

RADIAL STABILITY OF PERIODIC SOLUTIONS OF THE GYLDEN-MESHCHERSKII-TYPE PROBLEM

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ABSTRACT. For the Gylden-Meshcherskii-type problem with a periodically changing gravitational parameter, we prove the existence of radially periodic solutions with high angular momentum, which are Lyapunov stable in the radial direction.

1. Introduction. In Celestial Mechanics and Astrophysics, problems involving gravitating bodies with variable mass arise with relative frequency. Perhaps the best known of them is the two-body problem with variable masses, known as the Gylden-Meshcherskii problem. If such variation is assumed to be periodic, the mathematical model under consideration is

$$\ddot{x} = -h(t) \frac{x}{|x|^3}, \quad (1)$$

where h is a continuous, T -periodic function. Originally, the Gylden-Meshcherskii problem was proposed by Gylden to explain the secular acceleration observed in the Moon's longitude, but nowadays it is used to describe a variety of phenomena including the evolution of binary stars, dynamics of particles around pulsating stars, photogravitational effects and many others. See [1, 8, 21, 24, 25, 26, 27] and the

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references therein. The famous Newtonian equation for the motion of a particle subjected to the gravitational attraction of a sun which lies at the origin

$$\ddot{x} = -\frac{cx}{|x|^3} \quad (2)$$

corresponds to (1) with the choice $h(t) = c$ for some positive constant $c > 0$. In a different line of research, the Newtonian motion of a particle under a central force field which may depend periodically on time has been recently studied by Fonda and his coworkers [9, 10, 11, 12, 13, 14].

In this paper, we continue this topic and study the radial stable periodic orbits of (1). Generally, we will consider the following differential equation

$$\ddot{x} = -h(t)\frac{x}{|x|^{\alpha+1}}, \quad \alpha > 0, \quad (3)$$

which becomes the Gylden-Meshcherskii problem (1) when $\alpha = 2$.

It is well known that for this kind of central force fields every orbit lies in a plane, then passing to polar coordinates

$$x(t) = r(t)(\cos \theta(t), \sin \theta(t))$$

with amplitude $r(t) > 0$ and angle $\theta(t) \in \mathbb{R}$ for every t , system (3) is equivalent to

$$\ddot{r} = \frac{\mu^2}{r^3} - \frac{h(t)}{r^\alpha}. \quad (4)$$

A simple computation shows that

$$\frac{d}{dt}(r^2(t)\dot{\theta}(t)) = 0, \quad \text{for every } t,$$

which implies that

$$\mu = r^2\dot{\theta}$$

is constant along any solution x . It is called angular momentum of x . We shall say that a solution $x : \mathbb{R} \rightarrow \mathbb{R}^2 \setminus \{0\}$ of (3) is *radially T -periodic* if the radial component $r(t)$ is T -periodic. In this case, the number $\omega = \frac{\theta(T) - \theta(0)}{T}$ can be interpreted as the average angular speed of x and will be called the *rotation number* of x and denoted by $\omega = \text{rot}x$. Then, a radially T -periodic solution x is T -periodic if and only if $\text{rot}x$ is an integer multiple of $2\pi/T$. If $\text{rot}x = (m/n)(2\pi/T)$ for some relatively prime integers $m \neq 0 \neq n$, then x will be subharmonic with minimal period nT . In other case, x is quasiperiodic with two natural frequencies.

Note that (3) and (4) are singular differential equations. During the last few decades, based on different methods in nonlinear analysis, the existence of periodic solutions for singular differential equations has been studied by many researchers. See [3, 7, 15, 16, 22, 29, 31, 32, 37, 39] and the references therein. Compared with many existence results, only a few works [2, 4, 30, 33, 35] focus on the further dynamics properties, such as ellipticity and stability. Our main objective is to find conditions for the existence of solutions of (1) which are Lyapunov-stable in the radial direction. The precise definition of radial stability is as follows. We say that a solution $x(t) = r(t)e^{i\theta(t)}$ is *radially stable in the Lyapunov sense* if for every $\epsilon > 0$, there exists $\delta > 0$ such that, given any other solution $\tilde{x}(t) = \tilde{r}(t)e^{i\tilde{\theta}(t)}$, the condition $|x(0) - \tilde{x}(0)| + |x'(0) - \tilde{x}'(0)| < \delta$ implies that $|r(t) - \tilde{r}(t)| < \epsilon$ for every $t > 0$.

Throughout the paper, it is assumed that h is a continuous, T -periodic and positive function and denote

$$m = \min_{[0,T]} h(t), \quad M = \max_{[0,T]} h(t), \quad \Delta = \frac{M}{m}.$$

This quantity Δ can be regarded as a measure of the variation of the gravitational parameter and will play a key role in our main results. The mean value of h is denoted by

$$\bar{h} = \frac{1}{T} \int_0^T h(t) dt.$$

Furthermore, we always assume that $0 < \alpha < 3$. For simplicity, we introduce a new parameter

$$\gamma = \frac{1}{3 - \alpha}.$$

Obviously, $\gamma \in (1/3, +\infty)$.

Our aim is to prove the following main result.

Theorem 1.1. *There exists a constant $\mu_0 > 0$ such that for any $\mu > \mu_0$, system (3) has a radially T -periodic orbit with angular momentum μ which is radially stable in the Lyapunov sense. Moreover, there exists a constant Δ_0 such that if $\Delta < \Delta_0$, then we have the following explicit bound*

$$\mu_0^{2\gamma(\alpha+1)} \leq \left(\frac{T}{\pi}\right)^2 (3M^{4\gamma} - \alpha m^{4\gamma}).$$

Theorem 1.1 establishes the existence of stable radially T -periodic orbits for large angular momentum, and under an assumption over Δ , an explicit bound is given on how large must be the angular momentum. We apply Theorem 1.1 to the Gylden-Meshcherskii problem (1) and we can easily get the following result.

Corollary 1. *There exists $\mu_0 > 0$ such that for any $\mu > \mu_0$, the Gylden-Meshcherskii problem (1) has a radially T -periodic orbit with angular momentum μ which is radially stable in the Lyapunov sense. Moreover, if $\Delta < \Delta'_0 \simeq 1.00828$, then the following explicit bound holds*

$$\mu_0^6 \leq \left(\frac{T}{\pi}\right)^2 (3M^4 - 2m^4). \quad (5)$$

As a practical example of Corollary 1, let us consider $h(t) = \delta(1 + \varepsilon \cos \omega t)$. Then the condition $\Delta < \Delta'_0$ is equivalent to

$$\varepsilon < \frac{\Delta'_0 - 1}{\Delta'_0 + 1} \simeq 0.00412293,$$

and orbitally stable radially T -periodic orbits exist with angular momentum

$$\mu > \frac{4\delta^4}{\omega^2} (3(1 + \varepsilon)^4 - 2(1 - \varepsilon)^4) = \frac{4\delta^4}{\omega^2} (1 + 20\varepsilon + 6\varepsilon^2 + 20\varepsilon^3 + \varepsilon^4).$$

In particular, we can modulate the frequency in order to obtain orbitally stable T -periodic orbits with arbitrary small angular momentum. The family of radially T -periodic orbits that we found will be in general quasiperiodic orbits of two frequencies and include a sequence of nT -periodic orbits (subharmonics) with n tending to $+\infty$.

To prove Theorem 1.1, we will use a combination of ideas coming from different papers. The existence of radially T -periodic orbits for large angular momentum

was proved in [9]. In Section 2, we use a different method based on the theory of upper and lower solutions, that has the advantage of giving explicit bounds on the solutions. Also in [9], it is observed that such orbits approach to circular ones when the angular momentum goes to $+\infty$. We obtain a more precise information on how this limit behaves by means of averaging theory, very much inspired by the arguments exposed in [19]. As a second step, in Section 3 we analyze the Lyapunov stability of the solutions of equation (4). Related scalar equations with singularities like Lazer-Solimini or Brillouin equation have been profusely studied (see for instance [2, 4, 30, 33, 34, 35]) in the recent years. Once Section 3 is completed, Section 4 presents a brief proof of Theorem 1.1 and Corollary 1.

As a final remark, let us observe that the condition $0 < \alpha < 3$ is natural in some sense. In fact, if $\alpha > 3$ the upper and lower solutions constructed in Section 2 are in the right order, leading to an unstable solution (see [5]). In the limiting case $\alpha = 3$, equation (4) is a singular equation with indefinite weight, with rather different properties. For instance, no periodic solutions exist for large values of the angular momentum μ .

2. Existence and estimation of radially T -periodic solutions. The existence of T -periodic solutions for large angular momentum μ was proved in [9, Corollary 1] for a more general equation. The first result of this section provides a more precise quantitative information by using a different method of proof based on the technique of upper and lower solutions in the reversed order (see for instance [6, 34]).

Proposition 1. *Assume that*

$$\mu^{2\gamma(\alpha+1)} > \left(\frac{T}{\pi}\right)^2 (3M^{4\gamma} - \alpha m^{4\gamma}). \quad (6)$$

Then equation (4) has at least one T -periodic solution such that

$$\left(\frac{\mu^2}{M}\right)^\gamma < r(t) < \left(\frac{\mu^2}{m}\right)^\gamma. \quad (7)$$

Proof. We apply the method of upper and lower functions in the reversed order (see for instance [6, 34]). Note that $\beta_1(t) = (\frac{\mu^2}{m})^\gamma$ is a constant strict lower function and $\beta_2(t) = (\frac{\mu^2}{M})^\gamma$ is a constant strict upper function in the reserved order $\beta_1 > \beta_2$. In general, if a second order differential equation $\ddot{r} = f(t, r)$ has a couple of upper and lower functions $\beta_2 < \beta_1$ on the reversed order, a sufficient condition for the existence of T -periodic solution is

$$f_r(t, r) \geq -\left(\frac{\pi}{T}\right)^2 \quad \text{for all } t, \beta_2(t) < r < \beta_1(t). \quad (8)$$

On the case of equation (4) is

$$f_r(t, r) = -\frac{3\mu^2}{r^4} + \frac{\alpha h(t)}{r^{\alpha+1}} \geq -\frac{3\mu^2}{\beta_2^4} + \frac{\alpha h(t)}{\beta_1^{\alpha+1}} \geq -\left(\frac{\pi}{T}\right)^2.$$

After some algebra, it is easy to verify that this latter inequality is equivalent to (7). \square

In [9], it is also proved that the periodic solutions of (3) approach to a constant as the angular momentum tends to infinity. In the original system (3), the corresponding orbits approach to circular ones. Inspired by [19], we are going to obtain a more precise information by using the averaging theory.

In the following, we will write $r(t; \mu)$ to denote the periodic solution obtained in Proposition 1.

Proposition 2. *Let $r(t; \mu)$ be the solution found in Proposition 1. Then we obtain the following result*

$$\lim_{\mu \rightarrow +\infty} \frac{r(t; \mu)}{\mu^{2\gamma}} = \frac{1}{(\bar{h})^\gamma}, \quad \text{uniformly in } t. \quad (9)$$

Proof. The proof is a basic application of the averaging theory. For a general introduction to averaging theory one can consult for instance the books [23, 36], or the arguments exposed in [19].

The first step is to write equation (4) as a perturbative system. To this aim, we rename the variables and fix a small parameter as

$$u(t) = \mu^{-2\gamma} r(t), \quad y = \mu^{\gamma(\alpha-1)} \dot{r}(t), \quad \varepsilon = \mu^{-\gamma(1+\alpha)}.$$

In the new variables, equation (4) is equivalent to the system

$$\begin{aligned} \dot{u} &= \varepsilon y, \\ \dot{y} &= \varepsilon \left(\frac{1}{u^3} - \frac{h(t)}{u^\alpha} \right). \end{aligned} \quad (10)$$

The averaged system corresponding to system (10) is just

$$\begin{aligned} \dot{\xi} &= \varepsilon \nu, \\ \dot{\nu} &= \varepsilon \left(\frac{1}{\xi^3} - \frac{\bar{h}}{\xi^\alpha} \right). \end{aligned} \quad (11)$$

It is a matter of simple computations to verify that the averaged system (11) has a unique constant solution

$$(\xi_0, \nu_0) = (\bar{h}^{-\gamma}, 0),$$

which is non-degenerate, that is, the determinant of the Jacobian matrix evaluated on (ξ_0, ν_0) is different from zero. Then, the equilibrium (ξ_0, ν_0) is continuable for small ε , that is, there exists ε_0 such that system (10) has a T -periodic solution $(u(t, \varepsilon), y(t, \varepsilon))$ for $0 < \varepsilon < \varepsilon_0$, tending uniformly to (ξ_0, ν_0) as $\varepsilon \rightarrow 0^+$. Going back to the original variables, one gets (9). \square

3. Stability. In this section, we use the information collected in the previous section to find sufficient conditions for the twist character (and hence the stability in the sense of Lyapunov) of the solutions given by Proposition 1.

We first summarize some basic facts about the method of the third approximation and the twist coefficient. Consider the scalar equation

$$u'' + f(t, u) = 0, \quad (12)$$

where $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is T -periodic in t and of class $C^{0,4}$ in (t, u) . Let $\psi(t)$ be a T -periodic solution of (12). By translating the periodic solution $\psi(t)$ of (12) to the origin, we obtain the third order approximation

$$u'' + a(t)u + b(t)u^2 + c(t)u^3 + o(u^3) = 0, \quad (13)$$

where

$$a(t) = f_u(t, \psi(t)), \quad b(t) = \frac{1}{2} f_{uu}(t, \psi(t)), \quad c(t) = \frac{1}{6} f_{uuu}(t, \psi(t)).$$

The linearized equation of (13) is the Hill equation

$$u'' + a(t)u = 0. \quad (14)$$

We say (14) is elliptic if its multipliers λ_1, λ_2 satisfy $\lambda_1 = \bar{\lambda}_2$, $|\lambda_1| = 1$, $\lambda_1 \neq \pm 1$. The T -periodic solution ψ of (12) is called 4-elementary if the multipliers λ of (14) satisfy $\lambda^q \neq 1$ for $1 \leq q \leq 4$. The rotation number θ is defined by the relation $\lambda = \exp(\pm i\theta)$. The T -periodic solution $\psi(t)$ is said to be of twist type if the first twist coefficient

$$\beta = \iint_{[0, T]^2} b(t)b(s)R^3(t)R^3(s)\chi_\theta(|\varphi(t) - \varphi(s)|)dtds - \frac{3}{8} \int_0^T c(t)R^4(t)dt \quad (15)$$

is non-zero, where $\Psi(t) = R(t)(\exp)(i\varphi(t))$ is the complex solution of (14) with initial conditions $\Psi(0) = 1, \Psi'(0) = i$ and the kernel χ is given by

$$\chi_\theta(\sigma) = \frac{3 \cos(\sigma - \theta/2)}{16 \sin(\theta/2)} + \frac{\cos 3(\sigma - \theta/2)}{16 \sin(3\theta/2)}, \quad \sigma \in [0, \theta].$$

This formulation is a compact form, obtained in [38] (see also [17, 18]), of the original Ortega's formula [20]. As a consequence of Moser's invariant curve theorem [28], a solution of twist type is Lyapunov stable. Moreover, the Moser twist theorem asserts also a complicated dynamics near the periodic solution such as the existence of infinitely many subharmonics with minimal periods tending to infinity and the existence of infinitely many quasi-periodic solutions.

Based on the above theory, the following result for (13) has been established in [35]. Although [35, Theorem 3.1] deals with the equation with the period 2π , the following condition (iii) remains unchanged for any period T if we check the details of its proof.

Lemma 3.1. [35, Theorem 3.1] *Assume that there exists a T -periodic solution ψ of (12) such that*

- (i) $0 < a_* \leq a^* < (\frac{\pi}{2T})^2$,
- (ii) $c_* > 0$,
- (iii) $10b_*^2 a_*^{3/2} > 9c^*(a^*)^{5/2}$,

where the constants are given as

$$a_* = \inf_{t \in [0, T]} a(t), \quad b_* = \inf_{t \in [0, T]} |b(t)|, \quad c_* = \inf_{t \in [0, T]} c(t), \quad a^* = \sup_{t \in [0, T]} a(t), \quad c^* = \sup_{t \in [0, T]} c(t).$$

Then the solution $\psi(t)$ of (12) is of twist type.

Now we use the above result to obtain our first stability result for equation (4).

Theorem 3.2. *Assume that*

$$\mu^{2\gamma(\alpha+1)} > \left(\frac{2T}{\pi}\right)^2 (3M^{4\gamma} - \alpha m^{4\gamma}). \quad (16)$$

Then there exists a constant $\Delta_0 > 0$ such that the T -periodic solution $r(t)$ of equation (4) obtained in Proposition 1 is of twist type if $\Delta < \Delta_0$.

Proof. We will apply Lemma 3.1. For simplicity, we use $r(t)$ to denote the periodic solution $r(t; \mu)$. Let us fix

$$f(t, r) = \frac{h(t)}{r^\alpha} - \frac{\mu^2}{r^3}.$$

Then the coefficients of the third-order approximation are

$$a(t) = a(t; \mu) = -\frac{\alpha h(t)}{r^{\alpha+1}} + \frac{3\mu^2}{r^4}, \quad (17)$$

$$b(t) = b(t; \mu) = \frac{\alpha(\alpha+1)h(t)}{2r^{\alpha+2}} - \frac{6\mu^2}{r^5}, \quad (18)$$

and

$$c(t) = c(t; \mu) = -\frac{\alpha(\alpha+1)(\alpha+2)h(t)}{6r^{\alpha+3}} + \frac{10\mu^2}{r^6}. \quad (19)$$

Using the estimates (7), we have

$$\begin{aligned} \frac{3m^{4\gamma} - \alpha M^{4\gamma}}{\mu^{2\gamma(\alpha+1)}} < a(t) < \frac{3M^{4\gamma} - \alpha m^{4\gamma}}{\mu^{2\gamma(\alpha+1)}}, \\ \frac{\alpha(\alpha+1)m^{5\gamma} - 12M^{5\gamma}}{2\mu^{2\gamma(\alpha+2)}} < b(t) < \frac{\alpha(\alpha+1)M^{5\gamma} - 12m^{5\gamma}}{2\mu^{2\gamma(\alpha+2)}}, \end{aligned} \quad (20)$$

and

$$\frac{-\alpha(\alpha+1)(\alpha+2)M^{6\gamma} + 60m^{6\gamma}}{6\mu^{2\gamma(3+\alpha)}} < c(t) < \frac{-\alpha(\alpha+1)(\alpha+2)m^{6\gamma} + 60M^{6\gamma}}{6\mu^{2\gamma(3+\alpha)}}.$$

First, note that if

$$\Delta < \left(\frac{3}{\alpha}\right)^{1/4\gamma} =: \Delta_1,$$

then

$$a_* > \frac{3m^{4\gamma} - \alpha M^{4\gamma}}{\mu^{2\gamma(\alpha+1)}} = \frac{3 - \alpha\Delta^{4\gamma}}{m^{4\gamma}\mu^{2\gamma(\alpha+1)}} > 0.$$

Using this estimate together with (16) in (20), it is easy to verify that (i) of Theorem 3.1 holds. Moreover, (ii) of Theorem 3.1 is satisfied if

$$\left(\frac{m}{M}\right)^{6\gamma} > \frac{\alpha(\alpha+1)(\alpha+2)}{60},$$

which is equivalent to

$$\Delta^{6\gamma} < \frac{60}{\alpha(\alpha+1)(\alpha+2)},$$

and it holds if

$$\Delta < \left(\frac{60}{\alpha(\alpha+1)(\alpha+2)}\right)^{1/6\gamma} =: \Delta_2.$$

On the other hand, if

$$\Delta < \left(\frac{12}{\alpha(\alpha+1)}\right)^{1/5\gamma} =: \Delta_3,$$

we have

$$\frac{\alpha(\alpha+1)M^{5\gamma} - 12m^{5\gamma}}{2\mu^{2\gamma(\alpha+2)}} = \frac{\alpha(\alpha+1)\Delta^{5\gamma} - 12}{2m^{5\gamma}\mu^{2\gamma(\alpha+2)}} < \frac{\alpha(\alpha+1)\Delta_3^{5\gamma} - 12}{2m^{5\gamma}\mu^{2\gamma(\alpha+2)}} < 0,$$

which means that $b(t) < 0$ for all t , and in consequence,

$$b_* > \frac{12m^{5\gamma} - \alpha(\alpha+1)M^{5\gamma}}{2\mu^{2\gamma(\alpha+2)}} > 0.$$

Therefore, (iii) of Theorem 3.1 holds if we have the following inequality

$$\begin{aligned} 10 \left[\frac{12m^{5\gamma} - \alpha(\alpha+1)M^{5\gamma}}{2\mu^{2\gamma(\alpha+2)}} \right]^2 \left[\frac{3m^{4\gamma} - \alpha M^{4\gamma}}{\mu^{2\gamma(\alpha+1)}} \right]^{3/2} > \\ 9 \left[\frac{-\alpha(\alpha+1)(\alpha+2)m^{6\gamma} + 60M^{6\gamma}}{6\mu^{2\gamma(3+\alpha)}} \right] \left[\frac{3M^{4\gamma} - \alpha m^{4\gamma}}{\mu^{2\gamma(\alpha+1)}} \right]^{5/2}, \end{aligned}$$

which is equivalent to the inequality

$$g_1(\Delta) > g_2(\Delta), \quad (21)$$

where

$$g_1(\Delta) = 5 [12 - \alpha(\alpha + 1)\Delta^{5\gamma}]^2 (3 - \alpha\Delta^{4\gamma})^{3/2},$$

and

$$g_2(\Delta) = 3 [60\Delta^{6\gamma} - \alpha(\alpha + 1)(\alpha + 2)] (3\Delta^{4\gamma} - \alpha)^{5/2}.$$

Observe that $g_1(0) > g_2(0)$, therefore by continuity there exists Δ_4 such that (21) holds whenever $\Delta < \Delta_4$. Let

$$\Delta_0 = \min\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}.$$

Note that Δ_0 depends only on the exponent α . Therefore we have verified all the conditions of Theorem 3.1 if $\Delta < \Delta_0$. \square

Remark 1. Consider the Gylden-Meshcherskii problem (1). Then $\alpha = 2$ and $\gamma = 1$. In this case, it is easy to see that $\Delta_1 = \sqrt[4]{3/2}$ and

$$\Delta_1 < \Delta_2, \quad \Delta_1 < \Delta_3.$$

The constant Δ_0 can be chosen as

$$\Delta_0 = \min\{\Delta_1, \Delta_4\},$$

where Δ_4 is the unique positive solution of the equation

$$10(6 - 3\Delta^5)^2(3 - 2\Delta^4)^{3/2} = 9(10\Delta^6 - 4)(3\Delta^4 - 2)^{5/2}.$$

A numerical computation gives $\Delta_4 = 1.00828$, which is the value consigned in Corollary 1.

Admittedly, this first stability result imposes a very conservative bound for Δ , but the advantage of the result is that we get an explicit lower bound for the angular momentum μ . In this sense, Theorem 3.2 is of global nature. On the other hand, if we look for a perturbative scenario and let the angular momentum to be ‘‘large enough’’, we can find a different result by exploiting the asymptotic behavior observed in Proposition 2.

Theorem 3.3. *The T -periodic solution $r(t)$ obtained in Proposition 2.1 is of twist type if μ is large enough.*

Proof. We will analyze the formula (15) and prove that β is positive. The coefficients of the third-order approximation are given as (17)-(19). By inserting the limit (9) into (17)-(19), we have

$$\lim_{\mu \rightarrow +\infty} \mu^{2\gamma(\alpha+1)} a(t) = \bar{h}^{\gamma(\alpha+1)} [3\bar{h} - \alpha h(t)], \quad (22)$$

$$\lim_{\mu \rightarrow +\infty} \mu^{2\gamma(\alpha+2)} b(t) = \bar{h}^{\gamma(\alpha+2)} \left[\frac{\alpha(\alpha+1)}{2} h(t) - 6\bar{h} \right], \quad (23)$$

and

$$\lim_{\mu \rightarrow +\infty} \mu^{2\gamma(\alpha+3)} c(t) = \bar{h}^{\gamma(\alpha+3)} \left[10\bar{h} - \frac{\alpha(\alpha+1)(\alpha+2)}{6} h(t) \right]. \quad (24)$$

Moreover, by straightforward computations, we obtain

$$\begin{aligned} \lim_{\mu \rightarrow +\infty} \mu^{2\gamma(\alpha+1)} \bar{a} &= (3 - \alpha)\bar{h}^{4\gamma}, \\ \lim_{\mu \rightarrow +\infty} \mu^{\gamma(\alpha+1)} \theta &= T\sqrt{3 - \alpha}\bar{h}^{2\gamma}, \end{aligned} \quad (25)$$

and

$$\lim_{\mu \rightarrow +\infty} \frac{R(t)}{\mu^{\gamma(\alpha+1)/2}} = \frac{1}{\sqrt[4]{3-\alpha} \bar{h}^\gamma}. \quad (26)$$

We refer to [4, Corollary 4.1] for the asymptotical behavior (25) and (26). Moreover, it follows from [17, Lemma 3.6] that (14) is elliptic and 4-elementary if μ is large enough.

Moreover, when μ is large enough, we have

$$\begin{aligned} \chi_\theta(|\varphi(t) - \varphi(s)|) &\geq \min_{u \in [0, \theta]} \chi_\theta(u) \\ &= \chi_\theta(0) \\ &= \frac{5}{8 \sin(3\theta/2)} \frac{(1 + 4 \cos \theta) \cos(\theta/2)}{5} \\ &= \frac{5}{12\theta} (1 + O(\theta^2)) \\ &= \frac{5}{12} (T\sqrt{\bar{a}})^{-1} + O(\bar{a}), \end{aligned}$$

here we have used the fact $\theta = T\rho$ and the rotation number

$$\rho = \sqrt{\bar{a}} + O(\bar{a}), \quad \text{when } \bar{a} \rightarrow 0^+.$$

Let

$$\beta_1 = \iint_{[0, T]^2} b(t)b(s)R^3(t)R^3(s)\chi_\theta(|\varphi(t) - \varphi(s)|)dt ds,$$

and

$$\beta_2 = \int_0^T c(t)R^4(t)dt.$$

Using (22)-(24), (25), (26) and the above facts, we obtain

$$\begin{aligned} \lim_{\mu \rightarrow +\infty} \mu^{4\gamma} \beta_2 &= \lim_{\mu \rightarrow +\infty} \int_0^T \mu^{2\gamma(\alpha+3)} c(t) \cdot \left(\frac{R(t)}{\mu^{\gamma(\alpha+1)/2}} \right)^4 dt \\ &= \int_0^T \bar{h}^{-\gamma(\alpha+3)} \left[10\bar{h} - \frac{\alpha(\alpha+1)(\alpha+2)}{6} h(t) \right] \frac{1}{(3-\alpha)\bar{h}^{4\gamma}} dt \\ &= \frac{T}{3-\alpha} \bar{h}^{2\gamma} \left[10 - \frac{\alpha(\alpha+1)(\alpha+2)}{6} \right] \\ &= \frac{\alpha^2 + 6\alpha + 20}{6} T \bar{h}^{2\gamma}. \end{aligned}$$

Using the similar way, we obtain

$$\begin{aligned}
\lim_{\mu \rightarrow +\infty} \mu^{4\gamma} \beta_1 &\geq \lim_{\mu \rightarrow +\infty} \iint_{[0, T]^2} \mu^{4\gamma} b(t) b(s) R^3(t) R^3(s) \chi_\theta(0) dt ds \\
&= \iint_{[0, T]^2} \bar{h}^{2\gamma(\alpha+2)} \left[\frac{\alpha(\alpha+1)}{2} h(t) - 6\bar{h} \right] \left[\frac{\alpha(\alpha+1)}{2} h(s) - 6\bar{h} \right] \\
&\quad \frac{1}{(3-\alpha)^{3/2} \bar{h}^{6\gamma}} \frac{5}{12T} \frac{1}{\sqrt{3-\alpha} \bar{h}^{2\gamma}} dt ds \\
&= \frac{5\bar{h}^{2\gamma(\alpha-2)}}{12T(3-\alpha)^2} \iint_{[0, T]^2} \left[\frac{\alpha(\alpha+1)}{2} h(t) - 6\bar{h} \right] \left[\frac{\alpha(\alpha+1)}{2} h(s) - 6\bar{h} \right] dt ds \\
&= \frac{5\bar{h}^{2\gamma(\alpha-2)}}{12T(3-\alpha)^2} T^2 \bar{h}^2 \left(\frac{\alpha(\alpha+1)}{2} - 6 \right)^2 \\
&= \frac{5T(\alpha+4)^2}{48} \bar{h}^{2\gamma}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{\mu \rightarrow +\infty} \mu^{4\gamma} \beta &= \lim_{\mu \rightarrow +\infty} \mu^{4\gamma} [\beta_1 - \frac{3}{8}\beta_2] \\
&\geq T\bar{h}^{2\gamma} \left[\frac{5(\alpha+4)^2}{48} - \frac{\alpha^2 + 6\alpha + 20}{16} \right] \\
&= \frac{(\alpha+1)(\alpha+10)T}{24} \bar{h}^{2\gamma} > 0,
\end{aligned}$$

which means that the twist coefficient β is positive when μ is large enough. Now the proof is finished. \square

4. Proof of Theorem 1.1. Let $\mu > \mu_0$ and $r(t)$ be the solution of equation (4). It corresponds to a radially T -periodic orbit given by $x(t) = r(t; \mu)e^{i\theta(t)}$, where

$$\theta(t) = \int_0^t \frac{\mu}{r^2(s; \mu)} ds.$$

Note that the choice of the initial time is not restrictive because the rotational invariance of the system. Our objective is to prove that $x(t)$ is radially stable in the Lyapunov sense.

Let $\tilde{x}(t) = \tilde{r}(t)e^{i\tilde{\theta}(t)}$ be a new solution with angular momentum $\tilde{\mu}$. The uniparametric family $r(t; \mu)$ given by Proposition 1 is a continuous branch of solutions, so there exists $\delta_1 > 0$ such that

$$|\mu - \tilde{\mu}| < \delta_1 \implies |r(t, \mu) - r(t, \tilde{\mu})| < \frac{\epsilon}{2}, \quad \text{for all } t.$$

On the other hand, the twist character is robust under small variations of the parameters. In other words, if $\tilde{\mu}$ is close enough to μ , then $r(t; \tilde{\mu})$ is of twist type, hence Lyapunov stable. In consequence, there exists $\delta_2 > 0$ such that

$$|\tilde{r}(0) - r(0, \tilde{\mu})| < \delta_2 \implies |\tilde{r}(t) - r(t, \tilde{\mu})| < \frac{\epsilon}{2}, \quad \text{for all } t > 0.$$

Now, by continuity of the angular momentum of the solutions with respect to initial conditions, there exists $\delta > 0$ such that $|x(0) - \tilde{x}(0)| + |x'(0) - \tilde{x}'(0)| < \delta$ implies $|\mu - \tilde{\mu}| < \delta_1$ and $|\tilde{r}_1(0) - r(0, \tilde{\mu})| < \delta_2$. In consequence,

$$|r(t, \mu) - \tilde{r}(t)| < |r(t, \mu) - r(t, \tilde{\mu})| + |\tilde{r}(t) - r(t, \tilde{\mu})| < \epsilon,$$

and the proof is done.

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