Chaos in Predator Prey Systems With/Without Impulsive Effect

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Abstract

In this paper we prove analytically that the seasonal effect can cause chaos in predator prey systems. Our method of proof is based on some recent results on topological horseshoes. Some applications in systems with impulsive effect are given.

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1 Introduction

Chaos is one of the most important phenomena in dynamical systems. Intuitively, a system is chaotic if there exists an invariant set Λ semi conjugated to Bernoulli shift on two symbols with infinitely many period points, and sensitive dependence on the initial conditions (see [8], [10], [17], [31], [32]). In the literature, there are many tools to detect the presence of chaotic dynamics. Among them we can point out Lyapunov exponents, Mel'nikov method, Sil'nikov method, the Conley-Ważewski theory or some fixed point indices, just to mention a few approaches, see for instance [20], [21], [22],[31], [32]. However, it is not an easy task to apply these tools in some concrete examples.

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In this paper we give sufficient conditions for the presence of chaotic dynamics in the system

$$\begin{cases} x' = x(a(t) - b(t)x + c(t)y) \\ y' = y(d(t) - e(t)x - f(t)y) \end{cases} \tag{1.1}$$

where all coefficients are T-periodic and $b(t), c(t), e(t), f(t) > 0$. This system corresponds with the classical Lotka-Volterra model and describes the evolution of two species, one predator and one prey, sharing the same habitat. The dependence on the time in (1.1) is introduced in order to model the seasonal effect of the environment. This consideration is essential for the presence of chaos since in the autonomous case, it is easy to check that every solution converges to an equilibrium. Another key ingredient in our results is the type of interaction. To observe this fact, we recall that de Mottoni and Schiaffino in [30] proved that, in the competitive case (i.e. when the coefficients in (1.1) satisfies $c(t) < 0$ and $b(t), e(t), f(t) > 0$), every solution converges to solution with period T. This shows that there is no chaos in the competitive case even in the presence of seasonal effects.
Many ecological systems in the real life suffer abrupt changes caused by some human exploit activities such as planting, harvesting, etc. In this context, impulsive differential equations can be considered as a natural framework. In the recent years, impulsive effects have been introduced in many models of population dynamics, see [23], [24], [29], and several aspects such as permanence, extinction or global stability have been extensively studied, see [25], [27], [28], [29], [26]. However, to the best of the author's knowledge, there are few analytic criteria to detect chaos in this scenario. Motivated by this fact, in this paper we derive a mechanism to generate chaos in impulsive differential equations and as an application we prove the presence of chaotic dynamics in the system considered by Wang, Chen and Nieto in [16].
Our method of proof combines the notion of Stretching Along Paths (see [8]) with the notion of topological horseshoes (see [14]). With the term topological horseshoe we understand the adaptation of Smale's theory to a topological setting. This approach has the following advantages:

- we do not have to check any hyperbolicity condition,
- we do not work with asymptotic and small parameters,
- our results are robust under small perturbations.

The last property is very important from the biological point of view since all systems in Ecology are subject to small errors in modelling. Notice that this technique has been already employed in different contexts, see for instance [7],[6],[4],[3],[9]. However in most situations, the authors study hamiltonian systems.

The structure of the paper is as follows. In section 2 we recall the results about chaotic dynamics which we are going to use throughout the paper. In section 3 we prove the existence of chaos for the system (1.1) and compare our results with those in the literature. In section 4, we expose a mechanism to generate chaos from impulses.

2 Topological Tools

The aim of this section is to give the definitions and tools that we use in this paper. Firstly we fix what we understand by chaos or chaotic dynamics.

Definition 2.1. *Let $\psi : \mathcal{D}_\psi \longrightarrow \mathbb{R}^2$ be a continuous map and let $\mathcal{D} \subset \mathcal{D}_\psi$. We say that ψ induces* ***chaotic dynamics on two symbols*** *on the set $\mathcal{D}$ if there exist two disjoint compact sets*

$$\mathcal{K}_0, \mathcal{K}_1 \subset \mathcal{D},$$

such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}$, there exists a corresponding sequence $(\omega_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ such that

$$\omega_i \in \mathcal{K}_{s_i} \quad and \quad \omega_{i+1} = \psi(\omega_i) \qquad for \;\; all \;\; i \in \mathbb{Z} \tag{2.2}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i$) for some $k \geq 1$, there exists a k-periodic sequence $(\omega_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (2.2). To put the emphasis on the sets $\mathcal{K}_j$'s, we may also say that ϕ induces chaotic dynamics on two symbols on the set $\mathcal{D}$ relatively to $\mathcal{K}_0$ and $\mathcal{K}_1$.

In what follows, we will say that a continuous map ψ is **chaotic** if there exist $\mathcal{D}, \mathcal{K}_0$ and $\mathcal{K}_1$ as in the previous definition. In contrast with other definitions of chaos, we can say that if a map is chaotic according Definition 2.1, then it is also chaotic in the sense of Block-Coppel and also in the sense of "coin-tossing", (see [10]). The following result gives some important implications of our notion of chaos.

Theorem 2.1. *([8, Theorem 2.2]) Let ψ be a continuous map that induces chaotic dynamics on two symbols on a set $\mathcal{D}$ and is continuous on*

$$\mathcal{K} := \mathcal{K}_1 \cup \mathcal{K}_2 \subset \mathcal{D}$$

where $\mathcal{K}_0, \mathcal{K}_1$ and $\mathcal{D}$ are as in the definition 2.1. Defining the nonempty compact set

$$\mathcal{I}_\infty = \bigcap_{n=0}^{\infty} \psi^{-n}(\mathcal{K}), \tag{2.3}$$

then there exists a nonempty compact set

$$\mathcal{I} \subset \mathcal{I}_\infty \subset \mathcal{K},$$

on which the following are fulfilled:

i) *$\mathcal{I}$ is invariant for ψ, (i.e. $\psi(\mathcal{I}) = \mathcal{I}$).*

ii) *$\psi|_{\mathcal{I}}$ is semi-conjugate to the Bernoulli shift on two symbols, that is there exists a continuous map g of $\mathcal{I}$ onto $\Sigma_2^+ := \{0,1\}^{\mathbb{N}}$, endowed with the distance*

$$d(s', s'') := \sum_{i\in\mathbb{N}} \frac{\widetilde{d}(s'_i, s''_i)}{2^{i+1}}, \quad \text{for } s' = (s'_i)_{i\in\mathbb{N}},\; s'' = (s''_i)_{i\in\mathbb{N}} \in \Sigma_2^+$$

(where $\widetilde{d}(\cdot,\cdot)$ *is the discrete distance on* $\{0,1\}$: $\widetilde{d}(s'_i,s''_i)=0$ *for* $s'_i=s''_i$ *and* $\widetilde{d}(s'_i,s''_i)=1$ *for* $s'_i\neq s''_i$*), such that the diagram*

$$\begin{tikzcd} \Lambda \arrow[r, "\Phi"] \arrow[d, "g"'] & \Lambda \arrow[d, "g"] \\ \Sigma_2^+ \arrow[r, "\sigma"'] & \Sigma_2^+ \end{tikzcd}$$

commutes, where $\sigma:\Sigma_2^+\longrightarrow\Sigma_2^+$ *is the Bernoulli shift defined by* $\sigma((s_i)_i):=(s_{i+1})_i$ *for all* $i\in\mathbb{N}$.

iii) *The set* $\mathcal{P}$ *of the periodic points of* $\psi\mid_{\mathcal{I}_\infty}$ *is dense in* $\mathcal{I}$ *and the pre-image* $g^{-1}(s)\subset\mathcal{I}$ *of every* k*-periodic sequence* $s=(s_i)_{i\in\mathbb{N}}\in\sum_2^+$ *contains at least one* k*-periodic point.*

Furthermore, from property $ii)$ *it follows that:*

iv) $h_{top}(\psi)\geq h_{top}(\psi\mid_{\mathcal{I}})\geq h_{top}(\sigma)\geq\log(2)$, *where* h_{top} *is the topological entropy.*

v) *There exists a compact invariant set* $\Lambda\subset\mathcal{I}$ *such that* $\psi\mid_\Lambda$ *is semi-conjugate to the Bernoulli shift on two symbols, topologically transitive and has sensitive dependence on initial conditions.*

After that, the following step is to give criteria ensuring that a concrete map induces chaotic dynamics on two symbols. To this end we need the next concepts.

Definition 2.2. *Let* $\mathcal{R}=[a_1,b_1]\times[a_2,b_2]$ *be a rectangle in* $\mathbb{R}^2$. *We will say that the pair* $\widetilde{\mathcal{R}}=(\mathcal{R},\mathcal{R}^-)$ *is an* ***oriented rectangle*** *if* $\mathcal{R}^-=\mathcal{R}_l^-\cup\mathcal{R}_r^-$ *where* $\mathcal{R}_l^-$, $\mathcal{R}_r^-$ *are two disjoint compact arcs contained in the boundary of* $\mathcal{R}$.

Definition 2.3. *Suppose that* $\psi:\mathcal{D}_\psi\longrightarrow\mathbb{R}^2$ *is a continuous map defined on a set* $\mathcal{D}_\psi$ *and consider* $\widetilde{\mathcal{A}}:=(\mathcal{A},\mathcal{A}^-)$ *and* $\widetilde{\mathcal{B}}:=(\mathcal{B},\mathcal{B}^-)$ *two oriented rectangles of* $\mathbb{R}^2$ *and a compact set* $\mathcal{K}\subset\mathcal{A}\cap\mathcal{D}_\psi$. *We say that* $(\mathcal{K},\psi)$ ***stretches*** $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths and write*

$$(\mathcal{K},\psi):\widetilde{\mathcal{A}}\Leftrightarrow\!\!\!\rightarrow\widetilde{\mathcal{B}},$$

if the following conditions hold:

- ψ *is continuous on* $\mathcal{K}$.
- *For every path* $\gamma:[0,1]\longrightarrow\mathcal{A}$ *such that* $\gamma(0)\in\mathcal{A}_l^-$ *and* $\gamma(1)\in\mathcal{A}_r^-$ *there exists a subinterval* $[t',t'']\subset[0,1]$ *so that*

  $$\gamma(t)\in\mathcal{K},\ \ \psi(\gamma(t))\in\mathcal{B}$$

  for all $t\in[t',t'']$ *and moreover,* $\psi(\gamma(t'))$ *and* $\psi(\gamma(t''))$ *belong to different components of* $\mathcal{B}^-$.

The next theorem links the definition of stretching along paths with the notion of chaotic dynamics on two symbols.

Theorem 2.2. *([8, Theorem 2.3]) Let $\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ be an oriented rectangle of $\mathbb{R}^2$ and let $\mathcal{D} \subset \mathcal{R} \cap \mathcal{D}_\psi$, with $\mathcal{D}_\psi$ the domain of a continuous map $\psi : \mathcal{D}_\psi \longrightarrow \mathbb{R}^2$. If $\mathcal{K}_0, \mathcal{K}_1$ are two disjoint compact sets with $\mathcal{K}_0 \cup \mathcal{K}_1 \subset \mathcal{D}$ and*

$$(\mathcal{K}_i, \psi) : \widetilde{\mathcal{R}} \Leftrightarrow\!\!\!\!\to \widetilde{\mathcal{R}}, \qquad \textit{for all } i = 0, 1$$

then ψ induces chaotic dynamics on two symbols on $\mathcal{D}$ relatively to $\mathcal{K}_0$ and $\mathcal{K}_1$. It follows that the map ψ has the properties of the theorem 2.1.

Finally, in order to study chaotic behaviors of small perturbations of ψ, we use the following result.

Corollary 2.1. *Consider $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ with $\mathcal{R} = [a_1, b_1] \times [a_2, b_2]$ and $\mathcal{R}_l^- = \{a_1\} \times [a_2, b_2]$ $\mathcal{R}_r^- = \{b_1\} \times [a_2, b_2]$. Assume that $\mathcal{K}_0$ and $\mathcal{K}_1$ are compact sets contained in $\mathcal{R}$ and $\psi : \mathcal{R} \to \mathbb{R}^2$ is a continuous map. If $(\mathcal{K}_i, \psi) : \widetilde{\mathcal{R}} \Leftrightarrow\!\!\!\!\to \widetilde{\mathcal{R}}$, with*

$$\mathcal{K}_0\,,\ \mathcal{K}_1 \subset]a_1, b_1[\times [a_2, b_2] \quad \textit{and} \quad \psi(\mathcal{K}_0) \cap \mathcal{R},\ \psi(\mathcal{K}_1) \cap \mathcal{R} \subset \mathbb{R} \times]a_2, b_2[,$$

then there exists $\varepsilon > 0$ such that every continuous map $\phi : \mathcal{R} \to \phi(\mathcal{R})$ is chaotic provided that

$$||\psi(x) - \phi(x)|| \leq \varepsilon \quad \textit{for all } x \in \mathcal{R}$$

where $\| \cdot \|$ denotes the Euclidean norm.

3 Chaotic Dynamics in the system (1.1)

The purpose of this section is to prove the presence of chaotic dynamics in (1.1). Firstly we consider the T-periodic system

$$\begin{cases} x' = x(-a_1 + c_1 y) \\ y' = y(d_1 - e_1 x) \qquad \text{for } t \in [nT, nT + T_1[\end{cases} \tag{3.4}$$

$$\begin{cases} x' = x(-a_2 - b_2 x + c_2 y) \\ y' = y(d_2 - f_2 y) \qquad \text{for } t \in [nT + T_1, (n+1)T[\end{cases} \tag{3.5}$$

where all parameters are strictly positive and $0 < T_1 < T$. For convenience, we employ the notation (S) to denote the previous system and $T_2 := T - T_1$. From a mathematical point of view, the dynamics of (S) is described in the following way. If $(x(t, (p_1, p_2)), y(t, (p_1, p_2)))$ denotes the maximal solution with initial condition (p_1, p_2), we have that $(x(t, (p_1, p_2)), y(t, (p_1, p_2)))$ is solution of (3.4) for $t \in [0, T_1[$, it is solution of (3.5) for $t \in [T_1, T[$ and the same transition is repeated in a T-periodic manner. Therefore, using that the system (3.5) is autonomous, the Poincaré map associated with the system (S), that is

$$\Phi : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$$

$$\Phi(p_1,p_2)=(x(T,(p_1,p_2)),y(T,(p_1,p_2))),$$

satisfies that $\Phi=\Phi_2\circ\Phi_1$ with

$$\begin{cases}\Phi_i:\mathbb{R}^2_+\longrightarrow\mathbb{R}^2_+\\ \Phi_i(p_1,p_2)=(x_i(T_i,(p_1,p_2)),y_i(T_i,(p_1,p_2)))\end{cases}\tag{3.6}$$

where $(x_1(t,(p_1,p_2)),y_1(t,(p_1,p_2)))$ (resp. $(x_2(t,(p_1,p_2)),y_2(t,(p_1,p_2))))$ is the solution of the system (3.4) (resp. (3.5)) with initial condition $p=(p_1,p_2)$. In the previous definition we have employed the notation $\mathbb{R}^2_+=\{(x,y):x,y\geq 0\}$.

Proposition 3.1. *Consider the system (S) with all parameters fixed except d_1,T_1,T_2 and suppose that*

$$\frac{3d_2}{4f_2}<\frac{a_2}{c_2}<\frac{5d_2}{4f_2}.\tag{3.7}$$

Then there exist a constant T_2^* and two maps $d_1^*(\widetilde{T}_2)$, $T_1(\widetilde{d}_1,\widetilde{T}_2)$ such that if $0<T_2<T_2^*$, $d_1>d_1^*(T_2)$, $T_1>T_1^*(d_1,T_2)$; the Poincaré map associated to (S) with parameters T_2,d_1 and T_1 is chaotic.

Remark 3.1. *In the proof we give precise estimates of T_2^*, $d_1^*(T_2)$ and $T_1^*(d_1,T_2)$ depending on the coefficients of the system.*

Proof. We split the proof into three steps.

Step 1: Construction of the rectangle.
Consider the second equation of the second system, namely

$$Y'=Y(d_2-f_2Y)\tag{3.8}$$

and define

$$T_2^*:=\max\{T_2:\left|Y(t,\frac{d_2}{2f_2})-\frac{d_2}{f_2}\right|,\left|Y(t,\frac{3d_2}{2f_2})-\frac{d_2}{f_2}\right|\geq\frac{d_2}{4f_2}\text{ for all }t\in[0,T_2]\}$$

where $Y(t,Y_0)$ denotes the maximal solution of (3.8) with initial condition Y_0. Notice that the equation (3.8) can be easily integrated and so we can explicitly compute T_2^*. After that we fix three constants T_2,α,l such that $0<T_2<T_2^*$, $\alpha>l>0$ and

$$-a_2-b_2(\alpha+l)+\frac{5c_2d_2}{4f_2}>0,\tag{3.9}$$

$$-a_2-b_2(\alpha-l)+\frac{3c_2d_2}{4f_2}<0,\tag{3.10}$$

$$-b_2\alpha^2+\alpha(-a_2+\frac{5c_2d_2}{4f_2})+la_2+l^2b_2-\frac{5lc_2d_2}{4f_2}-\frac{2l}{T_2}>0,\tag{3.11}$$

$$-b_2\alpha^2+\alpha(2b_2l-a_2+\frac{3c_2d_2}{4f_2})+la_2-l^2b_2-\frac{3lc_2d_2}{4f_2}+\frac{2l}{T_2}<0.\tag{3.12}$$

To prove the existence of these constants, we observe that the previous inequalities hold for $l = 0$ and

$$0 < \alpha < \frac{-a_2 + \frac{5c_2 d_2}{4f_2}}{b_2},$$

(see (3.7)). At this point we define the rectangle

$$\mathcal{S} = [\alpha - l, \alpha + l] \times \left[\frac{d_2}{2f_2}, \frac{3d_2}{2f_2}\right].$$

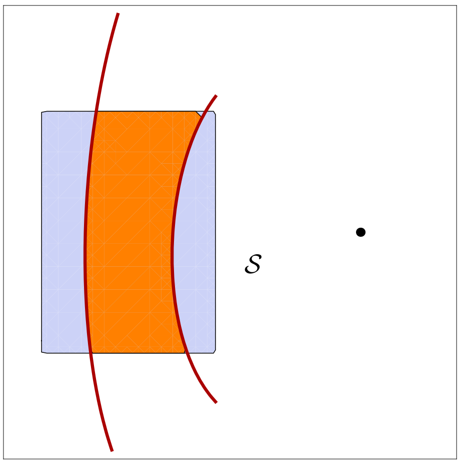
Finally we estimate d_1^*. Consider d_1 satisfying that $\frac{d_1}{e_1} > (\alpha + l)$. It is clear that if $(x_1(t), y_1(t))$ is solution of (3.4) then we can compute the derived of the first component with respect to the second component provided $(x_1(t), y_1(t)) \in \mathcal{S}$. In this case

$$\left|\frac{dx_1}{dy_1}\right| = \left|\frac{x_1(-a_1 + c_1 y_1)}{y_1(d_1 - e_1 x_1)}\right| \leq \frac{(\alpha + l)\max\{| -a_1 + \frac{3c_1 d_2}{2f_2} |, | -a_1 + \frac{c_1 d_2}{2f_2} |\}}{\frac{d_2}{2f_2}(d_1 - e_1(\alpha + l))} = K.$$

Thus, if

$$K\frac{d_2}{f_2} < \frac{l}{2}, \tag{3.13}$$

we deduce that the solutions of (3.4) with initial conditions at $(\alpha - \frac{l}{2}, \frac{d_2}{l_2})$ and $(\alpha + \frac{l}{2}, \frac{d_2}{l_2})$ leave the rectangle $\mathcal{S}$ across the faces $[\alpha - l, \alpha + l] \times \{\frac{d_2}{2f_2}\}$ and $[\alpha - l, \alpha + l] \times \{\frac{3d_2}{2f_2}\}$. We illustrate this behavior with the following figure.



A straightforward computation shows that if $d_1 > d_1^*$ with

$$\frac{4(\alpha + l)\max\{| -a_1 + \frac{3c_1 d_2}{2f_2} |, | -a_1 + \frac{c_1 d_2}{2f_2} |\}}{l} + e_1(\alpha + l) = d_1^*,$$

the condition (3.13) holds. Notice that d_1^* depends on the coefficients of the system (3.4) except d_1, T_1 and on the coefficients of the system (3.5), including T_2.
In the rest of the proof, we fix a constant d_1 such that $d_1 > d_1^*$.

Step 2: Stretching property for the map Φ_1.
In this step we prove the existence of a constant $T_1^* > 0$ such that if $T_1 > T_1^*$ then there exists two compact sets $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{S}$ satisfying that

$$(\mathcal{K}_i, \Phi_1) : \widetilde{\mathcal{S}_1} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{S}_2}, \qquad \text{for } i = 1, 2$$

where Φ_1 is the Poincaré map associated with (3.6) at the instant T_1, and the orientated rectangles are defined as $\widetilde{\mathcal{S}_1} = (\mathcal{S}, \mathcal{S}_1^-)$ with $(\mathcal{S}_1)_l^- = \{\alpha - l\} \times [\frac{d_2}{2f_2}, \frac{3d_2}{2f_2}]$, $(\mathcal{S}_1)_r^- = \{\alpha + l\} \times [\frac{d_2}{2f_2}, \frac{3d_2}{2f_2}]$ and $\widetilde{\mathcal{S}_2} = (\mathcal{S}, \mathcal{S}_2^-)$ with $(\mathcal{S}_2)_l^- = [\alpha - l, \alpha + l] \times \{\frac{d_2}{2f_2}\}$, $(\mathcal{S}_2)_r^- = [\alpha - l, \alpha + l] \times \{\frac{3d_2}{2f_2}\}$.
Before starting the proof of this claim, we introduce the concept of **rotation number** for the system (3.4). It is well known that given $q = (q_1, q_2) \in Int\mathbb{R}^2_+ \backslash \{(\frac{d_1}{e_1}, \frac{a_1}{c_1})\}$, the solution of (3.4) with this initial condition, namely $(x_1(t, q), y_1(t, q))$, is a periodic orbit determined by the energy function

$$\mathcal{E}(x_1, y_1) = c_1 y_1 + e_1 x_1 - a_1 \log y_1 - d_1 \log x_1.$$

Using this fact, we can define the rotation number associated to the system (3.4) in the following way

$$\operatorname{rot}(q, \tau) := \frac{1}{2\pi} \int_0^\tau \frac{\Big(y_1(t; q) - \frac{a_1}{b_1}\Big) X_1(t) - \Big(x_1(t, q) - \frac{d_1}{e_1}\Big) X_2(t)}{\Big(x_1(t, q) - \frac{d_1}{e_1}\Big)^2 + \Big(y_1(t, q) - \frac{a_1}{b_1}\Big)^2}\, dt,$$

where

$$X_1(t) := x_1(t; q)(-a_1 + c_1 y_1(t; q)), \quad X_2(t) := y_1(t, q)(d_1 - e_1 x_1(t, q))$$

The rotation number counts the number of winds around the equilibrium along the interval $[0, \tau]$ in the clockwise sense. Moreover, we can point out the following properties:

- $\operatorname{rot}(q, \tau)$ is a strictly increasing function of τ.
- $\operatorname{rot}(q, \tau) \lesseqgtr m \Longleftrightarrow \tau \lesseqgtr mP(q)$ where $P(q)$ is the minimal period of the solution $(x_1(t, q), y_1(t, q))$.

At this point we recall that by [1, Theorem 2], the minimal period of $(x_1(t, q), y_1(t, q))$ is a strictly increasing function with respect to the energy and so using that $d_1 > d_1^*$,

$$P_1 = P(\alpha - \frac{l}{2}, \frac{d_2}{f_2}) > P_2 = P(\alpha + \frac{l}{2}, \frac{d_2}{f_2}).$$

For convenience we introduce $\theta(\tau, q)$ as the angular coordinate at time τ of the solution of (3.4) departing from $q \in \mathcal{S}$ and such that for all $q \in \mathcal{S}$

$$\theta(\tau, q) = \theta(0, q) + 2\pi\mathrm{rot}(q, \tau) \in [2\pi\mathrm{rot}(q, \tau) - \pi/2, 2\pi\,\mathrm{rot}(q, \tau) + \pi/2].$$

Now we are ready to prove the aim of this step. Indeed, take

$$T_1 > \frac{5P_1P_2}{P_1 - P_2} = T_1^*.$$

By the choice of T_1, we can find two integers m_1, m_2 so that $m_1 \leq \frac{T_1}{P_1}$, $m_2 \geq \frac{T_1}{P_2}$ and $m_2 - m_1 \geq 3$. From these inequalities, we can check that

$$\mathrm{rot}(q, T_1) \leq m_1 \quad \text{for all } \ q \in \mathcal{S} \cap \mathcal{B}$$

$$\mathrm{rot}(q, T_1) \geq m_2 \quad \text{for all } \ q \in \mathcal{S} \cap \mathcal{A}$$

where $\mathcal{B} = \{(x, y) \in \mathcal{S} : \mathcal{E}(x, y) = \mathcal{E}(\alpha - l, \frac{d_2}{f_2})\}$ and $\mathcal{A} = \{(x, y) \in \mathcal{S} : \mathcal{E}(x, y) = \mathcal{E}(\alpha + l, \frac{d_2}{f_2})\})$. Therefore, for $k_1, k_2 \in [m_1 + 1, m_2 - 1]$,

$$[2k_i\pi - \frac{\pi}{2}, 2k_i\pi + \frac{\pi}{2}] \subset [\max_{q\in\mathcal{B}} \theta(T_1, q), \min_{q\in\mathcal{A}} \theta(T_1, q)]. \tag{3.14}$$

Finally, we define the sets

$$\mathcal{K}_1 := \{q \in \mathcal{S} : \theta(T_1, q) \in [2\pi k_1 - \frac{\pi}{2}, 2\pi k_1 + \frac{\pi}{2}], \Phi_1(q) \in \mathcal{S},$$
$$\mathcal{E}(q) \in [\mathcal{E}(\alpha + l, \frac{d_2}{f_2}), \mathcal{E}(\alpha - l, \frac{d_2}{f_2})]\}$$
$$\mathcal{K}_2 := \{q \in \mathcal{S} : \theta(T_1, q) \in [2\pi k_2 - \frac{\pi}{2}, 2\pi k_2 + \frac{\pi}{2}], \Phi_1(q) \in \mathcal{S},$$
$$\mathcal{E}(q) \in [\mathcal{E}(\alpha + l, \frac{d_2}{f_2}), \mathcal{E}(\alpha - l, \frac{d_2}{f_2})]\}.$$

Once these comments have been done, we prove the stretching property for Φ_1. Indeed, consider $\gamma : [0, 1] \longrightarrow \mathcal{S}$ a continuous path satisfying that $\gamma(0) \subset \{(x, y) \in \mathcal{S} : x = \alpha - l\}$ and $\gamma(1) \subset \{(x, y) \in \mathcal{S} : x = \alpha + l\}$. Firstly we take a subinterval $[\Gamma_1, \Gamma_2] \subset [0, 1]$ with $\gamma([\Gamma_1, \Gamma_2]) \subset \{(x, y) \in \mathcal{S} : \mathcal{E}(\alpha + \frac{l}{2}, \frac{d_2}{f_2}) \leq \mathcal{E}(x, y) \leq \mathcal{E}(\alpha - \frac{l}{2}, \frac{d_2}{f_2}\}$ and

$$\gamma(\Gamma_1) \in \{(x, y) \in \mathcal{S} : \mathcal{E}(x, y) = \mathcal{E}(\alpha - \frac{l}{2}, \frac{d_2}{f_2})\}$$

$$\gamma(\Gamma_2) \in \{(x, y) \in \mathcal{S} : \mathcal{E}(x, y) = \mathcal{E}(\alpha + \frac{l}{2}, \frac{d_2}{f_2})\}.$$

By continuity and using (3.14), we can find two subintervals $[\Gamma_0', \Gamma_1'], [\Gamma_0'', \Gamma_1'']$ satisfying that

$$-\frac{\pi}{2} + 2k_1\pi \leq \theta(T_1, \gamma(s)) \leq \frac{\pi}{2} + 2k_1\pi \qquad \text{for all } s \in [\Gamma_0', \Gamma_1']$$

$$-\frac{\pi}{2} + 2k_2\pi \leq \theta(T_1, \gamma(s)) \leq \frac{\pi}{2} + 2k_2\pi \quad \text{for all } s \in [\Gamma_0'', \Gamma_1''].$$

From these inequalities we obtain easily the desired subintervals $[\sigma_0', \sigma_1'] \subset [\Gamma_0', \Gamma_1']$ and $[\sigma_0'', \sigma_1''] \subset [\Gamma_0'', \Gamma_1'']$.

Step 3: Stretching property for the map Φ_2.
In this step we prove the following stretching property: given a continuous path $\gamma : [0,1] \longrightarrow \mathcal{S}$ with $\gamma(0) \subset \{(x,y) \in \mathcal{S} : y = \frac{d_2}{2f_2}\}$ and $\gamma(1) \subset \{(x,y) \in \mathcal{S} : y = \frac{3d_2}{2f_2}\}$, there exists a subinterval $[\zeta_1, \zeta_2] \subset [0,1]$ so that $\Phi_2(\gamma([\zeta_1,\zeta_2])) \subset \mathcal{S}$ with $\Phi_2(\gamma(\zeta_1)) \subset \{(x,y) \in \mathcal{S} : x = \alpha - l\}$ and $\Phi_2(\gamma(\zeta_2)) \subset \{(x,y) \in \mathcal{S} : x = \alpha + l\}$. Indeed, using the dynamics of the equation (3.8), $\Phi_2(\gamma([0,1])) \subset \{(x,y) : \frac{d_2}{2f_2} \leq y \leq \frac{3d_2}{f_2}\}$. At this moment, it is enough to prove that $\Phi_2(\gamma(1)) \subset \{(x,y) : x > \alpha + l\}$ and $\Phi_2(\gamma(0)) \subset \{(x,y) : x < \alpha - l\})$. To see the first claim, we recall that by the definition of T_2, $\Phi_2(\gamma(1)) \subset \{(x,y) : \frac{5d_2}{4f_2} \leq y \leq \frac{3d_2}{2f_2}\}$. On the other hand, by (3.9) we know that

$$x(-a_2 - b_2x + c_2y) > 0 \quad \text{for all } (x,y) \in [\alpha - l, \alpha + l] \times [\frac{5d_2}{4f_2}, \frac{3d_2}{2f_2}].$$

To conclude the proof, we notice that by (3.11),

$$(\alpha - l) + T_2 \min\{x(-a_2 - b_2x + c_2y) : (x,y) \in \mathcal{S} \cap \{\frac{5d_2}{4f_2} \leq y \leq \frac{3d_2}{2f_2}\}\} > \alpha + l.$$

Another claim is proved analogously.

Finally we apply the theorem 2.2 to $\mathcal{K}_1$, $\mathcal{K}_2$, $\widetilde{\mathcal{S}_1}$ and Φ. □

Next we give the main result of this section.

Theorem 3.1. *Fix all parameters in (S) verifying the conditions of proposition 3.1, i.e. (3.7) and $0 < T_2 < T_2^*$, $d_1 > d_1^*(T_2)$, $T_1 > T_1^*(d_1, T_2)$. Then there exists $\epsilon > 0$ such that if the distance in L_T^1 between the previous parameters in (S) and the coefficients of (1.1) is smaller than ϵ, the Poincaré map associated to (1.1) is chaotic.*

Given two T-periodic integrable functions $f(t)$ and $g(t)$, their distance in L_T^1 is given by $\int_0^T |f(t) - g(t)|\, dt$. In our setting, the assumptions of theorem 3.1 means that

$$\int_0^{T_1} |a(t) + a_1| dt + \int_{T_1}^{T} |a(t) + a_2| dt < \epsilon,$$

and so on (for the other coefficients).
Proof. Notice that by the construction, the conditions of the corollary 2.1 hold. □

In [2], Amine and Ortega give a system of the type (1.1) with three T-periodic solutions in the $Int\mathbb{R}_+^2$. Theorem 3.1 can be considered as a generalization of this fact since we give sufficient conditions ensuring the existence of infinitely many periodic solutions. In contrast with [9], we observe that in our construction, all the parameters can be large (in integral sense), however, in [9], the parameters $b(t), f(t)$ must be small (in sense of perturbation). In fact, the technique of Stretching Along Paths has been mainly applied in hamiltonian systems, (see [3], [4], [6], [7], [9]).

4 Systems with Impulsive Effect

Consider the system

$$\begin{cases} I' = I(-\alpha + \beta S) \\ S' = S(\gamma - \delta I) \qquad \text{for } t \neq nT \end{cases} \tag{4.15}$$

$$\Delta I = a + uS + qI \qquad \text{for } t = nT$$

where $\Delta I = I(T^+) - I(T)$, all parameters are positive and also assume that $0 \leq q \leq 1$. It is well known that the dynamics of (4.15) is completely determined by $\Phi = \Phi_2 \circ \Phi_1$ with Φ_1 the Poincaré map at the instant T associated to the system

$$\begin{cases} I' = I(-\alpha + \beta S) \\ S' = S(\gamma - \delta I) \end{cases} \tag{4.16}$$

(See (3.6)) and

$$\Phi_2(I, S) = (a + (1 - q)I + uS, S).$$

The next result provides us sufficient conditions for the presence of chaotic dynamics in the system (4.15).

Theorem 4.1. *Fix all parameters in (4.15) except T, u, a, q. Then, there exist $T^* > 0$ and three closed intervals with non-empty interior I_1, I_2, I_3 so that if $T > T^*$ and $(u, a, q) \in I_1 \times I_2 \times I_3$, the map Φ associated to (4.15) with parameters T, a, u, q is chaotic.*

Remark 4.1. *In the proof we give precise estimates of T^*, I_1, I_2, and I_3 depending on the coefficients of the system.*

Proof. We divide the proof into three steps.

Step 1: Construction of the rectangle.
Denote by $\mathcal{E}(I, S)$ the energy function of the system (4.16), namely

$$\mathcal{E}(I, S) = \delta I + \beta S - \alpha \log S - \gamma \log I.$$

By the dynamics of the system (4.16), we can take a constant $\widetilde{X}_0$ with $\widetilde{X}_0 < \frac{\gamma}{\delta}$ such that the curve μ_0 given by

$$\mu_0 = \{(I, S) : \mathcal{E}(I, S) = \mathcal{E}(\widetilde{X}_0, \frac{\alpha}{\beta})\}$$

satisfies that

$$\mu_0 \cap \{(I,S) : S = \frac{\alpha}{\beta}\} = \{(\widetilde{X}_0, \frac{\alpha}{\beta}), (X_0, \frac{\alpha}{\beta})\}$$

with $X_0 > \frac{\alpha}{\beta}$. Next, take $l > 0$ small enough satisfying the following properties:

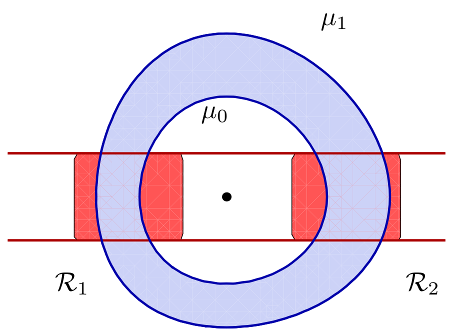
- $\widetilde{X}_0 + l < \frac{\gamma}{\delta} < X_0 - l$,
- for the rectangles

$$\mathcal{R}_1 = [\widetilde{X}_0 - l, \widetilde{X}_0 + l] \times [\frac{\alpha}{\beta} - l, \frac{\alpha}{\beta} + l],$$

$$\mathcal{R}_2 = [X_0 - l, X_0 + l] \times [\frac{\alpha}{\beta} - l, \frac{\alpha}{\beta} + l],$$

  the curve μ_0 leaves $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) across the faces $]\widetilde{X}_0 - l, \widetilde{X}_0 + l[\times\{\frac{\alpha}{\beta} - l\}$ and $]X_0 - l, X_0 + l[\times\{\frac{\alpha}{\beta} + l\}$ (resp. $]X_0 - l, X_0 + l[\times\{\frac{\alpha}{\beta} - l\}$ and $]X_0 - l, X_0 + l[\times\{\frac{\alpha}{\beta} + l\}$).

By continuity, we can find $\widetilde{X}_1$ so that $\widetilde{X}_0 - l < \widetilde{X}_1 < \widetilde{X}_0$ and the curve $\mu_1 = \{(I,S) : \mathcal{E}(I,S) = \mathcal{E}(\widetilde{X}_1, \frac{\beta}{b})\}$ leaves the rectangle $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) across the faces $]\widetilde{X}_0 - l, \widetilde{X}_0 + l[\times\{\frac{\alpha}{\beta} - l\}$ and $]\widetilde{X}_0 - l, \widetilde{X}_0 + l[\times\{\frac{\alpha}{\beta} + l\}$ (resp. $]X_0 - l, X_0 + l[\times\{\frac{\alpha}{\beta} - l\}$ and $]X_0 - l, X_0 + l[\times\{\frac{\alpha}{\beta} + l\}$). This construction is illustrated in the figure below.



Step 2: Stretching property for Φ_1.
Consider $\widetilde{\mathcal{R}}_2 = (\mathcal{R}_2, \mathcal{R}_2^-)$ with $\mathcal{R}_2^- = (\{X_0 - l\} \times [\frac{\alpha}{\beta} - l, \frac{\alpha}{\beta} + l]) \cup (\{X_0 + l\} \times [\frac{\alpha}{\beta} - l, \frac{\alpha}{\beta} + l])$ and $\widetilde{\mathcal{R}}_1 = (\mathcal{R}_1, \mathcal{R}_1^-)$ with $\mathcal{R}_1^- = ([\widetilde{X}_0 - l, \widetilde{X}_0 + l] \times \{\frac{\alpha}{\beta} - l\}) \cup ([\widetilde{X}_0 - l, \widetilde{X}_0 + l] \times \{\frac{\alpha}{\beta} + l\})$. In this step we prove the existence of a constant T_1^* such that if $T_1 > T_1^*$ then there exist two compact sets $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{R}_2$ so that

$$(\mathcal{K}_i, \Phi_1) : \widetilde{\mathcal{R}}_2 \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathcal{R}}_1$$

where Φ_1 is the Poincaré map associated with the system (4.16) at the instant T_1. Firstly, we define rot$(\cdot)$ the rotation number for the system (4.16) as in the previous section. Again, by [1, Theorem 2], we know that

$$P_1 = P(\widetilde{X}_1, \frac{\alpha}{\beta}) > P_2 = P(\widetilde{X}_0, \frac{\alpha}{\beta})$$

where $P(I_0, S_0)$ is the minimal period of the orbit of (4.16) with initial condition at (I_0, S_0). For convenience, we introduce $\widetilde{\theta}(\tau, q)$ as the angular coordinate at time τ of the solution of (4.16) departing from $q \in \mathcal{R}_2$ such that for all $q \in \mathcal{R}_2$

$$\widetilde{\theta}(\tau, q) = \widetilde{\theta}(0, q) + 2\pi\mathrm{rot}(q, \tau) \in [2\pi\mathrm{rot}(q, \tau) - \frac{\pi}{2}, 2\pi\mathrm{rot}(q, \tau) + \frac{\pi}{2}].$$

At this moment, we are ready to prove the aim of this step. Indeed, take

$$T_1 > \frac{5P_1P_2}{P_1 - P_2} = T_1^*.$$

By the choice of T_1, we can find two integers m_1, m_2 so that $m_1 \le \frac{T_1}{P_1}$, $m_2 \ge \frac{T_1}{P_2}$ and $m_2 - m_1 \ge 3$. For these inequalities

$$\mathrm{rot}(q, T_1) \le m_1 \quad \text{for all } q \in \mathcal{R}_2 \cap \mathcal{A}$$

$$\mathrm{rot}(q, T_1) \ge m_2 \quad \text{for all } q \in \mathcal{R}_2 \cap \mathcal{B}$$

where $\mathcal{B} = \{(I, S) \in \mathcal{R}_2 : \mathcal{E}(I, S) = \mathcal{E}(\widetilde{X}_0, \frac{\alpha}{\beta})\}$ and $\mathcal{A} = \{(I, S) \in \mathcal{R}_2 : \mathcal{E}(I, S) = \mathcal{E}(\widetilde{X}_1, \frac{\alpha}{\beta})\}$. Therefore, for $k_1, k_2 \in [m_1 + 1, m_2 - 1]$,

$$[2k_i\pi + \frac{\pi}{2}, 2k_i\pi + \frac{3\pi}{2}] \subset [\max_{q\in\mathcal{B}} \widetilde{\theta}(T_1, q), \min_{q\in\mathcal{A}} \widetilde{\theta}(T_1, q)]. \tag{4.17}$$

Finally, we define the sets

$$\mathcal{K}_1 = \{q \in \mathcal{R}_2 : \widetilde{\theta}(T_1, q) \in [2\pi k_1 + \frac{\pi}{2}, 2\pi k_1 + \frac{3\pi}{2}], \Phi_1(q) \in \mathcal{S}$$
$$\mathcal{E}(q) \in [\mathcal{E}(\widetilde{X}_0, \frac{\alpha}{\beta}), \mathcal{E}(\widetilde{X}_1, \frac{\alpha}{\beta})]\}$$
$$\mathcal{K}_2 = \{q \in \mathcal{R}_2 : \widetilde{\theta}(T_1, q) \in [2\pi k_2 + \frac{\pi}{2}, 2\pi k_2 + \frac{3\pi}{2}], \Phi_1(q) \in \mathcal{S}$$
$$\mathcal{E}(q) \in [\mathcal{E}(\widetilde{X}_0, \frac{\alpha}{\beta}), \mathcal{E}(\widetilde{X}_1, \frac{\alpha}{\beta})]\}.$$

Once these comments have been done, we prove the stretching property for Φ_1. Indeed, consider $\gamma : [0, 1] \longrightarrow \mathcal{R}_2$ a continuous path satisfying that $\gamma(0) \subset \{(I, S) \in \mathcal{R}_2 : I = X_0 - l\}$ and $\gamma(1) \subset \{(I, S) \in \mathcal{R}_2 : I = X_0 + l\}$. Firstly, we take a subinterval $[\Gamma_1, \Gamma_2] \subset [0, 1]$ with $\gamma([\Gamma_1, \Gamma_2]) = \{(I, S) \in \mathcal{R}_2 : \mathcal{E}(\widetilde{X}_0, \frac{\alpha}{\beta}) \le \mathcal{E}(I, S) \le \mathcal{E}(\widetilde{X}_1, \frac{\alpha}{\beta})\}$ and

$$\gamma(\Gamma_1) = \{(I, S) \in \mathcal{R}_2 : \mathcal{E}(\widetilde{X}_0, \frac{\alpha}{\beta}) = \mathcal{E}(I, S)\}$$

$$\gamma(\Gamma_2) = \{(I,S) \in \mathcal{R}_2 : \mathcal{E}(\widetilde{X}_1, \frac{\alpha}{\beta}) = \mathcal{E}(I,S)\}.$$

Now, by continuity and using (4.17), we can find two subintervals $[\Gamma_0', \Gamma_1']$, $[\Gamma_0'', \Gamma_1'']$ so that

$$\frac{\pi}{2} + 2k_1\pi \leq \theta(T_1, \gamma(s)) \leq \frac{3\pi}{2} + 2k_1\pi \qquad \text{for all } s \in [\Gamma_0', \Gamma_1']$$
$$\frac{\pi}{2} + 2k_2\pi \leq \theta(T_1, \gamma(s)) \leq \frac{3\pi}{2} + 2k_2\pi \qquad \text{for all } s \in [\Gamma_0'', \Gamma_1''].$$

From these inequalities we obtain easily the desired subintervals $[\sigma_0', \sigma_1'] \subset [\Gamma_0', \Gamma_1']$ and $[\sigma_0'', \sigma_1''] \subset [\Gamma_0'', \Gamma_1'']$.

Step 3: Stretching property for Φ_2.
Firstly we take three intervals I_1, I_2 and I_3 so that for all $(a,u,q) \in I_1 \times I_2 \times I_3$

$$a + u(\frac{\alpha}{\beta} - l) + (1-q)(\widetilde{X}_0 + l) < X_0 - l \tag{4.18}$$

$$a + u(\frac{\alpha}{\beta} + l) + (1-q)(\widetilde{X}_0 - l) > X_0 + l. \tag{4.19}$$

To prove the existence of these intervals we notice that the previous inequalities hold for $q = 1$, $a = 0$ and u satisfying

$$\frac{X_0 + l}{\frac{\alpha}{\beta} + l} < u < \frac{X_0 - l}{\frac{\alpha}{\beta} - l},$$

(this inequality makes sense by the condition $X_0 > \frac{\alpha}{\beta}$). At this moment, we can deduce that given a continuous path $\gamma : [0,1] \longrightarrow \mathcal{R}_1$ with $\gamma(0) \subset \{(I,S) \in \mathcal{R}_1 : S = \frac{\alpha}{\beta} - l\}$ and $\gamma(1) \subset \{(I,S) \in \mathcal{R}_1 : S = \frac{\alpha}{\beta} + l\}$, there exists a subinterval $[\zeta_1, \zeta_2] \subset [0,1]$ so that $\Phi_2(\gamma[\zeta_1, \zeta_2]) \subset \mathcal{R}_2$ with $\Phi_2(\gamma(\zeta_1)) \subset \{(I,S) \in \mathcal{R}_2 : I = X_0 - l\}$ and $\Phi_2(\gamma(\zeta_2)) \subset \{(I,S) \in \mathcal{R}_2 : I = X_0 + l\}$.

To conclude the proof we apply theorem 2.2 to $\widetilde{\mathcal{R}}_2$, $\mathcal{K}_1$, $\mathcal{K}_2$ and Φ. □

As a direct consequence of the previous theorem and corollary 2.1, we are able to prove the presence of chaotic dynamics in the system

$$\begin{cases} I' = I(-\alpha^\epsilon + \beta^\epsilon S - \omega^\epsilon I)) \\ S' = S(\gamma^\epsilon - \delta^\epsilon I) \qquad \text{for } t \neq nT \end{cases} \tag{4.20}$$
$$\begin{array}{l} \Delta I = a^\epsilon + u^\epsilon S - q^\epsilon I \\ \Delta S = -\kappa^\epsilon S \qquad\qquad \text{for } t = nT \end{array}$$

where all parameters are positive and $0 < q^\epsilon, \kappa^\epsilon \leq 1$. The next result shows this fact.

Corollary 4.1. *Fix all the parameters in (4.15) verifying the conditions of Theorem 3.1, i.e. $T > T^*$ and $(u,a,q) \in I_1 \times I_2 \times I_3$. Then there exists $\epsilon > 0$ such that if the distance between the previous parameters in (4.15) and the coefficients of (4.20) is smaller than ϵ, the map Φ^ϵ associated to (4.20) is chaotic.*

In [16], Wang, Chen, Nieto prove numerically that the system (4.20) can be chaotic. From the previous corollary we analytically confirm these evidences.

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