this problem was first discussed in [26], although a broader literature about the subject is starting to emerge. In morphogenesis, the fact that cells form clusters connected by cytonemes and vesicles transmitting information, as well as their ability of organizing their dynamics according to a strategy (based on nonlinear additive actions in the group), is attractive and needs further exploration.

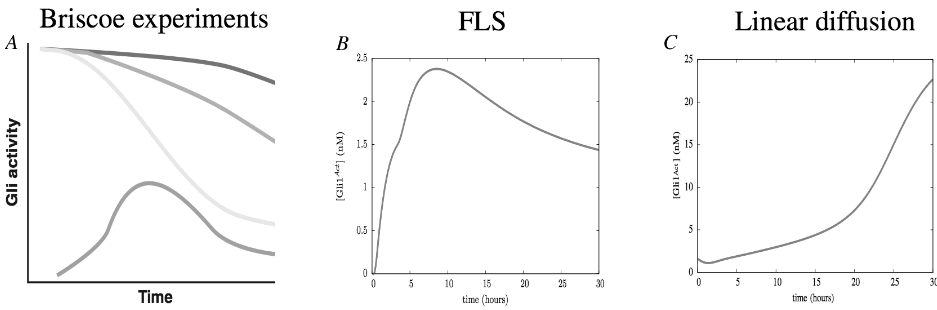

Figure 6. (A) Briscoe's group experiment on desensitization. (B,C) Evolution of Gli1^{Act} over time at a distance of 1 μm from the floor plate resulting from signal accumulation and subsequent desensitization in our FLS model (B). This behavior is not observed in the linear diffusion model (C). The FLS curve reproduces a temporal adaptation mechanism: After 12 hours cells become desensitized to Shh signal and the response decreases.

The mathematical description of cytonemes requires, at least, a bidimensional version of the model introduced in [20, 4, 49, 52, 61, 62, 67, 115]. Such a description must incorporate the potential generated by Ptc1, as well as other interactions (for instance, that with Cdon) as a source of directionality. This is a challenging line for future research.

References

[1] M. Agueh, *Existence of solutions to degenerate parabolic equations via the Monge–Kantorovich theory*, PhD Thesis, Georgia Tech, Atlanta, 2001. MR 2703679

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 2000. Zbl 0957.49001 MR 1857292

[3] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows: in metric spaces and in the space of probability measures*, Birkhäuser 2008. Zbl 1145.35001 MR 2401600

[4] F. Andreu, J. Calvo, J. M. Mazón and J. Soler, On a nonlinear flux–limited equation arising in the transport of morphogens, *J. Differential Equations* **252** (2012), 5763–5813. Zbl 1263.35166 MR 2902135

[5] F. Andreu, V. Caselles and J. M. Mazón, A parabolic quasilinear problem for linear growth functionals, *Rev. Mat. Iberoamericana* **18** (2002), 135–185. Zbl 1010.35063 MR 1924690

[6] F. Andreu, V. Caselles and J. M. Mazón, Existence and uniqueness of solution for a parabolic quasilinear problem for linear growth functionals with L^1 data. *Math. Ann.* **322** (2002), 139–206. Zbl 1056.35085 MR 1883392

[7] F. Andreu, V. Caselles and J. M. Mazón, The Cauchy Problem for Linear Growth Functionals. Dedicated to Philippe Bénilan. *J. Evol. Equat.* **3** (2003), 39–65. Zbl 1028.35054 MR 1977428

[8] F. Andreu, V. Caselles and J. M. Mazón, *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*. Progress in Mathematics, vol. 223, 2004. Birkhäuser. Zbl 1053.35002 MR 2033382

[9] F. Andreu, V. Caselles and J. M. Mazón. A Strongly Degenerate Quasilinear Equation: the Elliptic Case, *Annali della Scuola Norm. Sup. di Pisa. Serie V. Vol III, Fasc.* **3** (2004), 555–587. Zbl 1117.35022 MR 2099249

[10] F. Andreu, V. Caselles and J. M. Mazón. A Strong Degenerate Quasilinear Equation: the Parabolic Case, *Arch. Rat. Mech. Anal.* **176** (2005), 415–453. Zbl 1112.35111 MR 2185664

[11] F. Andreu, V. Caselles and J. M. Mazón, A Strongly Degenerate Quasilinear Elliptic Equation, *Nonlinear Analysis*, TMA **61** (2005), 637–669. Zbl 1190.35100 MR 2126618

[12] F. Andreu, V. Caselles and J. M. Mazón, The Cauchy Problem for a Strongly Degenerate Quasilinear Equation, *Journal European Math. Society* **7** (2005), 361–393. Zbl 1082.35089 MR 2156605

[13] F. Andreu, V. Caselles and J. M. Mazón. Some regularity results on the 'relativistic' heat equation, *Journal of Differential Equations* **245** (2008), 3639–3663. Zbl 1160.35016 MR 2462697

[14] F. Andreu, V. Caselles and J. M. Mazón. Limited flux diffusion equations in *Functional Analysis and Evolutions Equations, The Günter Lumer Volume*, H. Amann, W. Arendt, M. Hieber, F. Neubrander, S. Nicaise, J. von Below., editiors, 17–34, Birkhäuser, Basel, 2008. Zbl 1170.35465 MR 2402719

[15] F. Andreu, V. Caselles and J. M. Mazón. A Fisher-Kolmogorov equation with finite speed of propagation, *Journal of Differential Equations* **248** (2010), 2528–2561. Zbl 1198.35121 MR 2600967

[16] F. Andreu, V. Caselles and J. M. Mazón. Some flux-limited quasi-linear elliptic equations without coercivity in *On the notions of solution to nonlinear elliptic problems: results and developments*, Angelo Alvino, Anna Mercaldo, Francois Murat and Ireneo Peral, editors, Quad. Mat., **23**, 25–66, Dept. Math., Seconda Univ. Napoli, Caserta, 2008. Zbl 1226.35039 MR 2762160

[17] F. Andreu, V. Caselles, J. M. Mazón and S. Moll. Finite Propagation Speed for Limited Flux Diffusion Equations, *Arch. Ration. Mech. Anal.* **182** (2006), 269–297. Zbl 1142.35455 MR 2259334

[18] F. Andreu, V. Caselles, J. M. Mazón and S. Moll. A Diffusion Equation in Transparent Media, *J. Evol. Equat.* **7** (2007), 113–143. Zbl 1116.35072 MR 2305728

[19] F. Andreu, V. Caselles, J. M. Mazón and S. Moll, The Dirichlet problem associated to the relativistic heat equation, *Math. Ann.* **347** (2010), no. 1, 135–199. Zbl 1188.35108 MR 2593288

[20] F. Andreu, V. Caselles, J. M. Mazón, J. Soler and M. Verbeni, Radially symmetric solutions of a tempered diffusion equation. A porous media flux-limited case, *SIAM J. Math. Anal.* **44** (2012), 1019–1049. Zbl 1276.35109 MR 2914259

[21] S. B. Angenent, Local Existence and Regularity for a Class of Degenerate Parabolic Equations, *Math. Ann.* **280** (1988), 465–482. Zbl 0619.35114 MR 936323

[22] G. Anzellotti, Pairings Between Measures and Bounded Functions and Compensated Compactness, *Ann. di Matematica Pura ed Appl. IV* **135** (1983), 293–318. Zbl 0572.46023 MR 750538

[23] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study, *Proc. Nat. Acad. Sci.*, **105** (2008), no. 4, 1232–1237.

[24] N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, Multiscale biological tissue models and flux-limited chemotaxis for multicellular growing systems, *Math. Models Methods Sci.*, **20** (2010), 1179–1207. Zbl 05781037 MR 2673415

[25] N. Bellomo, D. Knoppof and J. Soler, On the difficult interplay between life, "complexity", and mathematical sciences, *Math. Models Methods Appl. Sci.*, **23** (2013), 1861–1914. Zbl 06239134 MR 3078677

[26] N. Bellomo and J. Soler. On the mathematical theory of the dynamics of swarms viewed as a complex system, *Math. Models Methods Appl. Sci.*, **22** (2012), Paper No. 1140006, 29 pages. Zbl 1242.92065 MR 2974186

[27] Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J. L. Vazquez, An L^1-Theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations, *Ann. Scuola Normale Superiore di Pisa IV*, **Vol. XXII** (1995), 241–273. Zbl 0866.35037 MR 1354907

[28] Ph. Bénilan and M. G. Crandall, Regularizing effects of homogeneous evolution equations in *Contributions to Analysis and Geometry*, 23–39, Johns Hopkins Univ. Press, Baltimore, Md. 1981. Zbl 0556.35067 MR 648452

[29] Ph. Bénilan and M. G. Crandall, Completely Accretive Operators, in *Semigroups Theory and Evolution Equations*, Ph. Clement et al., editors, 41–76, Lecture Notes in Pure and Appl. Math., 135, Marcel Dekker, 1991. Zbl 0895.47036 MR 1164641

[30] Ph. Bénilan, M. G. Crandall and A. Pazy, *Evolution Equations Governed by Accretive Operators*, Book, preprint.

[31] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems. I - Periodic framework, *J. European Math. Soc.*, **7** (2005), 173–213. Zbl 1142.35464 MR 2127993

[32] H. Berestycki and L. Rossi, Reaction-diffusion equations for population dynamics with forced speed, I - The case of the whole space, *Discrete and Continous Dynamical Systems*, **21** (2008), 41–67. Zbl 1173.35542 MR 2379456

[33] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, *Comm. Pure Appl. Math.*, **55** (2002), 949–1032. Zbl 1024.37054 MR 1900178

[34] F. Berthelin, P. Degond, M. Delitala and M. Rascle, A model for the formation and evolution of traffic jams, *Arch. Rat. Mech. Anal.*, **187** (2008), 185–220. Zbl 1153.90003 MR 2366138

[35] M. Bertsch and R. Dal Passo, Hyperbolic Phenomena in a Strongly Degenerate Parabolic Equation, *Arch Rational Mech. Anal.*, **117** (1992), 349–387. Zbl 0785.35056 MR 1148213

[36] M. Bertsch and R. Dal Passo, A Parabolic Equation with Mean-Curvature Type Operator in *Progress Nonlinear Differential Equation Appl.* **7**, F. Browder, editor, 89–97, Birkhäuser, 1992. Zbl 0795.35064 MR 1167831

[37] M. Bischoff, A. C. Gradilla, I. Seijo, G. Andrés, C. Rodríguez-Navas, L. González-Méndez and I. Guerrero. Cytonemes are required for the establishment of a normal Hedgehog morphogen gradient in Drosophila epithelia, *Nature Cell Biology*, **15** (2013), 1269–1283.

[38] P. Blanc, On the regularity of the solutions of some degenerate parabolic equations, *Comm. in Partial Diff. Equat.*, **18** (1993), 821–846. Zbl 0818.35050 MR 1218520

[39] P. Blanc. *Sur une classe d'equations paraboliques degenerées a une dimension d'espace possedant des solutions discontinues*, Ph.D. Thesis, number 798, Ecole Polytechnique Federale de Lausanne, 1989.

[40] L. L. Bonilla, C. J. Pérez-Vicente, F. Ritort and J. Soler, Exactly solvable phase oscillators models with synchronization dynamics, *Phys. Rev. Lett.*, **81** (1998), 3643–3646.

[41] G. Bouchitté and A. Chambolle, in preparation.

[42] Y. Brenier, Extended Monge-Kantorovich. Theory in *Optimal Transportation and Applications*, Lectures given at the C.I.M.E. Summer School help in Martina Franca, L. A. Caffarelli and S. Salsa, editors, 91–122, Lecture Notes in Math. 1813, Springer-Verlag, 2003. Zbl 1064.49036 MR 2006306

[43] H. Brezis. *Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert*, North Holland, Amsterdam, 1973. Zbl 0252.47055 MR 348562

[44] F. Browder, Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains, *Proc. Nat. Acad. Sci. USA*, **74** (1977), 2659–2611. Zbl 0358.35034 MR 445124

[45] E. Bouin, V. Calvez and G. Nadin, Hyperbolic traveling waves driven by growth, *Math. Models Methods Appl. Sci.*, **24** (2014), 1165–1195. Zbl 1292.35036 MR 3187195

[46] L. A. Caffarelli and J. L. Vázquez. Nonlinear porous medium flow with fractional potential pressure, *Arch. Rational Mech. Anal.*, **202** (2011), 537–565. Zbl 1264.76105 MR 2847534

[47] L. A. Caffarelli, F. Soria and J. L. Vázquez. Regularity of solutions of the fractional porous medium flow, **15** (2013), 1701–1746. Zbl 1292.35312 MR 3082241

[48] J. Calvo, *Analysis of a class of diffusion equations with a saturation mechanism*, preprint, arXiv:1312.4034, 2014. Accepted for publication in *SIAM Journal on Mathematical Analysis*.

[49] J. Calvo, J. Campos, V. Caselles, O. Sánchez and J. Soler, *Pattern formation in a flux limited reaction-diffusion equation of porous media type*, arXiv:1309.6789, 2013.

[50] J. Calvo, J. Campos, V. Caselles, O. Sánchez and J. Soler, *Qualitative behaviour for flux-saturated mechanisms: traveling waves, waiting time and smoothing effects*, preprint, arXiv:1402.1679, 2014. Accepted for publication in *Journal of the European Mathematical Society*.

[51] J. Calvo and V. Caselles, Local-in-time regularity results for some flux-limited diffusion equations of porous media type, *Nonlinear Analysis TMA*, **93** (2013), 236–272. Zbl 1282.35112 MR 3117162

[52] J. Calvo, J. Mazón, J. Soler and M. Verbeni. Qualitative properties of the solutions of a nonlinear flux-limited equation arising in the transport of morphogens, *Math. Models Methods Appl. Sci.*, **21** (2011), 893–937. Zbl 1223.35057 MR 3090597

[53] A. Callejo, J. Culi and I. Guerrero, Patched, the receptor of Hedgehog, is a lipoprotein receptor, *Proc. Natl. Acad. Sci. USA*, **105** (2008), 912–917.

[54] J. Campos, P. Guerrero, O. Sánchez and J. Soler. On the analysis of travelling waves to a nonlinear flux limited reaction-diffusion equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **30** (2013), 141–155. Zbl 1263.35059 MR 3011295

[55] M. J. Cardozo, L. Sánchez-Arrones, A. Sandonis, C. Sánchez-Camacho, G. Gestri, S. W. Wilson, I. Guerrero and P. Bovolenta, Cdon acts as a Hedgehog decoy receptor during proximal-distal patterning of the optic vesicle, *Nature Communications* (2014) DOI: 10.1038/ncomms5272.

[56] J. Carrillo, V. Caselles, S. Moll. On the relativistic heat equation in one space dimension, *Proc. London Math. Soc.* **107** (2013), 1395–1423. Zbl 1282.35199 MR 3149851

[57] J. Carrillo, S. Moll, Numerical simulation of diffusive and aggregation phenomena in nonlinear continuity equations by evolving diffeomorphisms, *SIAM Journal on Scientific Computing* **31** (2010), no. 6, 4305–4329. Zbl 1205.65265 MR 2566595

[58] J. Carrillo, P. Wittbold. Uniqueness of Renormalized Solutions of Degenerate Elliptic-Parabolic problems, *Journal Differential Equations* **156** (1999), 93–121. Zbl 0932.35129 MR 1701806

[59] V. Caselles. Convergence of the 'relativistic' heat equation to the heat equation as $c \to \infty$, *Publicacions Matemàtiques*, **51** (2007), 121–142. Zbl 1140.35466 MR 2307149

[60] V. Caselles, An existence and uniqueness result for flux limited diffusion equations, *Discrete and Continuous Dynamical Systems* (special volume in honor of Ennio de Giorgi and Guido Stampacchia), **31** (2011), no. 4, 1151–1195. Zbl 1252.35163 MR 2836347

[61] V. Caselles. On the entropy conditions for some flux limited diffusion equations, *Journal Differential Equations* **250** (2011), 3311–3348. Zbl 1231.35101 MR 2772392

[62] V. Caselles. Flux limited generalized porous media diffusion equations, *Publ. Mat.*, **57** (2013), 155–217. Zbl 1282.35205 MR 3058932

[63] V. Caselles. Convergence of flux limited porous media diffusion equations to its classical counterpart, To appear in *Ann. Scuola Norm. Sup. di Pisa*, DOI Number 10.2422/2036-2145.201204_004.

[64] C. R. Cattaneo, Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée, *Comptes Rendus*, **247** (1958), no. 4, 431.

[65] A. Chambolle, V. Caselles, D. Cremers, M. Novaga and T. Pock. An introduction to Total Variation for image analysis in *Theoretical Foundations and Numerical Methods for Sparse Recovery*, 263–340, vol. 9 of Radon Series Comp. Appl. Math., De Gruyter, 2010. Zbl 1209.94004 MR 2731599

[66] G. Q. Chen and H. Frid. Divergence-Measure Fields and Hyperbolic Conservation Laws, *Arch. Rational Mech. Anal.* **147** (1999), 89–118. Zbl 0942.35111 MR 1702637

[67] A. Chertock, A. Kurganov and P. Rosenau. Formation of discontinuities in flux-saturated degenerate parabolic equations, *Nonlinearity*, **16** (2003), 1875–1898. Zbl 1049.35111 MR 2012846

[68] A. Chertock, A. Kurganov and P. Rosenau. On degenerate saturated-diffusion equations with convection, *Nonlinearity*, **18** (2005), 609–630. Zbl 1067.35037 MR 2122677

[69] A. Chertock, A. Kurganov, X. Wang and Y. Wu, On a chemotaxis model with saturated chemotactic flux, *Kinetic and Related Models*, **5** (2012), 51–95. Zbl 06075645 MR 2875735

[70] J.-F. Coulombel, F. Golse and T. Goudon. Diffusion Approximation and Entropy-based Moment Closure for Kinetic Equations, *Asymptotic Analysis*, **45** (2005), 1–39. Zbl 1091.35038 MR 2181257

[71] I. D. Couzin. Collective minds, *Nature*, **445** (2007), 715.

[72] M. G. Crandall. Nonlinear Semigroups and Evolution Governed by Accretive Operators in *Proc. of Symp. in Pure Mat., Part I*, F. Browder, editor, A.M.S., 305–338, Providence 1986. Zbl 0637.47039 MR 843569

[73] M. G. Crandall and T. M. Liggett. Generation of Semigroups of Nonlinear Transformations on General Banach Spaces, *Amer. J. Math.*, **93** (1971), 265–298. Zbl 0226.47038 MR 287357

[74] F. Crick, Diffusion in embryogenesis, *Nature*, **40** (1970), 561–563.

[75] G. Dal Maso. Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals, *Manuscripta Math.*, **30** (1980), 387–416. Zbl 0435.49016 MR 567216

[76] R. Dal Passo, Uniqueness of the entropy solution of a strongly degenerate parabolic equation, *Comm. in Partial Diff. Equat.*, **18** (1993), 265–279. Zbl 0815.35056 MR 1211734

[77] V. De Cicco, N. Fusco and A. Verde. On L^1-lower semicontinuity in BV, *J. Convex Anal.* **12** (2005), 173–185. Zbl 1115.49011 MR 2135805

[78] E. Dessaud, L. L. Yang, K. Hill, B. Cox, F. Ulloa, A. Ribeiro, A. Mynett, B. G. Novitch and J. Briscoe. Interpretation of the Sonic Hedgehog morphogen gradient by a temporal adaptation mechanism, *Nature* **450** (2007), 717–720.

[79] J. J. Duderstadt and G. A. Moses. *Inertial Confinement Fusion*, John Wiley and Sons, 1982.

[80] J. Dolbeault, O. Sánchez and J. Soler. Asymptotic behaviour for the Vlasov–Poisson system in the stellar dynamics case, *Arch. Ration. Mech. Anal.*, **171**, 301–327 (2004). Zbl 1057.70009 MR 2038342

[81] A. Einstein. Zür Theorie der Brownschen Bewegung, *Annalen der Physik* **19** (1906), 371–381. Zbl 37.0814.02

[82] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, 1992. Zbl 0804.28001 MR 1158660

[83] L. Giacomelli. *Finite speed of propagation and waiting-time phenomena for degenerate parabolic equations with linear growth Lagrangian*. Preprint, 2013. Available at: http://cvgmt.sns.it/papers/?year=2013

[84] G. Godlewski and P. A. Raviart. *Hyperbolic systems of conservation laws*, Ellipses, 1991. Zbl 0768.35059 MR 1304494

[85] S. Goldstein. On diffusion by discontinuous movements, and on the telegraph equation, *Q. J. Mech. Appl. Math.*, **4** (1951), no. 2, 129–156. Zbl 0045.08102 MR 47963

[86] T. Goudon, J. Nieto, J. Soler and F. Poupaud. Multidimensional high-field limit of the electrostatic Vlasov–Poisson–Fokker–Planck system, *J. Diff. Eq.*, **213** (2005) 418–442. Zbl 1072.35176 MR 2142374

[87] A. C. Gradilla and I. Guerrero. Cytoneme-mediated cell-cell signaling during development, *Cell Tissue Res* **352** (2013), 59–66.

[88] V. Greco, M. Hannus, S. Eaton. Argosomes: a potential vehicle for the spread of morphogens through epithelia, *Cell* **106** (2001), 633–645.

[89] I. Guerrero and C. Chiang. A conserved mechanism of Hedgehog gradient formation by lipid modifications, *Trends Cell Biol.* **17** (2007), 1–5.

[90] J. B. Gurdon, A. Mitchell and D. Mahony. Direct and continuous assessment by cells of their position in a morphogen gradient, *Nature* **376** (1995), 520–521.

[91] T. M. Jessell. Neuronal specification in the spinal cord: inductive signals and transcriptional codes, *Nature Reviews Genetics* **1** (2000), 20–29.

[92] G. S. Jiang and C. W. Shu. Efficient implementation of weighted ENO schemes, *J. Comp. Phys.* **126** (1996), 202–228. Zbl 0877.65065 MR 1391627

[93] D. R. Jordan, D. Kinderlehrer and F. Otto. The variational formulation of the Fokker–Planck equation, *SIAM J. Math. Anal.* **29** (1998), 1–17. Zbl 0915.35120 MR 1617171

[94] S. Kondo and T. Miura. Reaction-Diffusion model as a framework for understanding biological pattern formation, *Science* **329** (2010), 1616–1620. Zbl 1226.35077 MR 2732467

[95] T. B. Kornberg. The Imperatives of Context and Contour for Morphogen Dispersion, *Biophysical Journal* **103** (2012), 2252–2256.

[96] S. N. Kruzhkov. First order quasilinear equations in several independent variables, *Math. USSR-Sb.* **10** (1970), 217–243. Zbl 0215.16203

[97] A. Kurganov and P. Rosenau. On reaction processes with saturating diffusion, *Nonlinearity* **19** (2006), 171–193. Zbl 1094.35063 MR 2191624

[98] A. D. Lander, Q. Nie and F. Y.-M. Wan. Do morphogen gradients arise by diffusion?, *Developmental Cell* **2** (2002), 785–796

[99] C. D. Levermore. *A Chapman–Enskog approach to flux limited diffusion theory*, Technical Report, Lawrence Livermore Laboratory, UCID-18229, 1979.

[100] C. D. Levermore. Relating Eddington factors to flux limiters, *J. Quant. Spectrosc. Radiat. Transfer* **31** (1984), 149–160.

[101] C. D. Levermore and G. C. Pomraning. A Flux-Limited Diffusion Theory, *Astrophys. J.* **248** (1981), 321–334.

[102] A. Marquina. Diffusion front capturing schemes for a class of Fokker–Planck equations: Application to the relativistic heat equation, *Journal of Computational Physics* **229** (2010), 2659–2674. Zbl 05680406 MR 2586208

[103] R. McCann and M. Puel. Constructing a relativistic heat flow by transport time steps, *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* **26**(6) (2009), 2539–2580. Zbl 1181.80001 MR 2569908

[104] H. Meinhardt. Space-dependent cell determination under the control of a morphogen gradient, *J. Theor. Biol.* **74** (1978), 307–321.

[105] A. M. Meirmanov, V. V. Pukhnachov and S. I. Shmarev. *Evolution equations and Lagrangian coordinates*, De Gruyter expositions in mathematics, 24, 1997. Zbl 0876.35001 MR 1440181

[106] D. Mihalas and B. Mihalas. *Foundations of radiation hydrodynamics*, Oxford University Press, 1984. Zbl 0651.76005 MR 781346

[107] J. E. Morel. Diffusion-limit asymptotics of the transport equation, the $P_{1/3}$ equations, and two flux-limited theories, *Journal of Quantitative Spectroscopy & Radiative Transfer* **65** (2000), 769–778.

[108] J. Nieto, F. Poupaud and J. Soler. High field limit for the Vlasov–Poisson–Fokker–Planck system, *Arch. Rational Mech. Anal.* **158** (2001), no. 1, 29–59. Zbl 1038.82068 MR 1834113

[109] G. L. Olson, L. H. Auer and M. L. Hall. Diffusion, P_1, and other approximate forms of radiation transport, *Journal of Quantitative Spectroscopy & Radiative Transfer* **64** (2000), 619–634.

[110] D. Panáková, H. Sprong, E. Marois, C. Thiele and S. Eaton. Lipoprotein particles are required for Hedgehog and Wingless signalling, *Nature* **435** (2005), 58–65.

[111] T. Pock, D. Cremers, H. Bischof and A. Chambolle. Global solutions of variational models with convex regularization, *SIAM Journal on Imaging Sciences* **3** (2010), no. 4, 1122–1145. Zbl 1202.49031 MR 2763711

[112] F. Poupaud and J. Soler, Parabolic Limit and Stability of the Vlasov–Poisson–Fokker–Planck system, *Math. Models Methods Appl. Sci.* **10** (2000) 1027–1045. Zbl 1018.76048 MR 1780148

[113] J. Rinzel and J. B. Keller. Travelling wave solutions of a nerve conduction equation, *Biophysical Journal* **13** (1973), 1313–1336. MR 2623547

[114] P. Rosenau. Free Energy Functionals at the High Gradient Limit, *Phys. Review A* **41** (1990), 2227–2230.

[115] P. Rosenau. Tempered Diffusion: A Transport Process with Propagating Front and Inertial Delay, *Phys. Review A* **46** (1992), 7371–7374.

[116] P. Rosenau, P. S. Hagan, R. L. Northcutt and D. S. Cohen. Delayed diffusion due to flux limitation, *Physics Letters A* **142** (1989), no. 1, 26–30.

[117] S. Roy, F. Hsiung, T. B. Kornberg. Specificity of Drosophila cytonemes for distinct signaling pathways, *Science* **15** (2011), 354–358.

[118] M. B. Rubin. Hyperbolic Heat Conduction and the Second Law, *Int. J. Engng. Sci.* **30** (1992), 1665–1676. Zbl 0764.73006 MR 1182891

[119] K. Saha and D. V. Schaffer. Signaling dynamics in Sonic hedgehog tissue patterning, *Development* **133** (2006), 889–900.

[120] O. Sánchez and J. Soler. Orbital stability for polytropic galaxies, *Annales de l'IHP- Analyse non linéaire* **23** (2006), no. 6, 781–802. Zbl 1110.35011 MR 2271693

[121] T. A. Sanders, E. Llagostera and M. Barna. Specialized filopodia direct long-range transport of Shh during vertebrate tissue patterning, *Nature* doi: 10.1038/nature12157 (2013).

[122] S. Serna and A. Marquina. Fronts propagating with signal dependent speed in Limited Diffusion and related Hamilton–Jacobi formulations, *Applied Numerical Mathematics* **73** (2013), 48–62. Zbl 06362732 MR 3146866

[123] B. Stecca and A. Ruiz i Altaba. Context-dependent regulation of the GLI code in cancer by Hedgehog and non-Hedgehog signals, *J. Mol. Cell Biol.* **2** (2010), no. 2, 84–95.

[124] W. Thomson. On the theory of the electric telegraph, *Proc. R. Soc. Lond. Ser. I* **7** (1854–1855), 382–399.

[125] C. Tsallis. Possible generalization of Boltzmann-Gibbs statistics, *Journal of Statistical Physics* **52** (1988), 479–487. Zbl 1082.82501 MR 968597

[126] A. M. Turing, The chemical basis of morphogenesis. *Philos. Trans. R. Soc. Lond. Ser. B Biol. Sci.* **237** (1952), 37–72.

[127] J. L. Vázquez. *The porous medium equation: mathematical theory*, Oxford University Press, 2007. Zbl 1107.35003 MR 2286292

[128] J. L. Vázquez. *Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type*, Oxford Lecture Series in Mathematics and its Applications, 2006. Zbl 1113.35004 MR 2282669

[129] M. Verbeni, O. Sánchez, E. Mollica, I. Siegl-Cachedenier, A. Carleton, I. Guerrero, A. Ruiz i Altaba and J. Soler. Morphogenetic action through flux-limited spreading, *Phys. Life Rev.* **10** (2013), 457–475.

[130] C. Villani. *Topics in Optimal Transportation*, American Mathematical Society, 2003. Zbl 1106.90001 MR 1964483

[131] N. Vyas, D. Goswami, A. Manonmani, P. Sharma, H. Ranganath, K. Vijay Raghavan, L. Shashidhara, R. Sowdhamini and S. Mayor. Nanoscale organization of Hedgehog is essential for long-range signaling, *Cell* **133** (2008), 1214–1227.

[132] X. Zeng, J. A. Goetz, L. M. Suber, W. J. Scott Jr, C. M. Schreiner and D. J. Robbins. A freely diffusible form of Sonic Hedgehog mediates long-range signalling, *Nature* **411** (2011), 716–720.

[133] W. P. Ziemer. *Weakly Differentiable Functions*, GTM 120, Springer-Verlag, 1989. Zbl 0692.46022 MR 1014685

Received 13 March, 2015