CONES OF RANK 2 AND THE POINCARÉ-BENDIXSON PROPERTY FOR A NEW CLASS OF MONOTONE SYSTEMS

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Abstract. We study systems that are monotone in a generalized sense with respect to cones of rank 2. The main result of the paper is the existence of a Poincaré-Bendixson property for some solutions of those systems.

1. Introduction

Monotone dynamical systems refer to difference and differential equations whose solutions respect an order structure. Though they have been studied for a long time, it was Hirsch in [5, 6, 7, 8, 9, 10] who started the full description of their main dynamical properties in the setting of cooperative and competitive systems. Later extensions to other classes of equations (as periodic or delay equations) and the integration with some existing monotonicity results for parabolic partial differential equations have built up an extensive theory to which some monographs (as [11] and [20]) have been devoted (see also [4] for applications in the setting of parabolic partial differential equations). In these references one can appreciate the richness of this subject in what concerns as much the amount of sharp theoretical results as the wide range of applications to real models.

Let us make a brief description of that theory for continuous semiflows. A convex cone K in phase space is considered, and hence the partial ordering induced by it. A semiflow is monotone if ordered initial states remain ordered as they evolve along the flow. From this it is shown that the dynamical behavior of the semiflow is strongly determined by the properties of its set of equilibrium points. In fact the most outstanding result is the convergence of almost every positive semiorbit to equilibrium points (see [11]).

A geometrical insight of the theory of monotone semiflows will help to motivate the present paper. Basically the long-term behavior of solutions is conveniently projected either over straight lines contained in $K \cup -K$ or over hyperplanes outside $K \cup -K$. In the first case we get a one-dimensional, and hence trivial, dynamics. In the second case complicated behavior may

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appear, but the point is that it is highly unstable. Of course it is the further usage of the order structure what leads to the extremely precise description of the dynamics achieved in this theory. However it is conceivable that weaker structures which induce similar well-behaved projections would allow to establish dynamical properties for other classes of semiflows. Actually by projecting over linear subspaces of dimension greater than 1 it is expectable to capture less stringent dynamical phenomena.

In this paper we try to carry out this task by employing the so-called cones of rank k. These sets were already considered in [13] in connection with generalization of Krein-Rutman theory, and they are defined as closed sets that consist of straight lines and which contain a linear subspace of dimension k and no linear subspace of higher dimension. A usual convex cone K for example defines the generalized cone $K \cup -K$ that is of rank 1. By introducing then a new notion of monotone semiflows with respect to these cones we are able to prove the existence of good projections for them as just a convex cone does. Concretely by using cones of rank 2 we are able to project part of the dynamics into planes. From this we deduce the Poincaré-Bendixson property for some orbits, that is, we prove that some compact omega-limits sets without equilibrium points consist just of one closed orbit.

Some preceding works have already exploited similar ideas. The classical theory of monotone systems itself produces a Poincaré-Bendixson theorem for competitive three-dimensional systems. We shall reinterpret it in our setting by noticing that competitiveness can be seen as a generalized monotonicity with respect to the cone of rank 2 complementary to $K \cup -K$. The class of monotone cyclic feedback systems (in the finite dimensional case [14] and in the infinite-dimensional one [15]) were also shown to verify a similar property. In that case the very particular structure of the systems implies another kind of monotonicity with respect to a sequence of nested cones. Finally R. A. Smith succeeded in providing a Poincaré-Bendixson theorem for systems having a sort of Lyapunov function. Again we shall show that his theory is strongly related to our work. Actually our main aim in this paper is to single out perhaps the essential ingredient in order to achieve multidimensional versions of the Poincaré-Bendixson theorem and give in this way a unified view of the preceding works.

We now describe how the paper is organized. We first introduce all the basic definitions about cones of rank k and the corresponding generalized monotonicity notion for semiflows. We define also a class of systems that extend the classical cooperative systems. They enjoy several monotonicity properties in this generalized sense and this fact enables us to state a Poincaré-Bendixson theorem for them. This theorem will be proved along sections 3 and 4.

In section 3 a location theorem on the omega-limit sets of our generalized cooperative systems is given. The main tool is a perturbation argument based in the well-known closing lemma (see [1, 18]). From this we shall deduce that some omega-limit sets of monotone flows with respect to cones of rank k are in some sense k-dimensional.

In section 4 we employ the extension of Krein-Rutman theory carried out in [2] and [13] in order to estimate the dimension of the local invariant manifolds associated to closed orbits. In particular for monotone flows with respect to cones of rank 2 some closed orbits are shown to have center-unstable manifolds of dimension at most 2. Finally the theory of conjugations around partially hyperbolic fixed points of [12] provides a sharp description of the local behavior around those closed orbits that leads to the proof of the Poincaré-Bendixson property.

The last section is devoted to discuss our extension of the cooperativeness conditions and to relate it with previous works.

2. Basic definitions and Main Theorem

Many of the definitions of this section are taken from [13]. We begin with a generalization of the concept of classical convex cone.

Definition 1. *A set $C \subset \mathbb{R}^n$ is a cone of rank k if:*

(1) *C is closed.*
(2) $x \in C,\ \alpha \in \mathbb{R}\ \Rightarrow\ \alpha x \in C$.
(3) $\max\{\dim W :\ C \supset W \quad \text{linear subspace}\} = k$

The closure of the set $\mathbb{R}^n - C$ is also a cone. We shall call it the complementary cone of C, and it will be denoted by C^c. In order to avoid trivial situations we always suppose that C and C^c are nonempty.

As a first example consider a usual convex cone K, that is, K is a convex closed subset consisting of rays starting at the zero vector and satisfying $K \cap -K = \{0\}$. It is easy to prove that the set $C_K = K \cup -K$ is a cone of order 1 (see [2]). These cones will be important in this paper since they will act as a bridge between classical monotone systems and our extended monotone systems.

If K is the convex cone of vectors with nonnegative coordinates then C_K is the cone of rank 1 of vectors with no sign changes in their coordinates. In order to generalize this define, for a vector $(x_1, \ldots, x_n)$ with nonzero coordinates, the function $N(x)$ as the number of sign changes in the sequence $\{x_1, \ldots, x_n\}$. Then the sets

$$T(k, \mathbb{R}^n) = \overline{\{x \in \mathbb{R}^n :\ N(x) \leq k-1\}}$$

are cones of rank k (see page 71 in [13]). These cones play a big role in the theory of oscillatory matrices and are related to the integer-valued Lyapunov functionals used in [14, 15].

As a final example we show what we call quadratic cones. Let P be a symmetric inversible matrix of order n having k negative eigenvalues and $n-k$ positive eigenvalues. Then the sets

$$C^-(P) = \{x \in \mathbb{R}^n :\ \langle x, Px \rangle \leq 0\}$$

and

$$C^+(P) = \{x \in \mathbb{R}^n : \langle x, Px \rangle \geq 0\}$$

are easily shown to be cones of order k and $n-k$ respectively. In fact they are complementary cones.

Definition 2. *The cone C of order k is solid if $\overset{\circ}{C} \neq \emptyset$. C is k-solid if there is a linear subspace W of dimension k such that $W - \{0\} \subset \overset{\circ}{C}$.*

It is not difficult to see that the quadratic cones $C^-(P), C^+(P)$ are respectively k-solid and $n-k$-solid. C_K is 1-solid as long as K has nonempty interior. In [13] it is proved that cones $T(k, \mathbb{R}^n)$ are k-solid too.

Fix a cone C of rank k. We want now to extend several concepts based on the order induced by convex cones to our general setting.

First recall that, given the convex cone K, the order in $\mathbb{R}^n$ is defined in the form

$$x \leq y \Leftrightarrow x - y \in K$$

for any $x, y \in \mathbb{R}^n$. The fact that this definition provides an order comes from the convexity of K and the property $K \cap -K = \{0\}$. Since these two properties are not fulfilled by general cones of rank k no natural order relation can be induced. Nevertheless the idea of points x, y to be ordered (or related), meaning this that either $x \leq y$ or $y \leq x$, can be written just as $x - y \in C_K$. This justifies next definition.

Definition 3. *Two points x and y are said to be ordered if $x - y \in C$. They are said to be strongly ordered if $x - y \in \overset{\circ}{C}$.*

From now on by $x \sim y$ we denote that x and y are ordered, and by $x \approx y$ that x and y are strongly ordered.

In the theory of monotone systems an important role is played by two types of sets: ordered and balanced sets (see [11] for definitions). These notions are extended as follows:

Definition 4. *A set $S \subset \mathbb{R}^n$ is ordered if $p \sim q$ for any $p, q \in S$. It is strongly ordered if $p \approx q$ for any $p, q \in S$ with $p \neq q$. The set S is balanced if there are no points $p, q \in C$ such that $p \approx q$, and it is strongly balanced if there are no $p, q \in C$ with $p \neq q$ such that $p \sim q$.*

Consider now $M : \mathbb{R}^n \to \mathbb{R}^n$ any map.

Definition 5.
1. *M is positive if $M(C) \subset C$.*
2. *M is strongly positive if $M(C - \{0\}) \subset \overset{\circ}{C}$.*
3. *M is monotone if $x \sim y$ implies $M(x) \sim M(y)$.*
4. *M is strongly monotone if $x \sim y$, $x \neq y$ implies $M(x) \approx M(y)$.*

Finally consider an autonomous equation

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where F is C^1. We suppose that equation (1) induces a semiflow $\Phi(t, x)$ defined for all $t \geq 0$.

The monotonicity of the semiflow $\Phi(t, x)$ is understood with respect to the x-variable and for positive t. To be precise, $\Phi(t, x)$ is monotone if

$$x \sim y \Rightarrow \Phi(t, x) \sim \Phi(t, y) \quad \text{for } t > 0,$$

and strongly monotone if

$$x \sim y,\ x \neq y \Rightarrow \Phi(t, x) \approx \Phi(t, y) \quad \text{for } t > 0.$$

We also say that $\Phi(t, x)$ is (strongly) infinitesimally monotone if the spatial derivative $D\Phi(t, x)$ is a (strongly) positive operator.

It is easy to prove that the monotonicity implies the infinitesimal monotonicity, but the strong monotonicity does not have to imply the strong infinitesimal monotonicity. More interesting would be to know if the converse implications hold true. Dealing with the monotonicity induced by convex cones these implications are deduced from the integral mean value theorem and the convexity of the cone (see section 3.1 in [11]). Since cones of rank k ($k > 1$) are not convex in general we cannot assert that these two notions are not independent.

We now give a condition over the vector field F which will imply the two preceding notions of monotonicity. To do that we introduce, for any $p, q \in \mathbb{R}^n$, the matrices

$$A^{pq}(t) = \int_0^1 DF(s\Phi(t, p) + (1 - s)\Phi(t, q))ds$$

and $U^{pq}(t)$ the solution of

$$\dot{U} = A^{pq}(t)U,\ U(0) = I.$$

Definition 6. *We say that system (1) is C-cooperative if for every $p, q \in \mathbb{R}^n$, the matrix $U^{pq}(t)$ is strongly positive for $t > 0$.*

Proposition 1. *If system (2) is C-cooperative then the semiflow $\Phi(t, x)$ is strongly monotone and strongly infinitesimally monotone.*

Proof Given $p, q \in \mathbb{R}^n$ we have that

$$\Phi(t, p) - \Phi(t, q) = U_{pq}(t)(p - q)$$

if $p \neq q$, and

$$D\Phi(t, p) = U_{pp}(t).$$

From this and definition 6 our assertion follows.

We are now ready to state our Poincaré-Bendixson theorem.

Theorem 1. *Let $C \subset \mathbb{R}^n$ be a cone of rank 2 such that C is 2-solid and C^c is $(n-2)$-solid. Let us assume that equation (2) is C-cooperative. Let Ω be the compact omega-limit set of a solution $x(t)$ and suppose that:*

i) *Ω has no equilibrium point.*
ii) *$\dot{x}(t_0) \in C$ for some $t_0 \in \mathbb{R}$.*

Then Ω is a closed orbit.

We delay a discussion about C-cooperative systems and theorem 1 until the last section. Nevertheless we make now a couple of remarks in order to grasp what theorem 1 says. Firstly the C-cooperativeness will be implied by pointwise conditions over $DF(x)$ and it will not depend on any explicit knowledge of solutions of system (1). Secondly the Poincaré-Bendixson property will hold only for certain solutions, and so we cannot preclude that other solutions behave in a more complicated manner. However we give some indication in the direction that such orbits are non-stable (see proposition 4).

3. Omega-limit sets of monotone semiflows

Along this section we assume that system (1) is C-cooperative with respect to a cone C of rank k. Let $x(t)$ be any nonconstant solution. If there exists t_0 such that $\dot{x}(t_0) \in C$ then $\dot{x}(t) \in \overset{\circ}{C}$ for every $t > t_0$. This follows from the identity $\dot{x}(t) = U^{x(t_0)x(t_0)}(t)(\dot{x}(t_0))$. Hence we can classify every nonconstant solution into two types:

Type I: $\dot{x}(t) \in \overset{\circ}{C}$ for t sufficiently large.
Type II: $\dot{x}(t) \notin C$ for any t.

Remark 1. *This classification only depend on the corresponding semiorbit, so we shall use it for both solutions and semiorbits.*

In case $x(t)$ is periodic the preceding distinction is expressed as $\dot{x}(t) \in \overset{\circ}{C}$ or $\dot{x}(t) \notin C$ for all $t \in \mathbb{R}$. This can be read as that any closed orbit is either locally ordered or locally balanced. Next proposition gives a global version of this fact.

Proposition 2. *Let γ be a closed orbit associated to a T-periodic solution $p(t)$. If $p(t)$ is of type I then γ is strongly ordered. Similarly if $p(t)$ is of type II then γ is strongly balanced.*

Proof: We only consider the case that $p(t)$ is of type I, the other case being similar. Let us assume that there are $p, q \in \gamma$ with $p - q \notin \overset{\circ}{C}$ and let us reach a contradiction. First of all being $p(t)$ of type I ensures that $p - r \in \overset{\circ}{C}$ for r near enough to p. This and our momentary assumption shows that we can take in fact p and q ($p \neq q$) such that $p - q \in \partial C$. We write $p = p(0)$ and $q = p(t_1)$ with $t_1 > 0$. Applying the strong monotonicity and the T-periodicity of $p(t)$ we obtain that $p(0+T) - p(t_1+T) = p - q \in \overset{\circ}{C}$, a contradiction.

The next natural step is to prove a similar dichotomy for compact omega-limit sets. To do that we are going to approximate these omega-limit sets by closed orbits by means of the closing lemma. This enforces us first to study how the C-cooperativeness is preserved under small perturbations of F.

Consider a sequence F_n of C^1 vector field converging on compact sets to F in the C^1 topology and let

$$\dot{x} = F_n(x). \tag{2}$$

We call $\Phi_n(t,p)$ to the corresponding induced semiflow.

Lemma 1. *Let $K \subset \mathbb{R}^n$ convex and compact and $\tau > 0$. There is $m \in \mathbb{N}$ such that for all $n > m$ it holds that, given $p, q \in K$ with $\Phi_n(t,p), \Phi_n(t,q) \in K$ for all $t > 0$, the operator $U_n^{pq}(t)$ is strongly positive for $t \geq \tau$.*

Proof: We first prove the positiveness of $U_n^{pq}(t)$ in the interval $I = [\tau, 2\tau]$. We do that by reduction to the absurd and so suppose that there exist $p_n, q_n \in K$, $v_n \in C$ with $\|v_n\| = 1$, and $\tau_n \in I$ such that $U_n^{p_n q_n}(\tau_n)v_n \in \partial C$. By extracting a subsequence we can assume that p_n and q_n converge to p_0 and q_0, v_n converges to $v_0 \in C$ and that τ_n tends to $\tau_0 \in I$. Obviously $\Phi_n(t,p_n)$ and $\Phi_n(t,q_n)$ converge uniformly in $[\tau, 2\tau]$ to $\Phi(t,p_0)$ and $\Phi(t,q_0)$ respectively. From this $A_n^{p_n q_n}(t)$ tends uniformly in I to $A^{p_0,q_0}(t)$, and so the same convergence occurs of $U_n^{p_n q_n}(t)$ to $U^{p_0 q_0}(t)$. This would imply that $U^{p_0 q_0}(t)v_0 \in \partial C$, contradicting the strong positiveness of this operator.

To prove now that this is valid for any $t \geq \tau$ let us write $t = t_0 + k\tau$ with $t_0 \in [\tau, 2\tau]$ and $k \in \mathbb{N}$. Let us call $t_j = t_0 + j\tau$, and p_j and q_j to $\Phi(t_j, p)$ and $\Phi(t_j, q)$ respectively. It is not difficult to see that

$$U_n^{pq}(t) = U_n^{p_{k-1}q_{k-1}}(t_0) \dots U_n^{p_1 q_1}(\tau) U_n^{pq}(\tau).$$

From the preceding proof each individual factor on the right is strongly positive, and so $U_n^{pq}(t)$ also is.

Remark 2. *In the preceding setting let $n > m$ and take any T-periodic solution $p(t)$ of system (2) whose orbit lies in the compact set K. We assert that $p(t)$ verifies proposition 2. To see that observe that $\Phi_n(t,x)$ is now strongly monotone and strongly infinitesimal monotone for $t > \tau$. Thus in the proof of the proposition we reach the contradiction $p(0+jT) - p(t_1+jT) = p - q \in \overset{\circ}{C}$ by taking $j \in \mathbb{N}$ with $jT > \tau$.*

Let us state now our desired result on omega-limit sets.

Theorem 2. *Let Ω be the omega-limit set of a solution $x(t)$ and let $y(t)$ a nonconstant solution whose orbit γ is contained in Ω. If $x(t)$ is of type I then γ is ordered. Similarly if $x(t)$ is of type II then γ is balanced.*

Proof:

Again we only prove the theorem in case that $x(t)$ is of type I. Let us fix $p, q \in \gamma$, $p \neq q$. We suppose that $p = y(0)$ and $q = y(t_1)$ with $t_1 > 0$. Notice that $y(t)$ is also of type I. Using the version of the closing lemma stated in chapter II of [1] we can find a sequence F_n of C^1 vector fields verifying the following properties:

(1) $F_n = F$ excepting in a small of radius $1/n$ centered at a fixed omega-limit point of γ.
(2) $\|F_n - F\| + \|DF_n - DF\| \to 0$ as $n \to +\infty$.
(3) System (2) has a closed orbit γ_n through $p_n \in B(p, 1/n)$.
(4) γ_n is $1/n$-close to a segment of the semiorbit $S = \{x(t) : t \geq 0\}$.

Thus we can fix a large radius $R >$ such the the compact ball $K = \overline{B}(0, R)$ contains γ_n for all n.

Let us call $p_n(t)$ the periodic solution generating the closed orbit γ_n with $p_n(0) = p_n$. It is immediate that for n large $p_n(t)$ is of type I. Therefore lemma 1 and remark 2 imply that $r \approx s$ for any pair of different points r, s in γ_n. In particular $p_n(0) - p_n(t_1) \in \mathring{C}$. Since $p_n(0)$ and $p_n(t_1)$ tend to p and q respectively as $n \to +\infty$, we deduce that $p - q \in C$, that is, $p \sim q$ as desired.

Remark 3. *The C-cooperativeness is used in the proof of theorem 2 only because of the perturbation lemma 1. We do not know whether a similar property is satisfied by the monotonicity of $\Phi(t, x)$ (the infinitesimal monotonicity does verify it). If this were so the C-cooperativeness will remain solely as a condition to be checked in applications.*

We are not able to prove this property for any two points in Ω. However in case that C is of order 2, we shall reach a much stronger result. This is the aim of next section.

4. The Poincaré-Bendixson property for C-cooperative systems

We assume henceforth that we are in the setting described in theorem 1. Observe that hypothesis iii) in that theorem means that $x(t)$ is a solution of type I.

Let $y(t)$ be a nonconstant solution whose orbit is contained in the omega-limit set Ω of $x(t)$, and call $\Omega_1 \subset \Omega$ to the closure of that orbit. Since the orbit of $y(t)$ is ordered Ω_1 is ordered too.

Proposition 3. *The dynamics on Ω_1 is topologically conjugate to the dynamics of a compact invariant set of a Lipschitz-continuous vector field in $\mathbb{R}^2$.*

Proof: The argument is exactly the same as the one in theorem 3.17 of [11]. The starting point is to choose H and H^c subspaces of dimension 2 and $n-2$ respectively satisfying $H - \{0\} \subset \mathring{C}$ and $H^c - \{0\} \subset \mathring{C}^c$. This can be done since C and C^c are 2-solid and $(n-2)$-solid respectively. Now take $\Pi : \mathbb{R}^n \to H$ the linear projection onto H parallel to H^c. Since Ω_1 is ordered the restriction of Π to Ω_1 is one-to-one. Now the proof of theorem 3.17 of [11] directly applies.

Corollary 1. *If Ω_1 has no equilibrium point, then it consists of closed orbits and orbits connecting two of such closed orbits.*

Proof: This is deduced from the Poincaré-Bendixson theory for planar autonomous systems and proposition 3.

We see in particular that Ω has a closed orbit provided that it does not have equilibrium points. In fact we shall go further by using the extension of Krein-Rutman theory developed in [2] and [13]. We just state the main theorem in [2].

Theorem 3. *Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a strongly positive linear map with respect to the 2-solid cone C. Let the spectrum of M be $Sp(M) = \{\lambda_1, \dots \lambda_n\}$ ordered such that $|\lambda_i| \geq |\lambda_j|$ for $i > j$. Then*

$$|\lambda_i| > |\lambda_j| \ \ \textit{for } i = 1, 2, \ j = 3, \dots n. \tag{3}$$

Moreover there are two unique subspaces V and W verifying:

1. $V - \{0\} \subset \overset{\circ}{C}$, $\qquad W \cap C = \{0\}$.
2. $M(V) \subset V$, $\qquad M(W) \subset W$.
3. *The spectrum of M restricted to V is $\{\lambda_1, \lambda_2\}$ and the spectrum of M restricted to W is $\{\lambda_3, \dots, \lambda_n\}$.*

Consider now a closed orbit γ associated to a T-periodic solution $p(t)$ of system (1) (T is the minimal period of $p(t)$). Let M be the monodromy operator associated to $p(t)$, that is, $M = D\Phi(T, p)$. We write its spectrum

$$Sp(M) = \{\lambda_1, \dots, \lambda_n\}$$

(repeating each eigenvalue as many times as its multiplicity) as in the preceding theorem. It is well known that there is $\alpha \in \{1, \dots, n\}$ such that $\lambda_\alpha = 1$. Moreover $\dot{p}(0)$ is an eigenvector associated to λ_α.

By hypothesis M is strongly positive with respect to C, and so theorem 3 applies. We keep on calling V and W to the eigenspaces associated to $\{\lambda_1, \lambda_2\}$ and $\{\lambda_3, \dots, \lambda_n\}$ respectively.

If $p(t)$ is of type II then $\dot{p}(0) \notin C$. Therefore, in view of properties (1),(2),(3) in theorem 3, $\lambda_\alpha = 1$ for some $\alpha = 3 \dots, n$. So we can assert:

Proposition 4. *If $p(t)$ is T-periodic of type II then*

$$|\lambda_i| > 1 \ \textit{for } i = 1, 2. \tag{4}$$

In particular $p(t)$ is unstable.

Let us suppose now that $p(t)$ is of type I. We have that $\dot{p}(0) \in \overset{\circ}{C}$, and a similar reasoning to the preceding one shows that, for example, $\lambda_1 = 1$. From this we state:

Proposition 5. *If $p(t)$ is T-periodic of type I then*

$$1 > |\lambda_j| \ \textit{for } j = 3, \dots, n. \tag{5}$$

To get further into the local behavior around the type I periodic solution $p(t)$ we shall recall the theory of invariant manifolds for closed orbits. The proofs can be found in [3] and [16].

We take a Poincaré section (Π, P) for $p(t)$. Here Π is an hyperplane through a point p in the orbit of $p(t)$ which is transversal to $F(p)$. P is the first return map defined in a neighborhood of p in Π. It is well-known that P is a C^1-diffeomorphism, and that the spectrum of $DP(p)$ is

$$Sp(DP(p)) = \{\lambda_2, \lambda_3, \ldots, \lambda_n\}.$$

Actually we choose Π containing the eigenspace W. This can be done because $\dot{p}(0) \in \overset{\circ}{C}$ (see theorem 3). We can proceed in a similar way than in [16] and consider a system of coordinates with respect to a basis whose elements are a basis of W, a vector in the complementary of W in Π and vector $F(p)$. In these coordinates $M = D\Phi(T, p)$ is written as

$$\begin{pmatrix} DP(p) & 0 \\ v & 1 \end{pmatrix} \tag{6}$$

Here v is a row vector of $n-1$ components and 0 stands for the zero column vector of dimension $n-1$. From this it is immediate that W is also invariant for $DP(p)$ and the spectrum of $DP(p)$ restricted to W is just $\{\lambda_3, \ldots, \lambda_n\}$.

We call W^s the local C^1 invariant manifold of P at p associated to those eigenvalues of $DP(p)$. This manifold is tangent at p to the linear subspace W and $P^n(q)$ tends to p as n tends to $+\infty$ for every $q \in W^s$.

Remark 4. *We know that $W \cap C = \{0\}$ by (1) in theorem 3. So for every $q \in W^s$ it holds that $P^n(q)$ and p are not ordered for n large enough.*

We are now ready to prove our main theorem.

Proof of theorem 1:

Proof: Consider an ordered set Ω_1 as constructed at the beginning of this section. This set should contain at least one closed orbit γ (that is of type I). We suppose that $\Omega_1 \neq \gamma$ and let us reach a contradiction.

Firstly from corollary 1 we can assume that either

(C1) γ is the omega-limit set of an orbit $\bar{\gamma}$ in Ω_1,

or

(C2) γ is the limit of a sequence of closed orbits γ_n in Ω_1. In addition the (smaller) periods of γ_n tend to the (smaller) period of γ.

Notice that the additional assertion in (C2) is a consequence of proposition 3 and the theory of transversal segments for planar autonomous systems.

We construct the Poincaré section (Π, P) for γ and the manifold W^s described above. We claim that $|\lambda_2| = 1$. If this were not so, then either $|\lambda_2| < 1$ or $|\lambda_2| > 1$. In the first case γ would be orbitally asymptotically stable, what contradicts $\Omega_1 \neq \gamma$. In the second case W^s would be the local stable manifold associated to p and in addition there is an unstable manifold W^u of P at p that have dimension 1. Notice that $\bar{\gamma} \cap \pi$ in case (C1) would define a sequence of points $q_n \in \Pi$, $q_n = P^n(q_0)$ tending to p. Consequently

$q_n \in W^s$. This contradicts remark 4 and that Ω_1 is ordered. On the other hand the sequence γ_n in (C2) would define a sequence of fixed points q_n for P converging to p. The fact that they are fixed points and not periodic points of increasing period is deduced from the closeness of the periods of γ and γ_n remarked in (C2). This is impossible by the hyperbolicity of p.

We thus have $|\lambda_2| = 1$. Let us consider first the case $\lambda_2 = 1$.

There are coordinates (u, v) in a neighborhood O of $p \equiv (0, 0) \in \Pi$ where P has the form

$$P(u, v) = (u + U(u, v), Lv + V(u, v)), u \in \mathbb{R}, v \in \mathbb{R}^{n-2}$$

where U, V are C^1 functions verifying $U(0, 0) = V(0, 0) = 0$ and $DU(0, 0) = DV(0, 0) = 0$, and L is a square matrix whose spectrum is lesser than 1. Now W^s is tangent at p to the linear space $u = 0$.

We resort to the linearization theorem of Kirchgraber and Palmer (see page 46 in [12]) to suppose that actually P can be written in new coordinates as

$$P^1(u^1, v^1) = (u^1 + \phi(u^1), Lv^1)$$

in a certain neighborhood O^1 of $(0, 0)$. In these coordinates the stable manifold W^s has become the $(n-2)$-dimensional v^1-axis. On the other hand the u^1-axis is a center manifold for S^1, and the map

$$u^1 \to u^1 + \phi(u^1)$$

is a local increasing Lipschitz homeomorphism around zero. In addition it is immediate that

$$H_+ = \{(u^1, v^1) \in O^1 : u^1 > 0\}$$

and

$$H_- = \{(u^1, v^1) \in O^1 : u^1 < 0\}$$

are (locally) invariant sets for P^1.

Let us reformulate cases $(C1)$ and $(C2)$ in terms of the map P^1. In the first case the intersection of $\bar{\gamma}$ with the hyperplane Π provides a semiorbit for P tending to $(0, 0)$. This semiorbit cannot lie in the v^1-axis since $\bar{\gamma}$ was not in the stable manifold of γ. In addition u^1_n is monotone since $u^1 \to u^1 + \phi(u^1)$ is increasing around $(0, 0)$. In consequence (C1) changes into:

(C1') There is an orbit $(u^1_{n+1}, v^1_{n+1}) = P^1(u^1_n, v^1_n) \in O^1$ tending to $(0, 0)$ with (for example) $u^1_n > u^1_{n+1} > 0$.

Concerning (C2) the closed orbits γ_n define a sequence of fixed points for P^1 tending to $(0, 0)$. The monotonicity in u^1_n can be assumed, and this case turns into:

(C2') There is a sequence $(u^1_n, v^1_n) = (u^1_n, 0) \in O^1$ of fixed points of P^1 with $u^1_n > u^1_{n+1} > 0$.

The rest of the proof is common for both cases (C1') and (C2'). Consider the set

$$R = \{(u^1, v^1) \in \mathbb{R} \times \mathbb{R}^{n-2} : 0 < u^1 < u^1_n, \|v\| < r\}$$

where n is large and r is small so that $R \subset O^1$. It is immediate that R is positively invariant and that any semiorbit in R tends to a fixed point. Now consider the intersections of the orbit of the solution $x(t)$ with R. This set is nonempty since either the orbit $\bar{\gamma}$ or the sequence of closed orbits γ_n belong to the omega limit set Ω of $x(t)$. The positive invariance of R implies that this intersections form a positive semiorbit for the map P^1, and therefore it must converge to a fixed point. Then $x(t)$ has a omega-limit set just a closed orbit, in contradiction with (C1) and (C2).

The case $\lambda_2 = -1$ possibly cannot hold because of proposition 3. Instead of proving that we can consider the mapping $\bar{P} = P \circ P$ defined again in a neighborhood of $p \in \Pi$. It is immediate that $\bar{P}$ falls under the previous case, and the contradiction is reached in the same manner.

In consequence neither (C1) nor (C2) can occur and thus Ω_1 just consists of the closed orbit γ. The same reasoning applied to every set $\Omega_1 \subset \Omega$ shows that Ω has only closed orbits. If there were more than one and since Ω is connected, γ should be an accumulation orbit of a sequence γ_n of other closed orbits in Ω. Again no γ_n can lie in the stable manifold of γ, and the proof carried out above also yields to a contradiction. Therefore the theorem is proved.

Remark 5. *A similar result can be stated for alpha-limit sets. Any nonconstant solution $p(t)$ bounded in $]-\infty, 0]$ can be classified according to $\dot{x}(t) \notin C$ for t in a certain interval $]-\infty, t_0[$ or $\dot{x}(t) \in \overset{\circ}{C}$ for all t. Thus in the second case the alpha-limit set of $x(t)$ is projected into a plane and the same reasoning as above shows that also a Poincaré-Bendixson property holds.*

5. Discussion and related results

In remark 3 we have given a theoretical justification of our definition of C-cooperativeness in order to get the Poincaré-Bendixson theorem. We want in this section to show that, despite its technical appearance, the C-cooperativeness is easy to check for concrete systems and is a straight extension of the classical monotonicity assumptions.

To do that we first compare it with classical cooperative systems. Let

$$K = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \; x_i \geq 0\}$$

and $C_K = K \cup -K$.

Proposition 6. *System (1) is C_K-cooperative provided that it is cooperative and irreducible with respect to K.*

Proof: By hypothesis $DF(x)$ has nonnegative off-diagonal coefficients and is irreducible for all $x \in \mathbb{R}^n$. Hence for every pair $p, q \in \mathbb{R}^n$ the matrix $A^{pq}(t) = \int_0^1 DF(s\Phi(t,p) + (1-s)\Phi(t,q))ds$ has the same property for all t. The same proof of theorem 1.1 in [20] shows that the corresponding matrix solution $U^{pq}(t)$ is strongly positive in the classical sense for $t > 0$. Now we just notice that $U^{pq}(t)(C_K - \{0\}) \subset \overset{\circ}{C}_K$ as long as $U^{pq}(t)(K - \{0\}) \subset \overset{\circ}{K}$ to finish the proof.

We can now interpret the Poincaré-Bendixson theorem for classical competitive systems in $\mathbb{R}^3$ under a new perspective. Concretely in [11] it is proved that, given an irreducible competitive systems in $\mathbb{R}^3$, every bounded solution whose omega-limit set has no equilibrium point tends to a closed orbit. To make our interpretation just notice that, if $K \subset \mathbb{R}^3$ is a convex cone with nonempty interior, then $C = \overline{\mathbb{R}^3 - C_K}$ is a cone of rank 2 such that C and $C^c = C_K$ are 2-solid and 1-solid respectively. A similar argument to the one employed in the proof of 6 shows that system (1) is C-cooperative as long it is competitive and irreducible with respect to C_K. Then we can assert that any solution of type I whose omega-limit set has no equilibrium points tends to a closed orbit. On the other hand the omega-limit set of any solution of type II consists of equilibrium points. To prove that take $x(t)$ solution of type II and suppose that its omega-limit set has nontrivial orbits. The proof of theorem 2 would imply (for n large) the existence of closed orbits of type II for the perturbed systems (2). From the the analogue of proposition 3 and since C_K is of rank 1, these closed orbits should be injectively projected into a one-dimensional linear subspace. This is not possible and so a contradiction is reached.

As the second example we study the case $C = C^-(P)$ is a quadratic cone as described in section 2. Again we show that the $C^-(P)$-cooperativeness is rather computable since it is stated as a pointwise condition over $DF(x)$.

Proposition 7. *Assume that*

(7) $\quad < PDF(x)\xi, \xi >< 0 \;\; \textit{for} \;\; < P\xi, \xi >= 0,\ \xi \neq 0 \textit{ and all } x \in \mathbb{R}^n.$

Then equation (2) is $C^-(P)$-cooperative.

Proof: Let $p, q \in \mathbb{R}^n$ and $A^{pq}(t), U^{pq}(t)$ as above. Take $u_0 \in C^-(P)$ and define

$$a(t) = \langle PU^{pq}(t)u_0, U^{pq}(t)u_0\rangle \qquad t \geq 0.$$

Then $\dot{a}(t) = 2\langle PA^{pq}(t)U^{pq}(t)u_0, U^{pq}(t)u_0\rangle$. Using the definition of $A^{pq}(t)$ we get that

$$\dot{a}(t) = 2\langle P\int_0^1 DF(s\Phi(t,p) + (1-s)\Phi(t,q))dsU^{pq}(t)u_0, U^{pq}(t)u_0\rangle.$$

Now applying the linearity of the integral we obtain

$$\dot{a}(t) = 2\int_0^1 \langle PDF(s\Phi(t,p) + (1-s)\Phi(t,q))U^{pq}(t)u_0, U^{pq}(t)u_0\rangle ds.$$

This formula and (7) says that $\dot{a}(t) < 0$ when $a(t) = 0$. Since $u_0 \in C^-(P)$ and so $a(0) \leq 0$, we deduce that $a(t) < 0$ for all $t > 0$. This just means that $U^{pq}(t)u_0 \in \overset{\circ}{C}$. Since $u_0 \in C^-(P)$ is arbitrary we get that $U^{pq}(t)$ is a strongly positive operator.

Let us establish an interesting link with the Poincaré-Bendxson theory developed by R. A. Smith in [21, 22]. By applying lemma 1 of [17] in the same manner that therein one can show that condition (7) is equivalent to the existence of a (continuous) function

$$\lambda : \mathbb{R}^n \to \mathbb{R}$$

such that

$$DF(x)^*P + PDF(x) + \lambda(x)P < 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{8}$$

Here $DF(x)^*$ stands for the transpose of $DF(x)$ and $<$ refers to the usual order in the space of symmetric matrices. On the other hand R. A. Smith studied the class of systems satisfying

$$DF(x)^*P + PDF(x) + \lambda P < 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{9}$$

where now $\lambda > 0$ is a real constant, and proved that these systems verify the Poincaré-Bendixson property for all solutions. The reason for that is the existence of a Lipschitz-continuous 2-dimensional manifold that attracts all the orbits and which is, in our language, strongly ordered. Therefore we can say that the class of R. A. Smith consists of certain $C^-(P)$-cooperative systems for which solutions of type II have trivial dynamics.

We finally show a trivial example of a C-cooperative system displaying chaotic solutions. In fact let

$$\dot{y} = F(y),\ y \in \mathbb{R}^k \tag{10}$$

be any smooth dissipative chaotic system such that $DF(y)$ is bounded (for example the variation of Lorenz systems considered in [19]). It is immediate that $DF(y)^* + DF(y) + \lambda I_k$ is negative definite for λ near $-\infty$ (I_k will stand for the identity matrix of order k). Let $\alpha > -\lambda$ and consider the system

$$\begin{cases} \dot{x} = & \alpha x, & x \in \mathbb{R}^2 \\ \dot{y} = & F(y), & y \in \mathbb{R}^k. \end{cases} \tag{11}$$

Let

$$P = \left(\begin{array}{c|c} -I_2 & 0_{2\times k} \\ \hline 0_{k\times 2} & I_k. \end{array} \right)$$

It is immediate to check that system (11) is $C^-(P)$-cooperative. The dynamics of system (10) is embedded in the invariant balanced subspace $x = 0$. Any other solution is unbounded.

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