Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space*

Cristian BEREANU
Institute of Mathematics "Simion Stoilow", Romanian Academy
21, Calea Griviţei, RO-010702-Bucharest, Sector 1, România
cristian.bereanu@imar.ro

Petru JEBELEAN
Department of Mathematics, West University of Timişoara
4, Blvd. V. Pârvan RO-300223-Timişoara, România
jebelean@math.uvt.ro

Pedro J. TORRES
Departamento de Matemática Aplicada
Universidad de Granada, 18071 Granada, Spain
ptorres@ugr.es

Abstract

In this paper, by using Leray-Schauder degree arguments and critical point theory for convex, lower semicontinuous perturbations of C^1-functionals, we obtain existence of classical positive radial solutions for Dirichlet problems of type

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + f(|x|, v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R).$$

Here, $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$ and $f : [0, R] \times [0, \alpha) \to \mathbb{R}$ is a continuous function, which is positive on $(0, R] \times (0, \alpha)$.

MSC 2010 Classification : 35B09; 35J93; 47H11; 49J40

Key words : Dirichlet problem, positive radial solutions, mean curvature operator, Minkowski space, Leray-Schauder degree, Szulkin's critical point theory.

*The first author is partially supported by a GENIL grant YTR-2011-7 (Spain) and by the grant PN-II-RU-TE-2011-3-0157 (Romania). The second author is partially supported by the grant PN-II-RU-TE-2011-3-0157 (Romania). The third author is partially supported by Ministerio de Economía y Competitividad, Spain, project MTM2011-23652.

1 Introduction

The aim of this paper is to present some existence results for positive radial solutions of the Dirichlet problem with mean curvature operator in the flat Minkowski space

$$\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$$

endowed with the metric

$$\sum_{j=1}^{N}(dx_j)^2 - (dt)^2.$$

It is known (see [3, 26]) that the study of spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature leads to Dirichlet problems of the type

$$\mathcal{M}v = H(x,v) \quad \text{in} \quad \Omega, \quad v=0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where

$$\mathcal{M}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right),$$

Ω is a bounded domain in $\mathbb{R}^N$ and the nonlinearity $H : \Omega\times\mathbb{R} \to \mathbb{R}$ is continuous.

The starting point of this type of problems is the seminal paper [12] which deals with entire solutions of $\mathcal{M}v = 0$ (see also [1] for $N = 2$). The equation $\mathcal{M}v = \textit{constant}$ is then analyzed in [31], while $\mathcal{M}v = f(v)$ with a general nonlinearity f is considered in [8]. On the other hand, motivated by the study of stationary surfaces in Minkowski space, in [25] the author consider the Neumann problem

$$\mathcal{M}v = \kappa v + \lambda \quad \text{in} \quad \mathcal{B}(R), \quad \partial_\nu v = \mu, \quad \text{on} \quad \partial\mathcal{B}(R),$$

where $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $\lambda \neq 0, \kappa > 0, \mu \in [0,1)$ and $N = 2$. More general sign-changing nonlinearities are studied in [5].

If H is bounded, then it has been shown by Bartnik and Simon [3] that (1) has at least one solution $u \in C^1(\Omega) \cap W^{2,2}(\Omega)$. Also, when Ω is a ball or an annulus in $\mathbb{R}^N$ and the nonlinearity H has a radial structure (no boundedness assumptions), then it has been proved in [4] that (1) has at least one classical radial solution. This can be seen as an "universal" existence result for the above problem in the radial case. On the other hand, in this context the existence of positive solutions has been scarcely explored in the related literature. The importance of this type of study becomes apparent in many practical situations, for instance when the trivial solution is present.

In Section 2 we consider the Dirichlet problem

$$\mathcal{M}v + f(|x|, v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v=0 \quad \text{on} \quad \partial\mathcal{B}(R), \tag{2}$$

where $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$ and $f : [0, R] \times [0, \alpha) \to \mathbb{R}$ is a continuous function, which is positive on $(0, R] \times (0, \alpha)$. We prove that (2) has at least one classical positive radial solution provided that f is superlinear at 0 with respect to $\phi(s) = s/\sqrt{1 - s^2}$, that is

$$\lim_{s \to 0} \frac{f(r, s)}{s} = \infty \quad \text{uniformly for} \quad r \in [0, R] \tag{3}$$

and $R < \alpha$ (Corollary 1). In particular, we can consider nonlinearities of convex-concave type like those introduced in [2] (Example 2).

When $\alpha = R = 1$, (3) is satisfied and f is sublinear at 1 with respect to $\phi(s) = s/\sqrt{1 - s^2}$, that is

$$\lim_{s \to 1-} \sqrt{1 - s^2} f(r, s) = 0 \quad \text{uniformly for} \quad r \in [0, 1], \tag{4}$$

we prove that the same conclusion holds true (Corollary 1). Condition (3) has been considered by many authors, in connection with the existence of positive radial solutions for semilinear elliptic equations (see e.g. [21, 32]). Also, in the classical p-Laplacian case, for which $\phi_p(s) = |s|^{p-2}s$, usually, in order to prove the existence of positive solutions, condition (3) is considered together with the sublinear condition of f at infinity with respect to ϕ_p:

$$\lim_{s \to \infty} \frac{f(r, s)}{s^{p-1}} = 0 \quad \text{uniformly for} \quad r \in [0, R]. \tag{5}$$

In our situation, $\phi(s) = s/\sqrt{1 - s^2}$ and (5) is naturally replaced by (4). Note that, when $R < \alpha$ (f can be singular at α but α must be sufficiently large), condition (3) is sufficient to ensure the existence of a positive radial solution of (2).

In Section 3 we consider Dirichlet problems of the type

$$\mathcal{M}v + \mu(|x|)p(v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R), \tag{6}$$

where $\mu : [0, R] \to \mathbb{R}$ is continuous, positive on $(0, R]$ and $p : [0, \infty) \to \mathbb{R}$ is continuous with $p(0) = 0$ and $p(s) > 0$ for all $s > 0$. The form of the equation will allow us to consider, among others, nonlinearities of Hénon type [22] (Example 4), as well as of Brezis-Niremberg type [10] (Example 6). We prove (Corollary 3) that if

$$R^N < N \int_0^R r^{N-1} \mu(r) P(R - r) dr, \tag{7}$$

where P is the primitive of p with $P(0) = 0$, then problem (6) has at least one classical positive radial solution. This enables us to derive effective sufficient conditions for the existence of positive solutions for some problems of type (6), which appear as being in contrast with known results for similar Dirichlet problems involving the classical Laplacian.

Setting, as usual, $r = |x|$ and $v(x) = u(r)$, the Dirichlet problem (2) reduces to the mixed boundary value problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}f(r,u) = 0, \qquad u'(0) = 0 = u(R). \tag{8}$$

In Section 2, a fixed point operator is associated to problem (8) and the Leray-Schauder degree is applied to obtain the existence results. We adapt to our situation some ideas from the proof of the Krasnoselskii's fixed point theorem on compression-expansion of conical shells on a Banach space and from the papers [16, 18, 19, 20].

In Section 3, we first deal with a mixed boundary value problem, involving a more general nonlinearity than that in (6). To this aim, we attach an energy functional I defined on $C[0,R]$, which is the sum of a convex, lower semicontinuous function and of a C^1-function. So, I has the structure required by Szulkin's critical point theory [30]. We prove that I has always a minimizer on $C[0,R]$ for any continuous nonlinearity not necessarily positive, and, in our particular case, we provide a sufficient condition ensuring the non-triviality of the minimizers. We note that a variational approach has been also employed in [28] to prove various existence and multiplicity results concerning positive solutions of Dirichlet problems with mean curvature operators in the Euclidean space.

If Ω is an open bounded subset in a Banach space X and $T : \overline{\Omega} \to X$ is compact, with $0 \notin (I-T)(\partial\Omega)$, then $d_{LS}[I-T,\Omega,0]$ will denote the Leray-Schauder degree of T with respect to Ω and 0. For the definition and properties of the Leray-Schauder degree we refer the reader to e.g., [17].

2 A topological degree approach

Motivated by the model example from Introduction, in this Section we consider mixed boundary value problems of the type

$$(r^{N-1}\phi(u'))' + r^{N-1}f(r,u) = 0, \qquad u'(0) = 0 = u(R), \tag{9}$$

where $N \geq 1$ is an integer, $R > 0$ and the following main hypotheses hold true:

(H_ϕ) $\phi : (-a,a) \to \mathbb{R}$ $(0 < a < \infty)$ *is an odd, increasing homeomorphism with* $\phi(0) = 0$;

(H_f) $f : [0,R] \times [0,\alpha) \to [0,\infty)$ *is a continuous function with* $0 < \alpha \leq \infty$ *and such that* $f(r,s) > 0$ *for all* $(r,s) \in (0,R] \times (0,\alpha)$.

Below, the space $C := C[0,R]$ will be endowed with the usual supremum norm $||\cdot||$ and the corresponding open ball of center 0 and radius $\rho > 0$ will be denoted by B_ρ. Recall that by *a solution* of (9) we mean a function $u \in C^1[0,R]$ with $\|u'\| < a$, such that $\phi(u')$ is differentiable and (9) is satisfied.

We need the following elementary result.

Lemma 1 *Assume* (H_ϕ), (H_f) *and let* u *be a nontrivial solution of*

$$(r^{N-1}\phi(u'))' + r^{N-1}f(r,|u|) = 0, \qquad u'(0) = 0 = u(R). \tag{10}$$

Then $u > 0$ *on* $[0,R)$ *and* u *is strictly decreasing.*

Proof. From

$$r^{N-1}\phi(u') = -\int_0^r \tau^{N-1} f(\tau, |u(\tau)|)d\tau \tag{11}$$

it follows $u' \leq 0$ because $f(r,s) \geq 0$ for all $r \in [0,R]$ and $s \in [0,\alpha)$, so u is decreasing. Since $u(R) = 0$, we have $u \geq 0$ on $[0,R]$. As u is not identically zero, one has $u(0) > 0$ and, from (11) we deduce that $u' < 0$ on $(0,R]$, which ensures that actually u is strictly decreasing and $u > 0$ on $[0,R)$.

■

By virtue of Lemma 1, any nontrivial solution u of the mixed boundary value problem (10) is a strictly decreasing solution of (9).

Now, a fixed point operator is associated to problem (10) (see [4, 13] for slightly different operators). In this view, we shall use the compact linear operators

$$S : C \to C, \quad Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t)dt \quad (r \in (0,R]), \quad Su(0) = 0;$$

$$K : C \to C, \quad Ku(r) = \int_r^R u(t)dt, \quad (r \in [0,R]).$$

An easy computation shows that, for any $h \in C$, the mixed problem

$$(r^{N-1}\phi(u'))' + r^{N-1}h(r) = 0, \qquad u'(0) = 0 = u(R),$$

has an unique solution u given by

$$u = K \circ \phi^{-1} \circ S \circ h.$$

Moreover, from the compactness of S and K it follows that $K \circ \phi^{-1} \circ S : C \to C$ is compact. Consider now the Nemytskii type operator

$$N_f : B_\alpha \to C, \quad N_f(u) = f(\cdot, |u(\cdot)|).$$

Clearly, N_f is continuous and $N_f(\overline{B}_\rho)$ is a bounded subset of C for any $\rho < \alpha$. So, we have the following fixed point reformulation of the problem (10).

Lemma 2 *A function* $u \in C$ *is a solution of (10) if and only if it is a fixed point of the continuous nonlinear operator*

$$\mathcal{N} : B_\alpha \to C, \qquad \mathcal{N} = K \circ \phi^{-1} \circ S \circ N_f.$$

Moreover, $\mathcal{N}$ *is compact on* $\overline{B}_\rho$ *for all* $\rho \in (0,\alpha)$.

In the next Proposition we assume that f is superlinear with respect to ϕ at 0 and we prove that the Leray-Schauder degree $d_{LS}[I-\mathcal{N}, B_\rho, 0]$ is zero for all sufficiently small ρ.

Proposition 1 *Assume that*

$$\lim_{s\to 0+} \frac{f(r,s)}{\phi(s)} = +\infty \quad \textit{uniformly with} \quad r \in [0,R] \tag{12}$$

and

$$\limsup_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} < +\infty \quad \textit{for all} \quad \tau > 0. \tag{13}$$

Then there exists $0 < \rho_0 < \alpha$ *such that*

$$d_{LS}[I-\mathcal{N}, B_\rho, 0] = 0 \quad \textit{for all} \quad 0 < \rho \le \rho_0.$$

Proof. First, we show that there exists $\rho_0 \in (0,\alpha)$ such that the perturbed problem

$$(r^{N-1}\phi(u'))' + r^{N-1}[f(r,|u|) + M] = 0, \qquad u'(0) = 0 = u(R). \tag{14}$$

has at most the trivial solution in $\overline{B}_{\rho_0}$, for any $M \ge 0$. By contradiction, assume that there exist sequences $\{M_k\} \subset [0,\infty)$ and $\{u_k\} \subset C \setminus \{0\}$ with $||u_k|| \to 0$, such that u_k is a solution of (14) with $M = M_k$, for all $k \in \mathbb{N}$. By virtue of Lemma 1 one has that $u_k > 0$ on $[0,R)$ and u_k is strictly decreasing.

From (13) with $\tau = 3/R$, there is some $m > 0$ so that

$$\frac{m(R/3)^N}{N(2R/3)^{N-1}} > \limsup_{s\to 0} \frac{\phi(3s/R)}{\phi(s)} \tag{15}$$

and using (12), we can find $k_0 \in \mathbb{N}$ such that

$$f(r, u_k(r)) \ge m\phi(u_k(r)) \quad \text{for all} \quad r \in [0,R] \text{ and } k \ge k_0. \tag{16}$$

Then, integrating (14) with $M = M_k$ and $u = u_k$ over $[0,r]$ and taking into account (16), we get

$$-\phi(u_k') \ge mS[\phi(u_k)].$$

Hence, by the oddness of ϕ, we infer

$$-u_k' \ge \phi^{-1}(mS[\phi(u_k)]).$$

Integrating the above inequality on $[R/3, 2R/3]$ one obtains

$$u_k(R/3) - u_k(2R/3) \ge \int_{R/3}^{2R/3} \phi^{-1}\left(\frac{m}{t^{N-1}}\int_0^t \tau^{N-1}\phi(u_k(\tau))d\tau\right) dt.$$

Then, using that u_k is strictly decreasing on $[0,R]$ and $u_k > 0$ on $[0,R)$, it follows

$$
\begin{aligned}
u_k(R/3) &\geq \int_{R/3}^{2R/3} \phi^{-1}\left(\frac{m}{(2R/3)^{N-1}}\int_0^{R/3}\tau^{N-1}\phi(u_k(\tau))d\tau\right)dt \\
&\geq \frac{R}{3}\phi^{-1}\left(\frac{m(R/3)^N\phi(u_k(R/3))}{N(2R/3)^{N-1}}\right),
\end{aligned}
$$

for all $k \geq k_0$. This implies

$$\frac{\phi(u_k(R/3)3/R)}{\phi(u_k(R/3))} \geq \frac{m(R/3)^N}{N(2R/3)^{N-1}},$$

which together with $u_k(R/3) \to 0$ contradict (15).

Note that (14) has no solution in $\overline{B}_{\rho_0}$, for any $M > 0$.

Now, let $0 < \rho \leq \rho_0$ and consider the family of problems

$$(r^{N-1}\phi(u'))' + r^{N-1}[f(r,|u|) + \lambda] = 0, \qquad u'(0) = 0 = u(R), \tag{17}$$

where $\lambda \in [0,1]$. Let $\mathcal{H}(\lambda,\cdot) : B_\alpha \to C$ be the fixed point operator associated to (17) (see Lemma 2). Note that $\mathcal{H}(0,\cdot) = \mathcal{N}$ and $\mathcal{H} : [0,1]\times\overline{B}_\rho \to C$ is a compact homotopy. Also the Leray-Schauder condition on the boundary

$$u \neq \mathcal{H}(\lambda,u) \quad \text{for all} \quad (\lambda,u) \in [0,1]\times\partial B_\rho$$

is fulfilled. This implies that

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0].$$

But, from the previous arguments one has that

$$u \neq \mathcal{H}(1,u) \quad \text{for all} \quad u \in \overline{B}_\rho,$$

implying that

$$d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] = 0.$$

Consequently,

$$
\begin{aligned}
d_{LS}[I - \mathcal{N}, B_\rho, 0] &= d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] \\
&= d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] \\
&= 0
\end{aligned}
$$

which ends the proof. ■

In the following result we use that, in contrast with the classical case, the domain of ϕ is bounded.

Proposition 2 *If $aR < \alpha$, then one has*

$$d_{LS}[I - \mathcal{N}, B_{aR}, 0] = 1.$$

Proof. Consider the compact homotopy

$$\mathcal{H} : [0,1] \times \overline{B}_{aR} \to C, \quad \mathcal{H}(\lambda, u) = \lambda \mathcal{N}(u).$$

Note that $\mathcal{H}(0,\cdot) = 0$ and $\mathcal{H}(1,\cdot) = \mathcal{N}$. Let $(\lambda, u) \in [0,1] \times \overline{B}_{aR}$ be such that $\mathcal{H}(\lambda, u) = u$. It follows immediately that $||u'|| < a$, implying that $||u|| < aR$. So,

$$u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0,1] \times \partial B_{aR},$$

which implies that

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_{aR}, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_{aR}, 0].$$

Consequently,

$$d_{LS}[I - \mathcal{N}, B_{aR}, 0] = d_{LS}[I, B_{aR}, 0] = 1$$

and the proof is complete. ■

If $\alpha = a$, then in Proposition 2 one has that $R < 1$. We consider now the case $R = 1$, assuming that f is sublinear with respect to ϕ at a.

Proposition 3 *Assume that $a = \alpha$ and $R = 1$. If*

$$\lim_{s \to a-} \frac{f(r,s)}{\phi(s)} = 0 \quad \textit{uniformly with} \quad r \in [0,1], \tag{18}$$

then there exists $0 < \delta_1 < a$ such that

$$d_{LS}[I - \mathcal{N}, B_\delta, 0] = 1 \quad \textit{for all} \quad \delta_1 \leq \delta < a.$$

Proof. Consider the family of problems

$$(r^{N-1}\phi(u'))' + r^{N-1}\lambda f(r, |u|) = 0, \qquad u'(0) = 0 = u(R),$$

where $\lambda \in [0,1]$. Let $\mathcal{H}(\lambda, \cdot) : B_a \to C$ be the fixed point operator associated to (17) (see Lemma 2). Note that $\mathcal{H}(1,\cdot) = \mathcal{N}$ and $\mathcal{H} : [0,1] \times \overline{B}_\rho \to C$ is a compact homotopy for all $0 < \rho < a$. We show that there exists $0 < \delta_1 < a$ such that

$$u \neq \mathcal{H}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in [0,1] \times (B_a \setminus B_{\delta_1}). \tag{19}$$

By contradiction, assume that there exist sequences $\{\lambda_k\} \subset [0,1]$ and $\{u_k\} \subset C \setminus \{0\}$ such that $a > ||u_k|| \to a$ and $u_k = \mathcal{H}(\lambda_k, u_k)$. Clearly, $\lambda_k > 0$ for all $k \in \mathbb{N}$ and, from Lemma 1 one has that $u_k > 0$ on $[0,1)$ and u_k is strictly decreasing on $[0,1]$. Let $\{\varepsilon_k\} \subset (0,\infty)$ be such that $\varepsilon_k \to 0$. From (18) there exists $\{c_k\} \subset (0,\infty)$ such that

$$f(r,s) \leq \varepsilon_k \phi(s) + c_k \quad \text{for all} \quad (r,s) \in [0,1] \times [0,a). \tag{20}$$

As $||u_k|| \to a$, we can find a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ satisfying

$$\varepsilon_j \phi(||u_{k_j}||) \geq c_j \quad \text{for all} \quad j \in \mathbb{N}. \tag{21}$$

Using the monotonicity of ϕ^{-1} and $u_k = \mathcal{H}(\lambda_k, u_k)$ we obtain

$$||u_k|| \leq \int_0^1 \phi^{-1}\left(\frac{1}{r^{N-1}} \int_0^r t^{N-1} f(t, u_k(t))dt\right) dr.$$

This together with (20) and (21) imply

$$||u_{k_j}|| \leq \phi^{-1}(2\varepsilon_{k_j} \phi(||u_{k_j}||)/N),$$

and then

$$1 \leq 2\varepsilon_{k_j}/N \quad \text{for all} \quad j \in \mathbb{N}.$$

This contradicts $\varepsilon_{k_j} \to 0$, as $j \to \infty$. So, (19) holds true. It follows that, for any $\delta_1 \leq \delta < a$, one has

$$d_{LS}[I - \mathcal{H}(0, \cdot), B_\delta, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), B_\delta, 0].$$

Consequently,

$$d_{LS}[I - \mathcal{N}, B_\delta, 0] = d_{LS}[I, B_\delta, 0] = 1$$

and the proof is complete. ■

Theorem 1 *Assume that (H_ϕ), (H_f), (12) and (13) are fulfilled. Then problem (9) has at least one positive solution if either $aR < \alpha$ or $\alpha = a$, $R = 1$ and (18) holds true.*

Proof. Assume that $aR < \alpha$ and let ρ_0 be given in Proposition 1. We pick $\rho \in (0, \min\{\rho_0, aR\})$. From Propositions 1, 2 it follows that

$$d_{LS}[I - \mathcal{N}, B_{aR} \setminus \overline{B}_\rho, 0] = d_{LS}[I - \mathcal{N}, B_{aR}, 0] - d_{LS}[I - \mathcal{N}, B_\rho, 0] = 1,$$

which ensures the existence of some $u \in B_{aR} \setminus \overline{B}_\rho$, with $u = \mathcal{N}(u)$. Consequently, u is a solution of (9) and $u > 0$ on $[0, R)$.

If $\alpha = a$, $R = 1$ and (18) is satisfied, then the proof follows exactly as above but with Proposition 3 instead of Proposition 2. ■

Remark 1 In Theorem 1, if $\alpha < \infty$ and the function f is continuous on $[0, R] \times [0, \alpha]$, then condition $aR < \alpha$ can be replaced by $aR \leq \alpha$. This follows from the fact that, in this case,

$$d_{LS}[I - \mathcal{N}, B_\alpha, 0] = 1$$

(see Proposition 2).

Corollary 1 *Assume that (H_f) and condition (3) are fulfilled. Then problem (2) has at least one classical positive radial solution if either $R < \alpha$ or $\alpha = 1 = R$ and (4) holds true.*

Proof. It suffices to take in Theorem 1 the homeomorphism $\phi(s) = s/\sqrt{1-s^2}$ $(-1 < s < 1)$. Note that in this case $a = 1$. Then, conditions (H_ϕ) and (13) are satisfied. Actually, in this case one has that

$$\lim_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} = \tau \quad \text{for all} \quad \tau > 0.$$

$\blacksquare$

Example 1 Let the constants $0 \le q < 1$, $\alpha > 0$, $\gamma > 0$ and $\mu : [0, R] \to (0, \infty)$, $h : [0, R] \times [0, \infty) \to [0, \infty)$ be continuous functions.

(i) The Dirichlet problem

$$\mathcal{M}v + \mu(|x|)v^q + h(|x|, v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has at least one positive classical radial solution for any $R > 0$. It is interesting to note that, according to Theorem 3.2 and Remark 3.1 in [11], the problem

$$\mathcal{M}v + \mu(|x|)|v|^{q-1}v = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has infinitely many radial solutions with prescribed number of nodes (the positive case is not covered), provided that μ is continuously differentiable.

(ii) The Dirichlet problems

$$\mathcal{M}v + \frac{\mu(|x|)v^q}{\sqrt{\alpha^2 - v^2}} = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

and

$$\mathcal{M}v + \frac{\mu(|x|)v^q}{(\alpha - v)^\gamma} = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

have at least one positive classical radial solution for any $R < \alpha$.

(iii) From Remark 1, the Dirichlet problem

$$\mathcal{M}v + \mu(|x|)v^q(\alpha - v)^\gamma = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has at least one positive classical radial solution for any $R \le \alpha$.

(iv) If, in addition, $\gamma < \frac{1}{2}$, then the Dirichlet problem

$$\mathcal{M}v + \frac{\mu(|x|)v^q}{(1 - v^2)^\gamma} = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has at least one positive classical radial solution for any $R \le 1$.

Example 2 If $0 \leq q < 1 \leq p$ and $\lambda > 0$, then problem

$$\mathcal{M}v + \lambda v^q + v^p = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has at least one positive classical radial solution for any $R > 0$. In the classical case, using the upper and lower solutions method, it has been proved by Ambrosetti, Brezis and Cerami [2] that problem

$$\Delta v + \lambda v^q + v^p = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has a positive solution iff $0 < \lambda \leq \Lambda$ for some $\Lambda > 0$ $(0 < q < 1 < p)$.

Example 3 Assume that (H_f) and condition (3) are satisfied. Then the mixed boundary value problem

$$[r^{N-1}\tan(u')]' + r^{N-1}f(r,u) = 0, \qquad u'(0) = 0 = u(R). \tag{22}$$

has at least one positive solution if either $\pi R < 2\alpha$ or $2\alpha = \pi$, $R = 1$ and it holds

$$\lim_{s \to \frac{\pi}{2}-} f(r,s)\cos s = 0 \quad \text{uniformly with} \quad r \in [0,1],$$

3 Variational solutions

First, we deal with the general mixed boundary value problem

$$(r^{N-1}\phi(u'))' = r^{N-1}g(r,u), \qquad u'(0) = 0 = u(R), \tag{23}$$

where $g : [0,R] \times \mathbb{R} \to \mathbb{R}$ is continuous and ϕ satisfies the following potentiality hypothesis:

(H_Φ) $\Phi : [-a,a] \to \mathbb{R}$ *is continuous, of class* C^1 *on* $(-a,a)$, $\Phi(0) = 0$ *and* $\phi := \Phi' : (-a,a) \to \mathbb{R}$ *is an increasing homeomorphism such that* $\phi(0) = 0$.

Using the ideas from [6] (also see [7, 9, 26]) a variational approach is introduced for problem (23). With this aim let us denote $W^{1,\infty} := W^{1,\infty}(0,R)$. According to [7] we know that the convex set

$$K := \{v \in W^{1,\infty} \ : \ \|v'\| \leq a\}$$

is closed in C. This implies that

$$K_0 := \{v \in K \ : \ v(R) = 0\}$$

is also a convex, closed subset of C. On the other hand, as

$$||v|| \leq aR \quad \text{for all} \quad v \in K, \tag{24}$$

K_0 is bounded in $W^{1,\infty}$. Then, by the compactness of the embedding $W^{1,\infty} \subset C$, we infer that K_0 is compact in C.

Next, as in [7] we deduce that the functional $\Psi : C \to (-\infty, +\infty]$, defined by

$$\Psi(v) = \begin{cases} \displaystyle\int_0^R r^{N-1}\Phi(v')\,dr, & \text{if } v \in K_0, \\ +\infty, \quad \text{if } v \in C \setminus K_0 \end{cases}$$

is proper, convex and lower semicontinuous. Note that Ψ is bounded on K_0.

Setting

$$G(r,s) = \int_0^s g(r,\xi)d\xi, \qquad (r,s) \in [0,R] \times \mathbb{R},$$

we define $\mathcal{G} : C \to \mathbb{R}$ by

$$\mathcal{G}(v) = \int_0^R r^{N-1}G(r,v)dr, \qquad v \in C,$$

which is of class C^1 on C. Then, the energy functional $I := \Psi + \mathcal{G}$ has the structure required by Szulkin's critical point theory [30]. Accordingly, a function $u \in C$ is *a critical point* of I if $u \in K_0$ and

$$\Psi(v) - \Psi(u) + \langle \mathcal{G}'(u), v - u\rangle \geq 0 \quad \text{for all} \quad v \in C,$$

or, equivalently

$$\int_0^R r^{N-1}\left[\Phi(v') - \Phi(u') + g(r,u)(v-u)\right]dr \geq 0 \quad \text{for all} \quad v \in K_0. \tag{25}$$

Lemma 3 *Assume* (H_Φ). *Then, for every* $h \in C$, *the problem*

$$(r^{N-1}\phi(u'))' = r^{N-1}h(r), \qquad u'(0) = 0 = u(R), \tag{26}$$

has an unique solution u_h, *which is also the unique solution in* K_0 *of the variational inequality*

$$\int_0^R r^{N-1}\left[\Phi(v') - \Phi(u') + h(v-u)\right]dr \geq 0 \quad \textit{for all} \quad v \in K_0, \tag{27}$$

and the unique minimum over K_0 *of the strictly convex functional* $J : K_0 \to \mathbb{R}$ *defined by*

$$J(v) = \int_0^R r^{N-1}\left[\Phi(v') + hv\right]dr, \qquad v \in K_0.$$

Proof. A straightforward computation shows that problem (26) has as unique solution the function

$$u_h = -K \circ \phi^{-1} \circ S \circ h.$$

Now, if $u := u_h$ is the solution of (26), then, taking $v \in K_0$, multiplying each member of the differential equation by $v - u$, integrating over $[0, R]$, using the integration by parts formula and the boundary conditions, we get

$$\int_0^R r^{N-1}[\phi(u')(v' - u') + h(v - u)]dr = 0,$$

which, on account of the convexity inequality

$$\Phi(v') - \Phi(u') \geq \phi(u')(v' - u'),$$

gives (27). Next, it is clear that $u \in K_0$ is a solution of the variational inequality (27) if and only if it is a minimum of J on K_0. Moreover, using that Φ is strictly convex, it follows that J is strictly convex, implying the uniqueness of the minimum of J on K_0.

The proof is now completed. ■

Proposition 4 *Assume (H_Φ) and that $g : [0, R] \times \mathbb{R} \to \mathbb{R}$ is continuous. Then, each critical point of I is a solution of (23). Moreover, (23) has a solution which is a minimum point of I on C.*

Proof. Let $u \in K_0$ be a critical point of I. This means that u solves the variational inequality (27) with $h = g(\cdot, u)$ (see (25)). But, then Lemma 3 ensures that u is a solution of (23).

Next, from the definition of I one has that $I(v) \equiv +\infty$ on $C \setminus K_0$. This implies that

$$\inf_C I = \inf_{K_0} I =: c.$$

Using (24) we infer that I is bounded on K_0. Consider $\{u_k\} \subset K_0$ such that $I(u_k) \to c$. By the compactness of K_0 we may assume, passing to a subsequence if necessary, that there exists $u \in K_0$ such that $u_k \to u$ in C. It follows that $\mathcal{G}(u_k) \to \mathcal{G}(u)$ and $\Psi(u) \leq \liminf_{k\to\infty} \Psi(u_k)$. Consequently, one has that $I(u) \leq c$ and u is a minimum of I on C. By virtue of Proposition 1.1 in [30], u is a critical point of I, hence a solution of (23). ■

Remark 2 The above Proposition 4 reduces the search of solutions of problem (23) to finding critical points of the energy functional I. It is worth to point out that the potentiality hypothesis (H_Φ) plays here a crucial role. In this view, for instance, problems of type (22) can not be treated in this way, because the homeomorphism $\phi(s) = \tan(s)$ $(-\pi/2 < s < \pi/2)$ does not satisfy (H_Φ). On the other hand, this is a suitable way for problems of type (8), because in this case, as

$$\phi(s) = \frac{s}{\sqrt{1 - s^2}} \qquad (s \in (-1, 1)),$$

one takes

$$\Phi(s) = 1 - \sqrt{1 - s^2} \qquad (s \in [-1, 1]) \tag{28}$$

and it is easily seen that (H_Φ) is fulfilled.

Now, we come to study the problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}\mu(r)p(u) = 0, \qquad u'(0) = 0 = u(R), \tag{29}$$

under the hypotheses:

(H_μ) $\mu : [0, R] \to \mathbb{R}$ *is continuous and* $\mu(r) > 0$ *for all* $r > 0$*;*

(H_p) $p : [0, \infty) \to \mathbb{R}$ *is a continuous function such that* $p(0) = 0$ *and* $p(s) > 0$ *for all* $s > 0$*.*

Note that (29) is of type (8) with

$$f(r, s) = \mu(r)p(s), \qquad \text{for } (r, s) \in [0, R] \times [0, \infty)$$

and from (H_p) it is clear that $u \equiv 0$ is a solution of (29). We are interested on positive solutions.

It is a simple matter to see that any solution of problem (29) is nonnegative. So, we may assume that p is also defined on $(-\infty, 0)$ by setting $p(s) = 0$ for $s < 0$. Then, this fits with the variational setting from above by taking $g(r, s) = -\mu(r)p(s)$ for all $(r, s) \in [0, R] \times \mathbb{R}$. With Φ given in (28) and

$$G(r, s) = -\mu(r)P(s), \qquad P(s) = \int_0^s p(t)dt \qquad ((r, s) \in [0, R] \times \mathbb{R}),$$

the energy functional $I : C \to (-\infty, +\infty]$ associated to (29) will be

$$I(v) = \frac{R^N}{N} - \int_0^R r^{N-1}\sqrt{1-v'^2}dr - \int_0^R r^{N-1}\mu(r)P(v)dr \quad (v \in K_0)$$

and $I \equiv +\infty$ on $C \setminus K_0$.

The main result of this section is the following one.

Theorem 2 *Assume* (H_μ) *and* (H_p)*, together with*

$$\inf_{K_0} I < 0. \tag{30}$$

Then problem (29) has at least one solution u *such that* $u > 0$ *on* $[0, R)$ *and* u *is strictly decreasing.*

Proof. By Proposition 4 and $I(0) = 0$, it follows that (29) has a nontrivial (nonnegative) solution u. Clearly, this satisfies

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}\mu(r)p(|u|) = 0, \qquad u'(0) = 0 = u(R).$$

The conclusion follows now from Lemma 1. ■

Corollary 2 *If (H_μ), (H_p) and (7) are satisfied then problem (29) has at least one solution u such that $u > 0$ on $[0, R)$ and u is strictly decreasing.*

Proof. We show that condition (30) is fulfilled. Consider the function $v_R \in K_0$ given by

$$v_R(r) = R - r \quad \text{for all} \quad r \in [0, R].$$

Using (7), one gets

$$I(v_R) = \frac{R^N}{N} - \int_0^R r^{N-1}\mu(r)P(R-r)dr < 0,$$

and (30) holds true. ∎

Remark 3 Under the assumptions of Corollary 2, condition (7) can be replaced by $\mu_m := \min_{[0,R]} \mu > 0$ and

$$R^N < \mu_m \int_0^R (R-r)^N p(r)dr. \tag{31}$$

Indeed, we have

$$\begin{aligned} \frac{R^N}{N} - \int_0^R r^{N-1}\mu(r)P(R-r)dr &\leq \frac{R^N}{N} - \mu_m \int_0^R r^{N-1}P(R-r)dr \\ &= \frac{R^N}{N} - \mu_m \int_0^R \frac{(R-r)^N}{N} p(r)dr \\ &< 0, \end{aligned}$$

and (7) holds true.

Corollary 3 *Assume that (H_μ), (H_p) and (7) are fulfilled. Then problem (6) has at least one classical positive radial solution. The same is true if instead of (7) is $\mu_m > 0$ together with (31).*

Example 4 Given $m \geq 0$ and $q > 0$, let us consider the Hénon type problem

$$\mathcal{M}v + |x|^m v^q = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R). \tag{32}$$

It is easy to see that in this case inequality (7) becomes

$$1 < \frac{NR^{m+q+1}\Gamma(q+2)\Gamma(N+m)}{(q+1)\Gamma(N+m+q+2)}. \tag{33}$$

Consequently, if (33) holds then problem (32) has at least one classical positive radial solution.

Recall, the classical problem

$$\Delta v + |x|^m v^q = 0 \quad \text{in} \quad \mathcal{B}(1), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(1), \tag{34}$$

was introduced by Hénon [22] as a model to study spherically symmetric clusters of stars. Problems of this type have been widely studied (see for instance [23, 24, 27, 29] and the references therein). It is worth to point out that, as shown in [27], problem (34) has a positive radial solution provided that

$$q \in \left(1, \frac{N+2m+2}{N-2}\right) \quad (N \geq 3, m > 0).$$

Example 5 Let $q \geq 1$ (for $q \in (0,1)$ see Example 1 (i)) and $\lambda > 0$. Using (31), the Dirichlet problem

$$\mathcal{M}v + \lambda v^q = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R), \tag{35}$$

has at least one classical positive radial solution for any $R > 0$, such that

$$1 < \lambda R^{q+1} \frac{\Gamma(N+1)\Gamma(q+1)}{\Gamma(N+q+2)}.$$

Therefore, (35) has a positive solution provided that λ or R are large enough. The corresponding result in the Euclidean context is given in [14, 15]. On the other hand, this is in contrast with the classical case. Indeed, from the Pohozaev identity, the Dirichlet problem

$$\Delta v + \lambda v^q = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has no positive solutions if $q > (N+2)/(N-2)$, for any $\lambda, R > 0$ $(N \geq 3)$.

Also, if $q = 1$, setting $\Lambda := (N+1)(N+2)/R^2$, the problem

$$\mathcal{M}v + \lambda v = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has a classical positive radial solution for any $\lambda > \Lambda$. Such an Λ does not exist for the classical Laplacian case. To see this it suffices to take any λ outside of the spectrum of $-\Delta$ with homogeneous Dirichlet boundary condition and, $v \equiv 0$ will be the unique solution of problem

$$\Delta v + \lambda v = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R).$$

Example 6 Let $q \geq 1$ and $\lambda > 0$ be a parameter. From (31), the Brezis-Nirenberg type problem

$$\mathcal{M}v + v^q + \lambda u = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R),$$

has a classical positive radial solution for any $\lambda > 0$ such that

$$1 < \lambda \frac{R^2}{(N+1)(N+2)} + R^{q+1} \frac{\Gamma(q+1)N!}{\Gamma(N+q+2)}.$$

Again, this appears as being in contrast with the classical case. In this view, we only note that, as shown in the seminal paper [10], the problem

$$\Delta v + v^{\frac{N+2}{N-2}} + \lambda v = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R), \quad (N \geq 4)$$

has positive solution provided that $\lambda > 0$ is sufficiently small.

Acknowledgements. The authors express their gratitude to the anonymous referee for useful comments and remarks which led to the improvement of presentation and quality of the paper.

References

[1] L.J. Alías, B. Palmer, On the Gaussian curvature of maximal surfaces and the Calabi-Bernstein theorem, Bull. London Math. Soc. 33 (2001), 454-458.

[2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Functional Analysis 122 (1994), 519-543.

[3] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, Comm. Math. Phys. 87 (1982-83), 131-152.

[4] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces, Proc. Amer. Math. Soc. 137 (2009), 171-178.

[5] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowski spaces, Math. Nachr. 283 (2010), 379-391.

[6] C. Bereanu, P. Jebelean, J. Mawhin, Variational methods for nonlinear perturbations of singular ϕ-Laplacians, Rend. Lincei Mat. Appl. 22 (2011), 89-111.

[7] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions of Neumann problems involving mean extrinsic curvature and periodic nonlinearities, Calc. Var. PDE, DOI 10.1007/s00526-011-0476-x.

[8] M.F. Bidaut-Véron, A. Ratto, Spacelike graphs with prescribed mean curvature, Differential Integral Equations 10 (1997), 1003 -1017.

[9] H. Brezis, J. Mawhin, Periodic solutions of the forced relativistic pendulum, Differential Integral Equations 23 (2010), 801-810.

[10] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983), 437-477.

[11] A. Capietto, W. Dambrosio, F. Zanolin, Infinitely many radial solutions to a boundary value problem in a ball, Ann. Mat. Pura Appl. 179 (2001), 159-188.

[12] S.-Y. Cheng, S.-T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces, Ann. of Math. 104 (1976), 407-419.

[13] J.A. Cid, P.J. Torres, Solvability for some boundary value problems with ϕ-Laplacian operators, Discrete Continuous Dynamical Systems 23 (2009), 727-732.

[14] P. Clément, R. Manásevich, E. Mitidieri, On a modified capillary equation, J. Differential Equations 124 (1996), 343-358.

[15] C.V. Coffman, W.K. Ziemer, A prescribed mean curvature problem on domains without radial symmetry, SIAM J. Math. Anal. 22 (1991), 982-990.

[16] H. Dang, R. Manásevich, K. Schmitt, Positive radial solutions of some nonlinear partial differential equations, Math. Nachr. 186 (1997), 101-113.

[17] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, 1985.

[18] P. Drábek, M. García-Huidobro, R. Manásevich, Positive solutions for a class of equations with a p-Laplace like operator and weights, Nonlinear Analysis 71 (2009), 1281-1300.

[19] M. García-Huidobro, R. Manásevich, K. Schmitt, Positive radial solutions of quasilinear elliptic partial differential equations on a ball, Nonlinear Analysis 35 (1999), 175-190.

[20] M. García-Huidobro, R. Manásevich, J.R. Ward, Positive solutions for equations and systems with p-Laplace-like operators, Adv. Differential Equations 14 (2009), 401-432.

[21] D.D. Hai, On singular Sturm-Liouville boundary-value problems, Proc. Royal Soc. Edinburgh 140A (2010), 49-63.

[22] M. Hénon, Numerical experiments on the stability of spherical stelar systems, Astronomy and Astrophysics 24 (1973), 229-238.

[23] R. Kajikiya, Non-radial least energy solutions of the generalized Hénon equation, J. Differential Equations 252 (2012), 1987-2003.

[24] W. Long, J. Yang, Existence and asymptotic behaviour of solutions for Hénon type equations, Opuscula Math. 31 (2011), 411-424.

[25] R. López, Stationary surfaces in Lorentz-Minkowski space, Proc. Royal Soc. Edinburgh 138A (2008), 1067-1096.

[26] J. Mawhin, Radial solution of Neumann problem for periodic perturbations of the mean extrinsic curvature operator, Milan J. Math. 79 (2011), 95-112.

[27] W.M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications. Indiana Univ. Math. J. 31 (1982), 801-807.

[28] F. Obersnel, P. Omari, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation, J. Differential Equations 249 (2010), 1674-1725.

[29] D. Smets, J. Su, M. Willem, Non-radial ground states for the Hénon equation, Commun. Contemp. Math. 4 (2002), 467-480.

[30] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986), 77–109.

[31] A.E. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, Invent. Math. 66 (1982), 39-56.

[32] H. Wang, On the existence of positive solutions for semilinear elliptic equations in the annulus, J. Differential Equations 109 (1994), 1-7.