Existence of at least two periodic solutions of the forced relativistic pendulum

Cristian Bereanu
Institute of Mathematics "Simion Stoilow", Romanian Academy
21, Calea Griviţei, RO-010702-Bucharest, Sector 1, România
cristian.bereanu@imar.ro
Pedro J. Torres
Departamento de Matemática Aplicada
18071 Granada, Spain
ptorres@ugr.es

Abstract

Using Szulkin's critical point theory, we prove that the relativistic forced pendulum with periodic boundary value conditions

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least two solutions not differing by a multiple of 2π for any continuous function $h : [0,T] \to \mathbb{R}$ with $\int_0^T h(t)dt = 0$ and any $\mu \neq 0$. The existence of at least one solution has been recently proved by Brezis and Mawhin.

MSC 2010 Classification : 34B15; 49J52; 49J35.
Key words : relativistic pendulum, multiple solutions, Szulkin critical point theory.

1 Introduction and the main result

It is well known that the classical forced pendulum with periodic boundary value conditions

$$u'' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least two solutions not differing by a multiple of 2π for any continuous function $h : [0,T] \to \mathbb{R}$ with $\int_0^T h(t)dt = 0$ and any $\mu \neq 0$. The existence of at least one solution was proved by Hamel [9] and rediscovered independently by Dancer [7] and Willem [15]. Then, the existence of a second solution has been proved by Mawhin and Willem [11] using mountain pass arguments.

Motivated by those results, Brezis and Mawhin prove in [6] that the relativistic forced pendulum with periodic boundary value conditions

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \mu \sin u = h(t), \quad u(0)-u(T)=0=u'(0)-u'(T), \tag{1}$$

has at least one solution for any forcing term h with mean value zero and any $\mu \neq 0$. The above problem is reduced to finding a minimum for the corresponding action integral over a closed convex subset of the space of T-periodic Lipschitz functions, and then to show, using variational inequalities techniques, that such a minimum solves the problem.

In this paper we show that (1) has at least two solutions not differing by a multiple of 2π. Actually, we consider as in [2, 6], the more general periodic boundary value problem

$$(\phi(u'))' = f(t,u) + h(t), \quad u(0)-u(T)=0=u'(0)-u'(T), \tag{2}$$

where ϕ satisfies the hypothesis

(H_Φ) *there exists* $\Phi : [-a,a] \to \mathbb{R}$ *such that* $\Phi(0)=0$, Φ *is continuous, of class* C^1 *on* $(-a,a)$, *with* $\phi := \Phi' : (-a,a) \to \mathbb{R}$ *an increasing homeomorphism such that* $\phi(0)=0$,

$f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with its primitive

$$F(t,x) = \int_0^x f(t,\xi)d\xi, \qquad ((t,x) \in [0,T] \times \mathbb{R})$$

satisfying the hypothesis

(H_F) *there exists* $\omega > 0$ *such that* $F(t,x) = F(t,x+\omega)$ *for all* $(t,x) \in [0,T]\times\mathbb{R}$,

and finally the forcing term $h : [0,T] \to \mathbb{R}$ is supposed to be continuous and satisfies

(H_h) $\int_0^T h(t)dt = 0$.

Of course, by a solution of (2) we mean a function $u \in C^1[0,T]$ with $||u'||_\infty < a$, $\phi(u') \in C^1[0,T]$ and (2) is satisfied.

Our main result is the following one.

Theorem 1 *If the hypotheses* (H_Φ), (H_F) *and* (H_h) *are satisfied, then (2) has at least two solutions not differing by a multiple of* ω.

Taking in (2), $\phi(s) = \frac{s}{\sqrt{1-s^2}}$ so that $\Phi(s) = 1-\sqrt{1-s^2}$, and $f(t,x) = -\mu \sin x$ so that $F(t,x) = \mu(\cos x - 1)$ and $\omega = 2\pi$, one has the following

Corollary 1 *Problem (1) has at least two solutions not differing by a multiple of* 2π *for any forcing term* h *satisfying* (H_h) *and any* $\mu \neq 0$.

Our approach is variational and is based upon Szulkin's critical point theory [14] and some results given in [2]. The corresponding result for the one dimensional curvature operator has been recently proved, using also Szulkin's critical point theory, by Obersnel and Omari [12].

We point out that the approach of Mawhin and Willem [11] has an abstract formulation given by Pucci and Serrin in [13] and then the Pucci-Serrin's variant of the Mountain Pass Lemma has been generalized by Ghoussoub and Preiss in [8]. For Szulkin type functionals, the Ghoussoub - Preiss result is proved by Marano and Motreanu [10] assuming also the reflexivity of the space. In our case, we work in the space of continuous functions defined on a compact interval, which is not reflexive, and in order to avoid this difficulty we use a truncation strategy coming from upper and lower solutions method.

2 Auxiliary results and notation

In this section we state some results from [2] which are main tools in the proof of Theorem 1.

Let $g : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function with its primitive defined by

$$G(t,x)=\int_0^x g(t,\xi)d\xi, \qquad ((t,x)\in[0,T]\times\mathbb{R}),$$

and consider the periodic boundary value problem

$$(\phi(u'))'=g(t,u), \quad u(0)-u(T)=0=u'(0)-u'(T). \tag{3}$$

We set $C:=C[0,T]$, $L^\infty:=L^\infty(0,T)$ and $W^{1,\infty}:=W^{1,\infty}(0,T)$. The usual norm $\|\cdot\|_\infty$ is considered on C and L^∞, whereas in $W^{1,\infty}$ we consider the usual norm $\|u\|_{W^{1,\infty}}=\|u\|_\infty+\|u'\|_\infty$.

We decompose any $u\in C$ as follows

$$u=\overline{u}+\widetilde{u}, \quad \overline{u}=\frac{1}{T}\int_0^T u(t)dt \quad \text{and} \quad \int_0^T \widetilde{u}(t)dt=0.$$

Note that one has

$$||\widetilde{v}||_\infty\le T\|v'\|_\infty \quad \text{for all} \quad v\in W^{1,\infty}. \tag{4}$$

Let

$$K:=\{v\in W^{1,\infty}\ :\ \|v'\|_\infty\le a, \quad v(0)=v(T)\}$$

and $\Psi : C\to(-\infty,+\infty]$ be defined by

$$\Psi(v)=\begin{cases}\int_0^T\Phi(v'), & \text{if } v\in K,\\[2ex] +\infty, & \text{otherwise.}\end{cases}$$

Obviously, Ψ is proper and convex. On the other hand, as shown in [6] (see also [2]), Ψ is lower semicontinuous on C.

Next, let $\mathcal{G} : C \to \mathbb{R}$ be given by

$$\mathcal{G}(u) = \int_0^T G(t,u)\,dt, \quad u \in C.$$

A standard reasoning shows that $\mathcal{G}$ is of class C^1 on C and its derivative is given by

$$\langle \mathcal{G}'(u), v\rangle = \int_0^T g(t,u)v\,dt, \quad u,v \in C.$$

Following [2], we consider the energy functional associated to (3) given by

$$I : C \to (-\infty, +\infty], \qquad I = \Psi + \mathcal{G}.$$

Then, I has the structure required by Szulkin's critical point theory [14]. Accordingly, a function $u \in C$ is *a critical point* of I if $u \in K$ and

$$\Psi(v) - \Psi(u) + \langle \mathcal{G}'(u), v-u\rangle \geq 0 \quad \text{for all } v \in C.$$

It is shown in [2] that if u is a critical point of I, then u is a solution of (3).

On the other hand, $\{u_n\} \subset K$ is a *(PS)–sequence* if $I(u_n) \to c \in \mathbb{R}$ and

$$\int_{0)}^T [\Phi(v') - \Phi(u_n') + g(t,u_n)(v-u_n)]\,dt \geq -\varepsilon_n \|v-u_n\|_\infty \qquad \text{for all } v \in K,$$

where $\varepsilon_n \to 0_+$. According to [14], the functional I is said to satisfy the *(PS) condition* if any *(PS)*–sequence has a convergent subsequence in C. Note also that if $\{u_n\}$ is a (PS)–sequence, then, from [2] one has that

- the sequence $\{\int_0^T G(t,u_n)\,dt\}$ is bounded;
- if $\{\overline{u}_n\}$ is bounded, then $\{u_n\}$ has a convergent subsequence in C.

Next lemma is a direct consequence of [4, Theorem 3].

Lemma 1 *Let us assume that (3) has two solutions α, β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0,T]$. Let $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by*

$$\gamma(t,x) = \begin{cases} \beta(t), & \text{if } x > \beta(t), \\ x, & \text{if } \alpha(t) \leq x \leq \beta(t), \\ \alpha(t), & \text{if } x < \alpha(t). \end{cases}$$

Consider the modified problem

$$(\phi(u'))' = g(t, \gamma(t,u)) + u - \gamma(t,u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{5}$$

If u is a solution of (5), then

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \text{for all} \quad t \in [0,T],$$

and u is a solution of (3).

3 Proof of the main result

First of all, using the corresponding result for the periodic case of Corollary 1 in [2] one has that the energy functional I associated to (2) is bounded from below and there exists $u_0 \in K$ a minimizer for I, which is also a solution of (2). On the other hand, from (H_F) it follows that

$$I(u) = I(u + j\omega) \quad \text{for all} \quad u \in C, j \in \mathbb{Z}.$$

So, taking j sufficiently large, we can assume that u_0 is strictly positive and one has that $u_1 := u_0 + \omega$ is a minimizer of I and also a solution of (2).

We associate to (2) the corresponding modified problem

$$\begin{gathered} (\phi(u'))' = f(t, \gamma(t, u)) + h(t) + u - \gamma(t, u), \\ u(0) - u(T) = 0 = u'(0) - u'(T), \end{gathered} \tag{6}$$

where in this case $\gamma : [0, T] \times \mathbb{R} \to \mathbb{R}$ is given by

$$\gamma(t, x) = \begin{cases} u_1(t), & \text{if } x > u_1(t), \\ x, & \text{if } u_0(t) \le x \le u_1(t), \\ u_0(t), & \text{if } x < u_0(t). \end{cases}$$

So, if u is a solution of (6) then by Lemma 1,

$$u_0(t) \le u(t) \le u_1(t) \quad \text{for all} \quad t \in [0, T] \tag{7}$$

and u is a solution of (2).

Next, let $J : C \to (-\infty, \infty]$ be the energy functional associated to the modified problem (6). So,

$$J(u) = \int_0^T \Phi(u') + \int_0^T A(t, u)dt \quad \text{for all} \quad u \in K,$$

where $A : [0, T] \times \mathbb{R} \to \mathbb{R}$ is given by

$$A(t, x) = \int_0^x f(t, \gamma(t, \xi))d\xi + xh(t) + \frac{x^2}{2} - \int_0^x \gamma(t, \xi)d\xi,$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

Let us note that if u is a critical point of J, then u is a solution of (6), hence u satisfies (7) and u is also a solution of (2).

Lemma 2 *The following hold true.*

(i) $J(u_0) = J(u_1)$.

(ii) $\lim_{|x| \to \infty} A(t, x) = +\infty$ *uniformly in* $t \in [0, T]$.

(iii) The functional J is bounded from below and satisfies the (PS)-condition.

Proof. (i) From (H_F) and the definition of γ we infer that

$$A(t, u_0(t)) = u_0(t) f(t, u_0(t)) + u_0(t) h(t) - \frac{u_0^2(t)}{2},$$

and

$$A(t, u_1(t)) = u_0(t) f(t, u_0(t)) + u_1(t) h(t) - \frac{u_0^2(t)}{2},$$

for all $t \in [0, T]$. On the other hand, using (H_h) we deduce that

$$\int_0^T u_0(t) h(t) dt = \int_0^T u_1(t) h(t) dt.$$

Hence

$$\int_0^T A(t, u_0(t)) dt = \int_0^T A(t, u_1(t)) dt,$$

which together with

$$u_0' = u_1',$$

imply that (i) holds true.

(ii) Using that γ is bounded, it follows that there exists $c_1 > 0$ such that

$$A(t, x) \geq \frac{x^2}{2} - c_1 |x| \quad \text{for all} \quad (t, x) \in [0, T] \times \mathbb{R},$$

implying that (ii) holds true.

(iii) From (ii) we deduce immediately that J is bounded from below.

Now, let $\{u_n\}$ be a (PS)–sequence. Then, it follows that the sequence $\{\int_0^T A(t, u_n)\, dt\}$ is bounded. This together with (4) and (ii) imply that $\{\overline{u}_n\}$ is bounded. Again by (4) and the fact that $\{u_n\} \subset K$, we have that $\{u_n\}$ is bounded in $W^{1,\infty}$. By the compact embedding of $W^{1,\infty}$ into C (see for example [5]), it follows that $\{u_n\}$ has a convergent subsequence in C and J satisfies the (PS)-condition. ■

End of the proof of the main result. We conclude the proof by using an argument inspired in [12]. Using Lemma 2 (iii) and Theorem 1.7 from [14], we deduce that there exists u_2, a critical point of J such that

$$J(u_2) = \inf_C J.$$

We have two cases.

Case 1. If $u_2 \neq u_0$ and $u_2 \neq u_1$, then, using the fact that u_2 satisfies (7), it follows that u_2 is a solution of (2) such that $u_2 - u_0$ is not a multiple of ω.

Case 2. If $u_2 = u_0$ or $u_2 = u_1$, then using Lemma 2 (i), it follows that u_0 and u_1 are also minimizers of J. Hence, using Lemma 2 (iii) and [14, Corollary 3.3], we infer that there exists u_3 a critical point of J different to u_0 and u_1. Because u_3 is a critical point of J, one has that u_3 satisfies (7) and therefore u_3 is a solution of (2) such that $u_3 - u_0$ is not a multiple of ω.

4 Final remarks about the Neumann problem

Let us consider the Neumann problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[f(r,u) + h(r)], \quad u'(R_1) = 0 = u'(R_2), \tag{8}$$

where $0 \le R_1 < R_2$, $N \ge 1$ is an integer and ϕ, f and h satisfy hypothesis (H_Φ), (H_F) and (H_h). Then, using the same strategy as in the periodic case, without any change and the corresponding results from [2] and [1], one has that (8) has at least two solutions not differing by a multiple of ω. The existence of at least one solution has been proved in [3, 2].

In particular, the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \mu \sin u = h(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A},$$

where $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\}$, has at least two classical radial solutions not differing by a multiple of ω, for any $\mu \ne 0$ and any $h \in C$ such that

$$\int_{\mathcal{A}} h(|x|)\, dx = 0.$$

References

[1] C. Bereanu, P. Jebelean, and J. Mawhin, Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowsi spaces, Math. Nachr. 283 (2010), 379-391.

[2] C. Bereanu, P. Jebelean, and J. Mawhin, Variational methods for nonlinear perturbations of singular ϕ-Laplacians, preprint.

[3] C. Bereanu, P. Jebelean, and J. Mawhin, Radial solutions of Neumann problems involving mean extrinsic curvature and periodic nonlinearities, preprint.

[4] C. Bereanu, and J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, J. Differential Equations 243 (2007), 536-557.

[5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011.

[6] H. Brezis, and J. Mawhin, Periodic solutions of the forced relativistic pendulum, Differential Integral Equations 23 (2010), 801-810.

[7] E.N. Dancer, On the use of asymptotics in nonlinear boundary value problems, Ann. Mat. Pura Appl. 131 (1982), 167-185.

[8] N. Ghoussoub, and D. Preiss, A general mountain pass principle for locating and classifying critical points, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), 321-330.

[9] G. Hamel, Ueber erzwungene Schingungen bei endlischen Amplituden, Math. Ann. 86 (1922), 1-13.

[10] A.S. Marano, and D. Motreanu, A deformation theorem and some critical point results for non-differentiable functions, Topol. Methods Nonlinear Anal. 22 (2003), 139-158.

[11] J. Mawhin, and M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, J. Differential Equations 52 (1984), 264-287.

[12] F. Obersnel, and P. Omari, Multiple bounded variation solutions of a periodicaly perturbed sine-curvature equation, preprint.

[13] P. Pucci, and J. Serrin, Extensions of the mountain pass theorem, J. Funct. Anal. 59 (1984), 185-210.

[14] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986), 77-109.

[15] M. Willem, Oscillations forcées de l'équation du pendule, Pub. IRMA Lille, 3 (1981), V-1-V-3.