On existence of dark solitons in cubic-quintic nonlinear Schrödinger equation with a periodic potential

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Abstract

A proof of existence of stationary dark soliton solutions of the cubic-quintic nonlinear Schrödinger equation with a periodic potential is given. It is based on the interpretation of the dark soliton as a heteroclinic on the Poincaré map.

Key words: Dark soliton, heteroclinic, Nonlinear Schrödinger equation, periodic potential, upper and lower solutions, Brouwer degree

1 Introduction

In the present paper we consider stationary solutions of the cubic-quintic nonlinear Schrödinger equation (CQNLS)

$$i\psi_t + \psi_{xx} + V(x)\psi - g_1|\psi|^2\psi - g_2|\psi|^4\psi = 0 \tag{1}$$

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with an L-periodic symmetric potential: $V(x) = V(x+L) = V(-x)$ and real constants g_1 and g_2. More specifically, we are interested in solutions which allow the representation $\psi(t,x) = e^{-i\omega t}\phi(x)$ where the function ϕ solves the stationary equation

$$\phi_{xx} + \tilde{V}(x)\phi - g_1\phi^3 - g_2\phi^5 = 0 \tag{2}$$

with $\tilde{V}(x) \equiv \omega + V(x)$, and which can be identified as *dark solitons* due to their nonzero boundary conditions

$$\phi(x) \to \phi_\pm(x) \quad \text{as} \quad x \to \infty \tag{3}$$

with the functions $\phi_\pm(x)$ being real, sign definite, and L-periodic solutions of (2). As it is clear, $\phi(x)$ acquires a zero value at some point of space, and therefore without loss of generality can be searched real, what is taken into account in the passage from (1) to (2).

The model (1) having a general character, it describes weakly dispersive and weakly nonlinear wave processes, recently attracted considerable attention in connection with its application to the mean-field theory of Bose-Einstein condensates [1]. In this context, $\psi(x)$ is a macroscopic wave function, $|\psi(x)|^2$ described linear atomic density, and $V(x)$ is an optical lattice created by standing laser beams. In particular, existence of spatially localized pulses (also referred to as bright solitons) has been recently addressed in Refs. [2,3]. In Ref. [3] families of the solutions were presented and significant differences in behavior of stationary modes of the standard cubic nonlinear Schrödinger equation ($g_1 \neq 0$, $g_2 = 0$) and of the quintic nonlinear Schrödinger (QNLS) equation ($g_1 = 0$, $g_2 \neq 0$), have been found. Dark solitons of the NLS equation with a periodic potential have also been discussed in the small amplitude limit [4] and for a general case it has been studied numerically in Ref. [5] (see also [6] and references therein). The approach developed in [5] was based on the numerical study of Poincaré map generated by Eq. (2) considered at instants nL, the dark solitons appearing as *heteroclinics* on the map. The aim of the present paper is to extend earlier studies, providing for the first time a rigorous proof of the existence of a dark soliton solution of Eq. (1). From the mathematical point of view, the strategy of proof combines in a novel way several techniques from the classical theory of ODE's (upper and lower solutions [8] and truncature arguments) and dynamics of planar homeomorphisms (topological degree [9,10] and free homeomorphisms [11]).

We will restrict the consideration to the case $g_2 > 0$ only, what rules out any possibility of blowing up solutions. Then without loss of generality we can set $g_2 = 1$ through a rescalling, what is done in what follows.

2 Existence of one-signed periodic solutions

We start with a proof of existence of one-signed periodic solutions. To this end we impose the condition of boundness of $V(x)$: $V_{min} \leq V(x) \leq V_{max}$, consider $\omega > -V_{min}$, and introduce the notations $\lambda_1^2 = \omega + V_{min}$ and $\lambda_2^2 = \omega + V_{max}$. As it is clear $0 < \lambda_1^2 \leq \tilde{V}(x) \leq \lambda_2^2$. Next we consider two stationary equations $(j = 1, 2)$

$$\phi_{j,xx} + \lambda_j^2 \phi_j - g_1 \phi_j^3 - \phi_j^5 = 0 \tag{4}$$

Treating these equations as dynamical systems, one easily finds the (only) two nontrivial equilibria $\pm\rho_j$ where

$$\rho_j = \sqrt{\sqrt{g_1^2 + 4\lambda_j^2} - g_1}/\sqrt{2} \tag{5}$$

These are the hyperbolic points $+\rho_j$ and $-\rho_j$ which are connected by the heteroclinic orbits, which explicit forms read

$$\phi_j = \frac{\rho_j \alpha_j \tanh(k_j x)}{\sqrt{\rho_j^2 + \alpha_j^2 - \rho_j^2 \tanh^2(k_j x)}} \tag{6}$$

where

$$\alpha_j = \sqrt{2\rho_j^2 + \frac{3}{2} g_1}, \quad \text{and} \quad k_j = \rho_j \sqrt{\rho_j^2 + \frac{g_1}{2}} \tag{7}$$

Let us call $\rho_{1,2}$ the positive equilibria. As it is clear $\phi_1(0) = \phi_2(0) = 0$ and $\phi_1(x) < \phi_2(x)$ for $x > 0$.

At this point some considerations are required about the general second order equation

$$\phi_{xx} = f(x, \phi) \tag{8}$$

with $f(x, \phi)$ continuous with respect to the both arguments and L-periodic in x. The following definition is classical (see for instance [8] and its references).

Definition 1 *A function $\alpha : [x_0, +\infty) \to \mathbb{R}$ such that $\alpha_{xx}(x) > f(x, \alpha)$ ($\alpha_{xx}(x) < f(x, \alpha)$) for all $x > x_0$ is called a* lower *(*upper*) solution of eq. (8).*

Now we can formulate

Proposition 1 *ρ_1 and ρ_2 are respectively lower and upper solutions of equation (2). Hence here exists an unstable L-periodic solution between them.*

Proof: Let us observe that for $j = 1$

$$\rho_{1,xx} + \tilde{V}(x)\rho_1 - g_1\rho_1^3 - \rho_1^5 > \lambda_1^2\rho_1 - g_1\rho_1^3 - \rho_1^5 = 0 \tag{9}$$

and similarly for $j = 2$. Hence, $\rho_1 < \rho_2$ are a couple of well-ordered lower and upper solutions respectively, therefore there exists a periodic solution between them [8]. Such solution is unstable because the associated Brouwer index to the Poincaré map is -1 (see for instance [10]). □

Therefore we have a positive L-periodic solution of Eq. (2), we designate it as $\phi_+(x)$, such that $\rho_1 \le \phi_+(x) \le \rho_2$. By the symmetry of equation we also have a negative solution $\phi_-(x) = -\phi_+(x)$.

To give an example of a periodic solution, we consider the simplified QNLS model (1) with $g_1 = 0$ and with the potential

$$V(x) = \rho^4[2 - k^2\mathrm{sn}^2(\rho^2 x, k)]^2 \tag{10}$$

where $\mathrm{sn}(x, k)$ is the Jacobi elliptic function, $k \in [0, 1]$ is the elliptic modulus, and $\rho > 0$ (examples of the exact periodic solutions for the cubic nonlinear Schrödinger equation, $g_2 = 0$, and with a specific potential, were obtained in [14,15]). The respective positive definite solution reads

$$\phi_+(x) = \rho\,\mathrm{dn}(\rho^2 x, k) \tag{11}$$

and it corresponds to the frequency $\omega = \rho^4(k^2 - 3)$. To verify the stability of ϕ in the sense of the dynamical system (2) we consider small deviation $\psi(x) = \phi(x) - \phi_+(x)$ at $x \to \infty$, whose dynamics in the leading order is governed by the equation

$$\psi_{xx} - U(x, k)\psi = 0 \tag{12}$$

with

$$U(x, k) = \rho^4(4 - k^2 - 6k^2\mathrm{sn}(\rho^2 x, k)^2 + 4k^4\mathrm{sn}(\rho^2 x, k)^4) \ge \\ \ge \rho^4(\frac{7}{4} - k^2) > 0 \tag{13}$$

Thus, the obtained function $\phi_+(x)$ is a hyperbolic periodic solution of Eq. (2).

Considering now $\psi_+(x,t) = \phi_+(x)\exp(i(3-k^2)t)$ as a solution of Eq. (1), performing the stability analysis as in Ref. [14], only slightly modified due to presence of quintic nonlinearity, and taking into account that $\phi_+(x) > 0$ one verifies that $\psi_+(x,t)$ is linearly stable in the sense of the evolution problem (1).

More sophisticated models allowing exact sign definite periodic solutions can be constructed using a kind of "inverse engineering" (i.e. by obtaining potentials starting with given periodic solutions) as it is explained in [6].

3 Existence of a dark soliton

In this section we prove the existence of a heteroclinic orbit connecting the periodic solutions ϕ_- and ϕ_+. A battery of preparatory lemmas are necessary.

Lemma 1 *If $\phi : [x_0, +\infty) \to \mathbb{R}$ is a bounded solution of eq. (8), then the derivative ϕ_x is also bounded in $[x_0, +\infty)$.*

Proof: By the hypothesis $|\phi(x)| < M$, where M is a constant, for all $x \geq x_0$. Then, by the mean value theorem there exits $x_n \in (nL, (n+1)L)$ such that $\phi((n+1)L) - \phi(nL) = \phi_x(x_n)L$. From here $\phi_x(x_n) < 2M/L$ for all n. Applying the mean value theorem one more time one obtains

$$|\phi_x(x) - \phi_x(x_n)| < L \max_{|\phi|\leq M} |f(x,\phi)|, \quad \forall x \in (nL, (n+1)L) \tag{14}$$

because $|\phi_{xx}(x)| < \max_{|\phi|\leq M} |f(x,\phi)|$. Then

$$|\phi_x(x)| < L \max_{|\phi|\leq M} |f(x,\phi)| + \frac{2M}{L}, \quad \forall x \geq x_0. \tag{15}$$

$\square$

The following lemma is a key ingredient in our main result.

Lemma 2 *[12] Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism with a unique fixed point p_L such that $\gamma\{I - P, p_L\} \neq 1$. Then for any $p_0 \in \mathbb{R}^2$ one of the following possibilities holds*

i) $P^n(p_0) \to p_L$ *as* $n \to +\infty$

ii) $\|P^n(p_0)\| \to \infty$ *as* $n \to +\infty$

Here, $\gamma\{I-P, p_0\}$ is the local index associated to the Brouwer degree of p_0 as a fixed point of the homeomorphism P. The proof relies in a basic property of free homeomorphisms exposed in [11], namely the ω-limit set of a given point has to be a connected set of the fixed point set.

Lemma 3 *If $f(x,y)$ is strictly increasing in y, there exists a most one L-periodic solution of (8).*

Proof: By contradiction, let us assume that y_1, y_2 are two different L-periodic solutions of (8). First, let us suppose that y_1, y_2 intersect among themselves, that is, there should be t_1, t_2 such that $z(t) = y_1(t) - y_2(t)$ verifies $z(t_0) = 0 = z(t_1)$ and $z(t) > 0$ for $t \in (t_1, t_2)$. However, by substracting the corresponding equations and using that f is strictly increasing, we get that z should be convex in (t_1, t_2), which is a contradiction. Therefore, y_1, y_2 do not intersect and we assume without loss of generality that $y_1(t) > y_2(t)$ for all t. Again, z should be convex in the whole real line, but this is impossible because it is periodic. $\square$

With the help of these previous lemmas we are able to prove an abstract convergence result.

Theorem 1 *Let $\phi : [x_0, +\infty) \to \mathbb{R}$ a bounded solution of (8). Let us assume that*

$$\min_{\substack{x\in[0,L] \\ y\in[\inf_{x\geq x_0}\phi(x), \sup_{x\geq x_0}\phi(x)]}} \frac{\partial f(x,y)}{\partial y} > 0 \tag{16}$$

Then there exists an L-periodic solution $\varphi(x)$ such that

$$\lim_{x\to+\infty} (|\phi(x) - \varphi(x)| + |\phi_x(x) - \varphi_x(x)|) = 0 \tag{17}$$

Proof: Let us define $m = \inf_{x\geq x_0} \phi(x)$ and $M = \sup_{x\geq x_0} \phi(x)$, as well as the truncated function

$$\tilde{f}(x,y) = \begin{cases} f(x,y), & \forall y \in [m, M] \\ f(x,M) + f_y(x,M)(y-M), & \forall y > M \\ f(x,m) + f_y(x,M)(y-m), & \forall y < m \end{cases} \tag{18}$$

$\tilde{f}$ is strictly increasing in y. Note also that ϕ is a solution of the truncated equation

$$\phi_{xx} = \tilde{f}(x,\phi) \tag{19}$$

Obviously, $\lim_{y\to\pm\infty} \tilde{f}(x,y) = \pm\infty$ uniformly in x. Hence, there exist constants α and β, such that $\alpha < \beta$ and $\tilde{f}(x,\alpha) < 0 < \tilde{f}(x,\beta)$ for all x. Such α and β is a well-ordered coupled L-periodic lower and upper solutions, so there exists an L-periodic solution of (19) with index -1. This solution is unique by the previous lemma. Then, by Lemma 2, $\phi(x)$ must converge to $\varphi(x)$ since Lemma 1 excludes *ii*). As $\varphi(x) \in [m, M]$, it is a solution of (8). $\square$

Theorem 2 *Let us consider bounded functions* α, $\beta : [x_0, +\infty) \to \mathbb{R}$ *verifying*

1) $\alpha(x) < \beta(x)$, $\forall x > x_0$

2) $\alpha_{xx}(x) > f(x,\alpha)$ *and* $\beta_{xx}(x) > f(x,\beta)$, $\forall x > x_0$

Then there exists a solution $\phi(x)$ *of (8) such that*

$$\alpha(x) < \phi(x) < \beta(x) \tag{20}$$

If moreover, there exists x *such that*

3) $$\min_{\substack{x\in[0,L] \\ y\in[\inf_{x\geq x_0}\alpha(x), \sup_{x\geq x_0}\beta(x)]}} \frac{\partial f(x,y)}{\partial y} > 0$$

then there exists an L-periodic solution $\varphi(x)$ *such that*

$$\lim_{x\to+\infty} (|\phi(x) - \varphi(x)| + |\phi_x(x) - \varphi_x(x)|) = 0 \tag{21}$$

Besides, $\varphi(x)$ *is the unique L-periodic solution in the interval* $[\inf_{x\geq x_0}\alpha(x), \sup_{x\geq x_0}\beta(x)]$.

Proof: The first assertion is a classical result due to Opial [13]. The second conclusion is a corollary of Theorem 1. $\square$

In order to apply the above results to our model (1), we observe that i) now $f(x,y) \equiv y^5 + g_1 y^3 - \tilde{V}(x)y$, ii) due to parity of the potential one can consider $x \geq 0$ and extend the obtained solution $\phi(x)$ as an odd function to $x \leq 0$, iii) The functions $\phi_{1,2}(x)$ given by (6) satisfy the conditions 1) and 2) of the Theorem 2 where $\alpha(x) \equiv \phi_1(x)$ and $\beta(x) \equiv \phi_2(x)$. Hence, in order to prove that there exists a solution $\phi(x)$ of (1) converging to $\phi_\pm(x)$, found in Proposition 1, as $x \to \pm\infty$, one has to verify the condition 3) of Theorem 2.

As x_0 can be taken arbitrarily large, this last condition is equivalent to

$$\min_{\substack{y\in[\rho_1,\rho_2] \\ x\in[0,L]}} \left\{5y^4 + 3g_1 y^2 - \tilde{V}(x)\right\} > 0 \tag{22}$$

Starting with the case $g_1 \geq 0$ we observe that (22) is now equivalent to $5\rho_1^4 + 3g_1\rho_1^2 - \lambda_2^2 > 0$. The straightforward analysis of this last inequality, which takes into account the link (5), the definition of $\lambda_{1,2}$, the requirement $\omega > -V_{min}$ necessary for $\lambda_1^2 > 0$, gives the following estimate for the frequency

$$\omega - g_1\sqrt{g_1^2 + 4\omega + 4V_{min}} > V_{max} - 5V_{min} \tag{23}$$

Thus (23) is a sufficient condition for the existence of dark solitons at non-negative g_1.

Considering now $g_1 < 0$ the constrain (22) is reduced to $5\rho_1^4 - 3|g_1|\rho_2^2 - \lambda_2^2 > 0$ and subsequently to the following inequality constrain to the frequency

$$\begin{aligned} 8\omega + 2g_1^2 + 10V_{min} - 2V_{max} - 5g_1\sqrt{4\omega + g_1^2 + 4V_{min}} \\ +3g_1\sqrt{4\omega + g_1^2 + 4V_{max}} > 0 \end{aligned} \tag{24}$$

which must be satisfied simultaneously with $\omega > -V_{min}$.

In both cases considered before, the conclusion is that there exists an explicitly computable ω_0 such that the CQNLS has a different dark soliton for any $\omega > \omega_0$.

4 A concluding example

As a concluding remark we consider an example illustrating a dark soltion, as well as other concepts introduced in the paper. To this end we recall the potential (10) and construct a dark soliton which tends to ϕ_+ given by (11) (we thus consider now $g_1 = 0$). This cannot be done analytically, and that is why we employ numerics. An example is shown in Fig. 1.

For numerical obtaining the dark soliton, we have use shooting method. To this end we observed that (12) is a Hill's equation and thus taking into account (13), from Floquet's theorem one we have that $\psi(x) = P(x)\exp(-\alpha x)$ where $P(x) = P(x + 2\mathrm{K}(k)/\rho^2)$ is a periodic function and $\mathrm{K}(k)$ is a complete elliptic integral of the first kind. For given parameters k and ρ one can easily compute the respective Floquet's exponent α. In particular for our choice of $k = 0.5$ and $\rho = 1$, $\alpha \approx 2.014298$. Thus, starting with the points $x_{ini} = 2\mathrm{K}(k)/\rho^2$ where n is an integer, where $\phi(x_{ini}) = \phi_+(0) - C$ and $\phi_x(x_{ini}) = -C\alpha$, and by varying the parameter C one can meet the condition $\phi(0) = 0$. An example of implementation of this procedure is shown in Fig. 1.

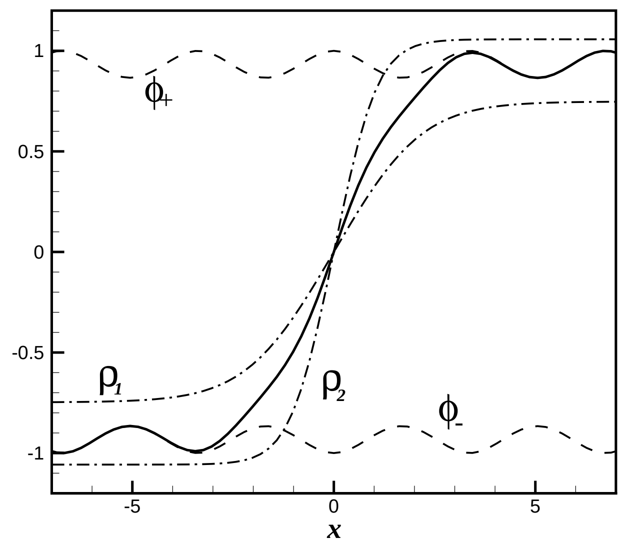

Fig. 1. The dark soliton – heteroclinic (solid line), lower and upper solutions $\rho_{1,2}$ (dashed-dotted line), and the hyperbolic periodic solutions $\phi_{\pm}(x)$ given by (11), for the parameters $k = 0.5$ and $\rho = 1$

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