Coexistence States for Cyclic 3-Dimensional Systems

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1 Introduction

In Ecology, the evolution of the species size is modelled by systems of differential equations of the type

$$\dot{x_i} = x_i\ f_i(t, x_1, \ldots, x_n) \qquad 1 \leq i \leq n \tag{1}$$

where $x_i = x_i(t) \geq 0$ is the size of the species i in the instant t. Depending on the properties of the functions f_i, the system can represent several kinds of interactions between the species. Due to seasonal fluctuations of the environment, it is usual to assume that the functions $f_i : \mathbb{R} \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ are T-periodic where $\mathbb{R}^n_+ = [0, +\infty[\times \ldots \times^{(n} \ldots \times [0, +\infty[$. A relevant class of solutions of these systems are the T-periodic solutions which remain in the interior of $\mathbb{R}^N_+$, we will refer to these solutions as the **coexistence states** of (1).

For $n = 3$, we can introduce the following cyclic behavior. The system (1) is **τ-cyclic** (resp. **σ-cyclic**) if the species x_i (resp. x_{i+1}) becomes extinct and the species x_{i+1} survives in the subsystem obtained from (1) by letting $x_{i-1} = 0$, for $i = 1, 2, 3$. (We always use the mod 3 notation.) Notice that we can pass from τ-cyclic to σ-cyclic systems by means of a permutation. There are many models of the type (1) which present a cyclic structure, for example May-Leonard's model [8]

$$\left\{ \begin{array}{l} \dot{x_1} = x_1(1 - x_1 - \alpha_1 x_2 - \beta_1 x_3) \\ \dot{x_2} = x_2(1 - \beta_2 x_1 - x_2 - \alpha_2 x_3) \\ \dot{x_3} = x_3(1 - \alpha_3 x_1 - \beta_3 x_2 - x_3) \end{array} \right. \tag{2}$$

where $0 < \alpha_i < 1 < \beta_i$ for $i = 1, 2, 3$. We observe that this system is σ-cyclic, competitive and admits a coexistence state that can be stable or unstable. The global behavior of this system is studied in [5] . Motivated by this example, Tineo proved in [11] the presence of a coexistence states for 3-dimensional competitive systems which are dissipative and have a cyclic behavior. The proof in [11] is based on the ideas of Hirsch [6] together with a result due to Campos, Ortega and Tineo in [4]. This problem for competitive systems has been also studied in [1] and [12]. The purpose of this paper is to show that in the result in [11] it is not essential that the system be competitive. This extension results interesting since apart from competitive systems, the cyclic structure can appear in many other interactions as it will be seen in the section 4. Our tools are the Brouwer's degree and the application of the Browder's results [2] for dissipative systems.
I would like to thank to my advisor, professor R. Ortega, for his help and suggestions in this paper.

2 Definitions and statement of the main result

Consider the system in $\mathbb{R}^3_+$

$$\begin{cases} \dot{x_1} = x_1 f_1(t, x_1, x_2, x_3) \\ \dot{x_2} = x_2 f_2(t, x_1, x_2, x_3) \\ \dot{x_3} = x_3 f_3(t, x_1, x_2, x_3). \end{cases} \tag{3}$$

We will assume, without further mention that the system (3) is T-periodic and has uniqueness of solution for the Cauchy problem.
It is usual to take to account the limitations of the environment, this is translated to our models in the following way.
We say that the system (3) is **dissipative** if there exists a constant M such that every solution of (3) verifies that

$$\limsup_{t\to\infty} \|x(t)\| \leq M$$

where $\|\cdot\|$ denote some norm in $\mathbb{R}^N$. Notice that we are implicitly assuming that all solutions can be continued in the future.
After these considerations, we need to introduce some definitions with clear biological meaning.

Definition 1 *A **coexistence state** for* (3) *is a T-periodic solution* $x(t) = (x_1(t), x_2(t), x_3(t))$ *such that* $x(t) \in Int(\mathbb{R}^3_+)$ *for all* t.

Definition 2 *The species i* ***becomes extinct*** *in the system* (3) *if*

$$\lim_{t\to\infty} x_i(t) = 0$$

for every solution $x(t) = (x_1(t), x_2(t), x_3(t))$ lying in $Int(\mathbb{R}^3_+)$.

Given $J \subset \{1,2,3\}$, we define the subsystem E_J as the system which is obtained when $x_i = 0$ for $i \notin J$. In particular, if $J = \{i\}$ we obtain the scalar equation

$$\dot{x}_i = x_i f_i(t, x_i e_i), \tag{4}$$

where e_1, e_2, e_3 denote the canonical basis of $\mathbb{R}^3$. It will be always assumed that these equations are of logistic growth type. By this we mean that they have an unique positive T-periodic solution $V_i(t)$ attracting all positive solutions; that is

$$\lim_{t\to\infty} [x_i(t) - V_i(t)] = 0$$

where $x_i(t) > 0$ is a positive solution of (4).
Now we are going to present the main result of this paper.

Theorem 3 *Assume that the system* (3) *is dissipative and verifies:*

i) *The subsystems $E_{\{1,2\}}$, $E_{\{2,3\}}$, $E_{\{1,3\}}$ do not have coexistence states,*

ii) *For each $i \in \{1,2,3\}$, the equation $\dot{x}_i = x_i f_i(t, x_i e_i)$ has logistic growth with periodic solution $V_i(t)$,*

iii) $\int_0^T f_i(t,0)dt > 0$ *for $i = 1,2,3$,*

iv) $\int_0^T f_{i+1}(t, V_i(t)e_i)dt > 0$ *for $i = 1,2,3$.*

Then, (3) admits a coexistence state.

Remark 4 *First, we observe that every cyclic system verifies* **i)**. *The condition* **iv)** *is considered in* [11] *for the case of τ-cyclic systems. In the section 4, we will interpret this condition and convince ourselves that it is verified for almost every τ-cyclic system. Moreover, we will notice that the hypotheses of the theorem 3 imply that the system is $\tau-$cyclic.*

We can obtain the same conclusion as the Theorem 3 if we replace **iv)** by **iv)*** where **iv)*** is

$$\int_0^T f_{i-1}(t, V_i(t)e_i)dt > 0.$$

In many concrete examples it is not easy to find an explicit formula for $V_i(t)$. This can make difficult the verification of **iv)** (or **iv)***). Next we introduce some explicit conditions implying **i)** and **iv)***.

$$\left\{ \begin{array}{lll} f_1(t,x_1,x_2,0) > f_2(t,x_1,x_2,0) & \text{if} & (x_1,x_2) \neq (0,0) \\ f_2(t,0,x_2,x_3) > f_3(t,0,x_2,x_3) & \text{if} & (x_2,x_3) \neq (0,0) \\ f_3(t,x_1,0,x_3) > f_1(t,x_1,0,x_3) & \text{if} & (x_1,x_3) \neq (0,0). \end{array} \right. \tag{5}$$

These can be interpreted in terms of dominance. For instance, the first condition says that in absence of species 3, 1 dominates 2.

Corollary 5 *Assume that the system* (3) *is dissipative and verifies* **ii)***,* **iii)** *and* (5)*. Then the system* (3) *has a coexistence state.*

3 Proof of the main Theorem

The main tools for the proof of the Theorem 3 are the Brouwer's degree and the following result due to Browder which can be found in [14] pag. 725. Indeed, we state the result only in finite dimension.

Theorem 6 *Let $T : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous map. Suppose that U is an open ball so that there exists K_0 such that $A^K(\overline{U}) \subset U$ for $K \geq K_0$. Then $deg_{\mathbb{R}^N}(id - T, U) = 1$.*

We will employ the notation $deg_{\mathbb{R}^N}(f, U)$ for Brouwer's degree of f in U where $U \subset \mathbb{R}^N$ is an open set and $f : \overline{U} \longrightarrow \mathbb{R}^N$ is a continuous map with $f(x) \neq 0$ for $x \in \partial U$. If $p \in U$ is an isolated fixed point for f then we define $ind_{\mathbb{R}^N}(f, p) = deg_{\mathbb{R}^N}(id - f, B_\delta(p))$ for $\delta > 0$ sufficiently small. (We are denoting by $B_\delta(p)$ the open ball centered at p and radius δ.) The reader who wishes a more precise definitions and its properties can consult [7].

Given $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3_+$, we denote by $x(t, \xi)$ the maximal solution of (3) such that $x(0) = \xi$.
The Poincaré map associated to (3) is defined as:

$$P : \mathbb{R}^3_+ \longrightarrow \mathbb{R}^3_+$$

$$P(\xi) = x(T, \xi).$$

We notice that the fixed points of P correspond with the periodic solutions of (3). Namely, the fixed points of P in $Int(\mathbb{R}^3_+)$ are associated with the coexistence states. From the expression of the system (3) and a straightforward computation it is clear that P admits the following expression:

$$P(\xi_1,\xi_2,\xi_3) = (\xi_1 e^{\int_0^T f_1(t,x(t,\xi))dt}, \xi_2 e^{\int_0^T f_2(t,x(t,\xi))dt}, \xi_3 e^{\int_0^T f_3(t,x(t,\xi))dt}).$$

We are going to denote by ξ_i^* the initial condition for the solution V_i, that is $\xi_i^* = V_i(0)$. The Poincaré map associated to (3) is defined in $\mathbb{R}^3_+$ but it admits a natural extension to the whole space $\mathbb{R}^3$ in the following way

$$\widehat{P}(\xi_1,\xi_2,\xi_3) = (\xi_1 e^{\int_0^T f_1(t,x(t,|\xi|))dt}, \xi_2 e^{\int_0^T f_2(t,x(t,|\xi|))dt}, \xi_3 e^{\int_0^T f_3(t,x(t,|\xi|))dt}) \quad (6)$$

where $|\xi| = (|\xi_1|,|\xi_2|,|\xi_3|)$. We observe that the formula above corresponds with the Poincaré map of the system

$$\dot{x_i} = x_i\ f_i(t,|x_1|,|x_2|,|x_3|) \qquad i = 1,2,3. \quad (7)$$

This extension commutes with symmetries with respect to the axes. More precisely

$$\widehat{P} \circ s_i = s_i \circ \widehat{P} \quad (8)$$

where $s_1(\xi_1,\xi_2,\xi_3) = (-\xi_1,\xi_2,\xi_3)$ and so on. Other important property is that $\widehat{P}(\mathcal{E}_J) \subset \mathcal{E}_J$ where $J \subset \{1,2,3\}$ and $\mathcal{E}_J = \{(x_1,x_2,x_3) : x_i = 0 \ \text{ if } i \notin J\}$. We denote by $\widehat{P}\mid_{\mathcal{E}_J}$ the restriction of $\widehat{P}$ to $\mathcal{E}_J$.
Now, let us begin the proof whose structure is the following. First, we compute the degree of $id - \widehat{P}$ in large balls. After that we compute the indexes of the fixed points of the axes. Finally, we deduce the result with an argument of additivity. We are going to proceed by steps.

Step1: Computation of $deg_{\mathbb{R}^3}(id - \widehat{P}, B_R(0))$ for $R > 0$ sufficiently large.
Using that the system (3) is dissipative we deduce there exists $M > 0$ such that for all x it verifies that

$$\limsup_{N\to\infty} \parallel \widehat{P}^N(x) \parallel < M.$$

Therefore, for each $x \in \mathbb{R}^3$, there exists N_x such that $\widehat{P}^{N_x}(x) \in \overline{B_M(0)}$. It is easy to check that

$$K = \bigcup_{n=1}^{\infty} \widehat{P}^n(\overline{B_M(0)})$$

is a compact set and positively invariant, i.e. $\widehat{P}(K) \subset K$. Then we can take $R > 0$ so that $K \subset B_R(0)$. Now, we deduce that there exists $N_0 > 0$ such that

$$\widehat{P}^N(\overline{B_R(0)}) \subset K \subset B_R(0)$$

for all $N > N_0$. This reasoning is standard for dissipative systems, for more details, see [7]. Finally, we apply the Theorem 6 and deduce that

$$deg_{\mathbb{R}^3}(id - \widehat{P}, B_R(0)) = 1.$$

Since every subsystem E_J is also dissipative, we can obtain with the same arguments that

$$\begin{aligned} deg_{\mathbb{R}^2}(id - \widehat{P}\mid_{\{1,2\}}, B_R(0)) = 1, \\ deg_{\mathbb{R}^2}(id - \widehat{P}\mid_{\{1,3\}}, B_R(0)) = 1, \\ deg_{\mathbb{R}^2}(id - \widehat{P}\mid_{\{2,3\}}, B_R(0)) = 1. \end{aligned} \tag{9}$$

Step 2: Indexes in two dimensions.
In this step we are going to compute the indexes of the fixed points of $\widehat{P}\mid_{\{1,2\}}$, analogously one might reason with the other restrictions. We notice that from the hypotheses **i)** of the Theorem 5 the unique fixed points of $\widehat{P}\mid_{\{1,2\}}$ are $(0,0), (\pm\xi_1^*, 0), (0, \pm\xi_2^*)$.

- $ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (0,0)) = 1.$
  By continuous dependence we can find $\delta > 0$, $\epsilon > 1$ so that if $|\xi_1|$, $|\xi_2| < \delta$ then
  $$e^{\int_0^T f_1(t,x(t,(|\xi_1|,|\xi_2|,0))dt} > \epsilon,$$
  $$e^{\int_0^T f_2(t,x(t,(|\xi_1|,|\xi_2|,0))dt} > \epsilon.$$
  Here we are using **iii)**.
  Now consider the following homotopy
  $$H : [0,1] \times [-\delta, \delta] \times [-\delta, \delta] \longrightarrow \mathbb{R}^2$$
  $$H(\lambda, (\xi_1, \xi_2)) = (\xi_1\alpha_1(\lambda, \xi_1, \xi_2), \xi_2\alpha_2(\lambda, \xi_1, \xi_2))$$
  where $\alpha_i(\lambda, \xi_1, \xi_2) = (1-\lambda)\epsilon + \lambda e^{\int_0^T f_i(t,x(t,(|\xi_1|,|\xi_2|,0))dt}$.
  H_λ does not have fixed points in the boundary of $[-\delta,\delta]\times[-\delta,\delta]$. This is deduced from
  $$(1-\lambda)\epsilon + \lambda e^{\int_0^T f_i(t,x(t,(|\xi_1|,|\xi_2|,0))dt} \geq \epsilon > 1.$$
  For $\lambda = 0$, we obtain that $H(0, (\xi_1, \xi_2)) = \epsilon(\xi_1, \xi_2)$ and so
  $$deg_{\mathbb{R}^2}(id - \widehat{P}\mid_{\{1,2\}},]-\delta,\delta[\times]-\delta,\delta[) = 1.$$

- $ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (\xi_1^*, 0)) = ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (-\xi_1^*, 0)) = -1$.
  From the invariance of the index by conjugation and the property (8), we deduce the first equality. Therefore we have just to prove that

  $$ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (\xi_1^*, 0)) = -1.$$

  By continuous dependence, we deduce that there exists $\delta > 0$, $\epsilon > 1$ such that if $\mid \xi_1 - \xi_1^* \mid < \delta$ and $\mid \xi_2 \mid < \delta$ then

  $$e^{\int_0^T f_2(t,x(t,(|\xi_1|,|\xi_2|,0)dt} > \epsilon.$$

  Here we have used **iv)**.
  Then we can consider the following homotopy

  $$H : [0,1] \times [\xi_1^* - \delta, \xi_1^* + \delta] \times [-\delta, \delta] \longrightarrow \mathbb{R}^2$$
  $$H(\lambda, (\xi_1, \xi_2)) = (\xi_1 \alpha_1(\lambda, \xi_1, \xi_2), \xi_2 \alpha_2(\lambda, \xi_1, \xi_2))$$

  where

  $$\alpha_1(\lambda, \xi_1, \xi_2) = e^{\int_0^T f_1(t,x(t,(|\xi_1|,\lambda|\xi_2|,0))dt}$$
  $$\alpha_2(\lambda, \xi_1, \xi_2) = (1-\lambda)\epsilon + \lambda e^{\int_0^T f_2(t,x(t,(|\xi_1|,|\xi_2|,0)dt}.$$

  H is an admissible homotopy (in the same sense as the previous apart) due to

  $$(1-\lambda)\epsilon + \lambda e^{\int_0^T f_2(t,x(t,(|\xi_1|,|\xi_2|,0))dt} \geq \epsilon > 1$$

  together with **ii)**. For $\lambda = 0$, we obtain

  $$H(0, (\xi_1, \xi_2)) = (\xi_1 e^{\int_0^T f_1(t,x(t,(|\xi_1|,0,0)dt}, \xi_2 \epsilon).$$

  From the product formula for the index and the logistic growth of the equation $\dot{x_1} = x_1 f_1(t, x_1 e_1)$ we conclude that

  $$ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (\xi_1^*, 0)) = -1.$$

- $ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (0, \xi_2^*)) = ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (0, \xi_2^*)) = 1$.
  Using the previous step we deduce that there exists $B_R(0) \subset \mathbb{R}^2$ such that $(0,0), (\pm\xi_1^*, 0), (0, \pm\xi_2^*) \in B_R(0)$ and

  $$deg_{\mathbb{R}^2}(id - \widehat{P}\mid_{\{1,2\}}, B_R(0)) = 1.$$

  From the additivity for the degree and the previous statements we deduce that

  $$ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (0, \xi_2^*)) = ind_{\mathbb{R}^2}(\widehat{P}\mid_{\{1,2\}}, (0, -\xi_2^*)) = 1.$$

  Here we have used that the unique fixed points for $\widehat{P}\mid_{\{1,2\}}$ are $(0,0)$, $(\pm\xi_1^*, 0)$, $(0, \pm\xi_2^*)$.

Step 3: Indexes in three dimensions.
Firstly we check that the fixed points $(0,0,0)$,$(\pm\xi_1^*,0,0)$, $(0,\pm\xi_2^*,0)$, $(0,0,\pm\xi_3^*)$ are isolated. We concentrate on $(\xi_1^*,0,0)$. By continuous dependence and **iv)**, there exists $\epsilon>1$, $\delta>0$ such that if $|\xi_1-\xi_1^*|<\delta$, $|\xi_2|<\delta$, $|\xi_3|<\delta$ then

$$e^{\int_0^T f_2(t,x(t,|\xi|)dt} > \epsilon.$$

From the formula (6), we deduce that there are no fixed points with $|\ \xi_2\ |>0$ if $|\xi_1-\xi_1^*|<\delta$, $|\xi_2|<\delta$, $|\xi_3|<\delta$. Now using the hypotheses **i)** we conclude that $(\xi_1^*,0,0)$ is an isolated fixed point. Analogously, we can deduce the same for the other fixed points. We remark $ind_{\mathbb{R}^3}(\widehat{P},(0,0,0))=-1$ reasoning in the same way as the previous step in the beginning. Now we compute $ind_{\mathbb{R}^3}(\widehat{P},(0,\xi_2^*,0))$. Using that the fixed point $(0,\xi_2^*,0)$ is isolated we can perfectly define the index in this point. By continuous dependence, there exists $\epsilon>1$, $\delta>0$ such that if $|\xi_1|<\delta$, $|\xi_2-\xi_2^*|<\delta$, $|\xi_3|<\delta$ then

$$e^{\int_0^T f_3(t,x(t,|\xi|)dt} > \epsilon.$$

Again **iv)** has been employed. Therefore, we can consider the following homotopy

$$H:[0,1]\times[-\delta,\delta]\times[\xi_2^*-\delta,\xi_2^*+\delta]\times[-\delta,\delta]\longrightarrow\mathbb{R}^3$$

$$H(\lambda,\xi_1,\xi_2,\xi_3)=(\xi_1\alpha_1(\lambda,\xi_1,\xi_2,\xi_3),\xi_2\alpha_2(\lambda,\xi_1,\xi_2,\xi_3),\xi_3\alpha_3(\lambda,\xi_1,\xi_2,\xi_3))$$

$$\alpha_i(\lambda,\xi_1,\xi_2,\xi_3)=e^{\int_0^T f_i(t,x(t,(|\xi_1|,|\xi_2|,\lambda|\xi_3|))dt}\quad \text{for}\ \ i=1,2$$

$$\alpha_3(\lambda,\xi_1,\xi_2,\xi_3)=(1-\lambda)\epsilon+\lambda e^{\int_0^T f_3(t,x(t,|\xi|)dt}$$

H is an admissible homotopy since as

$$(1-\lambda)\epsilon+\lambda e^{\int_0^T f_3(t,x(t,(|\xi_1|,|\xi_2|,|\xi_3|))dt}\geq\epsilon>1$$

then H_λ does not have fixed point for $\xi_3\neq0$ in the boundary of $[-\delta,\delta]\times[\xi_2^*-\delta,\xi_2^*+\delta]\times[-\delta,\delta]$ and when $\xi_3=0$ we obtain the same conclusion using the hypotheses **i)**. For $\lambda=0$ we have

$$H_0=\widehat{P}\mid_{\{1,2\}}\times\epsilon\ id_{\mathbb{R}}.$$

Using the product formula for the index and the previous step, we conclude that $ind_{\mathbb{R}^3}(\widehat{P},(0,\xi_2^*,0))=-1$. Reasoning analogously we can deduce the same conclusion for the other fixed points.
Conclusion:

We have seven fixed points, namely $(0,0,0)$, $(\pm\xi_1^*,0,0)$, $(0,\pm\xi_2^*,0)$, $(0,0,\pm\xi_3^*)$ with index equal to -1. We know that there exists $B_R(0) \subset \mathbb{R}^3$ which verifies that $(0,0,0)$,$(\pm\xi_1^*,0,0)$,$(0,\pm\xi_2^*,0)$,$(0,0,\pm\xi_3^*) \in B_R(0)$ and

$$deg_{\mathbb{R}^3}(id - \widehat{P}, B_R(0)) = 1.$$

Using the additivity of the degree and **i)**, we conclude that $\widehat{P}$ must a fixed point in some octant but from the expression of $\widehat{P}$ it is deduced that there exists a fixed point in $Int(\mathbb{R}^3_+)$.

4 Examples and some remarks

Now we are going to apply our results to concrete examples. In all cases it is assumed that the coefficient $\alpha_i(t)$ and $\beta_i(t)$ are continuous and T-periodic.
Competitive systems of May-Leonard type

$$\left\{\begin{array}{l} \dot{x_1} = x_1(1 - x_1 - \alpha_1(t)x_2 - \beta_1(t)x_3) \\ \dot{x_2} = x_2(1 - \beta_2(t)x_1 - x_2 - \alpha_2(t)x_3) \\ \dot{x_3} = x_3(1 - \alpha_3(t)x_1 - \beta_3(t)x_2 - x_3) \end{array}\right. \tag{10}$$

with $0 < \alpha_i(t) < 1 < \beta_i(t)$ for $i = 1, 2, 3$. From the inequalities

$$\dot{x_i} \leqslant x_i(1 - x_i) \quad \text{if} \quad i = 1, 2, 3$$

we easily deduce that the system (10) is dissipative. Applying the corollary 5, we deduce that (10) admits a coexistence state. This conclusion can be also obtained applying the result in [11].
Predator-prey with a common competitor

$$\left\{\begin{array}{l} \dot{x_1} = x_1(1 - x_1 - \alpha_1(t)x_2 - \beta_1(t)x_3) \\ \dot{x_2} = x_2(1 - \beta_2(t)x_1 - x_2 + \alpha_2(t)x_3) \\ \dot{x_3} = x_3(1 - \alpha_3(t)x_1 - \beta_3(t)x_2 - x_3) \end{array}\right. \tag{11}$$

with $0 < \alpha_1(t), \alpha_3(t) < 1$ and $\alpha_2(t) > 0$; $\beta_i(t) > 1$ for i=1,2,3. In the system (11), there is a predator-prey relationship between the species 2 and 3 and the species 1 is a competitor for both of them. From the inequalities

$$\dot{x_1} \leqslant x_1(1 - x_1)$$

$$\dot{x_3} \leqslant x_3(1 - x_3)$$

we deduce that for

$$\limsup_{t\to\infty} x_1(t) \leq 1$$

$$\limsup_{t\to\infty} x_3(t) \leq 1.$$

Hence, for $M > 1$ and large values of t, we deduce that

$$\dot{x_2} \leq x_2(1 - x_2 + M \parallel \alpha_2 \parallel_\infty)$$

we deduce that the system (11) is dissipative. Now, applying the corollary 5, we deduce that (11) admits a coexistence state.

Switching from cooperation to competition

$$\left\{ \begin{array}{l} \dot{x_1} = x_1(1 + (1 - x_1)x_1 + (\alpha_1(t) - x_2)x_2 + (\beta_1(t) - x_3)x_3) \\ \dot{x_2} = x_2(1 + (\beta_2(t) - x_1)x_1 + (1 - x_2)x_2 + (\alpha_2(t) - x_3)x_3) \\ \dot{x_3} = x_3(1 + (\alpha_3(t) - x_1)x_1 + (\beta_3(t) - x_2)x_2 + (1 - x_3)x_3) \end{array} \right. \tag{12}$$

with $0 < \alpha_i(t) < 1 < \beta_i(t)$ for $i = 1, 2, 3$. In this model there is cooperation relationship for small populations and competition for large ones. We can also apply the corollary 5 in this model and deduce that (12) admits a coexistence state.

After these examples we are going to interpret the condition **iv)** of the theorem 3. Consider the Poincaré map of the system (3)

$$P(\xi_1, \xi_2, \xi_3) = (\xi_1 e^{\int_0^T f_1(t,x(t,\xi))dt}, \xi_2 e^{\int_0^T f_2(t,x(t,\xi))dt}, \xi_3 e^{\int_0^T f_3(t,x(t,\xi))dt}).$$

For example, if $\int_0^T f_2(t, x(t, \xi_1^* e_1))dt > 0$ we deduce that in a neighborhood of $\xi_1^* e_1$ the species 2 increases its size. In fact, when the functions f_i are differentiable, we observe that $e^{\int_0^T f_{i+1}(t,x(t,\xi_i^* e_i))dt}$ and $e^{\int_0^T f_{i-1}(t,x(t,\xi_i^*))dt}$ are the Floquet multipliers associated to the eigenvectors e_{i+1}, e_{i-1}. Therefore, if the system (3) is τ-cyclic we have that $\int_0^T f_2(t, x(t, \xi_1^* e_1))dt \geq 0$. In addition the condition **iv)** is essential since in [11], it is given a competitive system verifying **i), ii), iii)** and does not have a coexistence state.

Now, we are going to prove that the hypotheses of the theorem 3 imply that the system is τ-cyclic. The proof of this fact is based on the theory of translation arcs. The reader who wishes information about translation arcs can consult [3], [9].

Assume that

$$\left\{ \begin{array}{l} \dot{x_1} = x_1 f_1(t, x_1, x_2) \\ \dot{x_2} = x_2 f_2(t, x_1, x_2). \end{array} \right. \tag{13}$$

is T-periodic, all the solutions are bounded in the future and does not have coexistence states. Then $\omega(x_0) \subset Fix(\widehat{P})$ and so it is connected. The ω-limit is defined as

$$\omega(x_0) = \{q \in \mathbb{R}^2 : \exists \sigma(n) \to \infty \text{ such that } \widehat{P}^{\sigma(n)}(x_0) \to q\}.$$

Arguing for contradiction. Assume that there exists $x_0 \in Int(\mathbb{R}^2_+)$ so that $y_0 \in \omega(x_0) \cap \left(\mathbb{R}^2_+ \backslash Fix(\widehat{P})\right)$, analogously one might reason with the other quadrants. Using that $y_0 \notin Fix(\widehat{P})$ we deduce that there exists D a disk centered at y_0 so that $D \cap \widehat{P}(D) = \emptyset$. Now, as $y_0 \in \omega(x_0)$, we deduce that there exists $z_0 \in Int(\mathbb{R}^2_+) \cap D$ such that $P^{N_0}(z_0) \in Int(\mathbb{R}^2_+) \cap D$ for some $N_0 > 1$. After these considerations, consider D_1 a topological disk contained in $Int(\mathbb{R}^2_+) \cap D$ so that $z_0,\ \widehat{P}^{N_0}(z_0) \in D_1$. Therefore, we can construct a translation arc $\gamma \subset Int(\mathbb{R}^2_+)$ so that $z_0, \widehat{P}^{N_0}(z_0) \in \gamma$. This is a contradiction since $\gamma \cup \widehat{P}(\gamma) \cup ... \cup \widehat{P}^{N_0}(\gamma) \subset Int(\mathbb{R}^2_+)$ and so it does not locate any fixed point. We deduce that $\omega(x_0)$ is connected using the proposition 9 of the chapter 3 in [9].
From the hypotheses of the theorem 3 and the previous result, we deduce that for $\widehat{P}\mid_{\{1,2\}}$ the set $\omega(x_0)$ is either $(0,0)$ or $(\pm\xi_1^*, 0)$ or $(0, \pm\xi_2^*)$. If we take $x_0 \in Int(\mathbb{R}^2_+)$, using the condition **iii)** we deduce that $\omega(x_0)$ can not be $(0,0)$, we obtain the same conclusion for $(\xi_1^*, 0)$ using the condition **iv)**. For the other subsystems we can reason analogously.

Under the hypotheses of the theorem 3, we can not provide any information on the stability of the coexistence state since in the May-Leonard system the positive equilibrium can be stable or unstable. We will study the problem of stability of the coexistence states in future works.

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