Translation arcs and stability in two dimensions

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Abstract Given a closed orbit γ of a system of differential equations in the plane

$$\dot{x} = X(x), \;\; x \in R^2,$$

the index of the vector field X around γ is one. This classical result has a counterpart in the theory of discrete systems in the plane. Consider the equation

$$x_{n+1} = h(x_n), \;\; x_n \in R^2,$$

where h is an orientation-preserving embedding and assume that there is a recurrent orbit that is not a fixed point. Then there exists a Jordan curve γ such that the fixed point index of h around this curve is one. The proof is based on the theory of translation arcs, initiated by Brouwer. These notes are dedicated to discuss some consequences of the above result, specially in stability theory. We will compute the indexes associated to a stable invariant object and show that Lyapunov stability implies persistence (in two dimensions). The invariant sets under consideration will be fixed points, periodic orbits and Cantor sets.

1 Introduction

We are going to discuss the dynamics of a system of the type

$$x_{n+1} = h(x_n), \;\; x_n \in R^2 \tag{1}$$

where $h : R^2 \to R^2$ is a one-to-one continuous map. Notice that h is not necessarily onto and so the image $h(R^2)$ can be a proper subset of R^2. In general orbits are

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defined in the future, $x_n = h^n(x_0)$, $n \geq 0$, but not always in the past. We will also assume that the map h is orientation preserving. This class of maps will be denoted by $\mathscr{E}_+$, indicating that h is an orientation preserving topological embedding. We refer to the appendix for more details on the topological aspects.

As a counterpoint to the discrete model (1) we can consider the system of differential equations

$$\dot{x} = X(x), \;\; x \in R^2 \tag{2}$$

where $X : R^2 \to R^2$ is a continuous vector field such that there is uniqueness for the initial value problem associated to (2). We all know that the behavior of the continuous system is much simpler. In particular Poincaré-Bendixson theory implies that the dynamics for (2) cannot be very intricate. In general this type of results cannot be extended to the discrete situation. In these notes we will be concerned with a result for continuous systems that is somehow exceptional because it can be extended to discrete systems. This result, already obtained by Poincaré, deals with the computation of the index around a closed orbit of (2). The discrete version is the so-called arc translation lemma, originally due to Brouwer. We will discuss these classical results and later derive some new consequences in stability theory.

2 The index around a closed orbit

Let Γ be a closed orbit of the system (2) and let $R_i(\Gamma)$ denote the bounded component of $R^2 \setminus \Gamma$. It is well known that the vector field X has an equilibrium on $R_i(\Gamma)$. Indeed a stronger conclusion can be obtained, namely

$$d(X, R_i(\Gamma), 0) = 1,$$

where d is the Brouwer degree on the plane. A proof of this result can be found in the book [18] and an alternative proof will be presented later. The book [15] contains a very complete presentation of degree theory. By now we just recall how to compute geometrically the degree $d(X, R_i(\Gamma), 0)$. The curve Γ can be parameterized by a continuous function $\beta : [0,1] \to R^2$ with $\Gamma = \beta([0,1])$, β is one-to-one in $[0,1[$, $\beta(0) = \beta(1)$. It is also assumed that β has positive orientation. The vector field X does not vanish on Γ and so we can find a continuous argument $\theta : [0,1] \to R$ satisfying

$$X(\beta(t)) = ||X(\beta(t))||(\cos\theta(t), \sin\theta(t)).$$

Then

$$d(X, R_i(\Gamma), 0) = \frac{1}{2\pi}(\theta(1) - \theta(0)).$$

3 Translation arcs

We go back to discrete systems. Our first task will be to construct an object that somehow plays the role of the orbits in continuous dynamics. An arc is a set $\alpha \subset R^2$ that is homeomorphic to the compact interval $[0,1]$. The end points of the arc will be denoted by $p \neq q$ and sometimes we will use the notation $\alpha = \widehat{pq}$ for the oriented arc. Given $h \in \mathscr{E}_+$ we say that $\alpha = \widehat{pq}$ is a translation arc for h if $h(p) = q$ and

$$h(\alpha \setminus \{q\}) \cap (\alpha \setminus \{q\}) = \emptyset.$$

We illustrate this definition with a typical situation.

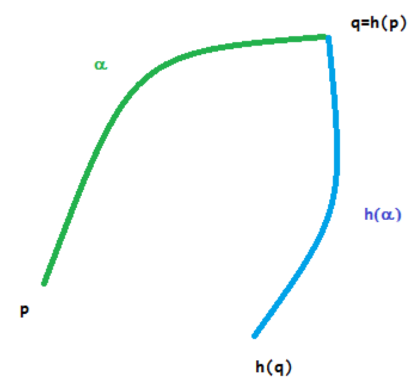

Fig. 1 A typical translation arc

Notice that the translation arc α cannot contain fixed points and the open arcs $\dot{\alpha} = \alpha \setminus \{p,q\}$ and $h(\dot{\alpha})$ cannot intersect. In some special cases the two arcs can have the same end points. This case can only occur if $\{p,q\}$ is a two cycle.

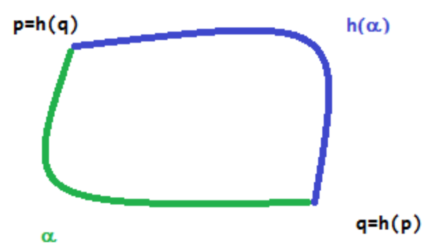

Fig. 2 A translation arc with 2-periodic end points

Next we discuss some simple examples:

1. Assume first that h is a translation, say

$$h(x,y) = (x+1,y).$$

The horizontal segment joining $p=(0,0)$ and $q=(1,0)$ is a translation arc. The broken line passing by $p=(0,0)$, $(0,1)$, $(1,1)$ and $q=(1,0)$ is not a translation arc, in this case $\alpha \cap h(\alpha)$ is the segment joining $(1,1)$ and q.

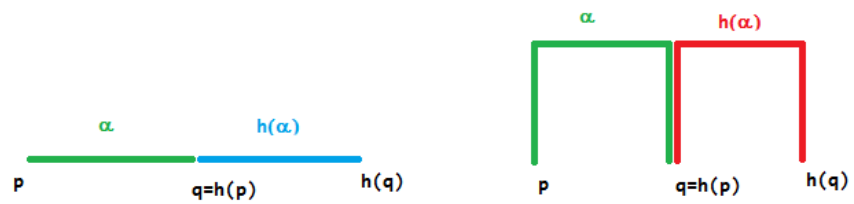

Fig. 3 In the left side $\alpha \cap h(\alpha) = \{q\}$ but in the right side this intersection is a segment.

2. Assume that h is the rotation of 180 degrees in the counterclockwise sense. The set $\alpha = \{(\cos\theta, \sin\theta) : \theta \in [0,\pi]\}$ is a translation arc. In this case the end points $p=(1,0)$ and $q=(-1,0)$ become two cycles.

3. Assume that the continuous system (2) defines a global flow $\{\phi_t\}_{t\in R}$ in the plane and take $h=\phi_\tau$ for some fixed $\tau>0$. We notice that ϕ_τ is a homeomorphism of the plane that is isotopic to the identity. In particular, $h \in \mathscr{E}_+$. Assume that p is a point in the plane that is not an equilibrium of X and let α be the piece of orbit running from p to $q=\phi_\tau(p)$, that is

$$\alpha = \{\phi_t(p) : t \in [0,\tau]\}.$$

This is an arc if the orbit is not closed or if it is closed and $\tau < T$, where $T>0$ is the minimal period. This segment of orbit is a translation arc whenever the orbit is not closed or $\tau \le \frac{1}{2}T$.

After these examples we go back to the general setting and ask ourselves about the existence of translation arcs. Next result shows that they exist for any $h \neq id$. By a topological disk we understand a subset D of the plane that is homeomorphic to the disk $\Delta = \{x \in R^2 : ||x|| \le 1\}$. Notice that ∂D is a Jordan curve and $\mathrm{int}(D) = R_i(\partial D)$.

Lemma 1. *Assume that $h \in \mathscr{E}_+$ and D is a topological disk with $D \cap h(D) = \emptyset$. In addition assume that the points $x_1,\dots,x_k$ belong to the interior of D. Then there exists a translation arc α with $x_1,\dots,x_k \in \dot{\alpha}$. See the figure below.*

This result is a variation of a lemma in [8]. A proof can be found in chapter 3 of [23].

Given $h \in \mathscr{E}_+$ and a translation arc α, the successive iterations of α can lead to two different situations: either $\alpha \cap h^n(\alpha) = \emptyset$ for each $n \ge 2$ or $\alpha \cap h^n(\alpha) \neq \emptyset$

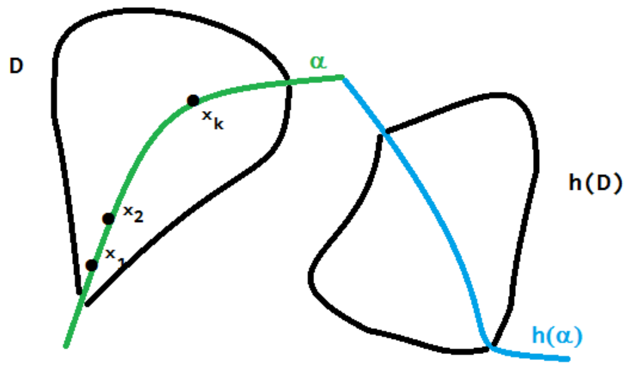

Fig. 4 The arc α passes through the points x_1,...,x_k, all lying in the disk D

for some $n \geq 2$. The prototype for the first situation is example 1 (translations), while the second appears in example 2 (rotations). The second situation is somehow analogous to the closed orbit in a continuous system. To obtain a result on the index for the discrete case we need to replace the vector field X by some map related to h. The simplest one is $f := id - h$, where id denotes the identity in the plane. The zeros of f are precisely the fixed points of h, in analogy with the equilibria of the continuous system which are the zeros of the vector field.

4 Lemma on translation arcs (Brouwer)

Given $h \in \mathscr{E}_+$, assume that for some $n \geq 2$ there exists a translation arc α with

$$\alpha \cap h^n(\alpha) \neq \emptyset.$$

Then there exists a Jordan curve Γ, contained in $\alpha \cup h(\alpha) \cup \cdots \cup h^n(\alpha)$, and such that

$$d(id - h, R_i(\Gamma), 0) = 1.$$

Notice that this degree is well defined because h cannot have fixed points on $\bigcup_{k\geq 0} h^k(\alpha)$. See the figure below. The proof of this lemma is delicate. Classical proofs (see for instance [25]) employ some intuitive method for computing the degree. This type of argument, very common in planar dynamics, is replaced by more rigorous arguments in [7, 13, 14]. In particular the proof by M. Brown is very elegant and, although it only covers the case of homeomorphisms h, with $h(R^2) = R^2$, it can be adapted to embeddings. See [20] and [23].

We are always assuming that h is orientation-preserving and this condition is essential for the previous lemma. This is shown by the following example. Consider

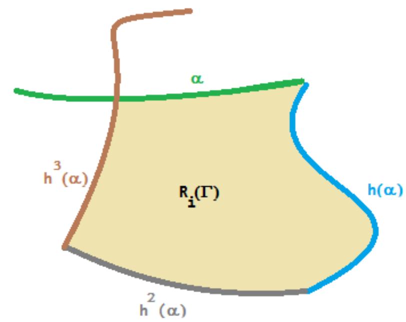

Fig. 5 Brouwer's lemma with $n=3$. The Jordan curve Γ is composed by $h(\alpha)$, $h^2(\alpha)$ and sub-arcs taken from α and $h^3(\alpha)$.

the map

$$h:\begin{cases} x_1=\lambda(y)x,\\ y_1=-y,\end{cases}$$

where $\lambda: R\to R$ is a continuous function satisfying

$$\lambda(y)>1 \ \text{ if } y\in]-2,2[,\ \ \lambda(y)=1 \ \text{ if } |y|\geq 2.$$

This is an orientation-reversing homeomorphism and the dynamics is easy to describe. The origin is a fixed point and the orbits lying on $x=0$ or $|y|\geq 2$ are two cycles, the remaining orbits are unbounded. The set

$$\alpha=\{(2\cos t, 2\sin t):\ t\in[-\frac{\pi}{2},\frac{\pi}{2}]\}$$

is a translation arc with $\alpha\cap h^2(\alpha)\neq\emptyset$. Notice that this intersection is composed by the end points of α. Since the origin is the only fixed point of h and lies in the unbounded component of $R^2\setminus(\alpha\cup h(\alpha)\cup h^2(\alpha))$, the degree of $id-h$ vanishes in any topological disk whose boundary is contained in $\alpha\cup h(\alpha)\cup h^2(\alpha)$. This shows that the conclusion of Brouwer's lemma cannot be valid if h is orientation-reversing.

It is interesting to recover the result stated in section 2 from Brouwer's Lemma. To this end we first recall two properties of degree. Given a bounded set Ω and a continuous function $f:\overline{\Omega}\to R^2$ with $f(x)\neq 0$ if $x\in\partial\Omega$, the *continuity property of degree* says that there exists $\eta>0$ such that if $g:\overline{\Omega}\to R^2$ is any continuous function with

$$||f(x)-g(x)||\leq\eta \ \text{ for each } x\in\partial\Omega,$$

then $d(f,\Omega,0)=d(g,\Omega,0)$. Notice that η must be small enough to guarantee that g does not vanish on $\partial\Omega$.
The second property refers to the composition with linear maps. Given a 2×2 matrix L,

$$d(Lf,\Omega,0)=\text{sign}(\det L)d(f,\Omega,0).$$

We are ready for the proof of the result in section 2. For simplicity we will assume that the vector field X defines a global flow $\{\phi_t\}_{t\in R}$ on the plane. First we prove that the degree of this vector field coincides with the degree of $id-\phi_t$ for t positive and small. To prove this we integrate the differential equation and obtain the formula

$$\frac{1}{t}(\phi_t(\xi)-\xi)=\frac{1}{t}\int_0^t X(\phi_s(\xi))ds.$$

Then $\frac{1}{t}(\phi_t-id)$ converges to X uniformly on compact sets. In consequence, if Ω is an open and bounded subset of the plane and X does not vanish on the boundary $\partial\Omega$, for small t the map ϕ_t will not have fixed points on $\partial\Omega$ and

$$d(X,\Omega,0)=d(\frac{1}{t}(\phi_t-id),\Omega,0). \qquad (3)$$

This last degree is the same as $d(id-\phi_t,\Omega,0)$. Assume now that Γ is a closed orbit. We fix $\xi\in\Gamma$ and consider the arc $\alpha=\{\phi_s(\xi):\ 0\le s\le t\}$. We know that α is a translation arc for $h=\phi_t$ if t is small. Moreover, since Γ is closed, some iterate of this arc will have a non-empty intersection with the initial arc α. We are now in the conditions of Brouwer's lemma and there exists a Jordan curve γ such that

$$d(id-h,R_i(\gamma),0)=1. \qquad (4)$$

But the curve γ is contained in Γ and so they have to coincide, $\gamma=\Gamma$. Putting together the two identities (3), (4) with $\Omega=R_i(\Gamma)$, we conclude that the degree of X is 1.

5 Stability and index of fixed points

Let us discuss some consequences of the previous result in stability theory. Assume that $h:\mathscr{U}\subset R^2\to R^2$ is a map defined on an open set $\mathscr{U}$ and let $p\in\mathscr{U}$ be a fixed point of h, $p=h(p)$. We say that p is *stable* if given any neighborhood $\mathscr{V}$ of p, there exists another neighborhood $\mathscr{W}\subset\mathscr{U}$ such that

$$h^n(\mathscr{W})\subset\mathscr{V}\ \text{ for each } n\ge 0.$$

We say that the fixed point p is *isolated* if there exists a neighborhood $\mathscr{N}$ such that p is the only fixed point of h in $\mathscr{N}$; that is,

$$\text{Fix }(h)\cap\mathscr{N}=\{p\},$$

where Fix $(h):=\{x\in\mathscr{U}:\ h(x)=x\}$.

To illustrate these notions we consider two simple examples. Assume first that h is a rotation around p, the disks centered at p are invariant under h and this implies that p is stable. Also, p is isolated because it is the only fixed point of h. In the second example h is a symmetry and p is a point in the axis of symmetry. For the same reasons the point p is stable but now it is not isolated because all points lying on the axis of symmetry are fixed.

In the next result we prove that stable and isolated fixed points have index one.

Theorem 1. *Assume that* $h : \mathscr{U} \subset R^2 \to R^2$ *is a continuous and one-to-one map that is orientation-preserving and let* p *be an isolated and stable fixed point of* h*, then*

$$d(id - h, \Omega, 0) = 1,$$

where Ω *is any open and bounded set with* $p \in \Omega$*,* $\overline{\Omega} \subset \mathscr{U}$ *and* $\mathrm{Fix}\,(h) \cap \overline{\Omega} = \{p\}$.

It is interesting to notice that this result can also be seen as a discrete counterpart of a classical result in the theory of continuous systems: given an isolated equilibrium of a vector field X, if this equilibrium is stable then the index of X around the equilibrium is one. A proof of theorem 1 using Brouwer's lemma was obtained by Dancer and myself in [12], previously Krasnoselskii stated the result (without proof) in [16]. We will present a simplified proof but first we discuss some consequences of the theorem. We also notice that the above theorem was extended to the orientation-reversing case by Ruiz del Portal [26]. In higher dimensions there are stable fixed points with arbitrary index, see [4] and the references therein.

6 Instability criteria

Traditionally the instability of fixed points is obtained using linearization or Lyapunov functions. In two dimensions the previous theorem allows to prove the instability of a fixed point via degree computations. As an example consider a smooth map that can be expressed (in complex notation) as

$$h(z, \bar{z}) = z + z^3 + R(z, \bar{z}),$$

where $R(z, \bar{z}) = o(|z|^3)$ as $z \to 0$. Then $z = 0$ is an unstable fixed point of

$$z_{n+1} = z_n + z_n^3 + R(z_n, \bar{z}_n).$$

To prove this we first apply the inverse function theorem and notice that h is in the conditions of theorem 1 when it is restricted to a small neighborhood of the origin. The expansion

$$z - h(z, \bar{z}) = -z^3 + o(|z|^3) \text{ as } |z| \to 0$$

implies that $z = 0$ is an isolated fixed point. Let us consider the parameterized curve $\gamma_\rho(t) = \rho e^{2\pi i t}$, $t \in [0, 1]$, with $\rho > 0$ very small. Then

$$(id - h)(\gamma_\rho(t)) = \rho^3 (e^{6\pi i t + \pi i} + o(1)) \text{ as } \rho \to 0.$$

The continuous argument of $(id - h) \circ \gamma_\rho$ satisfies

$$\theta_\rho(t) = 6\pi t + \pi + o(1) \text{ as } \rho \to 0.$$

From here it is easy to deduce that the degree on the disk $D_\rho = \{z \in C : |z| < \rho\}$ is

$$d(id - h, D_\rho, 0) = 3$$

for small ρ. This implies the instability of $z = 0$.

Many other results on unstable fixed points can be obtained using the known algorithms to compute the degree of a planar map.

7 Persistence and stability

In this section we work with the class of homeomorphisms of the plane, denoted by $\mathscr{H}$. The sub-class of orientation-preserving homeomorphisms will be denoted by $\mathscr{H}_+$. Notice that $\mathscr{H}_+ = \mathscr{H} \cap \mathscr{E}_+$. Given $h \in \mathscr{H}$ and a non-empty compact invariant set $\Lambda \subset R^2$, $\Lambda = h(\Lambda)$, we say that Λ is *persistent* if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\tilde{h} \in \mathscr{H}$ with

$$||h - \tilde{h}||_\infty < \delta$$

there exists a compact set $\tilde{\Lambda} \subset R^2$ that is invariant under $\tilde{h}$, $\tilde{\Lambda} = \tilde{h}(\tilde{\Lambda})$ and satisfies

$$D_H(\Lambda, \tilde{\Lambda}) \leq \varepsilon.$$

The notation D_H refers to the Haussdorf distance in the space of non-empty compact subsets of the plane and $||\cdot||_\infty$ is the uniform norm, that is

$$||h - \tilde{h}||_\infty = \sup_{x \in R^2} ||h(x) - \tilde{h}(x)||.$$

The theorem of the previous section implies the persistence of stable and isolated fixed points.

Corollary 1. *Given $h \in \mathscr{H}_+$ and $p = h(p)$ a fixed point that is stable and isolated, the invariant set $\Lambda = \{p\}$ is persistent.*

Proof. Since p is an isolated fixed point, for small $\varepsilon > 0$ we know that

$$\text{Fix } (h) \cap \overline{\Omega}_\varepsilon = \{p\},$$

where $\Omega_\varepsilon = \{x \in R^2 : ||x|| < \varepsilon\}$. From theorem 1,

$$d(f, \Omega_\varepsilon, 0) = 1,$$

where $f = id - h$. Let us take $g = id - \tilde{h}$ where $\tilde{h} \in \mathscr{H}$. Since $||f - g||_\infty = ||h - \tilde{h}||_\infty$, we can apply the continuity property of the degree that was stated in section 4 and select $\delta = \eta$ in order to find a fixed point $\tilde{p}$ of $\tilde{h}$ with $\tilde{p} \in \Omega_\varepsilon$. The proof is complete with $\tilde{\Lambda} = \{\tilde{p}\}$.

In the previous result it is essential that the fixed point is isolated. An example of a stable fixed point that is not persistent as invariant set is presented in [24].

8 Proof of the theorem

To prove theorem 1 we distinguish two cases:

(i) *p is asymptotically stable.*
In this case the theorem will be a consequence of degree theory and it is even valid in arbitrary (finite) dimension.

(ii) *p is not asymptotically stable.*
From our perspective this will be the more interesting case. We will prove the theorem using Brouwer's lemma. The proof and the result cannot be extended to higher dimensions.

8.1 Asymptotically stable fixed points

Although the contents of this subsection are valid in arbitrary dimension, we remain in the plane for notational convenience. We will consider a continuous map $h : \mathscr{U} \subset R^2 \to R^2$ defined on an open set. Notice that we do not need to assume that h is one-to-one.

A fixed point $p = h(p)$ is *asymptotically stable* if it is stable and there exists a neighborhood $\mathscr{R}$ of p such that

$$h^n(x) \to p \text{ as } n \to +\infty$$

for each $x \in \mathscr{R}$.

From this definition we notice that Fix $(h) \cap \mathscr{R} = \{p\}$. This implies that asymptotically stable fixed points are isolated. Also, it is not hard to prove that the attraction is uniform on compact subsets of $\mathscr{R}$. Next result is taken from Krasnoselskii's book [16].

Proposition 1. *In the previous assumptions for h, assume that $p = h(p)$ is an asymptotically stable fixed point. Then*

$$d(id - h, \Omega, 0) = 1$$

for each open and bounded set Ω satisfying $p \in \Omega$, $\overline{\Omega} \cap \text{Fix }(h) = \{p\}$.

Proof. Browder's asymptotic fixed point theorem (see [5]) implies that if D and Δ are closed disks in the plane with $\Delta \subset D$ and $h : D \to R^2$ is a continuous map such that all the iterates $h^n(\Delta)$ are well defined for $n \geq 0$ and, for some $N > 0$,

$$h^n(\Delta) \subset \Delta \quad \text{if} \quad n \geq N, \tag{5}$$

then h has a fixed point in Δ. A refinement of this result says that if the condition (5) is replaced by

$$h^n(\Delta) \subset \text{int}\,(\Delta) \;\; \text{if} \;\; n \geq N, \tag{6}$$

then

$$d(id - h, D, 0) = 1.$$

A complete proof of this result can be found in the book [17]. We are ready for the proof of the proposition.

Let us fix a small closed disk D centered at p and such that $D \subset \mathscr{U}$. Since p is stable we can find a smaller disk Δ centered at p and such that all forward iterates $h^n(\Delta)$ are well defined and contained in the interior of D; that is,

$$h^n(\Delta) \subset \text{int}\,(D) \;\; \text{if}\; n \geq 0.$$

After restricting the size of Δ we can also assume that it is contained in $\mathscr{R}$. Then the compact disk Δ is uniformly attracted by p. This means that

$$h^n(x) \to p \;\; \text{as}\; n \to +\infty,$$

uniformly in $x \in \Delta$. In consequence the condition (6) will hold if N is large enough. We conclude that $d(id - h, \text{int}\,(\Delta), 0) = 1$. The identity $d(id - h, \text{int}\,(\Delta), 0)) = d(id - h, \Omega, 0)$ follows by excision. Notice that $\Delta \cap \text{Fix}\,(h) \subset \mathscr{R} \cap \text{Fix}\,(h) = \{p\}$.

8.2 Stability without attraction

We are now assuming that h is continuous, one-to-one and orientation-preserving. However, since the map is not defined in the whole plane, we cannot say that it belongs to $\mathscr{E}_+$. Our first task will be to replace h by a map H in $\mathscr{E}_+$. To do this we first recall that a stable fixed point in the plane always possesses small neighborhoods that are positively invariant, open and simply connected. We refer to the appendix for more details. Since p is an isolated fixed point we find a neighborhood $\mathscr{N}$ with

$$\mathscr{N} \cap \text{Fix}\,(h) = \{p\}.$$

Let $\mathscr{W}$ be an open neighborhood of p satisfying

$$\mathscr{W} \subset \mathscr{U} \cap \mathscr{N}, \;\; \mathscr{W} \text{ is simply connected}\;, h(\mathscr{W}) \subset \mathscr{W}.$$

As a consequence of Riemann theorem on conformal maps we know that $\mathscr{W}$ is homeomorphic to the open disk or, equivalently, to the whole plane. It is possible to assume that the homeomorphism $\sigma : R^2 \cong \mathscr{W}$ satisfies $\sigma(p) = p$ and we define $H = \sigma^{-1} \circ h_{\mathscr{W}} \circ \sigma$ where $h_{\mathscr{W}} : \mathscr{W} \to \mathscr{W}$ is the restriction of h,

$$H: \ R^2 \cong \mathscr{W} \longrightarrow \mathscr{W} \cong R^2.$$

By definition H is continuous and one-to-one. Moreover, it is conjugate to the restriction of h to $\mathscr{W}$. This implies that H is orientation-preserving and the stability properties of p as a fixed point are the same for the two maps. We summarize the relevant properties of H,

- $H \in \mathscr{E}_+$
- p is a stable fixed point of H
- p is not asymptotically stable
- Fix $(H) = \{p\}$
- $d(id - h, \Omega, 0) = d(id - H, G, 0)$, where G is any open and bounded subset of the plane with $p \in G$.

Notice that the last property is a consequence of the excision property and the invariance of the fixed point index under conjugation. More precisely, given a homeomorphism $\psi : G_1 \cong G_2$ between two bounded and open subsets of R^2,

$$d(id - \psi^{-1} \circ f \circ \psi, G_1, 0) = d(id - f, G_2, 0)$$

for any continuous function $f : \overline{G}_2 \to \overline{G}_2$ with $\text{Fix}(f) \cap \partial G_2 = \emptyset$.

From now on we work with the map H. Since p is stable but not asymptotically stable, we can find a point $q \in R^2 \setminus \{p\}$ that is recurrent. This means that

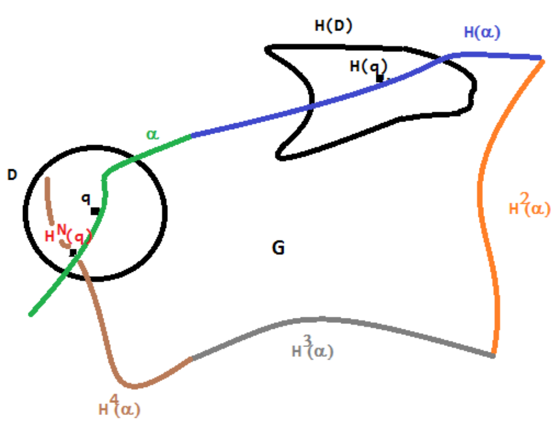

Fig. 6 In this example $N = 4$ and the degree is one in the region G

$q \in L_\omega(q, H)$, where $L_\omega(q, H)$ is the ω-limit set. We recall that this limit set is composed by all accumulation points of the sequence $\{H^n(p)\}_{n \geq 0}$. More details can be found in the appendix. The point q is not fixed under H and so we can find a small disk D around q satisfying

$$D \cap H(D) = \emptyset.$$

Since q is recurrent there exists an integer $N \geq 2$ such that $H^N(q) \in \text{int}\,(D)$. If we apply lemma 1, we find a translation arc α with $q, H^N(q) \in \dot{\alpha}$. After iterating N times the arc α under H, we observe that the point $H^N(q)$ belongs simultaneously to $\dot{\alpha}$ and $H^N(\dot{\alpha})$ and we are in the conditions of Brouwer's lemma.

9 Stability of invariant sets

From now on we assume that the map h is a homeomorphism of the plane and we indicate it by $h \in \mathscr{H}$. In particular it is assumed that h is onto, $h(R^2) = R^2$. Let Λ be a compact subset of R^2 that is invariant under h,

$$h(\Lambda) = \Lambda.$$

The notion of stability can be adapted to this general setting. We say that Λ is *stable* if given any neighborhood $\mathscr{U}$ of Λ, there exists another neighborhood $\mathscr{V}$ such that

$$h^n(\mathscr{V}) \subset \mathscr{U} \text{ for each } n \geq 0.$$

If Λ is a singleton, $\Lambda = \{p\}$, p is a fixed point and the two notions of stability coincide. When Λ is a finite set then it is composed by a finite number of periodic orbits, say

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$$

where

$$\Lambda_i = \{p_i, h(p_i), ..., h^{m_i-1}(p_i)\}$$

and $h^{m_i}(p_i) = p_i$, $i = 1, ..., r$. Each point p_i is fixed under the iterate h^{m_i} and Λ_i is stable if and only if p_i is a stable fixed point of h^{m_i}. In particular, if p_i is isolated as a fixed point of h^{2m_i} and Λ is stable we can apply theorem 1 to the orientation-preserving homeomorphism h^{2m_i} to deduce that

$$d(id - h^{2m_i}, G, 0) = 1$$

if G is a bounded and open subset of the plane with $p_i \in G$ and $\text{Fix}\,(h^{2m_i}) \cap \overline{G} = \{p_i\}$. This is a more or less trivial consequence of theorem 1 but we will obtain similar results when Λ is a Cantor set.

10 Stable Cantor sets

Assume that $\Lambda \subset R^2$ is a Cantor set that is invariant under h,

$$h(\Lambda) = \Lambda.$$

We recall that a Cantor set is a compact, perfect and totally disconnected metric space. We will also assume that Λ is *transitive*. This means that, for some $p \in \Lambda$,

$$L_\omega(p,h) = \Lambda.$$

There are many classical examples where transitive Cantor sets appear. In Smale's horseshoe there appear a Cantor set with the dynamics of the shift, see [1]. In the theory of twist maps, Aubry-Mather sets are invariant Cantor sets with Denjoy dynamics, see [19]. However, in these examples the set Λ is not stable. To present an example of stable Cantor set we consider the following construction, already described by Cartwright in [10]. Assume that D is a disk and inside D there are two disjoint disks D_0 and D_1. We repeat this process recursively so that D_0 contains two disks D_{00} and D_{01}, D_1 contains D_{10} and D_{11}, and so on. The intersection of the sets

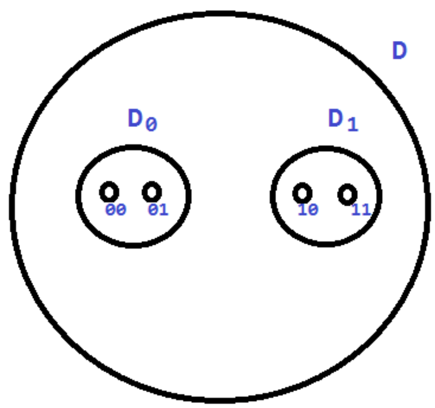

Fig. 7 Construction of a Cantor set as a limit of families of 2^n disks. The disks are labelled by the words of length n with the symbols 0 and 1. For $n = 2$, we have 00, 01, 10, 11.

D, $D_0 \cup D_1$, $D_{00} \cup D_{01} \cup D_{10} \cup D_{11}$... is a Cantor set Λ. We construct a homeomorphism h satisfying

$$h(D) = D$$

$$h(D_0) = D_1,\ \ h(D_1) = D_0$$

$$h(D_{00}) = D_{10},\ h(D_{10}) = D_{01},\ h(D_{01}) = D_{11},\ h(D_{11}) = D_{00}, \dots$$

Notice that, at each level of the construction, the iterates of h map every disk in all the others before returning. In particular $h(D) = D$, $h^2(D_0) = D_0$, $h^4(D_{00}) = D_{00}$,... Since the sets D, $D_0 \cup D_1$, $D_{00} \cup D_{01} \cup D_{10} \cup D_{11}$... are invariant under h, the same will happen to the intersection Λ. Moreover Λ is stable because it has a basis of invariant neighborhoods. Finally we notice that Λ is *minimal*. This means that, for each $p \in \Lambda$,

$$L_\omega(p,h) = \Lambda.$$

This is clear because the iterates of a disk visit all the disks at the same level. There can be some questioning as to whether this construction is feasible. This is easily solved after interpreting D as a section of the flow on a solid torus described in the next figure.

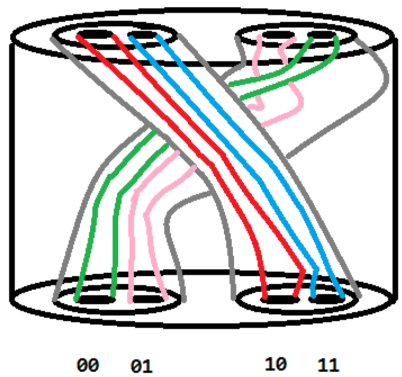

Fig. 8 The top and bottom of the cylinder are identified as a common disk. In this way we obtain a solid torus that is invariant under the flow. Inside this torus another invariant torus is obtained by gluing the two grey cylinders connecting the smaller disks. This torus winds twice before closing. The process is repeated recursively, in the next step there appear an invariant torus after gluing the four colored thinner cylinders.

Buescu and Stewart proved that the dynamics on any stable and transitive Cantor set is almost periodic, the so-called adding machine (see [9]). In particular Λ has to be minimal. The Bernoulli shift on a Cantor set is transitive but not minimal and so it is always unstable in the plane. Notice that this is the case in Smale's horseshoe. Cantor sets with Denjoy dynamics are minimal but not almost periodic (see [21]). For this reason Aubry-Mather sets cannot be stable.

In [3] Bell and Meyer proved that any stable and transitive Cantor set in the plane can be approximated by periodic points. In the same paper they constructed an example showing that this is false in three dimensions. Next we state the interesting result by Bell and Meyer.

Theorem 2. *Assume that $h \in \mathscr{H}$ and Λ is a Cantor set in R^2 that is invariant under h. In addition it is assumed that Λ is stable and transitive. Then, for each $p \in \Lambda$ there exists a sequence $\{x_n\}$, $x_n \in R^2$, converging to p and such that $h^{\sigma_n}(x_n) = x_n$ for some integer $\sigma_n \geq 1$.*

We know that Λ is minimal and so it cannot contain periodic points. For this reason the points x_n do not lie in Λ and $\sigma_n \to +\infty$. In particular Λ cannot be an isolated invariant set of h. The proof by Bell and Meyer combines the result by Buescu and Stewart mentioned above with the following classical fixed point theorem. We recall that a continuum is a non-empty compact and connected metric space.

Theorem 3. *Assume that $h \in \mathscr{H}_+$ and K is a continuum of the plane with $h(K) = K$. In addition assume that $R^2 \setminus K$ is connected. Then h has a fixed point in K.*

This result was proved by Cartwright and Littlewood in the beautiful paper [11]. Later Bell extended it to orientation-reversing homeomorphisms in [2]. In the recent paper [24], written in collaboration with Ruiz-Herrera, it is shown that additional information on stable Cantor sets can be obtained if one replaces the use of theorem 3 by Brouwer's lemma. In retrospect this seems natural because Cartwright-Littlewood theorem can be obtained as a consequence of Brouwer's lemma (see [6]). To state the next result we employ the notation $B(p,\delta)$ for the open ball of center p and radius δ.

Theorem 4. *Assume that the same conditions of theorem 2 hold. Then, for each $p \in \Lambda$ and each $\delta > 0$, there exists a topological disk D and an integer $N = N(p,\delta) \geq 1$ such that $D \subset B(p,\delta)$ and*

$$d(id - h^N, \mathrm{int}(D), 0) = 1.$$

As a consequence of this result we will deduce that stable and transitive Cantor sets have some remarkable properties: they are always persistent and non-isolated as invariant sets. We already mentioned that the non-isolated character follows from the theorem by Bell and Meyer. We can now see Bell-Meyer's result as a corollary of theorem 4, because h^N has a fixed point in D. Next we present a second corollary that seems to be new.

Corollary 2. *In the conditions of theorem 4, the set Λ is persistent as an invariant set.*

This is again a consequence of the continuity property of the degree that was discussed in section 4. Given $\varepsilon_1 > 0$ we find a finite set of points in Λ, $p_1, \dots, p_r$, such that for every $p \in \Lambda$,

$$\min_{i=1,\dots,r} ||p - p_i|| < \varepsilon_1.$$

Then we apply theorem 4 at each $p_i \in \Lambda$ and find topological disks D_i and integers $N_i \geq 1$ with $p_i \in \mathrm{int}(D_i)$, $D_i \subset B(p_i, \varepsilon_1)$ and

$$d(id - h^{N_i}, \mathrm{int}(D_i), 0) = 1.$$

Given $\tilde{h} \in \mathscr{H}$ with $||h - \tilde{h}||_\infty$ small, it is possible to prove that

$$\max_{x \in \partial D_i} ||h^{N_i}(x) - \tilde{h}^{N_i}(x)||$$

is also small for each $i = 1, \dots, r$. This implies that

$$d(id - \tilde{h}^{N_i}, \mathrm{int}(D_i), 0) = d(id - h^{N_i}, \mathrm{int}(D_i), 0) = 1.$$

In consequence $\tilde{h}$ has a periodic point $\tilde{p}_i$ lying in D_i and the set

$$\tilde{\Lambda} = \{\tilde{h}^k(\tilde{p}_i) : \ 0 \leq k < N_i, \ i = 1, \dots, r\}$$

is invariant under $\tilde{h}$. By some continuity arguments the number ε_1 can be adjusted so that the Haussdorf distance between Λ and $\tilde{\Lambda}$ satisfies $D_H(\Lambda,\tilde{\Lambda}) \le \varepsilon$. The details can be found in [24]. It is convenient to remark that the persistence of Λ is understood in the class of invariant sets, the perturbed set $\tilde{\Lambda}$ is not necessarily a Cantor set. Notice that in the above proof the perturbed set was finite.

We finish with a sketch of the proof of theorem 4. The complete details can be found in [24].

Proof. (Theorem 4). **First step:** Given $p \in \Lambda$ and $\delta > 0$ there exist an open and simply connected set $\omega \subset R^2$ and an integer $N \ge 1$ such that

$$p \in \omega \subset \overline{\omega} \subset B(p,\delta),\ \ h^N(\omega) \subset \omega.$$

The proof of this step combines the ideas of the proof of lemma 2 in the appendix with the minimality of Λ, that follows from Buescu-Stewart result.

Second step: There exists a point $q \in \omega \cap \Lambda$ that is recurrent under h^{2N}; that is, $q \in L_\omega(q,h^{2N})$.
To prove this step we consider the set

$$R_{2N} = \{q \in \Lambda :\ q \in L_\omega(q,h^{2N})\}$$

and prove that it is non-empty. Since Λ is invariant under h^{2N}, it must contain a minimal set M (with respect to h^{2N}). Then $\emptyset \ne M \subset R_{2N} \subset \Lambda$. Once we know that R_{2N} is non-empty we observe that it is invariant under h. From $h(R_{2N}) = R_{2N}$ we deduce that also the closure is invariant, $h(\overline{R}_{2N}) = \overline{R}_{2N}$. Since Λ is minimal with respect to h, $\overline{R}_{2N} = \Lambda$ and so the recurrent points (with respect to h^{2N}) are dense in Λ.

Third step: Conclusion via Brouwer's lemma.
Let $\sigma : R^2 \cong \omega$ be a homeomorphism with $\sigma(Q) = q$. Define $H = \sigma^{-1} \circ h_\omega^{2N} \circ \sigma$, where $h_\omega^{2N} : \omega \to \omega$ is the restriction of h^{2N}. Then $H \in \mathscr{E}_+$ and Q is recurrent under H, $Q \in L_\omega(Q,H)$. Since q belongs to Λ, it is not a fixed point of h^{2N}. Then Q is not a fixed point of H and a disk D around Q can be found such that $D \cap H(D) = \emptyset$. Let $M \ge 2$ be an integer such that $H^M(Q) \in D$. We draw a translation arc α passing through Q and $H^M(Q)$. Then $\alpha \cap H^M(\alpha)$ is non-empty and there exists a Jordan curve $\Gamma \subset \alpha \cup \cdots \cup H^M(\alpha)$ with

$$d(id - H, R_i(\Gamma), 0) = 1.$$

We transport this curve to ω, $\gamma = \sigma(\Gamma)$ and notice that $\overline{R_i(\gamma)} \subset \omega$ because ω is simply connected. The invariance of the fixed point index under conjugation implies that

$$d(id - h^{2N}, R_i(\gamma), 0) = d(id - H, R_i(\Gamma), 0) = 1$$

and the proof is complete with $D = \overline{R_i(\gamma)}$.

Acknowledgements Supported by the research project MTM2011-23652, Spain

Appendix

10.1 Remarks on embeddings

Given a continuous and one-to-one map $h : R^2 \to R^2$, the theorem of invariance of the domain implies that h is open. In particular $h(R^2)$ is an open subset of R^2.

The map h is orientation-preserving ($h \in \mathscr{E}_+$) if

$$d(h, G, h(p)) = 1,$$

where the degree is computed on any open and bounded set $G \subset R^2$ with $p \in G$. It can be proved that the value of this degree is independent of the choice of G and p. Moreover it can only take the values $+1$ or -1. In the second case h is orientation-reversing. Embeddings that are isotopic to the identity lie in $\mathscr{E}_+$. Those isotopic to the symmetry $S(x_1, x_2) = (x_1, -x_2)$ are orientation-reversing. We recall that two embeddings h_0 and h_1 are isotopic if there exists a continuous map

$$H : R^2 \times [0,1] \to R^2, \ \ (x, \lambda) \mapsto H_\lambda(x)$$

such that H_λ is one-to-one for each $\lambda \in [0,1]$, $H_0 = h_0$ and $H_1 = h_1$.

10.2 Lemmas for the proof of theorem 1

Lemma 2. *Assume that $\mathscr{U}$ is an open subset of the plane and $h : \mathscr{U} \subset R^2 \to R^2$ is a continuous and one-to-one map with a stable fixed point $p = h(p)$. Then, given any neighborhood $\mathscr{W}$ of p, there exists an open and simply connected set $\omega \subset \mathscr{U} \cap \mathscr{W}$ such that $p \in \omega$ and $h(\omega) \subset \omega$.*

Proof. It is extracted from [27], page 185. Let us fix a closed disk Δ centered at p and contained in $\mathscr{U} \cap \mathscr{W}$. The stability of the fixed point allows us to find an open and connected set G with $p \in G$ and $h^n(G) \subset \Delta$ for each $n \geq 0$. The open and connected set

$$\Omega := \bigcup_{n \geq 0} h^n(G)$$

is positively invariant under h and contained in Δ. At first sight Ω could be the searched set, because it is contained in $\mathscr{U} \cap \mathscr{W}$ and $h(\Omega) \subset \Omega$. However this set can have holes. Let $\widehat{\Omega}$ be the smallest simply connected domain containing Ω. Intuitively this set is constructed by filling the holes in Ω, a more formal construction can be found in lemma 2.6 of [22]. Since h is an embedding, $h(\widehat{\Omega}) = \widehat{h(\Omega)}$ and so we can take $\omega = \widehat{\Omega}$.

Lemma 3. *Assume that* $h : R^2 \to R^2$ *is a continuous and one-to-one map with a fixed point* $p = h(p)$ *that is stable but not asymptotically stable. Then, for each neighborhood* $\mathscr{W}$ *of* p *there exists a point* $q \in \mathscr{W} \setminus \{p\}$ *that is recurrent.*

Proof. By a contradiction argument assume that there exists a neighborhood $\mathscr{W}$ of p such that $\mathscr{W} \setminus \{p\}$ does not contain recurrent points. Since p is stable we can find two closed balls β and B centered at p and such that $\beta \subset B \subset \mathscr{W}$ and

$$h^n(\beta) \subset B \text{ for each } n \geq 0.$$

We claim that $p \in L_\omega(x,h)$ for each $x \in \beta$. Indeed, the positive orbit $\{h^n(x)\}_{n\geq 0}$ is contained in B and so it is bounded. In consequence the limit set $L_\omega(x,h)$ is non-empty, compact and invariant under h. This limit set has to contain a minimal set M. All points y in M must satisfy $L_\omega(y,h) = M$ and so they are recurrent. Summing up, we can say that $L_\omega(x,h)$ is contained in B and has at least one recurrent point. Since p is the only recurrent point in this ball, the claim $p \in L_\omega(x,h)$ has been proved. Now we observe that, due to the stability of p,

$$L_\omega(x,h) = \{p\}.$$

Then all points in β are attracted by p and this point should be asymptotically stable.

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