Exclusion and Dominance in Discrete Population Models via the Carrying Simplex

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Abstract

This paper is devoted to show that Hirsch's results on the existence of a carrying simplex are a powerful tool to understand the dynamics of Kolmogorov models. For two and three species we prove that there is exclusion for our models if and only if there are no coexistence states. The proof of this result is based on a result in planar topology due to Campos, Ortega and Tineo. For an arbitrary number of species, we will obtain dominance criteria following the notions of Franke and Yakubu. In this scenario, the crucial fact will be that the carrying simplex is an unordered manifold. Applications in concrete models are given.

Key words and phrases: Exclusion, Dominance, Carrying Simplex, Unordered manifold, Reduction of dimension.

1 Introduction

We study the class of systems

$$x_i(N+1) = x_i(N)f_i(x_1(N), ..., x_n(N)) \tag{1.1}$$

for $1 \leq i \leq n$ defined on the positive cone $K = \{x \in \mathbb{R}^n : x_i \geq 0,\ 1 \leq i \leq n\}$. The functions $f_i : K \longrightarrow]0, \infty[$ are continuous and strictly decreasing with respect to each variable x_j. These are natural assumptions in order to model the evolution

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of n competing species. As an instance, we can consider the case $f_i(x_1, ..., x_n) = \exp(A_i - \sum_{j=1}^{n} B_{ij} x_j)$, leading to the well-known model of May-Oster. By analogy with the continuous case, Hirsch developed in [10] a theory on the existence of a carrying simplex for the difference system (1.1). Let us recall this notion in an informal way. By a carrying simplex we understand a manifold Γ^{n-1} of dimension $n-1$, lying in K and invariant under the difference equation. Moreover, all non-trivial solutions of (1.1) are attracted by Γ^{n-1}. The existence of a carrying simplex is a powerful tool in the study of the dynamics of (1.1) and the purpose of this paper is to derive some consequences. First of all we will observe that, by passing from K to Γ^{n-1}, the dimension is reduced and this allows to apply techniques coming from planar topology in the study of three species ($n = 3$). Later we will obtain some results valid for an arbitrary number of species which are based on geometrical considerations. The crucial fact will be that the set Γ^{n-1} is unordered. This means that any two different points $x, y \in \Gamma^{n-1}$ can not satisfy $x_i \geq y_i$ for all $i = 1, ..., n$. The results that we are going to obtain are related to the notions of **Exclusion** and **Dominance**. Next we explain the exact meaning of these terms. We will say that there is exclusion for the system (1.1) if some of the species becomes extinct for every positive initial condition. This means that for each solution $x(N)$ with $x_i(0) > 0$, $i = 1, ..., n$, there exists an index j, $1 \leq j \leq n$, such that $x_j(N) \to 0$ as $N \to \infty$. Notice that j may depend on the initial condition and so the species j can become extinct for some initial conditions and survive for others. We say that the species j is dominated whenever $x_j(N) \longrightarrow 0$ for any initial condition lying in the interior of K. Finally, we say that the species j is dominant if there is $\delta > 0$ so that for all initial conditions in the interior of K, $x_i(N) \longrightarrow 0$ for all $i \neq j$ and $\liminf_{N \to \infty} x_j(N) > \delta > 0$. These notions are not new and have been analyzed in [2], [4], [8], [7], [9], [12], [1], [18], [13] and the references therein. The possible novelties of our results are linked to the use of topological tools and the geometrical point of view.

The structure of the paper is as follows. The main results are stated in Section 2. They include a characterization of exclusion for three species and a sufficient condition for dominance in systems with an arbitrary number of species. The result on exclusion can be seen as a discrete counterpart of a result obtained by Campos, Ortega and Tineo in [2] for periodic systems of differential equations of competitive type. The result on dominance makes use of a condition introduced by Franke and Yakubu in [8]. The proofs of these results are presented in Section 3. One of the main assumption in all the results is that the map T associated with the system (1.1) is retrotone. This notion was employed by Hirsch in his construction of the carrying simplex. We devote Section 4 to present a sufficient condition for the retrotone character of a map. This condition can be checked in practice and is satisfied in many examples. In Section 5 we apply our results in concrete examples. These examples are chosen in order to show the applicability of our results and also to compare with previous results in the literature. Finally we include an Appendix on the construction of the carrying simplex. This construction is essentially contained in [10] but we prefer to include complete proofs.

2 Assumptions and main results

The positive cone of $\mathbb{R}^n$ will be denoted by K. In coordinates,

$$K = \{x \in \mathbb{R}^n : \ x_i \geq 0\}.$$

The I' facet of K for I a subset of $\{1, ..., n\}$ will be

$$K_I = \{x \in K : x_j = 0 \text{ if } j \notin I\}.$$

According to the previous definition, the i-th positive coordinate axis will be denoted by $K_{\{i\}}$. After that we introduce the usual ordering in $\mathbb{R}^n$. For two vectors $x, y \in \mathbb{R}^n$, we write $x \preceq y$ if $x_i \leq y_i$ for all $i = 1, ..., n$. If $x \preceq y$ and $x \neq y$, we write $x \prec y$. Given $a \preceq b$, we can define the closed order interval as

$$[a, b] = \{x \in \mathbb{R}^n : a \preceq x \preceq b\}.$$

An important concept in this paper is the following.

Definition 2.1 *A map $F : K \longrightarrow K$ is retrotone in a subset $X \subset K$ if for $x, y \in X$ with $F(x) \succ F(y)$ we have that $x_i > y_i$ provided $x_i \neq 0$.*

In dimension 1 a map $F : [0, \infty[\longrightarrow [0, \infty[$ is retrotone if and only if it is monotone non-decreasing. Another example of retrotone map is the Poincaré map associated with the system

$$x_i' = x_i f_i(t, x) \quad \text{for all } i = 1, ..., n \tag{2.2}$$

where $f_i : \mathbb{R} \times K \longrightarrow \mathbb{R}$ is T-periodic in time and strictly decreasing in each variable, see [15]. Once these concepts have been introduced, we study the system (1.1). More precisely we consider the continuous map

$$T : K \longrightarrow K$$

$$x \mapsto (x_1 f_1(x), ..., x_n f_n(x))$$

with $f_i(x) > 0$ for all $x \in K$ and define the dynamical system

$$x(N + 1) = T(x(N)) \quad \text{with } \ x(0) \in K.$$

Throughout the paper we assume the following conditions for the map T:

C1) If $x \prec y$ then $f_i(y) < f_i(x)$ for all $i = 1, ..., n$,

C2) $T|_{K_{\{i\}}} : K_{\{i\}} \longrightarrow K_{\{i\}}$ admits a fixed point $(0, ...0, q_i, 0, ..., 0)$ with $q_i > 0$ for all $i = 1, ..., n$,

C3) T is retrotone and locally injective in a neighbourhood of $[0, q]$ where $q = (q_1, q_2, ..., q_n)$.

The biological interpretation of the previous conditions is as follows: the condition **C2)** says that each species has a coexistence state in the absence of the other species. The conditions **C1)** and **C3)** imply that our system is competitive and enjoys an additional monotonicity in the past. As we will see in Section 5, many models satisfy the previous conditions, see also the examples in [10].
Next we give our exclusion criteria for the system (1.1).

Theorem 2.1 *For* $n = 2$*, assume that (1.1) verifies* **C1), C2), C3)**. *Then the following statements are equivalent:*

i) *T has no fixed points in $IntK$.*

ii) *There is exclusion for the system (1.1).*

The previous result is well known, see for instance [16]. However this theorem motivates a similar result for $n = 3$. In this case we need to introduce an additional condition.

C4) T has a finite number of fixed points on ∂K .

Along the paper we use the notation ∂K to denote the boundary of K.

Theorem 2.2 *For* $n = 3$*, assume that (1.1) verifies* **C1), C2), C3), C4)**. *Then the following statements are equivalent:*

i) *T has no fixed points in $IntK$.*

ii) *There is exclusion for the system (1.1).*

As a direct consequence of the previous theorem, we can obtain the following result.

Corollary 2.1 *For* $n = 3$*, assume that (1.1) verifies* **C1), C2), C3)** *and* ***C4)*** *with* $\{p_1, p_2, ..., p_r\} = Fix(T) \cap \partial K$*. Then the following statements are equivalent:*

i) $\lim_{N\to\infty} x(N) = p_1$ *for all initial condition* $x(0) \in IntK$*.*

ii) *The system (1.1) does not have fixed points in $IntK$ and $W^s(p_n) \cap IntK = \emptyset$ for all $n \geq 2$.*

In the previous result $W^s(p)$ is defined as

$$W^s(p) = \{x(0) \in K : \lim_{N\longrightarrow\infty} x(N) = p\}.$$

We notice that the second condition of corollary 2.1 is easy to check in many models. For instance, if we assume that $p = (0, y_2, y_3)$ is a fixed point of T with $f_1(0, y_2, y_3) > 1$ then $W^s(p) \cap IntK = \emptyset$. At this point it is important to notice that the previous results are not true for $n > 3$ since we can adapt the example 3 in [2] to our context, (see also the last section in [2]). On the other hand, if we do not assume **C4)**, we can only ensure that $\omega(p)$ is a connected set contained in $Fix(T)$, (see example 2 in [2]), where $\omega(p)$ denotes the usual omega limit set.
Next we give a criterion guaranteeing the presence of dominant species. Motivated by [8], [7], [9] and [12], we introduce the following concept.

Definition 2.2 *The species j verifies F-Y condition if*

$$\bigcup_{i \in \{1,2,\ldots,n\} \setminus \{j\}} D_i^+ \subset D_j^*$$

where $D_j^* = \{x \in K : f_j(x) > 1\}$ *and* $D_i^+ = \{x \in K : f_i(x) \geq 1\}$.

The biological interpretation of the previous definition is as follows: if some species different from the species j does not decrease its size, then the species j increases strictly. Let us remark that

- F-Y condition for the species j does not imply, in general, $f_j(x) > f_i(x)$ for all $i \neq j$,
- F-Y condition does not ensure the presence of dominant species, (see [8], [18]).

Franke and Yakubu understand that a species is weakly dominant if it verifies F-Y condition. However this condition is far from being a necessary condition for the presence of dominant species. This will be shown with an example in section 5.

Theorem 2.3 *Assume that the system (1.1) verifies* **C1), C2), C3)**. *If the species j verifies F-Y condition then the species j is dominant.*

Finally we give a result ensuring the presence of dominated species in our system.

Theorem 2.4 *For n=3, assume that the system (1.1) verifies* **C1)**, **C2), C3)**. *If $D_1^+ \subset D_2^*$ then the species 1 is dominated.*

3 Proofs

The aim of this section is to prove the previous results. The key ingredient will be the existence of a carrying simplex for the system (1.1), i.e. the existence of a subset $\Gamma^{n-1} \subset [0, q] \setminus \{0\}$ having the following properties:

A1) Γ^{n-1} is homeomorphic to a $(n-1)$-simplex.

A2) Γ^{n-1} is unordered, i.e. if $x, y \in \Gamma^{n-1}$ and $x \succeq y$ then $x = y$.

A3) For every $x(0) \in K \setminus \{0\}$ the trajectory of $x(0)$ is asymptotic with the trajectory of some $y(0) \in \Gamma^{n-1}$, i.e. $\lim_{N \to \infty} x(N) - y(N) = 0$.

A4) $T(\Gamma^{n-1}) = \Gamma^{n-1}$ and $T : \Gamma^{n-1} \longrightarrow \Gamma^{n-1}$ is a homeomorphism.

By corollary 6.1 in the appendix, we can deduce that the conditions **C1), C2), C3)** guarantee that the system (1.1) admits a carrying simplex. Now we proceed to prove our results.

Proof of theorem 2.1.
ii) $\Rightarrow$ *i)*.

From the definition of exclusion we know that all solutions are attracted by the boundary of K. This excludes fixed points lying in $IntK$.
$i) \Rightarrow ii)$.
Since the system (1.1) admits carrying simplex,

$$T : \Gamma^1 \longrightarrow \Gamma^1$$

is a homeomorphism with Γ^1 homeomorphic to a closed interval. Now, using that T has no fixed points in $IntK \cap \Gamma^1$, we deduce that for all $(x, y) \in \Gamma^1 \cap IntK$ either $T^N(x, y) \to (q_1, 0)$ or $T^N(x, y) \to (0, q_2)$ where $(q_1, 0)$ and $(0, q_2)$ are the non trivial fixed points of T on the axes. Notice that we are dealing with monotone dynamics in one dimension. We conclude the proof by using that any orbit in $K\backslash\{0\}$ is asymptotic to an orbit in Γ^1, as stated in **A3)**.

Proof of theorem 2.2.
To prove this theorem we need the following result taken from [2]:

Theorem 3.1 *Let D be a topological disk and let h be an orientation preserving homeomorphism such that*

$$Fix(h) \subset \partial D.$$

Then, for each $p \in D$, $\omega(p)$ is a connected subset of $Fix(h)$.

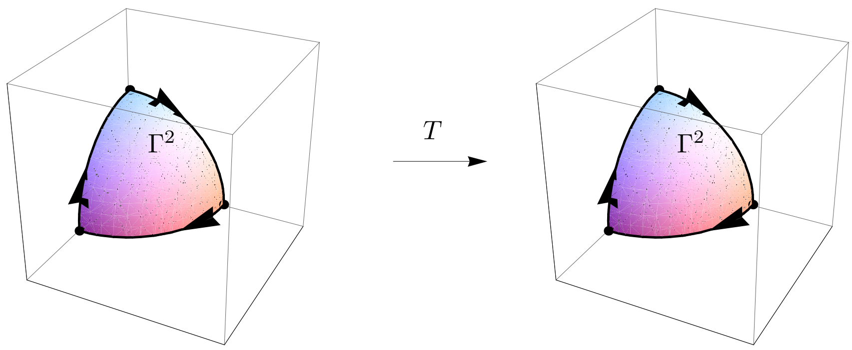
Firstly, we prove that $T|_{\Gamma^2}$ is an orientation-preserving homeomorphism. Indeed, on the boundary of the carrying simplex, we can define two orientations. Specifically if $\gamma : [0, 1] \longrightarrow \partial\Gamma^2$ is a parametrization of $\partial\Gamma^2$ with $\gamma(0) = (q_1, 0, 0)$ then either

$$t_2 < t_3 \tag{3.3}$$

or

$$t_3 < t_2, \tag{3.4}$$

where $\gamma(t_2) = (0, q_2, 0)$ and $\gamma(t_3) = (0, 0, q_3)$. Now it is clear that $T \circ \gamma$ is a parametrization of $\partial\Gamma^2$ verifying (3.3) (resp. (3.4)) provided γ verifies (3.3) (resp (3.4)). From these comments, we deduce that T is an orientation preserving homeomorphism in Γ^2. Next we illustrate the previous reasoning with a picture.



An alternative proof of this fact can be found in [14].
By theorem 3.1 and **C4)**, we deduce that for all $x(0) \in \Gamma^2$, $\omega(x(0)) = \{p_i\}$ with $p_i \in Fix(T)$. The proof is complete because we know that for all $y(0) \in K\backslash\{0\}$, there exists $x(0) \in \Gamma^2$ so that $\omega(y(0)) = \omega(x(0))$, (**A3)**).

Proof of corollary 2.1. By theorem 2.2 (see paragraph above), we deduce that for all $x(0) \in IntK$, $\omega(x(0)) = \{p_i\}$ for all $i = 1, ..., n$. We rule out the cases $i = 2, ..., n$ by using the condition **ii)**.

To prove theorem 2.3 we use some geometrical aspects of the carrying simplex. Specifically the key fact will be that Γ^{n-1} is an unordered manifold.
Proof of theorem 2.3.
Using that the species j verifies F-Y condition, we deduce that

$$Fix(T) \cap \{x_j \neq 0\} = \{(0, ..., 0, q_j, 0, ..., 0)\}$$

where $(0, ..., 0, q_j, 0, ..., 0)$ is the unique positive fixed point in $K_{\{j\}}$. After this remark we split the proof into two steps.
Step 1: Dominance in the carrying simplex.
In this step we prove that if $x(0) \in \Gamma^{n-1}$ with $x_j(0) \neq 0$ then

$$\lim_{N\to\infty} x(N) = (0, ..., 0, q_j, 0, ..., 0).$$

Indeed, take $x(0) \in \Gamma^{n-1}$ with $x(0) \neq (0, ..., 0, q_j, 0, ..., 0)$ and $x_j(0) \neq 0$. First of all, we prove that the sequence $\{x_j(N)\}_N$ is strictly increasing. Using that Γ^{n-1} is unordered and invariant, we can take an index i such that $x_i(0) < x_i(1)$. Now, by applying that the species j verifies F-Y condition, we deduce that $x_j(0) < x_j(1)$. Hence, by repeating this argument we obtain that the sequence $\{x_j(N)\}_N$ is strictly increasing. At this point it is clear that there exists $\alpha > 0$ such that $x_j(N) \nearrow \alpha$. Let us prove that $\alpha = q_j$. By contradiction, assume that $\alpha < q_j$. If this were the case, there would exist an orbit contained in $\{x \in \Gamma^{n-1} : x_j = \alpha\}$. This is impossible since given $y(0) \in \Gamma^{n-1} \cap \{x \in K : x_j = \alpha\}$, the sequence $\{y_j(N)\}_N$ is strictly increasing. To prove this claim, use that there are not fixed points for T in $\{x \in \Gamma^{n-1} : x_j = \alpha\}$ together with the previous argument. Finally, as $\Gamma^{n-1} \cap \{x_j = q_j\} = (0, ..., 0, q_j, 0, ..., 0)$ we conclude that

$$\lim_{N\to\infty} x(N) = (0, ..., 0, q_j, 0, ..., 0).$$

Step 2: $S = \Gamma^{n-1} \cap \{x_j = 0\}$ is a repellor.
In this step we prove that there exists $\epsilon > 0$ so that for all $x(0) \in IntK$ with $dist(x(0), S) < \epsilon$, there exists $N_0 := N_0(x(0)) > 0$ such that $dist(x(N_0), S) > \epsilon$. Indeed, take $(x_1, x_2, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) \in \Gamma^{n-1}$and distinguish two cases:

Case 1: The point $(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$ is a fixed point of T.
In this case, there exists an index i different from j such that

$$f_i(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) = 1.$$

Therefore, using that the species j verifies F-Y condition, we deduce that

$$f_j(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) > 1.$$

Case 2: The point $(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$ is not a fixed point for T.
In this case using that $\Gamma^{n-1} \cap \{x \in K : x_j = 0\}$ is unordered, we deduce that there exists an index i such that

$$f_i(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) > 1.$$

Then, as the species j verifies F-Y condition, we obtain that

$$f_j(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) > 1.$$

In short, from the previous comments and the compactness of the set $S = \Gamma^{n-1} \cap \{x \in K : x_j = 0\}$, we deduce that there exist $\epsilon > 0$ and $\delta > 0$ so that

$$f_j(x) > 1 + \delta \tag{3.5}$$

for all $x \in K$ with $dist(x, S) < \epsilon$. Thus we obtain that $x_j(N) > x_j(0)(1 + \delta)^N$ whenever $dist(x(N), S) < \epsilon$ and $x(0) \in IntK$. The proof of the claim in Step 2 is complete.
From the step 2 and **A3)** we deduce that for all $x(0) \in IntK$ there exists $y(0) \in \Gamma^{n-1} \backslash \mathcal{S}$ so that $\omega(x(0)) = \omega(y(0))$. To finish the proof we use the first step.

Proof of theorem 2.4.
From the hypotheses of theorem 2.4 we deduce that T has no fixed points in $IntK$. Combining theorem 3.1 and **A3)**, we deduce that given $p = (x_1, x_2, x_3) \in K$, $\omega(p)$ is a connected set contained in $Fix(T)$. Then we only need to prove that $A = Fix(T) \cap \{x \in \Gamma^2 : x_1 \neq 0\}$ is a repellor (in the same sense as in the previous proof). We proceed by steps:
Step 1: Fixed points in $\{x \in A, x_2 = 0\}$.
Firstly we observe that the set $F_2 = Fix(T) \cap \{x \in A : x_2 = 0\}$ is compact. Now we distinguish two cases:

- The fixed point $(0, 0, q_3)$ is not an accumulation point of F_2.
  In this case the set $\widetilde{F_2} = F_2 \backslash \{(0, 0, q_3)\}$ is compact and for all $(x_1, 0, x_3) \in \widetilde{F_2}$,

  $$f_1(x_1, 0, x_3) = 1.$$

  Thus, from $D_1^+ \subset D_2^*$, we deduce that

  $$f_2(x_1, 0, x_3) > 1$$

  for all $(x_1, 0, x_3) \in \widetilde{F_2}$. Finally we proceed as in the previous theorem in order to conclude that $\widetilde{F_2}$ is a repellor.

- The fixed point $(0, 0, q_3)$ is an accumulation point of F_2.
  In this case we can deduce that for all $(x_1, 0, x_3) \in \widetilde{F_2}$,

$$f_1(x_1, 0, x_3) = 1.$$

  Hence, by continuity, $f_1(0, 0, q_3) = 1$. Applying $D_1^+ \subset D_2^*$, we deduce that

$$f_2(x_1, 0, x_3) > 1$$

  for all $(x_1, 0, x_3) \in F_2$. From these facts we also deduce that F_2 is a repellor.

Step 2: Fixed points in $\{x \in \Gamma^2 : x_3 = 0\}$.
By $D_1^+ \subset D_2^*$, it is clear that $(0, q_2, 0)$ and $(q_1, 0, 0)$ are the unique fixed points of T in $\{x_3 = 0\}$. Moreover $f_2(q_1, 0, 0)$ is greater than 1.
The previous steps allow us to conclude the proof, (reason as in the previous theorems).

4 Retrotone maps

The aim of this section is to give criteria ensuring that a concrete map of the type

$$T(x_1, x_2, ..., x_n) = (x_1 f_1(x), ..., x_n f_n(x))$$

with $f_i(x) > 0$ is retrotone in $C = [0, r]$ for $r \in IntK$. Following this purpose we introduce the next result.

Proposition 4.1 *Consider U a neighbourhood of C. If $T \in \mathcal{C}^1(U)$ verifies that for each $x \in C \backslash \{0\}$,*

$$[DT(x)]^{-1}_{i,j} > 0 \quad \textit{with } i, j \in I(x) = \{j : x_j \neq 0\}, \tag{4.6}$$

then T is retrotone and one-to-one in C.

To prove this proposition we need the following result.

Lemma 4.1 *(Lemma 2.3.4 in [3]) Assume that $C \subset \mathbb{R}^n$ is a compact set and*

$$f : C \longrightarrow f(C)$$

is a local homeomorphism. Then for all $y \in f(C)$, the cardinal of $f^{-1}(y)$ is finite. If $f(C)$ is also connected then there exists a constant r so that the cardinal of $f^{-1}(y)$ is exactly r for all $y \in f(C)$.

Proof of proposition 4.1 Use that $T^{-1}(\{0\}) = \{0\}$ to conclude that $T : C \longrightarrow T(C)$ is a homeomorphism. Finally, we use proposition 2.1 in [11].

Next we present a criterion given in [10] to prove the condition (4.6). First of all we compute the jacobian matrix of T,

$$DT(x) = [F(x)]^{diag} + [x]^{diag} DF(x) \text{ then}$$

$$DT(x) = [F(x)]^{diag}(Id - M(x))$$

where $M(x) = -[\frac{x}{F(x)}]^{diag}DF(x)$ and $F(x) = (f_1(x), f_2(x), ..., f_n(x))$. If for all $x \in K\backslash\{0\}$, we assume that $DF(x)$ has strictly negative entries then $M(x)$ verifies that

$$M_{i,j}(x) := -\frac{x_i}{f_i(x)}\frac{\partial f_i}{\partial x_j}(x) =$$

$$= -x_i\frac{\partial \log f_i(x)}{\partial x_j} > 0.$$

By the previous computations, we deduce that if the spectral radius $\rho(M(x))$ is less than 1 then the matrix $Id - M(x)$ is invertible and $[DT(x)]^{-1}_{i,j\in I(x)} > 0$. For it, we have to use that

$$DT(x)^{-1} = (\sum_{k=0}^{\infty} M^k(x))([F(x)]^{diag})^{-1}.$$

Therefore, in order to prove that T is retrotone we need only check that $\rho(M(x)) < 1$. Consequently, if

$$\max\{\sum_i M_{i,j}(x),\ j = 1, 2..., n\} < 1 \quad \text{or} \tag{4.7}$$

$$\max\{\sum_j M_{ij}(x),\ i = 1, ..., n\} < 1, \tag{4.8}$$

for all $x \in C$, then T is retrotone in C.
The following lemma determines the function $M_{ij}(x)$ in some concrete examples.

Lemma 4.2 *If $f_i(x_1, ..., x_n) = \exp(B_i - \sum_j(A_{ij}x_j))$ with $A_{ij} > 0$ then $M_{ij}(x) = A_{ij}x_i$. If $f_i(x_1, ..., x_n) = \frac{B_i}{1+\sum_j A_{ij}x_j}$ with $A_{ij} > 0$ then $M_{ij}(x) = \frac{A_{ij}x_i}{1+\sum_j A_{ij}x_j}$.*

5 Examples

The aim of this section is to illustrate our results with concrete examples. First we consider the classical May Oster model and obtain new results on exclusion. In contrast, the conclusions on dominance that we obtain are known. This has lead us to present example 2. This model contains different types of growths, nevertheless our results on dominance apply. This is not the case for the results in [7]. Finally, in the third example we show that F-Y condition is not a necessary for the existence of dominant species.

Example 1: May Oster model.
Consider the system

$$x_i(N+1) = x_i(N)\exp\left(B_i - A_{i1}x_1(N) - A_{i2}x_2(N) - A_{i3}x_3(N)\right) \tag{5.9}$$

for $i = 1, 2, 3$, where the coefficients verify that $B_i, A_{ij} > 0$. It is clear that **C1), C2)** hold. Moreover if

$$\frac{B_i}{A_{ii}}(A_{i1} + A_{i2} + A_{i3}) < 1$$

for $i, j = 1, 2, 3$, then **C3)** also holds, (see lemma 4.2, (4.8) and proposition 4.1). Therefore, under these assumptions we can apply our results. Indeed, if

$$A_{ii}A_{jj} - A_{ij}A_{ji} \neq 0$$

for all $i \neq j$ then **C4)** holds and so, according to theorem 2.2 we deduce that there is exclusion for the system (5.9) if and only if the linear system

$$B_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3$$

$$B_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3$$

$$B_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3$$

has no solutions in $IntK$. Next we apply our dominance results. Indeed, the species 1 verifies F-Y condition if and only if

$$\frac{B_1}{A_{1i}} > \max\{\frac{B_2}{A_{2i}}, \frac{B_3}{A_{3i}}\} \quad \text{for } i = 1, 2, 3. \tag{5.10}$$

Thus if our system enjoys the condition (5.10), by theorem 2.3, the species 1 is dominant. If

$$\frac{B_1}{A_{1i}} > \frac{B_2}{A_{2i}} \quad \text{for } i = 1, 2, 3 \tag{5.11}$$

then $D_1^+ \subset D_2^*$ and so, by theorem 2.4, the species 2 is dominated.
As mentioned before, the conclusions on exclusion are new but the conclusions on dominance can be obtained using [8] and [7]. Actually, for this model, the conditions required by [8] and [7] are less restrictive than ours.
Example 2: Mixing exponential and rational functions.
If the system (1.1) has different types of growth functions, for instance $f_1(x) = \exp(B_1 - \sum_j A_{1j}x_j)$ and $f_2(x) = \frac{B_2}{1+\sum_j A_{2j}x_j}$, we cannot apply the results in [7]. Motivated by this fact we consider the system

$$x_1(N+1) = x_1(N)\exp(B_1 - A_{11}x_1(N) - A_{12}x_2(N) - A_{13}x_3(N)) \tag{5.12}$$

$$x_2(N+1) = \frac{B_2x_2(N)}{1 + A_{21}x_1(N) + A_{22}x_2(N) + A_{23}x_3(N)}$$

$$x_3(N+1) = \frac{B_3x_3(N)}{1 + A_{31}x_1(N) + A_{32}x_2(N) + A_{33}x_3(N)},$$

where the coefficients verify $B_1 > 0$, $B_2, B_3 > 1$ and $A_{ij} > 0$. To ensure that the system (5.12) verifies **C3)**, assume that

$$\frac{B_1}{A_{11}}(A_{11} + A_{12} + A_{13}) < 1$$

$$\frac{(B_i - 1)}{A_{ii}B_i}(\sum_{j=1}^{3} A_{ij}) < 1 \quad i = 2, 3,$$

(see lemma 4.2, (4.8) and proposition 4.1). Now it is clear that $D_2^+ \subset D_1^*$ if and only if

$$\frac{B_1}{A_{1i}} > \frac{B_2 - 1}{A_{2i}} \quad \text{for } i = 1, 2, 3. \tag{5.13}$$

Therefore, if (5.13) holds, by theorem 2.4, $x_2(N) \longrightarrow 0$ for all initial condition $x(0) \in IntK$.
If

$$\frac{B_1}{A_{1i}} > \max\{\frac{B_2 - 1}{A_{2i}}, \frac{B_3 - 1}{A_{3i}}\}$$

for $i = 1, 2, 3$ then the species 1 verifies F-Y condition and so, by theorem 2.4, $x_2(N), x_3(N) \longrightarrow 0$ for all initial condition $x(0) \in IntK$.
Example 3: Leslie Gower model, a concrete model.
The purpose of this example is to show the applicability of corollary 2.1. Moreover we will see that F-Y condition is not necessary for the existence of dominant species. Indeed, consider

$$x_1(N+1) = \frac{1.15x_1(N)}{1 + x_1(N) + x_2(N) + x_3(N)}$$

$$x_2(N+1) = \frac{1.1x_2(N)}{1 + x_1(N) + x_2(N) + 0.8x_3(N)}$$

$$x_3(N+1) = \frac{1.2x_3(N)}{1 + x_1(N) + 2.5x_2(N) + x_3(N)}.$$

This system verifies the conditions **C1), C2), C3)** (see lemma 4.2, (4.8) and proposition 4.1) and has no fixed points in $IntK$. The fixed points on ∂K are $(0.15, 0, 0)$,$(0, 0.1, 0)$, $(0, 0, 0.2)$ and $(0, 0.06, 0.05)$ and satisfy $f_3(0.15, 0, 0)$, $f_1(0, 0.1, 0)$, $f_1(0, 0.06, 0.05) > 1$. Thus, by corollary 2.1, the fixed point $(0, 0, 0.2)$ is a global attractor in $IntK$. On the other hand, it is clear that the species 3 does not verify F-Y condition. Furthermore, we observe that $D_3^* \not\supseteq D_1^+$ and $D_3^* \not\supseteq D_2^+$.

6 Appendix: Construction of the Carrying Simplex

Consider $C = [0, r]$ for $r \in IntK$ and $T : C \longrightarrow T(C) \subset C$ a continuous map with an expression of the type

$$T(x) = (T_1(x), ..., T_n(x)) = (x_1 f_1(x), x_2 f_2(x), ..., x_n f_n(x)) \tag{6.14}$$

with $f_i(x) > 0$ for all $i = 1, ..., n$. This kind of maps enjoys the following property:

$$T_j(x) > 0 \quad \text{if and only if} \quad x_j > 0. \tag{6.15}$$

Next we give the precise definition of a carrying simplex.

Definition 6.1 *We will say that $T : C \longrightarrow T(C)$ admits a* ***carrying simplex*** *if there exists a subset $\Gamma^{n-1} \subset C\backslash\{0\}$ having the following properties:*

A1) Γ^{n-1} *is homeomorphic to a $n-1$-simplex,*

A2) Γ^{n-1} *is unordered, i.e. if $x, y \in \Gamma^{n-1}$ and $x \succeq y$ then $x = y$,*

A3) *for every $x(0) \in C\backslash\{0\}$, there exists $y(0) \in \Gamma^{n-1}$ so that $\lim_{N\to+\infty}[x(N) - y(N)] = 0$,*

A4) Γ^{n-1} *is invariant, i.e. $T(\Gamma^{n-1}) = \Gamma^{n-1}$ and $T : \Gamma^{n-1} \longrightarrow \Gamma^{n-1}$ is a homeomorphism.*

We employ the notation $x(N) = T^N(x(0))$ where $x(0) \in K$ denotes the initial condition. Once this definition has been introduced, the next step will be to give criteria to guarantee the existence of a carrying simplex for the map T. This motivates the following result.

Theorem 6.1 *Assume that T verifies the following conditions:*

1. *$T |_{K_{\{i\}}}: C \cap K_{\{i\}} \longrightarrow T(C) \cap K_{\{i\}}$ admits a fixed point $q_i e_i$ with $q_i > 0$. Moreover, we assume that $q = (q_1, ..., q_n) \in IntC$,*
2. *T is retrotone and locally one to one in C,*
3. *for $x, y \in C$ with $T(x) \prec T(y)$, we have that, for each j, either $x_j = 0$ or $f_j(x) = \frac{T_j(x)}{x_j} > f_j(y) = \frac{T_j(y)}{y_j}$.*

Then the map T admits a carrying simplex.

Remark 6.1 *Firstly we note that by (6.15), the third condition always makes sense. Moreover if we assume that T is retrotone in C, this condition is weaker than the condition below*

$$f_i(y) < f_i(x) \text{ for all } i = 1, ..., n \text{ provided } x \prec y.$$

Remark 6.2 *For $n = 1$, if T is retrotone and locally injective then T is strictly increasing.*

Throughout this section, we will always assume without further mention that the conditions 1, 2, 3 of theorem 6.1 hold. Next we point out two simple properties of the map T. First we observe that $T : C \longrightarrow T(C)$ is a homeomorphism. Indeed, by the theorem of invariance of the domain, we deduce that T is a local homeomorphism. To see that T is one-to-one, we use that $T^{-1}(\{0\}) = \{0\}$ together with lemma 4.1. The second property of T is given in the following result.

Lemma 6.1 *For all $\lambda_0 > 1$ such that $\lambda_0 q \in IntC$ and for all $x(0) \in C$, there exists $N_0(x(0)) := N_0 \in \mathbb{N}$ so that $T^N(x(0)) = x(N) \in [0, \lambda_0 q]$ for all $N \geq N_0$.*

Proof. Firstly we prove that $[0, \lambda_0 q]$ is positively invariant. Indeed, given $x \in [0, \lambda_0 q]$, it is clear that $x_1 e_1$, $x_2 e_2$,...,$x_n e_n$ also belong to $[0, \lambda_0 q]$. Moreover, we can deduce that

$$T_i(x) \leq T_i(x_i e_i) \tag{6.16}$$

for all $i = 1, ..., n$. To prove these inequalities, we reason by contradiction and use that T is retrotone, $x_i e_i \preceq x$ and $T_j(x_i e_i) = 0$ for all $j \neq i$. Therefore, to conclude that $[0, \lambda_0 q]$ is positively invariant, we only need to prove that $T_i([0, \lambda_0 q_i e_i]) \subset [0, \lambda_0 q_i e_i]$. Let us now prove this fact. Since T is retrotone and locally one-to-one, the functions

$$h_i : [0, \lambda_0 q_i] \longrightarrow \mathbb{R}$$
$$h_i(x_i) := T_i(x_i e_i) = x_i f_i(x_i e_i)$$

are strictly increasing for all $i = 1, ..., n$, (see remark 6.2). Next we use that h_i is strictly increasing together with the condition 3 of theorem 6.1, to obtain that $x_i \mapsto f_i(x_i e_i)$ is strictly decreasing. This property enables us to obtain that

$$\begin{cases} f_i(x_i e_i) > 1 & \text{if } x_i < q_i \\ f_i(x_i e_i) < 1 & \text{if } x_i > q_i \end{cases} . \tag{6.17}$$

Combining (6.17) with the strict monotonicity of h_i, we deduce that $T_i([0, \lambda_0 q_i e_i]) = h_i([0, \lambda_0 q_i]) = [0, h_i(\lambda_0 q_i)] \subset [0, \lambda_0 q_i[$ and so $[0, \lambda_0 q]$ is positively invariant.
In fact, using that f_i is strictly decreasing and (6.16), we prove that

S1) if $x_i(N_0) < \lambda_0 q_i$ then for all $N \geq N_0$, $x_i(N) < \lambda_0 q_i$,

S2) $x_i(N+1) < x_i(N)$ provided $x_i(N) > q_i$.

As a second step we prove that every orbit must enter into $[0, \lambda_0 q]$. By contradiction, suppose that there exists a $x(0) \in C$ such that for all $N \in \mathbb{N}$, $x(N)$ does not belong to $[0, \lambda_0 q]$. From **S1)**, we can take an index $i \in \{1, ..., n\}$ so that $x_i(N) > \lambda_0 q_i$ for all $N \in \mathbb{N}$. Then, by **S2)**, the sequence $\{x_i(N)\}$ is strictly decreasing and so there exists $\beta \geq \lambda_0 q_i$ verifying $x_i(N) \searrow \beta$. At this moment we have found the contradiction. Indeed, any point in $\omega(x(0))$ is of the type $\widetilde{x} = (\widetilde{x_1}, ..., \widetilde{x_{i-1}}, \beta, \widetilde{x_{i+1}}, ..., \widetilde{x_n}))$ with $T_i(\widetilde{x}) = \beta$. On the other hand, if we consider $\widetilde{z} = \beta e_i$, we obtain that

$$\beta = T_i(\widetilde{x}) = \beta f_i(\widetilde{x}) \overbrace{\leq}^{(6.16)} \beta f_i(\widetilde{z}) \underbrace{<}_{\mathbf{S2)}} \beta.$$

This contradiction ends the proof of this lemma.

Next we present some lemmas for the proof of theorem 6.1. They are discrete counterparts of some of the lemmas in [14], which dealt with differential equations

Lemma 6.2 *Consider $y(0) \in C$ verifying that for all $N \in \mathbb{N}$, there exists $y(-N) = T^{-N}(y(0))$ and belongs to C. Then for $x(0) \in C$ with $x(0) \prec y(0)$, there exists $x(-N)$ for all $N \in \mathbb{N}$ and*

$$\lim_{N \longrightarrow +\infty} x(-N) = 0.$$

Proof. Firstly we prove the existence of $x(-N)$ for all $N \in \mathbb{N}$. This claim follows from the arguments of proposition 2.1 in [11].

Next we prove that $x_j(-N) \longrightarrow 0$ for all $j = 1, ..., N$. Indeed, fix an index j. Using that T is retrotone, we can deduce that $x_i(-N) < y_i(-N)$ for all $i = 1, .., n$ with $y_i(0) \neq 0$. Assume that $x_j(0) \neq 0$ because otherwise $x_j(-N) = 0$ for all $N \in \mathbb{N}$, see (6.14). Next, define $\Delta_N^j = \frac{x_j(-N)}{y_j(-N)}$. Notice that $x_i(-N) < y_i(-N)$ provided $y_i(0) \neq 0$. This fact together with the third condition of theorem 6.1 implies that Δ_N^j is strictly decreasing. Indeed,

$$\Delta_N^j := \frac{x_j(-N)}{y_j(-N)} = \frac{T_j(x(-N-1))}{T_j(y(-N-1))} = \frac{x_j(-N-1)f_j(x(-N-1))}{y_j(-N-1)f_j(y(-N-1))} > \Delta_{N+1}^j.$$

Hence, there exists $\beta \in [0, 1[$ such that $\Delta_N^j \searrow \beta$. If $\beta = 0$, the proof of the convergence is complete because $\Delta_j^N y_j(-N) = x_j(-N) \longrightarrow 0$. Now we prove that in the case $\beta > 0$, we also have $x_j(-N) \longrightarrow 0$. By contradiction, suppose that $\beta > 0$ and there exists $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ an increasing function so that $x_j(-\sigma(N)) > \delta > 0$. Under these assumptions, we consider the compact subset of $\mathbb{R}^N \times \mathbb{R}^N$

$$S = \overline{\{(x(-\sigma(n)), y(-\sigma(n))) : n \in \mathbb{N}\}}.$$

For each point $(\widetilde{x}, \widetilde{y}) \in S$, we have that $\widetilde{x} \preceq \widetilde{y}$. Actually we will prove the stronger property $\widetilde{x} \prec \widetilde{y}$. By contradiction, suppose that there is $(\widetilde{x}, \widetilde{y}) \in S$ so that $\widetilde{x} = \widetilde{y}$. In such case, there exists an strictly increasing function $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ so that

$$x(-\sigma(\tau(N))) \longrightarrow \widetilde{x}$$

$$y(-\sigma(\tau(N))) \longrightarrow \widetilde{x}.$$

This is a contradiction since $\Delta_{\sigma(\tau(N))}^j \longrightarrow 1$. Besides this property, it is clear that for all $(x, y) \in S$,

$$x_j \geq \delta > 0. \tag{6.18}$$

Next we consider the compact set

$$S_1 = (T^{-1} \times T^{-1})(S) = \overline{\{(x(-\sigma(n) - 1), y(-\sigma(n) - 1) : n \in \mathbb{N}\}}.$$

Combining (6.18) and (6.15) we obtain that for all $(x, y) \in S_1$, there is $\delta_1 > 0$ verifying that $x_j \geq \delta_1 > 0$. Therefore we deduce that $f_j(y) < f_j(x)$ for all $(x, y) \in S_1$. Here we have used that $x^0 \prec y^0$ for all $(x^0, y^0) \in S$ as well as the third condition of theorem 6.1. Now it is clear that

$$\min_{(x,y) \in S_1} \frac{f_j(x)}{f_j(y)} = \eta > 1.$$

Finally one can see that

$$\Delta_{\sigma(N)+1}^j < \Delta_1^j (\frac{1}{\eta})^N.$$

To prove this inequality, notice that $\Delta^j_{i+1} \leq \Delta^j_i$ for $i = 1, ..., \sigma(1)$ and $\Delta^j_{\sigma(1)+1} \leq \frac{1}{\eta}\Delta^j_{\sigma(1)}$. Hence $\Delta^j_N \longrightarrow 0 = \beta$. This contradiction ends the proof of this lemma.

Next, we define the following sets:

$$\Sigma^n = \{x(0) \in C : \exists T^{-N}(x(0)) \in C \text{ for all } N \in \mathbb{N}\},$$

$$\Sigma^n_0 = \{x(0) \in \Sigma^n : x(-N) \longrightarrow 0 \text{ as } N \longrightarrow \infty\},$$
$$\Gamma^{n-1} = \Sigma^n \backslash \Sigma^n_0.$$

Given an integer $k = 1, 2, ..., n$ we define

$$\Sigma^k = \{p = (p_1, p_2, ..., p_k, 0, ..., 0) : p \in \Sigma^n\},$$

$$\Sigma^k_0 = \{p \in \Sigma^k : p \in \Sigma^n_0\},$$
$$\Gamma^{k-1} = \Sigma^k \backslash \Sigma^k_0.$$

These sets are invariant under T. Next, we present some useful properties about these sets.

Lemma 6.3 *Σ^n is a compact set.*

Proof. By definition, Σ^n is contained in C and so it is bounded. Therefore it remains to prove that Σ^n is a closed set. Indeed, consider the sequence $\{z_N\}_N \subset \Sigma^n$ with $\{z_N\} \longrightarrow z_0$. According to the definition of Σ^n, we must to prove that $T^{-N}(z_0)$ exists and belongs to C for each $N \geq 1$. Using that C is compact, we conclude that there is a partial sequence verifying that $T^{-1}(z_{\sigma(N)}) \longrightarrow y_0 \in C$ and so $T(y_0) = z_0$. In this way we have proved the existence of $T^{-1}(z_0) \in C$. The proof is complete after an induction with respect to N.

Lemma 6.4 *Σ^n_0 is an open set (relative to C).*

Proof. We have already proven that $f_i(0) > 1$, (see proof of lemma 6.1). Then, by continuity, we deduce that there exist $\delta > 0$ and a ball B centered at zero so that $f_i(x) > 1 + \delta$ for all $x \in B \cap K$. These inequalities complete this lemma.

Remark 6.3 *Using that $f_i(0) > 1$ for all $i = 1, ..., n$, it is clear that the origin is a repellor for T.*

Lemma 6.5 *Suppose that there exist $x(0), y(0) \in C \backslash \{0\}$ so that $x(N) \prec y(N)$ for all $N \in \mathbb{N}$. Then, $\lim_{N \longrightarrow \infty}[x(N) - y(N)] = 0$.*

Proof. Using that the map T is retrotone, we can assume that $x_i(N) < y_i(N)$ for all $N \in \mathbb{N}$ provided $y_i(0) \neq 0$. Now, we fix an index j with $x_j(0) \neq 0$ and prove that

$$\lim_{N \to \infty} x_j(N) - y_j(N) = 0. \tag{6.19}$$

Indeed, consider $\Delta_j^N = \frac{x_j(N)}{y_j(N)}$. Reasoning in the same way as in lemma 6.2, one checks that Δ_j^N is an increasing sequence. If $\Delta_j^N \nearrow 1$ we have finished since $y_j(N) - x_j(N) = y_j(N)(1 - \Delta_j^N) \longrightarrow 0$. Next we prove that if $\Delta_j^N \nearrow \beta < 1$, the sequence $y_j(N) - x_j(N) \longrightarrow 0$. By contradiction, assume that $\Delta_j^N \nearrow \beta < 1$ and there is a partial sequence $\sigma(N)$ so that $y_j(\sigma(N)) - x_j(\sigma(N)) \longrightarrow \rho > 0$. In this case it is clear that there is $\eta > 0$ so that $x_j(\sigma(N)) \geq \eta > 0$, for otherwise $\Delta_j^N \longrightarrow 0$. Now consider

$$S = \overline{\{(x(\sigma(N)+1), y(\sigma(N)+1)) : N \in \mathbb{N}\}}$$

and reason as in lemma 6.2 to obtain a contradiction. To finish the proof suppose that we can take an index k verifying that $x_k(0) = 0$. In such case we must prove that $y_k(N) \longrightarrow 0$. By contradiction, assume that there is a point z in $\omega(y(0))$ so that $z_k > 0$. Then, by (6.19), there exists a point $\widetilde{z} \in \omega(x(0))$ verifying that $\widetilde{z}_j = z_j$ if $x_j(0) \neq 0$, $\widetilde{z}_j = 0 \leq z_j$ if $x_j(0) = 0$ and $0 = \widetilde{z}_k < z_k$. From the previous comments, we know that $\widetilde{z} \prec z$. Moreover by remark 6.3 there exists an index j_0 with $0 < \widetilde{z}_{j_0} = z_{j_0}$. This contradicts the first part of the proof of this lemma. Indeed, using that the map T is retrotone we obtain that $T_i^{-1}(\widetilde{z}) < T_i^{-1}(z)$ provided $z_i \neq 0$ and in particular, $T_{j_0}^{-1}(\widetilde{z}) < T_{j_0}^{-1}(z)$.

Proof of theorem 6.1. Firstly, we prove that there is a continuous map, strictly decreasing

$$\Psi : \Sigma^{n-1} \longrightarrow [0, q_n]$$

so that

$$\Sigma^n = \{(x,y) \in \Sigma^{n-1} \times [0, q_n] : 0 \leq y \leq \Psi(x)\},$$
$$\Gamma^{n-1} = \{(x, \Psi(x)) : x \in \Sigma^{n-1}\}.$$

The proof of this statement is the same as in theorem 2.2 in [14]. It will be given for completeness. Define

$$\Psi(x) = \max\{y : (x,y) \in \Sigma^n\}.$$

The function Ψ is strictly decreasing, that is $\Psi(x) < \Psi(y)$ if $y \prec x$. We prove this assertion by contradiction. Assume that $x \prec y$ with $\Psi(x) \leq \Psi(y)$. Then, by lemma 6.2 we obtain that $(x, \Psi(x)) \in \Sigma_0^n$. This is impossible since Σ_0^n is open and this point lies on the boundary. It is important to notice that from this property, we obtain that Γ^{n-1} is unordered.

Let us now prove that Ψ is continuous. Using that Ψ is bounded, it is sufficient to show that the graph of Ψ is closed. Indeed, consider the sequence $\{x_N\}_N \subset \Sigma^{n-1}$ with

$$\{(x_N, \Psi(x_N))\} \longrightarrow (x_0, y_0).$$

Using that Σ^n is compact, we deduce that $(x_0, y_0) \in \Sigma^n$. Assume that $(x_0, y_0) \in \Sigma_0^n$. In such a case, as Σ_0^n is an open set, one deduces that there exists $N \in \mathbb{N}$ so that $(x_N, \Psi(x_N)) \in \Sigma_0^n$. This contradiction proves that the graph of Ψ is closed.

The next step is to prove that Γ^{n-1} determines completely the dynamics of T. Indeed, for $p \in C \backslash \Gamma^{n-1}$, we prove that there exists $q \in \Gamma^{n-1}$ so that

$$\lim_{N \to \infty} [T^N(q) - T^N(p)] = 0. \tag{6.20}$$

We distinguish two cases: $p \in C \backslash \Sigma^n$ and $p \in \Sigma_0^n$. Suppose that we are in the first case. By lemma 6.5, it is sufficient to prove that there exists $q \in \Gamma^{n-1}$ verifying that $T^N(q) \preceq T^N(p)$ for all $N \in \mathbb{N}$. With this purpose, we define $\Gamma(N,p) = \{q \in \Gamma^{n-1} : T^N(q) \preceq T^N(p)\}$. Using that T is retrotone, we see that $\Gamma(N+1,p) \subset \Gamma(N,p)$. Next, we prove that $\Gamma(N,p)$ is non empty for all $N \in \mathbb{N}$. It is important to recall that $T^N : C \longrightarrow C$ is a homeomorphism onto its image and maps Γ^{n-1} and Σ^n to Γ^{n-1} and Σ^n respectively. For $s = T^N(p)$, there exists $\lambda_0 < 1$ such that $\lambda_0 s \in \Gamma^{n-1}$. Here we are using that Σ_0^n is an open and Σ^n is compact. From the previous comments, we deduce that $q = T^{-N}(\lambda_0 s) \in \Gamma^{N-1}$ and so $q \in \Gamma(N,p)$. Finally, one checks that

$$\bigcap_{N=1}^{\infty} \Gamma(N,p) \neq \emptyset$$

by using that the sequence $\{\Gamma(N,p)\}$ is a decreasing sequence of compact sets. Another case can be proved similarly reversing the ordering.

Corollary 6.1 *Assume that T verifies:*

C1) *If $x \prec y$ then $f_i(y) < f_i(x)$ for all $i = 1, ..., n$.*

C2) *$T|_{K_{\{i\}}} : K_{\{i\}} \cap K \longrightarrow K_{\{i\}} \cap K$ admits a fixed point $q_i e_i$ with $q_i > 0$.*

C3) *T is retrotone and locally one-to-one in $[0,q]$ where $q = (q_1, ..., q_n)$.*

Then T admits carrying simplex.

Proof. This corollary is immediate from theorem 6.1, (see Remark 6.1) and the following result.

Lemma 6.6 *Assume that C1), C2) and C3) hold. Given $\lambda > 1$, the set $C = [0, \lambda q]$ is positively invariant, i.e. $T(C) \subset C$ for all $N \in \mathbb{N}$ and for all $x(0) \in K$, there exists $N_0 \in \mathbb{N}$ so that $x(N_0) \in C$.*

Proof. The proof of this result is very similar to lemma 6.1. For this reason we only give a sketch. Firstly, we prove that C is positively invariant. Using C1), we directly obtain (3.5) and (3.6). By C3), we have that there exists $\lambda_0 > 1$ such that T is retrotone and locally injective in $[0, \lambda_0 q]$ and so the functions

$$h_i : [0, q_i] \longrightarrow \mathbb{R}$$

$$h_i(x_i) = x_i f_i(x_i e_i)$$

are strictly increasing. The rest of the proof of this statement is the same as in lemma 6.1

Remark 6.4 *If we replace **C1)** and **C3)** by the conditions*

C1') *$DF(x)$ has strictly negative entries,*

C3') *$\rho(M(x)) < 1$ for all $x \in [0,q] \backslash \{0\}$, $M(x)$ is defined in section 5.1* *we obtain Theorem 4 in [10].*

Finally we compare our construction with [5]. Firstly, we can drop the conditions on regularity, global injectivity and hiperbolicity of fixed points. Moreover we can replace T retrotone in the whole domain K by the conditions **C1)** and **C3)**. In applications, these assumptions are more suitable.

7 Discussion

Some criteria of exclusion and dominance for general Kolmogorov systems have been derived in this paper. More precisely, our main goal has been to find a broad class of systems with the following behaviors:

- This class contains classical systems such as May Oster model or Leslie Gower model.
- For two and three species, there is exclusion in the system (1.1) if and only if the system has no coexistence states.
- For any number of species, the notion of weak dominance introduced by Franke and Yakubu in [7], [8] implies dominance.

The main feature having our class of systems is the presence of a carrying symplex. Along the paper we have seen that this notion can be combined with low-dimensional topological theories to deduce new significant results for two and three species. In higher dimension we cannot obtain the same results and so it would be interesting to understand this new phenomena.

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