On a periodically forced Liénard differential equation with singular $\phi-$Laplacian

Manuel ZAMORA

Departamento de Matemática Aplicada
Universidad de Granada, 18071 Granada, Spain
`mzamora@ugr.es`

Abstract

Sufficient conditions are established in order to guarantee the existence of positive periodic solutions to

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + f(u)u' = \frac{m(t)}{u^\mu} - \frac{n(t)}{u^\lambda} + h(t)u^\delta,$$

where $f : (0, +\infty) \to \mathbb{R}$, $m, n : [0, T] \to \mathbb{R}_+$, $h : [0, T] \to \mathbb{R}$ are continuous functions and $\mu, \lambda, \delta \geq 0$.

1 Introduction

In the related literature a $\phi-$Laplacian operator is a increasing homeomorphism $\phi : (-a, a) \to (-b, b)$ with $\phi(0) = 0$ and $0 < a, b \leq +\infty$. Essentially there exists three type of $\phi-$Laplacian operators:

- The singular one: This is a $\phi-$Laplacian operator having bounded domain (that is $a < +\infty$). The paradigmatic model in this context is defined by

$$\phi(x) = \frac{x}{\sqrt{1-x^2}}, \qquad x \in (-1, 1).$$

- The regular one: It is a $\phi-$Laplacian operator having either unbounded domain and range. The classical model in this context is the p-Laplacian operator which is defined by

$$\phi_p(x) = |x|^{p-1}\operatorname{sgn}x, \qquad x \in \mathbb{R}, \quad p > 1.$$

- The bounded one: In this case the $\phi-$Laplacian operator has bounded range, and as a model one may consider

$$\phi_r(x) = \frac{x}{\sqrt{1+x^2}}, \qquad x \in \mathbb{R}.$$

There are a several number of references concerning to regular $\phi-$Laplacian operators [16, 24, 25, 26], most of them involving the case $p = 2$, e.g., $\phi_2 = Id$. This case is known as the classical case. For instance, the following type of equations

$$u'' = f(t, u, u'),$$

where $f : [0, T] \times D \to \mathbb{R}$ being D an open set of $\mathbb{R}^2$, are included in this case. Obviously we understand the reason why there are many papers concerning this case. However the number of papers decreases when one considers singular or bounded $\phi-$Laplacian operators.

In the present paper we will only consider the singular $\phi-$Laplacian operators. As some examples of interesting works in this framework we have [3, 4, 5, 6, 8, 9, 12, 17, 21, 22, 23]. One may observe that at the most of above references is only studied equations of type

$$(\phi(u'))' = f(t, u, u')$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, e.g., the nonlinearity of the differential equation has not singularities.

On the other hand, if one considers singularities at the nonlinearity we can cite [2, 23].

From applied point of view the singular $\phi-$Laplacian operator has relevance on the context of Special Relativity. More exactly when dealing with particles moving at speed close to that of light it may be important taking into account relativistic effects. In this line we cite, amount other ones, [1, 7, 11, 15, 18, 20, 23].

The objective of this paper is to continue studying the following family of periodic problems with singular nonlinearity

$$(\phi(u'))' + f(u)u' = \frac{m(t)}{u^\mu} - \frac{n(t)}{u^\lambda} + h(t)u^\delta, \qquad u(0) - u(T) = 0 = u'(0) - u'(T), \ (1)$$

where $f \in C\big((0, +\infty); \mathbb{R}\big)$ (it may have singularity at 0), $m, n \in C\big([0, T]; \mathbb{R}_+\big)$, $h \in C\big([0, T]; \mathbb{R}\big)$, μ, λ, δ are non-negative constants. More exactly we show a novel method of construction of lower and upper solutions using Theorem 2 in [3] and some recent results proved in [2] (see Theorem 1 and Theorem 2). In contrast with the results in [2] we can include the Liénard term $f(u)u'$, which is not possible consider it using similar arguments as there.

As a consequence of our main results we study the solvability of the following problem whose associated equation is known as Raleigh-Plesset equation (see [2, 13, 14])

$$(\phi(u'))' + f(u)u' = \frac{m}{u^\mu} - \frac{n}{u^\delta} + h(t)u^\delta, \qquad u(0) - u(T) = 0 = u'(0) - u'(T), \quad (2)$$

where $m, n > 0$, $\mu \geq 1$, $\mu > \delta > 0$ and h, f are defined as above, getting that if $\overline{h} := (1/T) \int_0^T h(s)ds < 0$ then (2) has at least one positive solution (see Theorem 3). Something similar was proven in [2], but there was necessary to assume that $h(t) \leq 0$ for $t \in [0, T]$.

The main tools employed explicitly or implicitly in this paper are lower and upper solutions and degree theory, in order to do a profound study on this techniques we refer to the reader to e.g., [10, 19].

The paper is organized as follows. In Section 2 we introduce some notation and auxiliary results (almost all taken from [3]). In Section 3 we show a new method for constructing lower and upper solutions of (1). Finally, in the last section, we offer some applications of our main results.

2 Some notations and auxiliary results

Let C denote the Banach space of continuous functions on $[0, T]$ endowed with the uniform norm $||\cdot||_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0, T]$ equipped with the norm

$$||u|| = ||u||_\infty + ||u'||_\infty \qquad (u \in C^1).$$

The following assumption upon ϕ is made throughout the paper:

(H_ϕ) $\phi :] - a, a[\rightarrow \mathbb{R}$ *is an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a < \infty$.*

If $u, v \in C$ are such that $u(t) \le v(t)$ for all $t \in [0, T]$, we write $u \le v$. Also, we write $u < v$ if $u(t) < v(t)$ for all $t \in [0, T]$.

Now it would be convenient to define the known concepts of lower and upper solutions to (1).

Definition 1 A *lower solution* α (resp. *upper solution* β) of (1) is a function $\alpha \in C^1([0, T]; \mathbb{R}_+)$ such that $||\alpha'||_\infty < a$, $\phi(\alpha') \in C^1$, $\alpha(0) = \alpha(T)$, $\alpha'(0) \ge \alpha'(T)$ (resp. $\beta \in C^1([0, T]; \mathbb{R}_+)$, $||\beta'||_\infty < a$, $\phi(\beta') \in C^1$, $\beta(0) = \beta(T)$, $\beta'(0) \le \beta'(T)$) and

$$(\phi(\alpha'))' + f(\alpha)\alpha' \ge \frac{m(t)}{\alpha^\mu} - \frac{n(t)}{\alpha^\lambda} + h(t)\alpha^\delta \tag{3}$$

$$\left(\text{resp.} \quad (\phi(\beta'))' + f(\beta)\beta' \le \frac{m(t)}{\beta^\mu} - \frac{n(t)}{\beta^\lambda} + h(t)\beta^\delta \right). \tag{4}$$

on whole the interval $[0, T]$. Such a lower or upper solution is called *strict* if (3) or (4) is a strict inequality.

At the first time we recall a criterion on solvability concerning to well-ordered lower and upper solutions proved in [3].

Lemma 1 *If (1) has a lower solution α and an upper solution β such that $\alpha \le \beta$, then (1) has a solution u such that $\alpha \le u \le \beta$. Moreover, if α and β are strict, then $\alpha < u < \beta$.*

An important fact throughout the paper is that the derivative of a solution u of (1) is uniformly bounded by a. This remark will be exploited in order to control the oscillation of solutions of (1). The next result is an elementary estimation of the oscillation of a periodic function.

Lemma 2 *If $u : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable and T-periodic function, then*

$$\max_{[0,T]} u - \min_{[0,T]} u \le \frac{T}{2} ||u'||_\infty.$$

Proof. Let $t_* \in [0, T)$ be such that $u(t_*) = \min_{[0,T]} u$ and $t^* \in [t_*, t_* + T]$ be such that $u(t^*) = \max_{[0,T]} u$. One has that

$$u(t^*) - u(t_*) = \int_{t_*}^{t^*} u'(s)ds \le ||u'||_\infty (t^* - t_*),$$

$$u(t^*) - u(t_*) = -\int_{t^*}^{t_* + T} u'(s)ds \le ||u'||_\infty (t_* + T - t^*).$$

3

Then, multiplying both inequalities and using that $xy \leq (x+y)^2/4$ for all $x, y \in \mathbb{R}$, it follows that

$$(u(t^*) - u(t_*))^2 \leq \frac{(\|u'\|_\infty T)^2}{4},$$

and the proof is completed. ∎

Now we will introduce a result proved as Theorem 1 in [2]. This result guarantees the solvability of (2) whenever it admits lower and upper solutions. Our main application will be supported on it.

Lemma 3 *Let us assume that there exists α and β lower and upper solutions to (2). Then there exists at least one solution u of (2) such that*

$$\min\{\alpha(t_u), \beta(t_u)\} \leq u(t_u) \leq \max\{\alpha(t_u), \beta(t_u)\}$$

for some $t_u \in [0, T]$.

3 Methods of construction of lower and upper solutions

Now we shall prove a general method to construct lower and upper solutions of (1).

At this moment will be convenient to introduce the following notation: for each $h \in L\big([0,T]; \mathbb{R}\big)$ we define the numbers

$$H = \int_0^T h(s)ds, \qquad H_+ = \int_0^T h^+(s)ds, \qquad H_- = \int_0^T h^-(s)ds$$

where for each $x \in \mathbb{R}$ its positive and negative part is denoted as $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

In order to prove the main Theorems we will need to introduce a continuous operator $\Pi : C^1\big([0,T]; \mathbb{R}\big) \to C^1\big([0,T]; \mathbb{R}\big)$. Let $x_1 > 0$, we define Π by

$$\Pi(u)(t) = x_1 + u(t) - \min_{[0,T]} u \qquad \text{for } t \in [0, T]. \tag{5}$$

Let us consider the auxiliar problem

$$(\phi(u'))' + f(\Pi(u))u' = q(t), \qquad u(0) = 0 = u(T). \tag{6}$$

The following lemma will allow us to establish a relationship between a periodic problem and a Dirichlet problem. The result was proved in [3] and it claims:

Lemma 4 *For each operator $F : C^1\big([0,T]; \mathbb{R}\big) \to C\big([0,T]; \mathbb{R}\big)$ continuous and takes bounded sets into bounded sets, the Dirichlet problem*

$$(\phi(u'))' = F(u), \qquad u(0) = 0 = u(T)$$

has at least one solution.

Remark 1 *For each $q \in C\big([0,T]; \mathbb{R}\big)$, defining the continuous operator $F(u)(t) = q(t) - f(\Pi(u))u'$ and using Lemma 4, it follows the solvability of (6).*

At this moment we are ready to prove our results in order to construct lower and upper solutions for (1).

Theorem 1 *Let us assume that there exist positive constants A_1, A_2, A_3, A_4 such that*

$$\max\left\{\left[\frac{1}{A_1}\right]^{\frac{1}{\mu}}, A_4^{\frac{1}{\delta}}\right\} + \frac{Ta}{2} \leq \min\left\{\left[\frac{1}{A_2}\right]^{\frac{1}{\lambda}}, A_3^{\frac{1}{\delta}}\right\}, \tag{7}$$

$$A_1 M - A_2 N + A_3 H_+ - A_4 H_- \leq 0 \tag{8}$$

are fulfilled. Then there exists $\alpha \in C^1([0,T]; \mathbb{R}_+)$ a lower solution of (1) verifying

$$\max\left\{A_4^{\frac{1}{\delta}}, \left[\frac{1}{A_1}\right]^{\frac{1}{\mu}}\right\} \leq \alpha(t) < \max\left\{A_4^{\frac{1}{\delta}}, \left[\frac{1}{A_1}\right]^{\frac{1}{\mu}}\right\} + \frac{aT}{2} \qquad \text{for } t \in [0,T]. \tag{9}$$

Proof. Let us define the operator Π as in (5) putting $x_1 = \max\left\{A_4^{\frac{1}{\delta}}, \left[\frac{1}{A_1}\right]^{\frac{1}{\mu}}\right\}$, and we consider, by Remark 1, $w \in C^1([0,T]; \mathbb{R})$ the solution to the Dirichlet problem (6) where

$$q(t) = m(t)A_1 - n(t)A_2 + h^+(t)A_3 - h^-(t)A_4 \qquad \text{for } t \in [0,T].$$

Now let us define $\alpha \in C^1([0,T]; \mathbb{R}_+)$ by $\alpha(t) = \Pi(w)(t)$ for $t \in [0,T]$. According to Lemma 2 it follows

$$\alpha(t) = x_1 + w(t) - \min_{[0,T]} w \leq x_1 + \max_{[0,T]} w - \min_{[0,T]} w < x_1 + \frac{aT}{2},$$

obtaining in this way (9). By virtue of (7), the above inequality implies $\alpha < \min\left\{\left[\frac{1}{A_2}\right]^{\frac{1}{\lambda}}, A_3^{\frac{1}{\delta}}\right\}$. Therefore, since $\alpha \geq x_1$, $x_1 \leq \alpha < \min\left\{\left[\frac{1}{A_2}\right]^{\frac{1}{\lambda}}, A_3^{\frac{1}{\delta}}\right\}$ holds.

On the other hand, according to the last inequality, one proves

$$m(t)A_1 - n(t)A_2 + h^+(t)A_3 - h^-(t)A_4 \geq \frac{m(t)}{\alpha^\mu} - \frac{n(t)}{\alpha^\lambda} + h^+(t)\alpha^\delta - h^-(t)\alpha^\delta.$$

In this way

$$q(t) \geq \frac{m(t)}{\alpha^\mu} - \frac{n(t)}{\alpha^\lambda} + h(t)\alpha^\delta. \tag{10}$$

Since $\alpha' = w'$, it verifies $(\phi(\alpha'))' + f(\alpha)\alpha' = q(t)$. Thus, from (10) it gets

$$(\phi(\alpha'))' + f(\alpha)\alpha' \geq \frac{m(t)}{\alpha^\mu} - \frac{n(t)}{\alpha^\lambda} + h(t)\alpha^\delta. \tag{11}$$

Finally, since $\alpha(0) = \alpha(T)$, from (11), in order to prove that α is a lower solution to (1) is sufficient proving that $\alpha'(0) \geq \alpha'(T)$, or equivalently that $\phi(\alpha'(0)) \geq \phi(\alpha'(T))$. Since $\phi(\alpha') = \phi(w')$, it implies

$$
\begin{aligned}
\phi(\alpha'(T)) - \phi(\alpha'(0)) &= \int_0^T (\phi(w'))' dt \\
&= -\int_0^T f(\Pi(w))w' dt + \int_0^T q(t) dt \\
&= A_1 M - A_2 N + A_3 H_+ - A_4 H_-,
\end{aligned}
$$

using (8) it obtains $\phi(\alpha'(0)) \geq \phi(\alpha'(T))$. ∎

Analogously one can prove a theorem in order to construct an upper solution for (1).

Theorem 2 *Let us assume that there exist positive constants B_1, B_2, B_3, B_4 such that*

$$\max\left\{\left[\frac{1}{B_2}\right]^{\frac{1}{\lambda}}, B_3^{\frac{1}{\delta}}\right\} + \frac{Ta}{2} \leq \min\left\{\left[\frac{1}{B_1}\right]^{\frac{1}{\mu}}, B_4^{\frac{1}{\delta}}\right\}, \tag{12}$$

$$B_1 M - B_2 N + B_3 H_+ - B_4 H_- \geq 0 \tag{13}$$

are fulfilled. Then there exists $\beta \in C^1\big([0,T];\mathbb{R}_+\big)$ an upper solution of (1) *verifying*

$$\max\left\{\left[\frac{1}{B_2}\right]^{\frac{1}{\lambda}}, B_3^{\frac{1}{\delta}}\right\} \leq \beta(t) < \max\left\{\left[\frac{1}{B_2}\right]^{\frac{1}{\lambda}}, B_3^{\frac{1}{\delta}}\right\} + \frac{aT}{2} \qquad for \ t \in [0,T]. \tag{14}$$

Remark 2 Following carefully the argument of Theorem 1 (resp. Theorem 2), one notes that if n is a strict positive function (resp. m is a strict positive function) the lower solution (resp. the upper solution) constructed above is also strict.

4 Applications

In order to study existence of solutions of (2) we shall introduce the continuous functions $\Psi_1, \Psi_2 : (0, +\infty) \to \mathbb{R}$ defined by

$$\Psi_1(x) = \frac{mT}{x^{\frac{\mu}{\delta}}} - \frac{nT}{\left(x^{\frac{1}{\delta}} + \frac{aT}{2}\right)^\delta} + \left(x^{\frac{1}{\delta}} + \frac{aT}{2}\right)^\delta H_+ - xH_-,$$

$$\Psi_2(x) = \frac{mT}{\left(x^{\frac{1}{\delta}} + \frac{aT}{2}\right)^\mu} - \frac{nT}{x} + xH_+ - \left(x^{\frac{1}{\delta}} + \frac{aT}{2}\right)^\delta H_-.$$

Now, we shall try to apply our Theorem 1 and 2, in order to study problem (2). For that reason, we consider that $\lambda = \delta$ and m, n are positive constants.

Theorem 3 *If $\overline{h} < 0$ then there exists at least one solution of* (2). *In particular, under this assumption, the periodic problem of*

$$\left(\frac{u'}{\sqrt{1 - u'^2}}\right)' + 4c\frac{u'}{u^{\frac{4}{5}}} = \frac{m}{u^\mu} - \frac{n}{u^\delta} + h(t)u^\delta$$

is solvable for $c \in \mathbb{R}$.

Proof. At the first time notice that, since all functions which involve to (2) are continuous, one may take β sufficiently small such that it is an strict upper solution for (2). On the other hand, since $\lim_{x \to +\infty} \Psi_1(x) < 0$, there exists $A_4 > 0$ sufficiently large such that $\Psi_1(A_4) \leq 0$. Next, let define the positive constants

$$A_1 = \left[\frac{1}{A_4}\right]^{\frac{\mu}{\delta}}, \qquad A_3 = \left(A_4^{\frac{1}{\delta}} + \frac{aT}{2}\right)^\delta, \qquad A_2 = \frac{1}{\left(A_4^{\frac{1}{\delta}} + \frac{aT}{2}\right)^\delta}. \tag{15}$$

One may check that (7) is fulfilled as an identity, and (8) is followed from the condition $\Psi_1(A_4) \leq 0$. Applying Theorem 1 and Remark 2 it follows the existence of α a strict lower solution such that (9) holds. Finally, by virtue of Lemma 3, the problem (2) has at least one solution. ∎

6

Now we will study the case when $\overline{h} > 0$. For that, notice that, since $\lim_{x \to 0+} \Psi_1(x) = \lim_{x \to +\infty} \Psi_1(x) = +\infty$, we may define $A_4 > 0$ such that $\Psi_1(A_4) = \min_{(0,+\infty)} \Psi_1$. Under this framework we have

Theorem 4 *If $\overline{h} > 0$ and $\Psi_1(A_4) \leq 0$, then (2) has at least two solutions. In particular, under this assumption, the periodic problem of*

$$\left(\frac{u'}{\sqrt{1 - u'^2}} \right)' + 4c \frac{u'}{u^{\frac{4}{5}}} = \frac{m}{u^\mu} - \frac{n}{u^\delta} + h(t)u^\delta$$

has at least two solutions.

Proof. Let us define the positive constant A_1, A_2 and A_3 by (15). In the same way as the previous theorem one may check that (7) is fulfilled and (8) is followed from $\Psi_1(A_4) \leq 0$. Thus, by Theorem 1 and Remark 2, there exists α a strict lower solution such that

$$A_4^{\frac{1}{\delta}} \leq \alpha < A_4^{\frac{1}{\delta}} + \frac{aT}{2} \qquad \text{for } t \in [0, T].$$

On the other hand, since $\lim_{x \to +\infty} \Psi_2(x) > 0$, there exists $B_3 > 0$ sufficiently large such that $\Psi_2(B_3) > 0$ and

$$B_3 \geq \left(A_4^{\frac{1}{\delta}} + \frac{aT}{2} \right)^\delta. \tag{16}$$

Let us define the positive constant

$$B_1 = \left[\frac{1}{B_3^{\frac{1}{\delta}} + \frac{aT}{2}} \right]^\mu, \qquad B_2 = \frac{1}{B_3}, \qquad B_4 = \left(B_3^{\frac{1}{\delta}} + \frac{aT}{2} \right)^\delta.$$

Analogously to the previous arguments it proves that (12) is fulfilled as an identity, and (13) is followed from $\Psi_2(B_3) > 0$. Thus, according to Theorem 2 and Remark 2, there exists a strict upper solution such that

$$B_3^{\frac{1}{\delta}} \leq \beta(t) < B_3^{\frac{1}{\delta}} + \frac{aT}{2} \qquad \text{for } t \in [0, T].$$

From (16) it follows that α and β are strict well-ordered lower and upper solutions. According to Lemma 1, the problem (2) has a solution which verifies $\alpha < u < \beta$.

On the other hand, if one takes $\beta_1 > 0$ a sufficiently small number in order to it would be a strict upper solution, since α and β_1 are reverse-ordered lower and upper solutions, according to Lemma 3, there exists v a solution of (2) such that $\beta_1 \leq v(t_v) \leq \alpha(t_v)$. Therefore, since $\alpha < u < \beta$, necessary, both solutions are different ones. ∎

Remark 3 Our main results do not cover the limit case $\overline{h} = 0$, thus it remains open.

Remark 4 One may use the same strategy in order to study the existence of $T-$periodic solutions to the following types of equations

$$(\phi(u'))' + f(u)u' + \frac{n(t)}{u^\lambda} = h(t),$$

where $f \in C((0, +\infty); \mathbb{R})$, $n, h \in C$ and $\lambda > 0$. The results are like in [2], but now we have achieved to include the Liénard term $f(u)u'$ in the equation.

References

[1] C.M. Andersen, H.C. Von Baeyer, On Classical Scalar Field Theories and the Relativistic Kepler Problem, **Annals of Physics** 62 (1971), 120-134.

[2] C. Bereanu, D. Gheorghe, M. Zamora, Periodic solutions for singular perturbations of the singular $\phi-$Laplacian operator, preprint.

[3] C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, **J. Differential Equations** 243 (2007) 536-557.

[4] C. Bereanu, J. Mawhin, Multiple periodic solutions of ordinary differential equations with bounded nonlinearites and $\phi-$Laplacian, **Nonlinear differ. equ. appl.** 15 (2008) 159-168.

[5] C. Bereanu, P. Jebelean, J. Mawhin, Multiple solutions for Neumann and periodic problems with singular ϕ-Laplacian, **J. Functional Analysis** 261 (2011) 3226-3246.

[6] C. Bereanu, P. Jebelean, J. Mawhin, Periodic solutions of pendulum-Like perturbations of singular and bounded $\phi-$Laplacian, **J. Dyn Diff Equat** 22 (2010) 463-471.

[7] T.H. Boyer, Unifamiliar trajectories for a relativistic particle in a Kepler or Coulomb potential, **Am J. Phys.** 72, Iss. 8 (2004), 992-997.

[8] H. Brezis, J. Mawhin, Periodic solutions of the forced relativistic pendulum, **Differential Integral Equations** 23 (2010) 801-810.

[9] J. Chu, J. Lei, M. Zhang, The stability of the equilibrium of a nonlinear planar system and application to the relativistic oscillator, **J. Differential Equations** 247 (2009) 530-542.

[10] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, 1985.

[11] R. Engelke, C. Chandler, Planetary Perihelion Precession with Velocity-Dependent Gravitational Mass, **Am. J. Phys.** 38, Iss. 1 (1970), 90-93.

[12] A. Fonda, R. Toader, Periodic solutions of pendulum-like Hamiltonian systems in the plane, preprint.

[13] R. Hakl, P.J. Torres, M. Zamora, Periodic solutions of singular second order differential equations: upper and lower functions, **Nonlin. Anal. TMA** 74 (2011) 7078-7093.

[14] R. Hakl, P.J. Torres, M. Zamora, Periodic solutions of singular second order differential equations: the repulsive case, **Topol. Methods Nonlinear Anal.** 39 (2012), 199–220.

[15] E.H. Hutten, Relativistic (non-linear) oscillator, **Nature** 205 (1965) 892.

[16] R. Manásevich, J. Mawhin, Periodic Solutions for Nonlinear Systems with $p-$Laplacian-Like Operators, **J. Differential Equations** 145 (1998), 367-393.

[17] R. Manásevich, J.R. Ward, On a result of Brezis and Mawhin, **Proc. Amer. Math. Soc.** 140 (2012) 531-539.

[18] G. Muñoz, I. Pavic, A Hamilton-like vector for the special-relativistic Coulomb problem, **Eur. J. Phys.** 27 (2006) 1007-1018.

[19] I. Rachunková, S. Staněk, M. Tvrdý, Solvability of nonlinear singular problems for ordinary differential equations, Hindawi, 2008.

[20] W.C. Saslaw, Motion around a source whose luminosity changes, **The Astrophysical Journal** 226 (1978), 240-252.

[21] P.J. Torres, Periodic oscillation of the relativistic pendulum with friction, **Phys. Lett.** A 372 (2008) 6386-6387.

[22] P.J. Torres, Nondegeneracy of the periodically forced Liénard differential equation with $\phi-$Laplacian, **Commun. Contemporary Math.** 13 (2011) 283-292.

[23] P. J. Torres, A. J. Ureña, M. Zamora,Periodic orbits of radially symmetric on the Relativistic machanic model. **Bulletin of the London Mathematical Society** (to appear)

[24] P. Yan, Nonresonance for one-dimensional p-Laplacian with regular restoring, **J. Math. Anal. Appl.** 285 (2003), 141-154.

[25] Y. Li, F. Zhang, Existence and uniqueness of periodic solutions for a kind of duffing type p-Laplacian equation, **Nonlinear Analysis: Real World Applications** 9 (2008), 985-989.

[26] M. Zhang, Nonuniform nonresonance at the first eigenvalue of p-Laplacian, **Nonlinear Anal.** 29 (1997), 41-51.