

Bifurcation of closed orbits from equilibria of Newtonian systems with Coriolis forces

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Abstract

We consider autonomous Newtonian systems with Coriolis forces in two and three dimensions and study the existence of branches of periodic orbits emanating from equilibria. We investigate both degenerate and nondegenerate situations. While Lyapunov's center theorem applies locally in the nondegenerate, nonresonant context, equivariant degree theory provides a global answer which is significant also in some degenerate cases.

We apply our abstract results to a problem from Celestial Mechanics. More precisely, in the three-dimensional version of the Restricted Triangular Four-body Problem with possibly different primaries our results show the existence of at least seven branches of periodic orbits emanating from the stationary points.

1 Introduction

Many mechanical problems can be modeled as the motion of a particle subjected to the conservative force created by a uniformly rotating source. This is for instance the case of the classical circular restricted three-body problem or, more generally, the restricted problem consisting of a massless particle moving under the attraction of a given family of primaries revolving solidly around the origin. In rotating coordinates we obtain a system of autonomous second-order equations which in the *planar* case reads

$$\begin{cases} \ddot{x} + 2\dot{y} + \frac{\partial V}{\partial x}(x, y) = 0, \\ \ddot{y} - 2\dot{x} + \frac{\partial V}{\partial y}(x, y) = 0, \end{cases} \quad [2d]$$

and in the *spatial* context becomes

$$\begin{cases} \ddot{x} + 2\dot{y} + \frac{\partial V}{\partial x}(x, y, z) = 0, \\ \ddot{y} - 2\dot{x} + \frac{\partial V}{\partial y}(x, y, z) = 0, \\ \ddot{z} + \frac{\partial V}{\partial z}(x, y, z) = 0, \end{cases} \quad [3d]$$

(the rotation is assumed to take place in the xy plane). One can condense both situations in a unified way as follows:

$$\ddot{q} - 2\alpha_N \dot{q} + V'(q) = 0, \quad q \in \Omega, \quad (1)$$

where $q = (x, y)$ or $q = (x, y, z)$ is a variable in \mathbb{R}^N with $N = 2$ or $N = 3$, respectively. We assume that the *effective potential* $V = V(q)$ is defined and has class C^2 on some open domain $\Omega \subset \mathbb{R}^N$, its gradient being denoted by $V' : \Omega \rightarrow \mathbb{R}^N$. Moreover, the skew-symmetric matrices $\alpha_2 \in \mathbb{R}^{2 \times 2}$, $\alpha_3 \in \mathbb{R}^{3 \times 3}$ are given by

$$\alpha_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_3 = \left(\begin{array}{c|c} & 0 \\ \alpha_2 & 0 \\ \hline 0 & 0 \end{array} \right).$$

In the spatial case **[3d]** we shall further assume that $V = V(x, y, z)$ satisfies

$$\frac{\partial V}{\partial z}(x, y, 0) = 0, \quad (x, y) \in \tilde{\Omega}, \quad \mathbf{[H_1]}$$

where $\tilde{\Omega} := \{(x, y) \in \mathbb{R}^2 : (x, y, 0) \in \Omega\}$. In this way, the restriction

$$\tilde{V}(x, y) := V(x, y, 0), \quad (x, y) \in \tilde{\Omega},$$

lies under the framework established in **[2d]**, and the corresponding closed orbits are actually closed orbits of the full spatial problem.

The equilibria of (1) coincide with the critical points of V . Throughout this paper it will always be assumed that:

$$\text{All critical points of } V \text{ are isolated.} \quad \mathbf{[V]}$$

Thus, we allow the possibility that V has infinitely many critical points, but in this case they must accumulate either at infinity or on the boundary of Ω . In addition, in the spatial case **[3d]** we shall always assume that

$$q_0 = (\tilde{q}_0, 0) \in \tilde{\Omega} \times \{0\}, \quad \beta_3(q_0) := \frac{\partial^2 V}{\partial z^2}(q_0) > 0, \quad \text{for every } q_0 \in (V')^{-1}(0). \quad \mathbf{[H_2]}$$

In combination with **[H₁]**, the second part of **[H₂]** roughly means that the force $-V'$ attracts our particle towards $\tilde{\Omega} \times \{0\}$ when it gets close to an equilibrium. The goal of this paper is to study the *existence of global branches of closed orbits of (1) emanating from equilibria*, both in the planar and the spatial cases.

A well-known *necessary* condition for the existence of closed orbits near a given equilibrium q_0 is that it must be nonhyperbolic. On the other hand, a *sufficient* condition is provided by Lyapunov's center theorem: under some nondegeneracy and nonresonance conditions, the existence of emanating families of closed orbits is guaranteed in the elliptic and elliptic-hyperbolic cases. Perhaps unsurprisingly, these situations can be described in terms of the spectrum of the Hessian matrix $V''(q_0)$. In addition, the equivariant degree theory for Hamiltonian systems provides powerful tools which will allow us to (a): replace the nondegeneracy condition with a milder condition on the Brouwer index of q_0 as a zero of V' ; (b): obtain the existence of *global* (rather than local) branches.

These results are formulated in a precise form in Sections 2-3. In the planar setting **[2d]** we characterize the existence of bifurcating branches of closed orbits in

terms of the eigenvalues β_1, β_2 of $V''(q_0)$. The degenerate situations where (β_1, β_2) lies on the boundary of the existence region while avoiding the boundary of the nonexistence one can still be dealt with if the Brouwer index of V' at the isolated zero q_0 does not vanish. On the other hand, in the spatial case [3d] these bifurcating branches of closed orbits do exist *in all nondegenerate cases*, and even in the degenerate ones when the corresponding Brouwer index does not vanish. Moreover, in some situations the bifurcating branch can be shown to be nonplanar. We illustrate our results with an example and formulate a number of open questions.

In Section 4 we apply our abstract results to a restricted four-body problem. Assuming that three (possibly different) positive masses revolve around their center of masses in a Lagrangian equilateral triangle, we study the motion of a massless test particle subjected to their gravitational attraction. For the three-dimensional problem it turns out that there are at least seven branches of closed orbits emanating from seven corresponding libration points.

In Section 5 we state, without proof, a bifurcation result taken from [7] which will be the main tool behind of our arguments. Based on degree theory for equivariant gradient maps, it applies to general (autonomous) Hamiltonian systems having a given equilibrium. Under the assumption that the Morse index of certain 8×8 or 12×12 matrices S_T changes as the parameter T varies on $(0, +\infty)$, Theorem 5.1 states the existence of a branch of closed orbits emanating from the equilibrium. We also recall De Gua's corollary of Descartes' rule of signs [14], which allows the exact computation of the Morse index of a symmetric matrix by looking at the number of sign changes in the sequence of coefficients of its characteristic polynomial.

Sections 6-9 are devoted to prove the general results announced in Section 3. More precisely, in Section 6 we observe that (1) can be rewritten in a Hamiltonian form and compare some properties that a given equilibrium may have in both contexts. Section 7 is devoted to study the spectrum of the matrices S_T , and in Section 8 we compute their Morse index. This will lead us to complete the proofs in Section 9. The paper closes with an Appendix where we discuss a couple of elementary facts from linear algebra needed in our computations of Sections 7-8.

Sumarizing, Sections 2-3 are devoted to present the general bifurcation results, in Section 4 we apply them in a problem of Celestial Mechanics, the expository Section 5 collects a couple of known theorems to be used later, and Sections 6-9 are concerned with the proofs. In addition, this latter 'proof block' resorts from time to time to material from the Appendix. The purpose of this scheme is to motivate the theory before going into the mathematical details, while keeping at the same time the pace of the exposition.

Predicting the existence of closed orbits near an equilibrium is a classical problem which goes back to Poincaré [29]. Coinciding with the space race, the seventies saw a renewed interest in this question [1, 23, 25, 33, 36, 38] that continues to this day [4, 5, 7, 8, 11, 12, 20, 22, 34, 37] (the lists are just an small sample and far from complete). The fact that the set of closed orbits is invariant under translations in the time variable introduces a degeneracy in the problem that has been solved in a variety of ways, including: geometrical index theories, symplectic reduction techniques, equivariant Ljusternik-Schnirelmann category theory, equivariant Morse theory and

equivariant topological degree theory. Our choice of the latter is motivated by the fact that it both applies to *degenerate* situations and produces *global* branches of closed orbits.

2 Equilibria and branches of closed orbits

Throughout this paper we study periodic solutions of (1) whose periods are not fixed in advance and, in general, may differ from one solution to another. The set of positive periods of a given periodic solution $q = q(t)$ is an infinite semigroup of real numbers. We shall follow the convention of counting each periodic solution infinitely many times -once for each positive period. After identifying the periodic solution $q : \mathbb{R}/T\mathbb{Z} \rightarrow \mathbb{R}^N$ with the pair (T, \bar{q}) where $\bar{q}(\theta) = q\left(\frac{T}{2\pi}\theta\right)$, we may see them as elements of the cartesian product $(0, +\infty) \times C(\mathbb{S}^1, \Omega)$. Here, and in what follows, $\mathbb{S}^1 = \{\theta : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. It motivates the following terminology: the pair $(T, \bar{q}) \in (0, +\infty) \times C(\mathbb{S}^1, \Omega)$ will be called a *closed orbit* if $q(t) := \bar{q}\left(\frac{2\pi}{T}t\right)$ is a periodic solution.

For instance, if $q_0 \in \Omega$ is a critical point of V (or equivalently, an equilibrium of (1)), then (T, q_0) is a closed orbit for each $T > 0$ -by abuse of notation we still denote by q_0 to the corresponding constant map defined on \mathbb{S}^1 . The closed orbits obtained in this way will be considered *trivial* and therefore the set of trivial closed orbits can be canonically identified with $(0, +\infty) \times (V')^{-1}(0)$.

On the other hand, a closed orbit (T, \bar{q}) will be called *nontrivial* if the function $\bar{q} : \mathbb{S}^1 \rightarrow \mathbb{R}^N$ is nonconstant. We denote by Λ the closure in $\mathbb{R} \times C(\mathbb{S}^1, \Omega)$ of the set of nontrivial closed orbits. By a result of Yorke [39], the set Λ is contained into $(0, +\infty) \times C(\mathbb{S}^1, \Omega)$. Therefore, all pairs $(T, \bar{q}) \in \Lambda$ are closed orbits but some trivial closed orbits may not belong to Λ . It motivates the notion of *branch*, a word that has been used quite loosely above. The following definition is inspired by Leray-Schauder's Continuation Theorem [19] (see also [31]).

Definition 2.1 (Branch of closed orbits). *A connected component \mathcal{B} of Λ will be called a branch of closed orbits if it satisfies at least one of the following assumptions:*

- (a) \mathcal{B} is unbounded in the Banach space $\mathbb{R} \times C(\mathbb{S}^1, \mathbb{R}^N)$,
- (b) the closure of the set $\{\bar{q}(\theta) : \theta \in \mathbb{S}^1, (T, \bar{q}) \in \mathcal{B}\}$ has nonempty intersection with $\partial\Omega$, or
- (c) \mathcal{B} is compact with the inherited $\mathbb{R} \times C(\mathbb{S}^1, \mathbb{R})$ -topology and contains at least two different trivial closed orbits.

The branch of closed orbits \mathcal{B} is said to emanate from the trivial closed orbit $(T, q_0) \in (0, +\infty) \times (V')^{-1}(0)$ provided that $(T, q_0) \in \mathcal{B}$. We shall simply say that \mathcal{B} emanates from the equilibrium q_0 provided that $(T, q_0) \in \mathcal{B}$ for some $T > 0$.

For short, one can name the three possibilities in Definition 2.1 by saying that a branch must, either: (a) be unbounded, or (b) go up to the boundary, or (c) be compact, containing at least two trivial closed orbits. A couple of remarks are in order here:

- (I) Possibility (a) may happen if either $\{\bar{q}(\theta) : \theta \in \mathbb{S}^1, (T, \bar{q}) \in \mathcal{B}\}$ is unbounded in \mathbb{R}^N or the set of periods $\{T : (T, \bar{q}) \in \mathcal{B}\}$ is unbounded from above. The second situation has been called the *blue sky catastrophe* in the literature. It can be excluded in some special cases, see [9, §7.6].
- (II) For the purposes of this paper, case (c) could have been stated in a stronger form: not only \mathcal{B} must be compact and contain at least two trivial closed orbits, but in addition the sum of the bifurcation numbers of the trivial closed orbits in \mathcal{B} must be zero. The *bifurcation number* is an integer $\gamma_N(T, q_0) \in \mathbb{Z}$ associated to any trivial closed orbit $(T, q_0) \in (0, +\infty) \times (V')^{-1}(0)$. More generally it can be defined for trivial closed orbits of arbitrary Hamiltonian systems, see (6) in Section 5. For system (1) these integers can be computed explicitly in terms of T and the spectrum of $V''(q_0)$ (and also the Brouwer index $i_B(q_0, V')$ in the degenerate cases). See (10)-(11) in Section 9. See also e.g. [6] for an introduction to the Brouwer index.

3 The general results

The main purpose of this section is to announce the main abstract bifurcation results of the paper. They are organized in Theorems 3.1-3.3. We shall subsequently apply them in a problem of Celestial Mechanics (in Section 4) and devote Sections 6-9 to their proofs.

Fix an equilibrium $q_0 \in \Omega$. We shall always denote by β_1, β_2 the eigenvalues of $V''(q_0)$ in the planar case [2d], or the eigenvalues of $\tilde{V}''(\tilde{q}_0)$ in the spatial situation [3d]. Inside the plane of couples $(\beta_1, \beta_2) \in \mathbb{R}^2$ we consider the closed set

$$C := (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}),$$

made up of both coordinate axis, and the open set

$$\mathcal{R}_0 := \left\{ (\beta_1, \beta_2) \in (-\infty, 0)^2 : \beta_1 + \beta_2 < \max(-4, -2 - (\beta_1 - \beta_2)^2/8) \right\}.$$

These sets are pictured in Figure 1 below. We have also labeled the four domains \mathcal{R}_i , $1 \leq i \leq 4$, in which the complementary open set $\mathbb{R}^2 \setminus (\bar{\mathcal{R}}_0 \cup C)$ is divided. Notice that \mathcal{R}_1 , \mathcal{R}_2 and \mathcal{R}_4 coincide, respectively, with the first, second, and fourth open quadrants, while \mathcal{R}_3 is the set of points (β_1, β_2) in the open third quadrant $(-\infty, 0)^2$ such that $\beta_1 + \beta_2 > \max(-4, -2 - (\beta_1 - \beta_2)^2/8)$.

We also consider the continuous, symmetric functions

$$T_- : (\mathbb{R}^2 \setminus \bar{\mathcal{R}}_0) \cup ((\partial\mathcal{R}_0) \setminus C) \rightarrow (0, +\infty), \quad T_+ : \mathcal{R}_1 \cup \mathcal{R}_3 \cup ((\partial\mathcal{R}_0) \setminus C) \rightarrow (0, +\infty),$$

defined by

$$T_{\pm}(\beta_1, \beta_2) := 2\pi\sqrt{2} \left(\beta_1 + \beta_2 + 4 \mp 2\sqrt{2}\sqrt{\beta_1 + \beta_2 + 2 + (\beta_1 - \beta_2)^2/8} \right)^{-1/2}. \quad (2)$$

It can be checked that $T_-(\beta_1, \beta_2) < T_+(\beta_1, \beta_2)$ on $\mathcal{R}_1 \cup \mathcal{R}_3$ and $T_-(\beta_1, \beta_2) = T_+(\beta_1, \beta_2)$ on $(\partial\mathcal{R}_0) \setminus C$, see Lemma 7.3. Our interest in these functions comes from

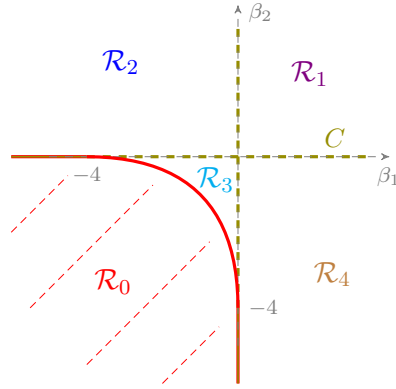


Figure 1: The plane (β_1, β_2) of eigenvalues of $V''(q_0)$ (in the planar case) or $\tilde{V}''(\tilde{q}_0)$ (in the spatial case).

the fact that they can be used to describe the nonzero, purely-imaginary part of the spectrum of the linearized system (1) at the equilibrium q_0 . In fact, in the planar case [2d] these characteristic exponents are $-i 2\pi/T_{\pm}(\beta_1, \beta_2)$, $i 2\pi/T_{\pm}(\beta_1, \beta_2)$, wherever defined. In addition, in the spatial situation [3d] it suffices to add $\pm\sqrt{\beta_3}i$ to the previous lists (see Lemma 7.4). We arrive to one of the main general results of this paper:

Theorem 3.1. *Assume [2d]. The following hold:*

- (i) *If $(\beta_1, \beta_2) \in \mathcal{R}_0$, then there are not closed orbits in a sufficiently small neighborhood of the equilibrium q_0 .*
- (ii) *If $(\beta_1, \beta_2) \notin \bar{\mathcal{R}}_0 \cup C$, then there is a branch of closed orbits emanating from $(T_-(\beta_1, \beta_2), q_0)$. The same conclusion holds if $(\beta_1, \beta_2) \in C \setminus \bar{\mathcal{R}}_0$ and $i_B(q_0, V') \neq 0$.*
- (iii) *If $(\beta_1, \beta_2) \in \mathcal{R}_1 \cup \mathcal{R}_3$, then there is also a branch of closed orbits emanating from $(T_+(\beta_1, \beta_2), q_0)$.*

The formulation of (i) is conveniently simple but needs some interpretation. It could be more precisely expressed as follows: If $(\beta_1, \beta_2) \in \mathcal{R}_0$ then there exists an open set $\Omega_1 \subset \Omega$ with $q_0 \in \Omega_1$ such that if $(T, \bar{q}) \in (0, +\infty) \times C(\mathbb{S}^1, \Omega_1)$ is a closed orbit of (1) then $\bar{q}(\theta) = q_0$ for every $\theta \in \mathbb{S}^1$.

Concerning the second part of (ii) we point out that if $(\beta_1, \beta_2) \notin C$ then

$$i_B(q_0, V') = \text{sign}(\beta_1\beta_2) = \begin{cases} 1 & \text{if } (\beta_1, \beta_2) \in (\bar{\mathcal{R}}_0 \setminus C) \cup \mathcal{R}_1 \cup \mathcal{R}_3, \\ -1 & \text{if } (\beta_1, \beta_2) \in \mathcal{R}_2 \cup \mathcal{R}_4. \end{cases} \quad (3)$$

Thus, assertion (ii) of Theorem 3.1 could be reformulated as follows: if $(\beta_1, \beta_2) \notin \bar{\mathcal{R}}_0$ and $i_B(q_0, V') \neq 0$, then there is a branch of closed orbits of (1) emanating from $(T_-(\beta_1, \beta_2), q_0)$. Two sufficient conditions implying, each of them, the inequality $i_B(q_0, V') \neq 0$, are: (a): V has a *local extremum* at q_0 ([16, Lemma 6.5], [32]), or (b): V is *even* with respect to q_0 (Borsuk-Ulam theorem). This observation leads to the following

Corollary 3.2. *If $(\beta_1, \beta_2) \notin \bar{\mathcal{R}}_0$ and either V attains a local extremum at q_0 , or V is even with respect to q_0 , then there is a branch of closed orbits of (1) emanating from $(T_-(\beta_1, \beta_2), q_0)$.*

In Section 7 it will be observed that if $(\beta_1, \beta_2) \in \mathcal{R}_0$ then the equilibrium q_0 is hyperbolic, if $(\beta_1, \beta_2) \in \mathcal{R}_1 \cup \mathcal{R}_3$ the equilibrium is elliptic, and if $(\beta_1, \beta_2) \in \mathcal{R}_2 \cup \mathcal{R}_4$ then the equilibrium is of elliptic-hyperbolic type (see Lemmas 7.3-7.4). Therefore, the branches of closed orbits described in (ii) and (iii) are the global continuations of the *short* and *long period* Lyapunov families, respectively. In the elliptic case $(\beta_1, \beta_2) \in \mathcal{R}_1 \cup \mathcal{R}_3$ it is well-known that if $T_+(\beta_1, \beta_2)/T_-(\beta_1, \beta_2)$ is an integer then both branches may actually correspond to the same periodic solutions of [2d] -just traveled through several times on each period. Even when the nonresonance condition $T_+(\beta_1, \beta_2)/T_-(\beta_1, \beta_2) \notin \mathbb{Z}$ holds we cannot exclude the possibility that the long and short period branches are connected, and so end up being the same branch. There is some additional local information at hand concerning these branches. For instance, when $(\beta_1, \beta_2) \notin \bar{\mathcal{R}}_0 \cup C$ the ‘short period’ closed orbits near $(T_-(\beta_1, \beta_2), q_0)$ have minimal period close to $T_-(\beta_1, \beta_2)$. Similarly, if $(\beta_1, \beta_2) \in \mathcal{R}_1 \cup \mathcal{R}_3$ and $T_+(\beta_1, \beta_2)/T_-(\beta_1, \beta_2) \notin \mathbb{Z}$, the ‘long period’ closed orbits near $(T_+(\beta_1, \beta_2), q_0)$ have minimal period close to $T_+(\beta_1, \beta_2)$. Moreover, the branch of closed orbits near the emanating point fills out a continuously embedded 2-dimensional disk, see e.g. [26, §2.2].

There are also some special situations where the dynamics are less understood. Even though these cases are non-generic, they are often hard to exclude in particular problems. For instance, the possibility $(\beta_1, \beta_2) \in (\partial\mathcal{R}_0) \setminus C$ corresponds to the strongly resonant situation in which there are double purely imaginary characteristic exponents $\pm i 2\pi/T_+(\beta_1, \beta_2) = \pm i 2\pi/T_-(\beta_1, \beta_2) \neq 0$, whereas if $(\beta_1, \beta_2) \in C$ then the equilibrium is degenerate, i.e., 0 is a characteristic exponent. Thus, when $(\beta_1, \beta_2) \in C \setminus (\partial\mathcal{R}_0)$ the usual versions of Lyapunov’s center theorem do not apply, but Theorem 3.1(ii) above states the existence of an emanating branch of closed orbits provided only that the Brouwer index $i_B(q_0, V')$ does not vanish. We emphasize that, in general, well-known examples of pathologies may occur when the assumptions of Lyapunov’s center theorem fail. See, e.g., examples 9.1 and 9.2 in [21, p. 210].

Let us now turn our attention to the spatial case [3d]. It seems reasonable that the additional dimension makes more space for the existence of emanating branches of closed orbits. As before, we denote by β_1, β_2 the eigenvalues of $\tilde{V}''(\tilde{q}_0)$ and set $\beta_3 := \beta_3(q_0) = \frac{\partial^2 V}{\partial z^2}(q_0) > 0$. One has

Theorem 3.3. *Assume [3d]. If $(\beta_1, \beta_2) \notin C$ then there is a branch of closed orbits emanating from $(\frac{2\pi}{\sqrt{\beta_3}}, q_0)$. The same conclusion holds when $(\beta_1, \beta_2) \in C$ provided that $i_B(\tilde{q}_0, \tilde{V}') \neq 0$.*

Remembering assertion (i) of Theorem 3.1 we see that if $(\beta_1, \beta_2) \in \mathcal{R}_0$ the emanating branch must be nonplanar. We emphasize that when $(\beta_1, \beta_2) \in C$ the critical point q_0 is degenerate and Lyapunov’s center theorem does not apply. The same arguments that lead us to Corollary 3.2 give rise now to a result without direct assumptions on the spectrum β_1, β_2 of $V''(q_0)$.

Corollary 3.4. *Assume [3d]. If either \tilde{V} attains a local extremum at q_0 , or \tilde{V} is even with respect to q_0 , then there is a branch of closed orbits of (1) emanating from q_0 .*

We point out that Theorems 3.1-3.3 extend [11, Proposition 3], which is concerned only with a particular family of potentials V and leaves aside the degenerate cases. Before closing this section we propose a pathological example for the planar case showing that bifurcation may not occur when some assumptions of Theorem 3.1 fail. Consider the system

$$\begin{cases} \ddot{x} + 2\dot{y} - x - x^3 - xy^2 = 0, \\ \ddot{y} - 2\dot{x} - y - y^3 - x^2y = 0, \end{cases}$$

on the domain $\Omega = \mathbb{R}^2$, with the only equilibrium $q_0 = (0, 0)$. Observe that it has the form [2d] with $V(x, y) = -(x^2 + y^2)/2 - (x^2 + y^2)^2/4$. In this case $(\beta_1, \beta_2) = (-1, -1) \in \partial\mathcal{R}_0$ and Theorem 3.1 gives no information. It turns out that the only periodic solution of this system is the equilibrium $q_0 = (0, 0)$. In fact, every solution $(x, y) \neq (0, 0)$ satisfies

$$\frac{d^2}{dt^2} \left(\frac{x^2 + y^2}{2} \right) = x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = (\dot{x} + y)^2 + (\dot{y} - x)^2 + (x^2 + y^2)^2 > 0,$$

and thus, it cannot be periodic.

In view of this example, an open question appears: in the planar case [2d], and assuming that $(\beta_1, \beta_2) \in (\partial\mathcal{R}_0) \setminus C$, is it possible to find sufficient conditions either implying or ruling out the existence of emanating branches of closed orbits? In case $(\beta_1, \beta_2) \in ((\partial\mathcal{R}_0) \cap C) \setminus \{(0, -4), (-4, 0)\}$ equality (3) leads us to conjecture that a bifurcating branch of closed orbits does exist if $i_B(q_0, V') < 0$ and does not exist if $i_B(q_0, V') > 0$; however, this is not proven in this paper. For the spatial case [3d] one can adapt the example above to show that there are situations where $(\beta_1, \beta_2) \in (\partial\mathcal{R}_0) \setminus C$ and the bifurcating branch of closed orbits, whose existence is granted by Theorem 3.3, is nonplanar. In the case where $(\beta_1, \beta_2) \in C \cap (\partial\mathcal{R}_0)$ and $i_B(\tilde{q}_0, \tilde{V}') \neq 0$ we do not know how to either guarantee or rule out that the branch of closed orbits emanating from $\left(\frac{2\pi}{\sqrt{\beta_3}}, q_0\right)$ is planar. On the other hand, we have no idea on whether the condition on the Brouwer index $i_B(\tilde{q}_0, \tilde{V}')$ (which coincides with $i_B(q_0, V')$, see Lemma 6.2) can be removed from Theorem 3.3.

4 The restricted triangular 4-body problem

In this section we apply the general results above to study the motion of a massless test particle subjected to the gravitational attraction of three primaries of (possibly different) masses $m_1, m_2, m_3 > 0$, which occupy the vertices of a Lagrangian equilateral triangle rotating solidly around their center of masses at constant angular speed.

The literature concerning this problem goes back to the dawn of the twentieth century. In [27, §8] one already reads that if the three primaries have equal mass then

the number of libration points must be 10. After some papers in the forties [15, 28], the development of computers allowed the introduction of numerical methods at the end of the seventies and beginning of the eighties [2, §3], [10], [35, §2]. For instance, in [35, p. 168] we find that for some choices of the masses the number of libration points can be just 8, while in [2, p. 14] it is announced that the number of libration points can be 8, 9 or 10 depending on the masses. Some geometrical insight in the nineties [24, §III] has been followed by a number of computer-assisted proofs in the new millennium [3, 17, 18], which have given a renewed interest to the problem.

After changing units in mass, space, and time, there is no loss of generality in assuming that the angular speed of the primaries is 1, the side of the equilateral triangle is $\sqrt{3}$, and $m_1 + m_2 + m_3 = 3\sqrt{3}$. Then, the gravitational constant must be $G = 1$, see [30, §2.8]. After choosing a synodic frame of reference we may assume that the primaries are fixed at the three cubic roots of unity: $q_1 = (1, 0)$, $q_2 = (-1/2, \sqrt{3}/2)$, $q_3 = (-1/2, -\sqrt{3}/2)$. We are led to the effective potential

$$V(q) := -\frac{|q - c|^2}{2} - \frac{m_1}{|q - q_1|} - \frac{m_2}{|q - q_2|} - \frac{m_3}{|q - q_3|}, \quad q \in \Omega := \mathbb{R}^2 \setminus \{q_1, q_2, q_3\}, \quad (4)$$

where $c := \frac{1}{3\sqrt{3}}(m_1 q_1 + m_2 q_2 + m_3 q_3)$ is the center of masses. It does not depend on time but, the three masses being possibly different, it may not coincide with the origin.

In the plane \mathbb{R}^2 we draw three circles of radius $\sqrt{3}$ around the positions of each of the primaries and extend in both directions the three sides of the triangle until they meet these circles again. In this way we obtain a compact set \mathcal{S} made up of three segments of common length $3\sqrt{3}$ and three circles of common radius $\sqrt{3}$. It divides the plane into seventeen open connected components, but we shall be interested in the open solid triangle \mathcal{T} , the three open circular sectors $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and the three open circular triangles $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$. See Fig. 2, a picture which is nowadays classical. To the best of our knowledge it appeared firstly in [28, p. 48], later in [35, Fig. 2(d), p. 168], and more recently in [18, p. 330] and [3, p. 1195]. The main result of this section is the following:

Theorem 4.1. *For any choices of the masses $m_1, m_2, m_3 > 0$ with $m_1 + m_2 + m_3 = 3\sqrt{3}$ the following hold:*

- (i) *There are at least 7 libration points for this problem: at least one in \mathcal{T} , at least one in each \mathcal{O}_i and at least one in each \mathcal{D}_i .*
- (ii) *After identifying the plane \mathbb{R}^2 with the horizontal plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and regarding the potential V in (4) as defined on $\mathbb{R}^3 \setminus \{q_1, q_2, q_3\}$, these seven planar libration points can be chosen so that there are (either planar or spatial) branches of closed orbits emanating from all of them.*

We emphasize that this result does not require nondegeneracy assumptions on the libration points. In particular, the classical versions of Lyapunov's center theorem do not seem directly applicable.

According to [2, p. 14] or [3, p. 1186], the minimal number of libration points in this problem is 8, and thus, our multiplicity result is not optimal; on the other

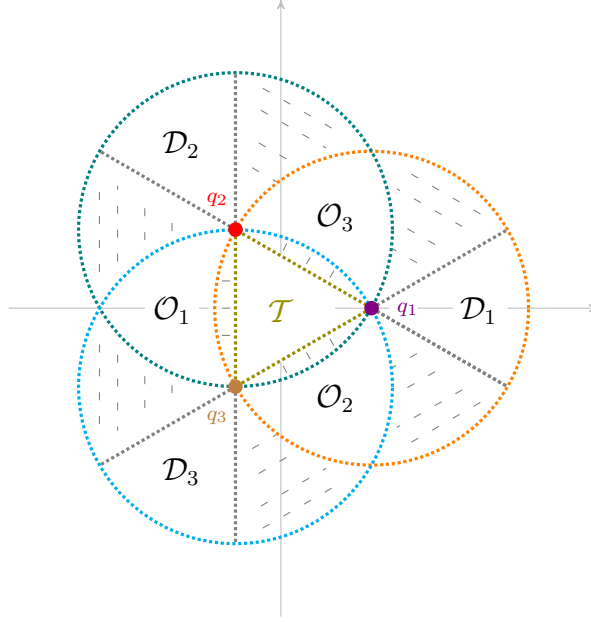


Figure 2: \mathcal{S} is pictured with dotted lines. Notice that $\mathbb{R}^2 \setminus \mathcal{S}$ has 17 open connected components, including the unbounded one.

hand we present a new proof which is based on Brouwer's degree theory and does not rely on computers.

The lemma below is well-known. To the best of our knowledge, the statement in (a) was first proved in [28, II. 7], see also [18, Lemma 3.2]. The statement in (b) can be obtained by combining the results in [15, §4] and, for instance, [11, Proposition 6(b)].

Lemma 4.2. *The following hold:*

- (a) *For any values of the masses $m_1, m_2, m_3 > 0$ with $m_1 + m_2 + m_3 = 3\sqrt{3}$ one has*

$$(V')^{-1}(0) \subset \mathcal{T} \cup \bigcup_{i=1}^3 \mathcal{D}_i \cup \bigcup_{i=1}^3 \mathcal{O}_i.$$

- (b) *In the particular case $m_1 = m_2 = m_3 = \sqrt{3}$ there are exactly ten critical points of V , all of which are nondegenerate. More precisely, there is a local maximum at the origin, three additional saddle points in \mathcal{T} , one local maximum in each set \mathcal{O}_i , and one saddle point in each set \mathcal{D}_i .*

In particular, Lemma 4.2(a) states that V does not have critical points on \mathcal{S} . Since $|V'|$ can be considered a continuous map from \mathbb{R}^2 to $(0, +\infty]$, a standard compactness argument shows that the minimal distance from \mathcal{S} to the set of critical points of V is bounded from below by some positive constant. It allows us to consider the *generalized Brouwer degrees*

$$d_B(V', \mathcal{T}), \quad d_B(V', \mathcal{O}_i), \quad d_B(V', \mathcal{D}_i), \quad i = 1, 2, 3,$$

by which we mean the Brouwer degrees of V' on slightly smaller open sets. For instance, setting $\mathcal{T}_\epsilon := \{q \in \mathcal{T} : \text{dist}(q, \partial\mathcal{T}) > \epsilon\}$, we define $d_B(V', \mathcal{T})$ as the Brouwer degree of V' on \mathcal{T}_ϵ for $\epsilon > 0$ small enough.

We observe that, even though the potential V depends on the masses m_1, m_2, m_3 , the above Brouwer indexes do not. To check this assertion we first point out that, if the masses are assumed to belong to some compact subset of $\{(m_1, m_2, m_3) \in \mathbb{R}^3 : m_i > 0 \forall i = 1, 2, 3, m_1 + m_2 + m_3 = 3\sqrt{3}\}$, then the positive lower bound for the distance between points of \mathcal{S} and critical points of V can be chosen uniformly with respect to the masses. Therefore our claim comes from the homotopy invariance of the Brouwer degree. We arrive to the following

Lemma 4.3. $d_B(V', \mathcal{T}) = -2$, $d_B(V', \mathcal{O}_i) = 1$ and $d_B(V', \mathcal{D}_i) = -1$ for $i = 1, 2, 3$. It holds true for any choice of the masses $m_1, m_2, m_3 > 0$ with $m_1 + m_2 + m_3 = 3\sqrt{3}$.

Proof. In view of the comments above, it suffices to show the result when $m_1 = m_2 = m_3 = \sqrt{3}$. In this case the result follows immediately from Lemma 4.2(b) and the additivity property of the Brouwer degree. \square

Proof of Theorem 4.1. Statement (i) follows directly Lemma 4.3. On the other hand, (ii) now follows from Theorem 3.3. In fact, assumptions $[\mathbf{H}_{1-2}]$ are easy to check, while assumption $[\mathbf{V}]$ was proven for this situation in [17, Theorem 2.1]. It concludes the argument. \square

5 Branches of closed orbits emanating from equilibria of Hamiltonian systems

Many results concerning the bifurcation of closed orbits from equilibria of Hamiltonian systems are available in the literature. In this section we present, without proof, a classic theorem in this direction that can be obtained by means of degree theory for equivariant gradient maps. We also recall De Gua's corollary of Descartes' rule of signs, a result which allows the computation of the total multiplicity of the positive roots of a real polynomial by looking at the number of sign changes in the list of its coefficients. The sole assumption is that the polynomial should have *only real roots*, which is the case for the characteristic polynomial of a symmetric matrix.

Consider a general (autonomous) Hamiltonian system:

$$\dot{u}(t) = J_N H'(u(t)). \quad (\text{HS})$$

Here $J_N = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix} \in \mathbb{R}^{2N \times 2N}$ is the standard symplectic matrix, $H : \mathcal{U} \rightarrow \mathbb{R}$ is a C^2 function defined on the open set $\mathcal{U} \subset \mathbb{R}^N \times \mathbb{R}^N$, and H' stands for its gradient. Through this section N could be any natural number, and is not restricted to 2 or 3. We further assume that all critical points of H are isolated. We pick an stationary point $u_0 \in \mathcal{U}$ and denote by $A := H''(u_0)$ the Hessian matrix of H at u_0 .

It is well-known that the eigenvalues of the Hamiltonian matrix $J_N A$ (usually referred to as the characteristic exponents at u_0) play an important role in the

dynamics of (HS) near the stationary solution. For instance, the Hartman-Grobman theorem implies that if $\sigma(J_N A) \cap (i\mathbb{R}) = \emptyset$ then there are no closed orbits of (HS) in a neighborhood of u_0 . This is the so-called hyperbolic case.

On the other hand, a sufficient condition is given by Lyapunov's center theorem: under the nondegeneracy condition $\det A \neq 0$, the presence of a purely-imaginary characteristic exponent λi with algebraic multiplicity 1 and no higher-order resonances implies the existence of an emanating local branch of closed orbits.

We are interested in generalizing this result in two directions: firstly, we would like to soften the condition on the purely-imaginary eigenvalue to be simple, and secondly we wish to obtain global (in the sense of Definition 2.1) rather than local branches. With this purpose we consider, for each $T > 0$, the symmetric matrix

$$S_T := \left(\begin{array}{c|c} -\frac{T}{2\pi}A & -J_N \\ \hline J_N & -\frac{T}{2\pi}A \end{array} \right) \in \mathbb{R}^{4N \times 4N}. \quad (5)$$

To the best of our knowledge, this kind of construction was introduced by Szulkin [37] while studying the local bifurcation of closed orbits via equivariant Morse theory; in fact, $S_T = T_1 \left(\frac{T}{2\pi}A \right)$ in Szulkin's symbols. We have opted for this change in the notation since the letter T stands for period throughout this paper.

Calling Υ the $4N$ -dimensional vector space

$$\Upsilon := \{(\sin \theta)v_1 + (\cos \theta)v_2 : v_1, v_2 \in \mathbb{R}^{2N}\} \subset C(\mathbb{S}^1, \mathbb{R}^{2N}),$$

denoting by $L_T : \Upsilon \rightarrow \Upsilon$ the linear map defined by

$$L_T \bar{\zeta} := -J_N \left[\left(\frac{d\bar{\zeta}}{d\theta} \right) - \frac{T}{2\pi} J_N A \bar{\zeta} \right], \quad \zeta \in \Upsilon,$$

and letting $\{e_1, \dots, e_{2N}\}$ be the canonical basis of \mathbb{R}^{2N} , one can think of S_T as being the matrix of L_T with respect to the basis of Υ given by

$$\left\{ (\sin \theta)e_1, (\sin \theta)e_2, \dots, (\sin \theta)e_{2N}, (\cos \theta)e_1, (\cos \theta)e_2, \dots, (\cos \theta)e_{2N} \right\}.$$

Therefore, via the linear reparametrization $\theta = 2\pi t/T$, the kernel of S_T corresponds to the set of sinusoidal closed curves of pure frequency $2\pi/T$ which solve the linearization of (HS) at $u(t) \equiv u_0$.

Given $T > 0$ it is well-known that S_T is singular if and only if $\frac{2\pi}{T}i \in \sigma(J_N A)$; we include a direct proof in Corollary 10.3 for completeness. Letting T vary on $(0, +\infty)$, the Morse index $m^-(S_T)$ may change only at these values. For any $T > 0$ the bifurcation number¹ $\gamma_H(T, u_0)$ is defined as follows:

$$\gamma_H(T, u_0) := i_B(u_0, H') \lim_{\epsilon \searrow 0} \left(\frac{m^-(S_{T+\epsilon}) - m^-(S_{T-\epsilon})}{2} \right). \quad (6)$$

¹The so-called *bifurcation index* has been more extensively studied in the literature, see, e.g., [7, 13]. The bifurcation number considered here is just the \mathbb{Z}_1 -component of the bifurcation index.

For instance, in the nondegenerate case $\det(H''(u_0)) \neq 0$ and $\gamma_H(T, u_0)$ is the sign of $\det(H''(u_0))$ times $(m^-(S_{T+\epsilon}) - m^-(S_{T-\epsilon}))/2$ for $\epsilon > 0$ small enough. We point out that the bifurcation number is an integer because the Morse index $m^-(S_T)$, which coincides with the total multiplicity of the negative eigenvalues of S_T , is always even, see Corollary 10.4. On the other hand, $\gamma_H(T, u_0)$ will certainly be zero if $\frac{2\pi}{T}i$ is not a characteristic exponent. The result from equivariant degree theory which we shall need in this paper is the following:

Theorem 5.1 (Dancer and Rybicki [7]). *If $\gamma_H(T, u_0) \neq 0$ for some $T > 0$, then there is a branch $\widehat{\mathcal{B}}$ of closed orbits of (HS) emanating from (T, u_0) .*

Throughout this paper and also in this result, the word *branch* should be understood in the sense of Definition 2.1. Thus, by a *branch* of solutions of (HS) we mean a connected component $\widehat{\mathcal{B}}$ of the closure $\widehat{\Lambda}$ of the set of nontrivial closed orbits of (HS) which, either: (a) is unbounded, or (b) goes up to the boundary of \mathcal{U} , or (c) is compact and contains at least two trivial closed orbits. Moreover, in this latter case the sum of the bifurcation numbers of the trivial closed orbits in $\widehat{\mathcal{B}}$ can be shown to vanish, an observation which will not be used in this paper.

We point out that, while the original literature deals with globally-defined Hamiltonians $H : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ (thus excluding possibility (b)), the more general situation considered in this paper can be dealt with by using the same arguments.

In the particular case of Hamiltonian systems, Theorem 5.1 extends the classical statement of Lyapunov's center theorem: it can be checked that if λi with $\lambda > 0$ is an algebraically simple characteristic exponent then $m^-(S_T)$ changes when T crosses $2\pi/\lambda$. In fact, this statement keeps its validity if λi has odd algebraic multiplicity, giving rise to results in the line of Krasnoselskii's celebrated theorem. We shall not use these facts in this paper.

In order to apply Theorem 5.1 one needs to compute the Morse indexes of the matrices S_T . Since these matrices are real and symmetric, their eigenvalues are real and De Gua's corollary of Descartes' rule of signs [14, Théorème III] will be useful. The precise statement of this result is given next:

Theorem 5.2 (De Gua [14]). *Let $p(\lambda) = d_k\lambda^k + d_{k-1}\lambda^{k-1} + \dots + d_1\lambda + d_0$ be a real polynomial without complex nonreal roots. Then the total multiplicity of the positive roots of p coincides with the number of sign changes in the ordered list of coefficients*

$$d_k, d_{k-1}, \dots, d_1, d_0,$$

where the zero elements that might possibly occur are to be removed.

6 From a second-order equation to a Hamiltonian system

From now on our goal will be to prove the results announced in Section 3. For this reason, and until the end of Section 9, we go back now to the general framework and notation of Sections 2-3. We shall start with the following observation: setting $p :=$

$\dot{q} - \alpha_N q$ and $u = (p, q) \in \mathbb{R}^N \times \mathbb{R}^N$, equation (1) can be rewritten as a Hamiltonian system. More precisely, one gets system (HS) for the Hamiltonian function

$$H(p, q) = \frac{1}{2}|p|^2 + \langle p, \alpha_N q \rangle + W(q), \quad (p, q) \in \mathcal{U} := \mathbb{R}^N \times \Omega, \quad (7)$$

the *amended potential* $W : \Omega \rightarrow \mathbb{R}$ being defined by

$$W(q) := V(q) - \frac{1}{2}\langle q, \alpha_N^2 q \rangle, \quad q \in \Omega,$$

or, more explicitly,

$$\begin{cases} W(x, y) = V(x, y) + (x^2 + y^2)/2, & (x, y) \in \Omega, & \text{in case [2d]}, \\ W(x, y, z) = V(x, y, z) + (x^2 + y^2)/2, & (x, y, z) \in \Omega, & \text{in case [3d]}. \end{cases}$$

From now on *it will always be assumed that the Hamiltonian H is given by (7)*.

The equilibria of (HS) are in a 1:1 correspondence with the equilibria of (1). More specifically, $u_0 = (p_0, q_0)$ is a critical point of H if and only if q_0 is a critical point of V and $p_0 = -\alpha_N q_0$. In this case, for any $T > 0$ we shall say that the pair (T, u_0) is a *trivial closed orbit* of (HS).

On the other hand, the map $\Phi(T, \bar{q}) := (T; \bar{q}, \bar{p})$ defined by $\bar{p} := (2\pi/T)(d\bar{q}/d\theta) - \alpha_N \bar{q}$ establishes a 1:1 correspondence between the closure Λ of the set of nontrivial closed orbits of (1), which we see as a subset of $(0, +\infty) \times C(\mathbb{S}^1, \Omega)$, and the closure $\hat{\Lambda}$ of the set of nontrivial closed orbits of (HS), regarded as a subset of $(0, +\infty) \times C(\mathbb{S}^1, \mathcal{U})$. There is no difficulty in translating Definition 2.1 to this context: by a *branch* of solutions of (HS) we shall mean a connected component $\hat{\mathcal{B}}$ of $\hat{\Lambda}$ which, either: (a) is unbounded, or (b) goes up to the boundary of \mathcal{U} , or (c) is compact and contains at least two trivial closed orbits.

There are a number of properties that q_0 may have as an equilibrium of (1) and are inherited by u_0 as an equilibrium of (HS); we collect some of them in Lemma 6.1 below. Assertion (ii) below should be read in the spirit of the comments following the statement of Theorem 3.1.

Lemma 6.1. *Let $u_0 = (p_0, q_0)$ be an equilibrium of (HS). Then, the following hold:*

- (i) u_0 is isolated as a critical point of H .
- (ii) System (1) does not have closed orbits in a sufficiently small neighborhood of q_0 if and only if (HS) does not have closed orbits in a sufficiently small neighborhood of u_0 .
- (iii) Given $T > 0$, there is a branch of closed orbits of (1) emanating from (T, q_0) if and only if there is a branch of closed orbits of (HS) emanating from (T, u_0) .

Proof. (i): The map $q \in \Omega \mapsto (-\alpha_N q, q)$ is an embedding sending q_0 into u_0 , and, both in the planar and the spatial cases, it carries the critical points of V in a neighborhood of q_0 into the critical points of H in a neighborhood of u_0 . Therefore, assumption [V] implies the statement. (ii): The nontrivial implication is contained

in the following claim: if $q_n : \mathbb{R} \rightarrow \mathbb{R}^N$ is a sequence of solutions of (1) uniformly converging to q_0 then $\dot{q}_n \rightarrow 0$ uniformly on \mathbb{R} . We check this statement by a contradiction argument and assume that $\{q_n\}$ is as above but there exists some sequence $\{t_n\} \subset \mathbb{R}$ such that $|\dot{q}_n(t_n)| \geq \epsilon$ for some $\epsilon > 0$ and every $n \in \mathbb{N}$. Since equation (1) is autonomous there is no loss of generality in assuming that $t_n = 0$ for every n . On the other hand, after possibly passing to a subsequence there is no loss of generality in assuming that either $\dot{q}_n(0) \rightarrow \dot{q}_0 \in \mathbb{R}^N$ or $|\dot{q}_n(0)| \rightarrow +\infty$ as $n \rightarrow +\infty$. In the first case, continuous dependence would imply that q_n converges uniformly on the compact interval $[0, 1]$ to the solution of $\ddot{q} - 2\alpha_N \dot{q} + V'(q) = 0$, $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$, which contradicts the fact that $q_n \rightarrow q_0$ uniformly on \mathbb{R} . In the second case one can repeat the argument with $r_n := (1/|\dot{q}_n(0)|)q_n$, which, for each $n \in \mathbb{N}$ solves the linear equation $\ddot{r} - 2\alpha_N \dot{r} + V'(q_n(t))/|\dot{q}_n(0)| = 0$, and passing to the limit as $n \rightarrow +\infty$ one arrives similarly to a contradiction. (iii): One immediately checks that a set $\mathcal{B} \subset \Lambda$ is a branch of closed orbits of (1) if and only if $\tilde{\mathcal{B}} := \Phi(\mathcal{B})$ is a branch of closed orbits of (HS). \square

We close this section with a result relating the Brouwer indexes of H' and V' at the isolated equilibria $u_0 = (p_0, q_0)$ and q_0 , respectively.

Lemma 6.2. *Let $u_0 = (p_0, q_0) \in \mathbb{R}^N \times \Omega$ be a critical point of H . Then, $i_B(u_0, H') = i_B(q_0, V')$. Moreover, in the spatial case this Brouwer index coincides with $i_B(\tilde{q}_0, \tilde{V}')$.*

Proof. We consider first the planar case [2d]. Setting $\Pi := \left(\begin{array}{c|c} I_2 & 0_2 \\ \hline \alpha_2 & I_2 \end{array} \right)$, which is a 4×4 matrix with determinant 1, we see that

$$\Pi H'(p, q) = (p + \alpha_2 q, V'(q)) = \mathcal{H}_0(p, q), \quad (p, q) \in \mathbb{R}^2 \times \Omega,$$

where $\mathcal{H}_\lambda(p, q) := (p + (1-\lambda)\alpha_2 q, V'(q))$. Now, witting $u_\lambda := ((1-\lambda)p_0, q_0)$ we notice that $\mathcal{H}_\lambda(u_\lambda) = (0, 0)$ for every $\lambda \in [0, 1]$. Moreover, q_0 being an isolated critical point of V , one easily checks that u_λ is an isolated zero of \mathcal{H}_λ and indeed the Brouwer index $i_B(u_\lambda, \mathcal{H}_\lambda)$ does not depend on $\lambda \in [0, 1]$. Thus, $i_B((p_0, q_0), H') = i_B((p_0, q_0), \mathcal{H}_0) = i_B((0, q_0), \mathcal{H}_1)$. Since $\mathcal{H}_1(p, q) = (p, V'(q))$, the multiplicative property of the Brouwer degree gives $i_B((0, q_0), \mathcal{H}_1) = i_B(q_0, V')$ and thus concludes the proof.

In the spatial case [3d] we use the notation $(p, q) \equiv (\tilde{p}, \tilde{q}, p_3, q_3) \in \mathbb{R}^4 \times \mathbb{R}^2$ and observe that

$$H'(p, q) = \left(\tilde{H}'(\tilde{p}, \tilde{q}); p_3, \frac{\partial V}{\partial z}(q) \right),$$

where \tilde{H} is the Hamiltonian corresponding to the planar case for the potential \tilde{V} . Remembering [H₁₋₂] and using the result for the planar case we see that

$$i_B(u_0, H') = i_B((\tilde{p}_0, \tilde{q}_0), \tilde{H}') = i_B(\tilde{q}_0, \tilde{V}') = i_B(q_0, V'),$$

proving the result. \square

7 On the spectrum of the linearization

Almost any study of the dynamics of (HS) near the equilibrium q_0 should begin with an analysis of the spectrum of the Hamiltonian matrix $J_N H''(u_0)$. Notice that

$$A := H''(u_0) = \left(\begin{array}{c|c} I_N & \alpha_N \\ \hline -\alpha_N & W''(q_0) \end{array} \right), \quad (8)$$

and we shall exploit this fact to reduce the order of certain determinants. The first result of this section, which is valid both in the planar and the spatial cases, shows that in the computation of the characteristic polynomial of $J_N A$ one can replace the $2N \times 2N$ matrix $J_N A - \lambda I_{2N}$ by the $N \times N$ matrix $V''(q_0) + \lambda^2 I_N - 2\lambda \alpha_N$.

Lemma 7.1. *For any $\lambda \in \mathbb{C}$ the following equality holds:*

$$\det(J_N A - \lambda I_{2N}) = \det(V''(q_0) + \lambda^2 I_N - 2\lambda \alpha_N).$$

Proof. It is well-known (and follows from Lemma 10.1) that $\det J_N = 1$. Moreover, one has:

$$\begin{aligned} J_N(J_N A - \lambda I_{2N}) &= -A - \lambda J_N = - \left(\begin{array}{c|c} I_N & \alpha_N \\ \hline -\alpha_N & W''(q_0) \end{array} \right) - \lambda \left(\begin{array}{c|c} 0_N & -I_N \\ \hline I_N & 0_N \end{array} \right) = \\ &= \left(\begin{array}{c|c} -I_N & -\alpha_N + \lambda I_N \\ \hline \alpha_N - \lambda I_N & -W''(q_0) \end{array} \right). \end{aligned}$$

If, in the last term, one adds to the last N rows the product of $\alpha_N - \lambda I_N$ by the first N rows, one obtains $\left(\begin{array}{c|c} -I_N & -\alpha_N + \lambda I_N \\ \hline 0_N & -W''(q_0) - (\alpha_N - \lambda I_N)^2 \end{array} \right)$; in particular, this latter matrix and $J_N A - \lambda I_{2N}$ have the same determinant. The result follows after remembering that $W''(q_0) + \alpha_N^2 = V''(q_0)$. \square

As in Section 3, we denote by $\beta_1, \beta_2 \in \mathbb{R}$ the eigenvalues of $V''(q_0)$ in the planar case, or the eigenvalues of $\tilde{V}''(\tilde{q}_0)$ in the spatial problem; in the latter situation we also set $\beta_3 := \beta_3(q_0) = \frac{\partial^2 V}{\partial z^2}(q_0) > 0$. Our next task is to compute explicitly the characteristic polynomial of $J_N A$ in terms of these numbers.

Corollary 7.2. *In the planar case the characteristic polynomial $\mathfrak{p}_2(\lambda) := \det(J_2 A - \lambda I_4)$ is given by*

$$\mathfrak{p}_2(\lambda) = \lambda^4 + (\beta_1 + \beta_2 + 4)\lambda^2 + \beta_1 \beta_2, \quad (9)$$

and in the three-dimensional case [3d] the characteristic polynomial $\mathfrak{p}_3(\lambda) := \det(J_3 A - \lambda I_6)$ is given by

$$\mathfrak{p}_3(\lambda) = (\lambda^2 + \beta_3) \mathfrak{p}_2(\lambda),$$

the fourth-order polynomial $\mathfrak{p}_2(\lambda)$ being defined by (9).

Proof. Let us start by considering the planar case [2d]. Lemma 7.1 gives

$$\mathfrak{p}_2(\lambda) = \det(V''(q_0) + \lambda^2 I_2 - 2\lambda \alpha_2).$$

On the other hand, $V''(q_0) \in \mathbb{R}^{2 \times 2}$ is symmetric and therefore, after possibly changing the order of the β_i 's, there exists a rotation $\mathcal{R} \in SO_2(\mathbb{R})$ such that $\mathcal{R}^T V''(q_0) \mathcal{R} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$. The matrices \mathcal{R} and α_2 commute, and therefore,

$$\begin{aligned} \mathbf{p}_2(\lambda) &= \det(\mathcal{R}^T V''(q_0) \mathcal{R} + \lambda^2 I_2 - 2\lambda \alpha_2) = \\ &= \det \begin{pmatrix} \beta_1 + \lambda^2 & 2\lambda \\ -2\lambda & \beta_2 + \lambda^2 \end{pmatrix} = \lambda^4 + (\beta_1 + \beta_2 + 4)\lambda^2 + \beta_1 \beta_2, \end{aligned}$$

thus showing (9).

In the three-dimensional situation one may differentiate in $[\mathbf{H}_1]$ to find

$$\frac{\partial^2 V}{\partial x \partial z}(x, y, 0) = 0 = \frac{\partial^2 V}{\partial y \partial z}(x, y, 0), \quad (x, y) \in \tilde{\Omega},$$

so that the Hessian of V at the point $q_0 = (\tilde{q}_0, 0) \in \tilde{\Omega} \times \{0\}$ has the form

$$V''(q_0) = \left(\begin{array}{c|c} \tilde{V}''(q_0) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & \beta_3 \end{array} \right).$$

The result follows from Lemma 7.1. \square

Set $\mathbf{q}(x) := x^2 + (\beta_1 + \beta_2 + 4)x + \beta_1 \beta_2$, which is a quadratic polynomial with $\mathbf{p}_2(\lambda) = \mathbf{q}(\lambda^2)$. In this way there is a 1 : 1 correspondence between the set of couples $\pm \lambda i \neq 0$ of purely-imaginary roots of \mathbf{p}_2 and the set of real negative roots of \mathbf{q} . We further observe that the discriminant of \mathbf{q} is given by $\Delta := (\beta_1 + \beta_2 + 4)^2 - 4\beta_1 \beta_2 = 8(\beta_1 + \beta_2 + 2 + (\beta_1 - \beta_2)^2/8)$. It follows that the nonzero, purely-imaginary roots of \mathbf{p}_2 are given by $\pm i 2\pi/T_-(\beta_1, \beta_2)$, $\pm i 2\pi/T_+(\beta_1, \beta_2)$, wherever defined. Here, the functions $T_- : (\mathbb{R}^2 \setminus \mathcal{R}_0) \cup ((\partial \mathcal{R}_0) \setminus C) \rightarrow (0, +\infty)$ and $T_+ : \mathcal{R}_1 \cup \mathcal{R}_3 \cup ((\partial \mathcal{R}_0) \setminus C) \rightarrow (0, +\infty)$ are as described in (2). Notice that:

Lemma 7.3. T_{\pm} are strictly positive in their respective domains. Moreover, $T_-(\beta_1, \beta_2) \leq T_+(\beta_1, \beta_2)$ on $\mathcal{R}_1 \cup \mathcal{R}_3 \cup ((\partial \mathcal{R}_0) \setminus C)$, and the equality holds if and only if $(\beta_1, \beta_2) \in (\partial \mathcal{R}_0) \setminus C$.

As announced in Section 3, the functions T_{\pm} can be used to draw a global picture of the purely-imaginary characteristic exponents at u_0 . More precisely, one has the following:

Lemma 7.4. *In the planar case the following hold:*

- (i) If $(\beta_1, \beta_2) \in \mathcal{R}_0$ then $\sigma(J_2 A) \cap (i\mathbb{R}) = \emptyset$.
- (ii) If $(\beta_1, \beta_2) \in \mathcal{R}_2 \cup \mathcal{R}_4$ then $\sigma(J_2 A) \cap (i\mathbb{R}) = \left\{ -\frac{2\pi}{T_-(\beta_1, \beta_2)} i, \frac{2\pi}{T_-(\beta_1, \beta_2)} i \right\}$.
- (iii) If $(\beta_1, \beta_2) \in \mathcal{R}_1 \cup \mathcal{R}_3$ then $\sigma(J_2 A) = \left\{ -\frac{2\pi}{T_-(\beta_1, \beta_2)} i, \frac{2\pi}{T_-(\beta_1, \beta_2)} i, -\frac{2\pi}{T_+(\beta_1, \beta_2)} i, \frac{2\pi}{T_+(\beta_1, \beta_2)} i \right\}$.
- (iv) If $(\beta_1, \beta_2) \in (\partial \mathcal{R}_0) \setminus C$ then $\sigma(J_2 A) = \left\{ -\frac{2\pi}{T_-(\beta_1, \beta_2)} i, \frac{2\pi}{T_-(\beta_1, \beta_2)} i \right\} = \left\{ -\frac{2\pi}{T_+(\beta_1, \beta_2)} i, \frac{2\pi}{T_+(\beta_1, \beta_2)} i \right\}$.

(v) If $(\beta_1, \beta_2) \in C \cap (\partial\mathcal{R}_0)$ then $\sigma(J_2A) \cap (i\mathbb{R}) = \{0\}$.

(vi) If $(\beta_1, \beta_2) \in C \setminus (\partial\mathcal{R}_0)$ then $\sigma(J_2A) = \left\{0, -\frac{2\pi}{T_-(\beta_1, \beta_2)}i, \frac{2\pi}{T_-(\beta_1, \beta_2)}i\right\}$.

Furthermore, in the spatial situation one obtains $\sigma(J_3A) \cap (i\mathbb{R})$ by adding $\pm\sqrt{\beta_3}i$ to the previously-described lists.

Proof. There are several of cases to be considered, but the arguments being similar, we shall only study in detail the nondegenerate situation $\beta_1 \neq 0 \neq \beta_2$ in which \mathfrak{p}_2 has exactly one pair $\pm\lambda i$ of purely imaginary roots. This is equivalent to say that \mathfrak{q} does not vanish at the origin and has exactly one negative root, which will happen if and only if, either $\mathfrak{q}(0) < 0$, or $\mathfrak{q}(0), \dot{\mathfrak{q}}(0) > 0$ and $\Delta = 0$. The first possibility corresponds to case (ii) and then, the purely-imaginary roots $\pm\lambda i = \pm\frac{2\pi}{T_-(\beta_1, \beta_2)}i$ are simple. The second option corresponds to case (iv) and then, the purely-imaginary roots $\pm\lambda i = \pm\frac{2\pi}{T_-(\beta_1, \beta_2)}i = \pm\frac{2\pi}{T_+(\beta_1, \beta_2)}i$ have algebraic multiplicity 2. \square

8 Computing Morse indexes via De Gua's corollary of Descartes' rule of signs

The purpose of this section is to compute explicitly the Morse index of S_T as a function of the parameter $T > 0$ and the eigenvalues of $V''(q_0)$. In the planar situation this latter plan will result in the following

Proposition 8.1. *In the planar case, the following hold for every $T > 0$:*

(i) If $(\beta_1, \beta_2) \in \mathcal{R}_0 \cup ((\partial\mathcal{R}_0) \cap C)$, then $m^-(S_T) = 4$, independently of the value of $T > 0$.

(ii) If $(\beta_1, \beta_2) \in (\partial\mathcal{R}_0) \setminus C$, then $m^-(S_T) = 4$ for any $T > 0$ with $T \neq T_-(\beta_1, \beta_2)$.

(iii) If $(\beta_1, \beta_2) \in \mathcal{R}_2 \cup \mathcal{R}_4 \cup (C \setminus (\partial\mathcal{R}_0))$, then $m^-(S_T) = \begin{cases} 4, & \text{if } 0 < T < T_-(\beta_1, \beta_2), \\ 6, & \text{if } T > T_-(\beta_1, \beta_2). \end{cases}$

(iv) If $(\beta_1, \beta_2) \in \mathcal{R}_1$, then $m^-(S_T) = \begin{cases} 4, & \text{if } 0 < T < T_-(\beta_1, \beta_2), \\ 6, & \text{if } T_-(\beta_1, \beta_2) < T < T_+(\beta_1, \beta_2), \\ 8, & \text{if } T > T_+(\beta_1, \beta_2). \end{cases}$

(v) If $(\beta_1, \beta_2) \in \mathcal{R}_3$, then $m^-(S_T) = \begin{cases} 4, & \text{if } 0 < T < T_-(\beta_1, \beta_2), \\ 6, & \text{if } T_-(\beta_1, \beta_2) < T < T_+(\beta_1, \beta_2), \\ 4, & \text{if } T > T_+(\beta_1, \beta_2). \end{cases}$

In order to prove Proposition 8.1 we need to apply Theorem 5.2 on the explicit expression of the characteristic polynomial of S_T . We obtain this polynomial below with some help from Lemma 10.1 and Corollary 10.4 in the Appendix.

Lemma 8.2. *In the planar situation, the following equality holds true for every $T > 0$:*

$$\det(S_T - \lambda I_8) = (d_4 \lambda^4 + d_3 \lambda^3 + d_2 \lambda^2 + d_1 \lambda + d_0)^2,$$

where

$$\begin{cases} d_4 := 1, \\ d_3 := c_1(\beta_1, \beta_2) \frac{T}{2\pi}, \\ d_2 := \left(c_0(\beta_1, \beta_2) + 3c_1(\beta_1, \beta_2) - 8 \right) \frac{T^2}{4\pi^2} - 2, \\ d_1 := \left(c_0(\beta_1, \beta_2) + c_1(\beta_1, \beta_2) - 4 \right) \frac{T^3}{4\pi^3} - c_1(\beta_1, \beta_2) \frac{T}{2\pi}, \\ d_0 := c_0(\beta_1, \beta_2) \frac{T^4}{16\pi^4} - c_1(\beta_1, \beta_2) \frac{T^2}{4\pi^2} + 1, \end{cases}$$

and $c_0(\beta_1, \beta_2) := \beta_1 \beta_2$; $c_1(\beta_1, \beta_2) := \beta_1 + \beta_2 + 4$.

Proof. Corollary 10.4 gives

$$\det(S_T - \lambda I_8) = p_T(\lambda)^2,$$

where

$$p_T(\lambda) = \det \left(-\frac{T}{2\pi} A + iJ_2 - \lambda I_4 \right) = \det \left(\begin{array}{c|c} -(\frac{T}{2\pi} + \lambda)I_2 & -\frac{T}{2\pi} \alpha_2 - iI_2 \\ \hline \frac{T}{2\pi} \alpha_2 + iI_2 & -\frac{T}{2\pi} W''(q_0) - \lambda I_2 \end{array} \right).$$

Remembering that $W''(q_0) = V''(q_0) + I_2$ and using Lemma 10.1 we see that

$$p_T(\lambda) = \det \left(\left(\lambda^2 + \frac{T}{\pi} \lambda - 1 \right) I_2 + \left(\frac{T}{2\pi} \lambda + \frac{T^2}{4\pi^2} \right) V''(q_0) + i \frac{T}{\pi} \alpha_2 \right).$$

The Hessian matrix $V''(q_0)$ is symmetric, and we deduce that, after possibly changing the order of the β_i 's there exists a rotation $\mathcal{R} \in SO(2)$ such that $\mathcal{R}^T V''(q_0) \mathcal{R} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$. The matrices α_2 and \mathcal{R} commute, and therefore

$$\begin{aligned} p_T(\lambda) &= \det \left(\left(\lambda^2 + \frac{T}{\pi} \lambda - 1 \right) I_2 + \left(\frac{T}{2\pi} \lambda + \frac{T^2}{4\pi^2} \right) \mathcal{R}^T V''(q_0) \mathcal{R} + i \frac{T}{\pi} \alpha_2 \right) = \\ &= \det \left(\begin{array}{cc} \lambda^2 + \frac{(2+\beta_1)T}{2\pi} \lambda + \frac{\beta_1 T^2}{4\pi^2} - 1 & -\frac{T}{\pi} i \\ \frac{T}{\pi} i & \lambda^2 + \frac{(2+\beta_2)T}{2\pi} \lambda + \frac{\beta_2 T^2}{4\pi^2} - 1 \end{array} \right), \end{aligned}$$

implying the result. \square

Given $T > 0$ such that $\det(S_T) \neq 0$, the negative Morse index $m^-(S_T)$ equals the order $4N = 8$ minus the total algebraic multiplicity of the positive roots of the characteristic polynomial of S_T . By combining Lemma 8.2 with Theorem 5.2 we see that $m^-(S_T)$ equals 8 minus twice the number of sign changes in the ordered sequence $\{d_4, d_3, d_2, d_1, d_0\}$. In particular, it depends only on T and the (unordered) eigenvalues β_1, β_2 . For this reason, in the following we shall assume, without loss of generality, that $V''(q_0) = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$ is diagonal, and write $S_T(\beta_1, \beta_2)$ instead of S_T .

Proof of Proposition 8.1. In view of Corollary 10.3 and Lemmas 7.3-7.4, the set

$$\Gamma = \left\{ S_T(\beta_1, \beta_2) : (\beta_1, \beta_2) \in \mathbb{R}^2, T > 0, \det(S_T(\beta_1, \beta_2)) \neq 0 \right\} \subset \mathbb{R}^{8 \times 8}$$

can be written as the union of three connected components $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. For instance, Γ_1 may be taken as the set of matrices $S_T(\beta_1, \beta_2)$ such that, either: $(\beta_1, \beta_2) \in \mathcal{R}_0 \cup (C \cap (\partial\mathcal{R}_0))$ and $T > 0$, or $(\beta_1, \beta_2) \in (\partial\mathcal{R}_0) \setminus C$ and $T \neq T_+(\beta_1, \beta_2)$, or $(\beta_1, \beta_2) \in \mathcal{R}_3$ and $T > T_+(\beta_1, \beta_2)$, or $(\beta_1, \beta_2) \in \mathbb{R}^2 \setminus \bar{\mathcal{R}}_0$ and $T < T_-(\beta_1, \beta_2)$. Similarly, one may take

$$\begin{cases} \Gamma_2 = \{ S_T(\beta_1, \beta_2) : (\beta_1, \beta_2) \in \mathcal{R}_1, T > T_+(\beta_1, \beta_2) \}, \\ \Gamma_3 = \{ S_T(\beta_1, \beta_2) : (\beta_1, \beta_2) \in \mathbb{R}^2 \setminus \bar{\mathcal{R}}_0, T_-(\beta_1, \beta_2) < T < T_+(\beta_1, \beta_2) \}, \end{cases}$$

where $T_+(\beta_1, \beta_2) := +\infty$ if $(\beta_1, \beta_2) \in \mathcal{R}_2 \cup \mathcal{R}_4 \cup (C \setminus (\partial\mathcal{R}_0))$. Moreover, Lemmas 7.3-7.4 imply that Γ does not contain singular matrices, and we deduce that the Morse index has a constant value on each Γ_i . All what remains to do is to pick some matrix in each set and to compute its Morse index.

In this task, the computation of the characteristic polynomial of $S_T(\beta_1, \beta_2)$ carried out in Lemma 8.2 will be useful. We start with the choice $(\beta_1, \beta_2) := (1, 1) \in \mathcal{R}_1$. Notice that $c_0(1, 1) = 1$ and $c_1(1, 1) = 6$, and so $\det(S_T(1, 1) - \lambda I_8) = (d_4\lambda^4 + d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0)^2$, the coefficients d_i being given by

$$d_4 = 1, \quad d_3 = \frac{3T}{\pi}, \quad d_2 = \frac{11T^2}{4\pi^2} - 2, \quad d_1 = \frac{3T^3}{4\pi^3} - \frac{3T}{\pi}, \quad d_0 = \frac{T^4}{16\pi^4} - \frac{3T^2}{2\pi^2} + 1.$$

For $T > 0$ small enough, $d_4, d_3, d_0 > 0$ while $d_2, d_1 < 0$, so that there are two sign changes in the ordered list d_4, d_3, d_2, d_1, d_0 and (by Theorem 5.2) we deduce that $m^-(S_T(\beta_1, \beta_2)) = 8 - 4 = 4$ if $S_T(\beta_1, \beta_2) \in \Gamma_1$. For $T > 0$ big enough, all these coefficients are positive, there are no sign changes in the ordered list d_4, d_3, d_2, d_1, d_0 and therefore the positive Morse index of $S_T(1, 1)$ is zero. Consequently, $m^-(S_T(\beta_1, \beta_2)) = 8 - 0 = 8$ if $S_T(\beta_1, \beta_2) \in \Gamma_2$. Similarly, one checks that the Morse index of $S_T(\beta_1, \beta_2)$ is 6 on Γ_3 , thus concluding the proof. \square

We are now ready to study the Morse index of S_T in the spatial case. In this situation S_T is a 12×12 symmetric matrix depending on the parameter $T > 0$. By combining Corollaries 7.2 and 10.3 we see that $\det S_T = 0$ if and only if either $T = \frac{2\pi}{\sqrt{\beta_3}}$ or $\det \tilde{S}_T = 0$. Here,

$$\tilde{S}_T = \left(\begin{array}{c|c} -\frac{T}{2\pi}\tilde{A} & -J_2 \\ \hline J_2 & -\frac{T}{2\pi}\tilde{A} \end{array} \right) \in \mathbb{R}^{8 \times 8}, \quad \tilde{A} = \left(\begin{array}{c|c} I_2 & \alpha_2 \\ \hline -\alpha_2 & \tilde{W}''(\tilde{q}_0) \end{array} \right) \in \mathbb{R}^{4 \times 4},$$

and $\tilde{W}''(\tilde{q}_0) = \tilde{V}''(\tilde{q}_0) + I_2 \in \mathbb{R}^{2 \times 2}$. On the other hand, the Morse index $m^-(\tilde{S}_T)$ was already computed in Proposition 8.1 as a function of $T > 0$ and the eigenvalues β_1, β_2 of $\tilde{V}''(q_0)$. Notice that

Proposition 8.3. *Let $T > 0$ be such that $\det(\tilde{S}_T) \neq 0$. Then,*

$$m^-(S_T) = \begin{cases} m^-(\tilde{S}_T) + 2 & \text{if } 0 < T < \frac{2\pi}{\sqrt{\beta_3}}, \\ m^-(\tilde{S}_T) + 4 & \text{if } T > \frac{2\pi}{\sqrt{\beta_3}}. \end{cases}$$

Proof. Corollary 10.4 states that $\det(S_T - \lambda I_{12}) = p_T(\lambda)^2$, where,

$$p_T(\lambda) := \det \left(-\frac{T}{2\pi} A + iJ_3 - \lambda I_6 \right) = \det \left(\begin{array}{c|c|c|c} -(\frac{T}{2\pi} + \lambda)I_2 & 0_{2 \times 1} & -\frac{T}{2\pi}\alpha_2 + iI_2 & 0_{2 \times 1} \\ \hline 0_{1 \times 2} & -\frac{T}{2\pi} - \lambda & 0_{1 \times 2} & i \\ \hline \frac{T}{2\pi}\alpha_2 - iI_2 & 0_{2 \times 1} & -\frac{T}{2\pi}\widetilde{W}''(q_0) - \lambda I_2 & 0_{2 \times 1} \\ \hline 0_{1 \times 2} & -i & 0_{1 \times 2} & -\frac{\beta_3 T}{2\pi} - \lambda \end{array} \right).$$

Here we have used the form (8) of the matrix A . Rearranging rows and columns we see that

$$p_T(\lambda) = \det \left(\begin{array}{c|c|c|c} -(\frac{T}{2\pi} + \lambda)I_2 & -\frac{T}{2\pi}\alpha_2 + iI_2 & 0_{2 \times 1} & 0_{2 \times 1} \\ \hline \frac{T}{2\pi}\alpha_2 - iI_2 & -\frac{T}{2\pi}\widetilde{W}''(q_0) - \lambda I_2 & 0_{2 \times 1} & 0_{2 \times 1} \\ \hline 0_{1 \times 2} & 0_{1 \times 2} & -\frac{T}{2\pi} - \lambda & i \\ \hline 0_{1 \times 2} & 0_{1 \times 2} & -i & -\frac{\beta_3 T}{2\pi} - \lambda \end{array} \right).$$

The four blocks in the upper-left corner constitute the matrix $-\frac{T}{2\pi}\tilde{A} + iJ_2 - \lambda I_4$, whose determinant will be denoted by $\tilde{p}_T(\lambda)$. On the other hand, the determinant of the 2×2 matrix in the lower-right corner is $\lambda^2 + \frac{(\beta_3+1)T}{2\pi}\lambda + \frac{\beta_3 T^2 - 4\pi^2}{4\pi^2}$. It follows that

$$p_T(\lambda) = \tilde{p}_T(\lambda) \left(\lambda^2 + \frac{(\beta_3 + 1)T}{2\pi} \lambda + \frac{\beta_3 T^2 - 4\pi^2}{4\pi^2} \right).$$

On the other hand, Corollary 10.4 states that $\det(\tilde{S}_T - \lambda I_8) = \tilde{p}_T(\lambda)^2$, leading us to the following connection between the characteristic polynomials of S_T and \tilde{S}_T :

$$\det(S_T - \lambda I_{12}) = \det(\tilde{S}_T - \lambda I_8) \left(\lambda^2 + \frac{(\beta_3 + 1)T}{2\pi} \lambda + \frac{\beta_3 T^2 - 4\pi^2}{4\pi^2} \right)^2.$$

In particular, the Morse index of S_T equals the Morse index of \tilde{S}_T plus twice the number of negative roots of the quadratic polynomial $\lambda^2 + \frac{(\beta_3+1)T}{2\pi}\lambda + \frac{\beta_3 T^2 - 4\pi^2}{4\pi^2}$. This latter number can be either computed directly or using Theorem 5.2, and the result follows. \square

9 Bifurcation numbers. Proofs of the main results

In this section we shall complete the proofs of Theorems 3.1-3.3. It will be done by combining Theorem 5.1 with the results of Section 6 and the explicit computation of the bifurcation number associated to any trivial closed orbit. The bifurcation number has been defined in (6) for stationary solutions of general Hamiltonian systems, but we shall now calculate them when the Hamiltonian has the form (7).

Thus, let $q_0 \in \Omega$ be a critical point of V and set $u_0 := (-\alpha_N q_0, q_0)$, which is a critical point of H . Given $T > 0$, the bifurcation index of the corresponding

closed orbit will be denoted, for simplicity, $\gamma_N(T, q_0) := \gamma_H(T, u_0)$. These are the bifurcation indexes mentioned at the end of Section 2. In the following result we compute them in all possible situations, both in the planar and the spatial cases.

Lemma 9.1. *Let $(T, q_0) \in (0, +\infty) \times (V')^{-1}(0)$ be a trivial closed orbit. In the planar case [2d], the bifurcation number $\gamma_2(T, q_0)$ is given as follows:*

$$\gamma_2(T, q_0) := \begin{cases} -1 & \text{if } (\beta_1, \beta_2) \in \mathcal{R}_2 \cup \mathcal{R}_4 \text{ and } T = T_-(\beta_1, \beta_2), \\ 1 & \text{if } (\beta_1, \beta_2) \in \mathcal{R}_1 \cup \mathcal{R}_3 \text{ and } T = T_-(\beta_1, \beta_2), \\ i_B(q_0, V') & \text{if } (\beta_1, \beta_2) \in C \setminus (\partial\mathcal{R}_0) \text{ and } T = T_-(\beta_1, \beta_2), \\ 1 & \text{if } (\beta_1, \beta_2) \in \mathcal{R}_1 \text{ and } T = T_+(\beta_1, \beta_2), \\ -1 & \text{if } (\beta_1, \beta_2) \in \mathcal{R}_3 \text{ and } T = T_+(\beta_1, \beta_2), \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

In the spatial case [3d], the bifurcation number $\gamma_3(T, q_0)$ is given by:

$$\gamma_3(T, q_0) := \begin{cases} \gamma_2(T, \tilde{q}_0) - 1 & \text{if } (\beta_1, \beta_2) \in \mathcal{R}_2 \cup \mathcal{R}_4 \text{ and } T = \frac{2\pi}{\sqrt{\beta_3}}, \\ \gamma_2(T, \tilde{q}_0) + 1 & \text{if } (\beta_1, \beta_2) \in (\bar{\mathcal{R}}_0 \setminus C) \cup \mathcal{R}_1 \cup \mathcal{R}_3 \text{ and } T = \frac{2\pi}{\sqrt{\beta_3}}, \\ \gamma_2(T, \tilde{q}_0) + i_B(\tilde{q}_0, \tilde{V}') & \text{if } (\beta_1, \beta_2) \in C \text{ and } T = \frac{2\pi}{\sqrt{\beta_3}}, \\ \gamma_2(T, \tilde{q}_0) & \text{if } T \neq \frac{2\pi}{\sqrt{\beta_3}}. \end{cases} \quad (11)$$

Proof. Just combine the definition of bifurcation numbers (6) with the form of $i_B(u_0, H')$ obtained in Lemma 6.2 (remember also (3)), and the computation of the Morse indexes carried out in Propositions 8.1-8.3. \square

We are now ready to complete the proof of the results announced in Section 3. We start with the proof of assertion (i) in Theorem 3.1.

Proof of Theorem 3.1. (i): The combination of Lemma 7.4(i) and the Hartman-Grobman theorem implies that (HS) does not have periodic solutions in a sufficiently small neighborhood of q_0 . The result follows from Lemma 6.1(ii). \square

It remains to establish Theorem 3.3 and assertions (ii)-(iii) of Theorem 3.1. By remembering Lemma 6.2 and in view of the bifurcation numbers computed above, we can combine all these statements into one single result. It is the following:

Theorem 9.2. *Let $(T, q_0) \in (0, +\infty) \times (V')^{-1}(0)$ be a trivial closed orbit of (1). If $\gamma_N(T, q_0) \neq 0$ then there is a branch of closed orbits of (1) emanating from (T, q_0) .*

Proof. It was seen in Lemma 6.1(i) that all critical points of H are isolated. Theorem 5.1 applies and provides the existence of a branch $\hat{\mathcal{B}}$ of closed orbits of (HS) bifurcating from u_0 . Therefore, Lemma 6.1(iii) implies the existence of a branch \mathcal{B} of closed orbits of (1) emanating from q_0 . The proof is complete. \square

10 Appendix

In this final section we present a couple of tricks allowing us to simplify some computations from linear algebra. Throughout what follows $N \in \mathbb{N}$ is an arbitrary natural number. Our first lemma is a general tool allowing us to reduce the order of some determinants; it is possibly known but we could not find a precise reference.

Lemma 10.1. *Let $B_1, B_2, B_3, B_4 \in \mathbb{R}^{N \times N}$ be square matrices such that $B_1 B_2 = B_2 B_1$. Then,*

$$\det \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array} \right) = \det(B_4 B_1 - B_3 B_2).$$

Proof. It suffices to prove this result when B_1 is nonsingular (in the general case it suffices to replace B_1 by $B_1 + \epsilon I_N$ and take limits when $\epsilon \rightarrow 0$). By subtracting to the last N columns the product of the first N columns by $B_1^{-1} B_2$ we get

$$\begin{aligned} \det \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array} \right) &= \det \left(\begin{array}{c|c} B_1 & 0_N \\ \hline B_3 & B_4 - B_3 B_1^{-1} B_2 \end{array} \right) = \\ &= \det(B_4 - B_3 B_1^{-1} B_2) \det B_1 = \\ &= \det(B_4 B_1 - B_3 B_1^{-1} B_2 B_1), \end{aligned}$$

and the result follows from the fact that B_1 and B_2 commute. \square

An important role in this paper is played by the matrices which we called S_T in Section 5. In the following lemma we provide a trick to simplify the computation of their determinants.

Lemma 10.2. *Let $A \in \mathbb{R}^{2N \times 2N}$ be symmetric and let $S_T \in \mathbb{R}^{4N \times 4N}$ be defined as in (5). Then, for every $T > 0$ one has*

$$\det S_T = \det \left(\frac{T}{2\pi} J_N A - i I_{2N} \right)^2 = \det \left(-\frac{T}{2\pi} A + i J_N \right)^2.$$

Proof. We first show the result for $T = 2\pi$. In this case, $S_{2\pi} = \left(\begin{array}{c|c} -A & -J_N \\ \hline J_N & -A \end{array} \right)$. By adding to the first N rows the product of i times the last N rows and then subtracting to the last N columns the product of i times the first N columns one gets

$$\begin{aligned} \det S_{2\pi} &= \det \left(\begin{array}{c|c} -A + i J_N & -J_N - i A \\ \hline J_N & -A \end{array} \right) = \det \left(\begin{array}{c|c} -A + i J_N & 0_N \\ \hline J_N & -A - i J_N \end{array} \right) = \\ &= \det(-A + i J_N) \det(-A - i J_N) = |\det(-A - i J_N)|^2 = \det(-A + i J_N)^2, \end{aligned}$$

the last equality coming from the fact that $A + i J_N$ is Hermitian so that its determinant is a real number (we have also used the identity $\det \bar{M} = \overline{\det M}$). Since $J_N^2 = -I_{2N}$ and $\det J_N = 1$ it follows that $\det S_{2\pi} = \det(J_N A - i I_{2N})^2$, thus proving the result for $T = 2\pi$.

In the general case we apply the above to the symmetric matrix $\frac{T}{2\pi} A$. The result follows. \square

We immediately obtain the following well-known result:

Corollary 10.3. *For $T > 0$, let S_T be defined as in (5). Then $\det S_T = 0$ if and only if $\frac{2\pi}{T}i$ is an eigenvalue of $J_N A$.*

Applying Lemma 10.2 to the matrix $A' := A + \frac{2\pi}{T}\lambda I_{2N}$ one obtains an equality simplifying the computation of the characteristic polynomial of S_T . Precisely:

Corollary 10.4. *Under the assumptions of Lemma 10.2, the characteristic polynomial of S_T is given by*

$$\det(S_T - \lambda I_{4N}) = p_T(\lambda)^2,$$

where $p_T(\lambda) := \det\left(-\frac{T}{2\pi}A + iJ_N - \lambda I_{2N}\right)$.

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References

- [1] Alexander, J.C.; Yorke, J.A., *Global bifurcations of periodic orbits*. Amer. J. Math. 100 (1978), no. 2, 263–292.
- [2] Arenstorf, R. F., *Central configurations of four bodies with one inferior mass*. Celestial Mech. 28 (1982), no. 1-2, 9–15.
- [3] Barros, J.F; Leandro, E., *Bifurcations and enumeration of classes of relative equilibria in the planar restricted four-body problem*. SIAM J. Math. Anal. 46 (2014), no. 2, 1185–1203.
- [4] Bartsch, T., *A generalization of the Weinstein-Moser theorems on periodic orbits of a Hamiltonian system near an equilibrium*. Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997), no. 6, 691–718.
- [5] Benci, V., *A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations*. Comm. Pure Appl. Math. 34 (1981), no. 4, 393-432.
- [6] Brown, R.F., *A Topological Introduction to Nonlinear Analysis*. Birkhauser Boston, (2004).
- [7] Dancer, E.N.; Rybicki, S., *A note on periodic solutions of autonomous Hamiltonian systems emanating from degenerate stationary solutions*. Differential Integral Equations 12 (1999), no. 2, 147–160.
- [8] Duistermaat, J.J., *Bifurcation of periodic solutions near equilibrium points of Hamiltonian systems*. Bifurcation theory and applications (Montecatini, 1983), 57-105, Lecture Notes in Math., 1057, Springer, Berlin, 1984.
- [9] Frauenfelder, U.; van Koert, O., *The Restricted Three-Body Problem and Holomorphic Curves*. Pathways in Mathematics. Birkhäuser/Springer, 2018.

- [10] Gannaway, J.R., *Determination of All Central Configurations in the Planar 4-Body Problem with One Inferior Mass*. Ph.D. Thesis, Vanderbilt University. 1981.
- [11] García-Azpeitia, C.; Ize, J., *Global bifurcation of planar and spatial periodic solutions in the restricted n-body problem*. *Celestial Mech. Dynam. Astronom.* 110 (2011), no. 3, 217–237.
- [12] Gęba, K.; Marzantowicz, W., *Global bifurcation of periodic solutions*. *Topol. Methods Nonlinear Anal.* 1 (1993), no. 1, 67–93.
- [13] Gołębiewska, A.; Rybicki, S., *Global bifurcations of critical orbits of G-invariant strongly indefinite functionals*. *Nonlinear Analysis* 74 (2011), 1823–1834.
- [14] Gua de Malves, J.P., *Démonstrations de la règle de Descartes, pour connoître le nombre des racines positives et negatives dans les équations qui n'ont point de racines imaginaires*. *Mémoires de Mathématique & de Physique de l'Académie Royale des Sciences*, 1741, pp. 72-96.
- [15] Hinrichsen, J.J., *The libration points in an n-body problem*. *Amer. Math. Monthly* 50(4) (1943), 231–237.
- [16] Krasnosel'skiĭ, M.A., *The Operator of Translation Along the Trajectories of Differential Equations*. *Translations of Mathematical Monographs*, Vol. 19 American Mathematical Society, Providence, R.I., 1968.
- [17] Kulevich, J.L.; Roberts, G.E.; Smith, C.J., *Finiteness in the planar restricted four-body problem*. *Qual. Theory Dyn. Syst.* 8 (2009), no. 2, 357–370.
- [18] Leandro, E., *On the central configurations of the planar restricted four-body problem*. *J. Differential Equations* 226 (2006), no. 1, 323–351.
- [19] Leray, J.; Schauder, J., *Topologie et équations fonctionnelles*. *Ann. Sci. École Norm. Sup. (3)* 51 (1934), 45-78.
- [20] Maciejewski, A.J.; Rybicki, S.M., *Global bifurcations of periodic solutions of the restricted three body problem*. *Celestial Mech. Dynam. Astronom.* 88 (2004), no. 3, 293–324.
- [21] Mawhin, J.; Willem, M., *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, New York Berlin Heidelberg, Applied Mathematical Sciences 74, 1989.
- [22] Meyer, K.R.; Palacián, J.F.; Yanguas, P., *The elusive Liapunov periodic solutions*. *Qual. Theory Dyn. Syst.* 14 (2015), no. 2, 381-401.
- [23] Meyer, K.R.; Schmidt, D.S., *Periodic orbits near L₄ for mass ratios near the critical mass ratio of Routh*. *Celestial Mech.* 4 (1971), 99-109.

- [24] Meyer, K.R.; Schmidt, D.S., *Bifurcations of central configurations in the N -body problem*. Mathematics of Nonlinear Science (Phoenix, AZ, 1989), 93–101, Contemp. Math., 108, AMS., Providence, 1990.
- [25] Moser, J., *Periodic orbits near an equilibrium and a theorem by Alan Weinstein*. Comm. Pure Appl. Math. 29 (1976), no. 6, 724–747.
- [26] Moser, J.; Zehnder, E.J., *Notes on Dynamical Systems*. Courant Lecture Notes in Mathematics, 12. Courant Institute of Mathematical Sciences, New York. AMS., Providence, 2005.
- [27] Moulton, F.R., *On a class of particular solutions of the problem of four bodies*. Trans. Amer. Math. Soc. 1 (1900), no. 1, 17–29.
- [28] Pedersen, P., *Librationspunkte im restringierten Vierkörperproblem*, Dan. Mat-Fys. Medd, 21 (6) (1944), pp. 1–80.
- [29] Poincaré, H., *Les Méthodes Nouvelles de la Mécanique Céleste*. Tome I, Gauthier-Villars, Paris, 1892.
- [30] Pollard, H., *Mathematical Introduction to Celestial Mechanics*. Prentice-Hall, 1966.
- [31] Rabinowitz, P.H., *Some global results for nonlinear eigenvalue problems*. J. Functional Analysis 7 (1971), 487-513.
- [32] Rabinowitz, P.H., *A note on topological degree for potential operators*. J. Math. Anal. Appl. 51 1975, no. 2, 483-492.
- [33] Rabinowitz, P.H., *Periodic solutions of Hamiltonian systems*. Comm. Pure Appl. Math. 31 (1978), no. 2, 157-184.
- [34] Radzki, W.; Rybicki, S., *Degenerate bifurcation points of periodic solutions of autonomous Hamiltonian systems*. J. Diff. Eq. 202 (2004), 284-305.
- [35] Simó, C., *Relative equilibrium solutions in the four-body problem*. Celestial Mech. 18 (1978), no. 2, 165–184.
- [36] Schmidt, D.S., *Periodic solutions near a resonant equilibrium of a Hamiltonian system*. Celestial Mech. 9 (1974), 81-103.
- [37] Szulkin, A., *Bifurcation for strongly indefinite functionals and a Liapunov type theorem for Hamiltonian systems*. Differential & Integral Equations 7 (1994), no. 1, 217-234.
- [38] Weinstein, A., *Normal modes for nonlinear Hamiltonian systems*. Invent. Math. 20 (1973), 47–57.
- [39] Yorke, J.A., *Periods of periodic solutions and the Lipschitz constant*. Proc. Amer. Math. Soc. 22 (1969), 509–512.