A note on stability criteria in the periodic Lotka-Volterra predator-prey model

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Abstract

We present a stability result for T-periodic solutions of the periodic predator-prey Lotka-Volterra model. In 2021, R. Ortega gave a stability criteria in terms of the L^1 norm of the coefficients of a planar linear system associated to the model. Previously, in 1994, Z. Amine and R. Ortega proved another stability criteria formulated in terms of the L^∞ norm. The present work gives a L^p criterion, building a bridge between the two previous results.

Keywords: Population dynamics, Lyapunov stability criterion, periodic planar systems, periodic predator-prey model

1. Introduction

In this work we consider the periodic predator-prey Lotka-Volterra model:

$$\begin{cases} \dot{u} = u(a(t) - b(t)u - c(t)v), \\ \dot{v} = v(d(t) + e(t)u - f(t)v) \end{cases} \tag{1}$$

with $u \geq 0$, $v \geq 0$. All the coefficients are T-periodic, $a, d \in L^p(\mathbb{T}_T)$, $p \in [1, \infty]$, and b, c, e and f are positive functions in $C(\mathbb{T}_T)$, where we denote the quotient set $\mathbb{R}/T\mathbb{Z}$ as $\mathbb{T}_T$. This model is a classical non-autonomous model for predator-prey interaction studied by many authors (see [2] and the references therein). In [2] the authors study the existence of coexistence states and in particular prove that if one among the trivial and semi-trivial states is linear stable then it attracts all the solutions with positive initial conditions. As an immediate consequence, for the existence of a coexistence state it is necessary that the trivial and the possible semi-trivial states are linearly unstable. In the same paper the authors prove that this is also a sufficient condition.

Assuming the existence of a coexistence state, to know if it is stable or not is an important problem. In [1] and in [5] the authors addressed this question and gave conditions for the existence of one stable coexistence state.

The stability of the coexistence state was obtained using a homotopy from a non-autonomous linear system to an autonomous one. In both results plays an important role to prove the non-existence of $2T$-periodic solutions for a linear system of the form

$$\begin{cases} \dot{x}_1 = -a_{11}(t)x_1 - a_{12}(t)x_2, \\ \dot{x}_2 = a_{21}(t)x_1 - a_{22}(t)x_2, \end{cases} \tag{2}$$

where the a_{ij} are non negative. In order to guarantee this nonexistence, in [1] a condition which implies

$$T\|a_{12}\|^{1/2}_{L^\infty(\mathbb{T}_T)} \; \|a_{21}\|^{1/2}_{L^\infty(\mathbb{T}_T)} + \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq \pi, \tag{3}$$

is given while in [5] the analogous expression

$$\|a_{12}\|^{1/2}_{L^1(\mathbb{T}_T)} \; \|a_{21}\|^{1/2}_{L^1(\mathbb{T}_T)} + \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq 2, \tag{4}$$

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but concerning the $L^1(0,T)$ norms, was obtained.

The main purpose of our paper is to extend these results allowing to use other L^p norms, see Proposition 2.1 below. With this result we connect the results in [1] and [5]. We also give an example of a case in which the results in [1] and in [5] do not apply but ours does with $p=2$. Finally, as in the previous papers, we give conditions for the local asymptotic stability of a coexistence state of (1).

2. Planar linear system: L^p stability result

Our aim is to give a L^p stability condition for the system

$$\begin{cases} \dot{x}_1 = -a_{11}(t)x_1 - a_{12}(t)x_2, \\ \dot{x}_2 = a_{21}(t)x_1 - a_{22}(t)x_2, \end{cases} \tag{5}$$

where the coefficients a_{ij} belong to $L^p(\mathbb{T}_T)$ with $p\in[1,\infty]$ and satisfy

$$\overline{a}_{11}\geq 0,\ \overline{a}_{22}\geq 0 \quad \text{and} \quad a_{12}(t)\geq\delta,\ a_{21}(t)\geq\delta \quad \text{a.e} \quad t\in\mathbb{R}, \tag{6}$$

for some $\delta>0$ where $\overline{a}_{ij}=\frac{1}{T}\int_0^T a_{ij}(t)\mathrm{d}t$.
In order to do that we are going to give conditions which guarantee the nonexistence of $2T$-periodic solutions for this linear system in the next Proposition.

Proposition 2.1. *The system* (5) *has no $2T$-periodic solutions except $x\equiv 0$ if the periodic coefficients a_{ij} satisfy (with $1/p+1/q=1$):*

$$T^{1/q}\|a_{12}\|^{1/2}_{L^p(\mathbb{T}_T)}\ \|a_{21}\|^{1/2}_{L^p(\mathbb{T}_T)}+\frac{1}{2}\|a_{11}-a_{22}\|_{L^1(\mathbb{T}_T)}\leq\frac{\mathcal{J}(q)}{2^{2-1/q}}, \tag{7}$$

where

$$\mathcal{J}(q)=\int_0^{2\pi}\frac{\mathrm{d}\theta}{\left(|\cos\theta|^{2q}+|\sin\theta|^{2q}\right)^{1/q}},\quad q\in[1,\infty[\quad \textit{and}\quad \mathcal{J}(\infty):=\lim_{q\to+\infty}\mathcal{J}(q).$$

Proof. We make a change of variables to the elliptic-polar coordinates with a weight $\mu>0$ that will be determined later:

$$x_1=\sqrt{\mu}r\cos\theta,\quad x_2=\frac{1}{\sqrt{\mu}}r\sin\theta.$$

Then the equation of motion for the variable θ is,

$$\dot{\theta}=\mu a_{21}(t)\cos^2\theta+\frac{1}{\mu}a_{12}(t)\sin^2\theta+(a_{11}(t)-a_{22}(t))\cos\theta\,\sin\theta. \tag{8}$$

From (8), we can write

$$\dot{\theta}\leq\left\langle\left(\mu a_{21}(t),\frac{1}{\mu}a_{12}(t)\right),(\cos^2\theta,\sin^2\theta)\right\rangle+|a_{11}(t)-a_{22}(t)|\left\langle\left(\frac{1}{2},\frac{1}{2}\right),(\cos^2\theta,\sin^2\theta)\right\rangle. \tag{9}$$

First, let us consider $p\in]1,\infty[$. By the Hölder inequality in $\mathbb{R}^2$, we have

$$\dot{\theta}\leq\left(\left((\mu a_{21}(t))^p+\left(\frac{1}{\mu}a_{12}(t)\right)^p\right)^{1/p}+|a_{11}(t)-a_{22}(t)|\left(\frac{2}{2^p}\right)^{1/p}\right)\left(|\cos\theta|^{2q}+|\sin\theta|^{2q}\right)^{1/q}. \tag{10}$$

Let us integrate on an interval $I\in\mathbb{R}$ where a solution of (5) is well defined,

$$\int_{\theta(I)}\frac{\mathrm{d}\theta}{\left(|\cos\theta|^{2q}+|\sin\theta|^{2q}\right)^{1/q}}\leq\int_I\left((\mu a_{21}(t))^p+\left(\frac{1}{\mu}a_{12}(t)\right)^p\right)^{1/p}\mathrm{d}t\,+\,2^{\frac{1-p}{p}}\int_I|a_{11}(t)-a_{22}(t)|\ \mathrm{d}t.$$

Now, by the Hölder inequality in L-spaces norms , we have

$$\int_{\theta(I)}\frac{\mathrm{d}\theta}{\left(|\cos\theta|^{2q}+|\sin\theta|^{2q}\right)^{1/q}}\leq|I|^{1/q}\left(\int_I(\mu a_{21}(t))^p\,\mathrm{d}t+\int_I\left(\frac{1}{\mu}a_{12}(t)\right)^p\mathrm{d}t\right)^{1/p}+\,2^{\frac{1-p}{p}}\int_I|a_{11}(t)-a_{22}(t)|\ \mathrm{d}t.$$

Let us assume that exists a non-trivial $2T$-periodic solution of (5) $(x_1(t), x_2(t))$. We claim that every non-trivial solution of (8) crosses the axes in a counter-clockwise sense. Intuitively, if the coefficients $a_{12}(t)$ and $a_{21}(t)$ are continuous and we consider small neighborhoods of the axes, the angular evolution is always positive since the sign in the right hand side of the equation (8) is positive. In [5] the author gives a proof for coefficients in $L^1(\mathbb{T}_T)$. Therefore we have in the angular variable $\theta(t+2T) = \theta(t) + 2\pi k$, being k a non-negative integer. Take $k = 0$ and let us consider that the solution $(x_1(t), x_2(t))$ cross an axis. Due to the periodicity it cannot cross the axis in a clockwise sense to came back. We conclude that the solution must lie in an open quadrant, and we have two possibilities, either $x_1(t) \cdot x_2(t) > 0$ or $x_1(t) \cdot x_2(t) < 0$ for $t \in \mathbb{R}$. In the first case we divide the first equation in (5) by x_1 and integrate over a $2T$-period to get $\overline{a}_{11} < 0$, in contradiction with (6). The second case has an analog treatment considering the second equation. See [5] for more details.
For $k \geq 1$, integrating from 0 to $2T$,

$$k\mathcal{J}(q) < (2T)^{1/q} \left(\int_0^{2T} (\mu a_{21}(t))^p \, \mathrm{d}t + \int_0^{2T} \left(\frac{1}{\mu} a_{12}(t) \right)^p \mathrm{d}t \right)^{1/p} + 2^{\frac{1-p}{p}} \int_0^{2T} |a_{11}(t) - a_{22}(t)| \, \mathrm{d}t.$$

Additionally, we have dropped the equal sign since the equality in (9) only occurs when $\theta = \pi/4 \pm \pi k$. Now, defining $\mu^p := \left(\int_0^T a_{12}(t)^p \mathrm{d}t / \int_0^T a_{21}(t)^p \mathrm{d}t \right)^{1/2}$, we arrive to a contradiction with the hypothesis (7) in the more restrictive case associated with $k = 1$.
Concerning the limiting cases, we can found a proof for the case $p = 1$ in [5] and for $p = \infty$ in [1]. □

In the next proposition we give a result which, together to the fact that $\mathcal{J}(1) = 2\pi$, allows us to conclude that (7) connects the results in [1] and [5].

Proposition 2.2. *We have that* $\lim_{q \to +\infty} \mathcal{J}(q) = 8$.

Proof. Let us study the limit of the integrand of $\mathcal{J}(q)$:

$$f(\theta; q) = \left(|\cos\theta|^{2q} + |\sin\theta|^{2q} \right)^{-1/q}. \tag{11}$$

It is useful to write the function in two different ways

$$f(\theta; q) = \cos^{-2}\theta \left(1 + |\tan\theta|^{2q} \right)^{-1/q} = \sin^{-2}\theta \left(1 + |\cot\theta|^{2q} \right)^{-1/q}.$$

For $\theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$ we have that

$$\lim_{q \to \infty} f(\theta; q) = \lim_{q \to \infty} \cos^{-2}\theta \left(1 + |\tan\theta|^{2q} \right)^{-1/q} = \cos^{-2}\theta. \tag{12}$$

Indeed, for $\theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$,

$$2^{-1/q} \leq \left(1 + \tan^2\theta \right)^{-1/q} \leq \left(1 + |\tan\theta|^{2q} \right)^{-1/q} \leq 1,$$

where we have used that $\tan^2\theta \leq 1$ and $q \geq 1$ and the result follows.
Analogously, for $\theta \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$, we have that

$$\lim_{q \to \infty} f(\theta; q) = \lim_{q \to \infty} \sin^{-2}\theta \left(1 + |\cot\theta|^{2q} \right)^{-1/q} = \sin^{-2}\theta. \tag{13}$$

Now, by Lebesgue's dominated convergence theorem,

$$\lim_{q \to \infty} \int_0^{2\pi} f(\theta; q) \mathrm{d}\theta = \int_{-\pi/4}^{\pi/4} \cos^{-2}\theta \, \mathrm{d}\theta + \cdots + \int_{5\pi/4}^{7\pi/4} \sin^{-2}\theta \, \mathrm{d}\theta = 2 + \cdots + 2 = 8.$$

□

From this convergence of $\mathcal{J}(q)$ we recover the condition in [5] for $p = 1$,

$$\|a_{12}\|_{L^1(\mathbb{T}_T)}^{1/2} \, \|a_{21}\|_{L^1(\mathbb{T}_T)}^{1/2} + \frac{1}{2} \|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq 2. \tag{14}$$

Additionally, we can check numerically that $\mathcal{J}(q) \in [2\pi, 8]$ for $q \in [1, \infty]$.

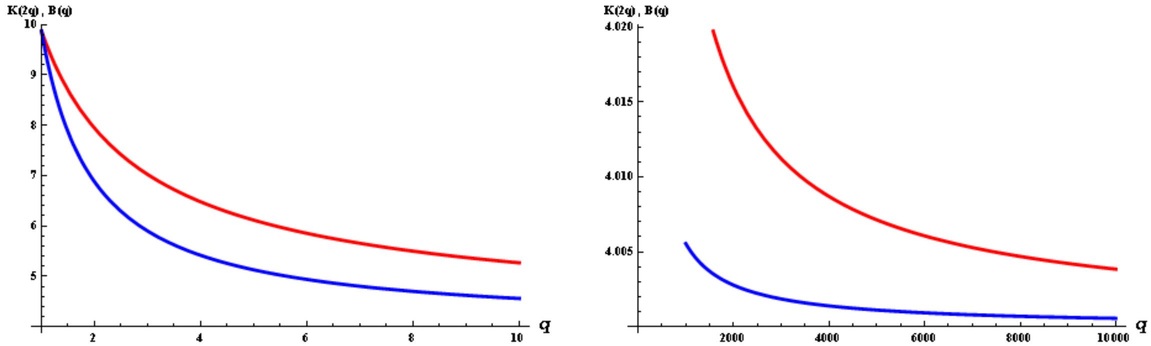

Figure 1: *Comparison of the Sobolev constant $K(2q)$ (in red) with the upper bound $B(q)$ (in blue). On the left, for $q \in [1, 10]$; on the right, asymptotic behavior for large q, note that $K(q)$ and $B(q)$ tend to 4 according to the classical Lyapunov criterion.*

Remark 2.1. In the Hill's equation

$$\ddot{x} + \alpha(t)\, x = 0, \qquad \alpha(t) > 0 \quad \text{a.e} \quad t \in \mathbb{R}, \tag{15}$$

our criterion becomes

$$\|\alpha\|_{L^p(\mathbb{T}_T)} \leq \left(\frac{\mathcal{J}(q)}{2^{2-1/q}}\right)^2 \frac{1}{T^{1+1/q}} := B_T(q). \tag{16}$$

The classical stability criterion due to Lyapunov (see [3] or [5]) follows for $q = \infty$. Additionally, if $q = 1$, we recover the condition in Lemma 4.4 of [1]. In [7], Zhang and Li extended the Lyapunov stability criterion using L^p norms, as follows

$$\|\alpha\|_{L^p(\mathbb{T}_T)} < K_T(2q) \text{ if } 1 < p \leq \infty, \quad \text{and} \quad \|\alpha\|_{L^1(\mathbb{T}_T)} < K_T(\infty) = \frac{4}{T}, \text{ if } p = 1. \tag{17}$$

Here, $K_T(q)$ is the Sobolev constant defined as the optimal constant for the following inequality

$$K_T(q)\|u\|^2_{L^q(\mathbb{T}_T)} \leq \|\dot{u}\|^2_{L^2(\mathbb{T}_T)},$$

where u is any function in the Sobolev space $H^1_0[0, T]$. This Sobolev constant is given by

$$K_T(q) = \begin{cases} \frac{2\pi}{q}\left(\frac{2}{2+q}\right)^{1-2/q}\left(\frac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)}\right)^2 \frac{1}{T^{1+2/q}}, & \text{if} \quad 1 \leq q < \infty, \\ \frac{4}{T}, \quad \text{if} \quad q = \infty. \end{cases} \tag{18}$$

Γ is the usual Gamma function (see [6] for the proof). The upper bounds in (17) are best possible in the sense that for any $\varepsilon > 0$, there is some α such that

$$\|\alpha\|_{L^p(\mathbb{T}_T)} < K_T(2q) + \varepsilon,$$

while (15) is unstable. In Figure 1 we compare numerically $B(q) := T^{1+2/q}B_T(q)$ with $K(q) := T^{1+2/q}K_T(q)$. It seems that our bound is not the best possible for the Hill's equation since $B(q) < K(2q)$ for $1 < q < \infty$. In the cases $q = 1$ and $q = \infty$ both are equal.

We are willing to get our stability result as in [5], that is, using a homotopy argument. We consider the family of continuous matrices $\{A_\lambda\}$ as

$$A_\lambda(t) = (1-\lambda)A(t) + \lambda\overline{A}$$

where $A(t)$ is the coefficient matrix of the system (5) whose elements satisfy (6), the elements of the matrix $\overline{A}$ are the average of a_{ij} and the matrix $A_\lambda(t)$ is the coefficient matrix of the associated system in the plane

$$\dot{x} = A_\lambda(t)x. \tag{19}$$

We have that $\forall\lambda \in [0, 1]$,

$$\overline{a}_{11}(\cdot, \lambda) \geq 0, \ \overline{a}_{22}(\cdot, \lambda) \geq 0 \quad \text{and} \quad a_{12}(t, \lambda) \geq \delta, \ a_{21}(t, \lambda) \geq \delta \quad \text{a.e} \quad t \in \mathbb{R}. \tag{20}$$

The continuous matrices A_0, A_1 will be called homotopic if the family of systems (19) is continuous on $\lambda \in [0,1]$ and has no $2T$-periodic solutions excepting $x \equiv 0$. The continuity of $\{A_\lambda\}$ means that for each element $a_{ij}(t,\lambda)$ of A_λ and $\lambda \in [0,1]$, $\lim_{h\to 0} \|a_{ij}(t,\lambda+h) - a_{ij}(t,\lambda)\|_{L^1(\mathbb{T}_T)} = 0$. It is important to note that requirement of non-existence of $2T$-periodic solutions is crucial to guarantee the stability properties of the family of systems (19) along the whole homotopy.

The following lemma establishes that $A(t)$ and $\overline{A}$ can be connected by the homotopy $A_\lambda(t)$.

Lemma 2.1. *Assume* (7)*, then $A_0 = A(t)$ and $A_1 = \overline{A}$ are homotopic.*

Proof. The family of systems (19) satisfies the criterion (7) of the previous section. The case $\lambda = 0$ is immediate. The case $\lambda = 1$, easily follows when $p = \infty$ and for $p \in [1, \infty[$ is a consequence of the Hölder inequality as

$$\|\overline{a}_{ij}\|_{L^p(\mathbb{T}_T)} \le T^{1/p-1}\|a_{ij}\|_{L^1(\mathbb{T}_T)} \le T^{1/p-1}T^{1/q}\|a_{ij}\|_{L^p(\mathbb{T}_T)} = \|a_{ij}\|_{L^p(\mathbb{T}_T)}.$$

The remaining details of the proof are easy to check. □

Finally, following the lines of the proof of Lemma 2.1 in [5] we obtain our main theorem, taking into account that the system of constant coefficients $\dot{x} = \overline{A}x$ is stable as we assume (6). The key idea of this proof is the relation between the eigenvalues of the monodromy matrix $X(T)$ associated to the periodic system (5) and the non-existence of $2T$-periodic solutions. It can be shown that the system (5) is asymptotically stable if and only if the trace of $X(T)$ satisfies

$$|\mathrm{tr}\, X(T)| < 1 + e^{-(\overline{a}_{11}+\overline{a}_{22})T}.$$

If $|\mathrm{tr}\, X(T)| = 1 + e^{-(\overline{a}_{11}+\overline{a}_{22})T}$, the system (5) has a non-trivial $2T$-periodic solution since in that case there is a real characteristic multiplier with absolute value equal to one. The homotopy between $A_0 = A(t)$ and $A_1 = \overline{A}$ applied to $\Delta(\lambda) := \mathrm{tr}\, X(T,\lambda)$, $\lambda \in [0,1]$ concludes the proof. The example 1.2 in [4] also helps to understand this relation between the stability properties of a periodic system and the trace of the associated monodromy matrix.

Theorem 2.1. *Assume* (7)*, then the system* (5) *is stable. Moreover, it is asymptotically stable if $\overline{a}_{11} + \overline{a}_{22} > 0$.*

2.1. An example

In the following example we ask if there are coefficients $a_{ij}(t)$ such that for some $p \in]1, \infty[$ and some $T > 0$ condition (7) is satisfied but the conditions (3) or (4) are not fulfilled.

Example 2.1. Let us consider $a_{12}(t) = 1 + \delta + \sin\left(\frac{2\pi t}{T}\right)$, $a_{21}(t) = 1 + \delta + \cos\left(\frac{2\pi t}{T}\right)$ and $p = 2$. Also we assume that $a_{11}(t) = a_{11}(\frac{2\pi t}{T})$ and $a_{22}(t) = a_{22}(\frac{2\pi t}{T})$ and both are in $L^1(\mathbb{T}_T)$. Then $D(T) := \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)}$ is a linear function, $D(T) = \alpha(a_{11}, a_{22})T$ with $\alpha \ge 0$. The conditions to fulfill become

$$\begin{cases} f(T) := \left[(1+\delta)^2 + \frac{1}{2}\right]^{1/2} T + \alpha(a_{11}, a_{22})\, T - \frac{\mathcal{J}(2)}{2^{3/2}} \le 0, & \left(\frac{\mathcal{J}(2)}{2^{3/2}} \approx 2,622\right) \\ g(T) := (1+\delta)\, T + \alpha(a_{11}, a_{22})\, T - 2 > 0, & \\ h(T) := (2+\delta)\, T + \alpha(a_{11}, a_{22})\, T - \pi > 0. & \end{cases} \tag{21}$$

Let us observe that the slope K of these linear functions is always positive and $K_h > K_f > K_g$. This guarantees that the functions intersect each other. Analyzing the ordering of the zeros and the cross-points of the previous functions with respect to the parameters δ and α, we can prove that there exists non-empty sets of possible values of T where the three previous conditions hold simultaneously. The zeros are given by the equations:

$$T_f = \frac{\frac{\mathcal{J}(2)}{2^{3/2}}}{\left[(1+\delta)^2 + \frac{1}{2}\right]^{1/2} + \alpha}, \quad T_g = \frac{2}{1+\delta+\alpha}, \quad T_h = \frac{\pi}{2+\delta+\alpha},$$

and we want that $T_f > T_g$ and $T_f > T_h$, simultaneously. We can check that

i) Since $\delta > 0$ and $\alpha \ge 0$, there are no restrictions to satisfy $T_f > T_g$.

ii) $T_f > T_h$ holds if $\psi(\delta) := \frac{\mathcal{J}(2)}{2^{3/2}}(2+\delta) - \pi\left[(1+\delta)^2 + \frac{1}{2}\right]^{1/2} > \left(\pi - \frac{\mathcal{J}(2)}{2^{3/2}}\right)\alpha \ge 0$. It is easy to find a set of the form $\mathcal{D} = \{(\alpha,\delta) : \alpha \in [0, \alpha^*[, \delta \in]\delta_1(\alpha), \delta_2(\alpha)[\subset]0, \delta_+[\}$, in which this inequality holds. Here, $\alpha^* = \frac{\psi(\delta_{max})}{\pi - \frac{\mathcal{J}(2)}{2^{3/2}}} \approx 2.692$, $\delta_{max} \approx 0,07145$ and $\delta_+ \approx 3,729$ is the positive root of $\psi(\delta)$.

iii) On the other hand, $T_h > T_g \Rightarrow \alpha > \frac{4-\pi}{\pi-2} \approx 0.752$ and $T_g > T_h \Rightarrow \alpha < \frac{4-\pi}{\pi-2}$, $\delta \in]0, \delta_3(\alpha)[\subset]0, \frac{4-\pi}{\pi-2}[$.

Consequently, it exists non-empty intervals of possible periods where the conditions (21) are satisfied simultaneously:

$$T \in]T_h, T_f[\text{ if } \alpha \in]\frac{4-\pi}{\pi-2}, \alpha^*[, \delta \in]\delta_1(\alpha), \delta_2(\alpha)[\subset]0, \delta_+[; \qquad T \in]T_g, T_f[\text{ if } \alpha \in [0, \frac{4-\pi}{\pi-2}[, \delta \in]0, \delta_3(\alpha)[\subset]0, \frac{4-\pi}{\pi-2}[.$$

3. Stability result in the predator-prey Lotka-Volterra model

Let us consider once again (1) and assume that $(u(t), v(t))$ is a coexistence state. In [2] the authors give sufficient and necessary conditions for the existence of such solutions of the system (1).
The following theorem gives an additional condition in L^p spaces to guarantee the uniqueness and the asymptotic stability of the T-periodic solution (coexistence state). Additionally, we give some estimates on the L^p norm of this solution $(u(t), v(t))$ in terms of the coefficients of system (1).

Theorem 3.1. *Assume that all possible coexistence states satisfy*

$$T^{1/q}\sqrt{\|eu\|_{L^p(\mathbb{T}_T)}\|cv\|_{L^p(\mathbb{T}_T)}} + \frac{T}{2}\|bu - fv\|_{L^1(\mathbb{T}_T)} \leq \frac{\mathcal{J}(q)}{2^{2-1/q}}, \tag{22}$$

where p and q are conjugate indices and $p, q \in [1, \infty]$. Then the T-periodic solution $(u(t), v(t))$ is unique and asymptotically stable. In addition, for $p \in [1, \infty[$ we have the following upper bounds of this solution:

$$\|u\|_{L^p(\mathbb{T}_T)} < \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}, \quad \|v\|_{L^p(\mathbb{T}_T)} < \frac{\|d\|_{L^p(\mathbb{T}_T)} + e_M \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}}{f_L}. \tag{23}$$

where $\varphi_L := \min_{t\in[0,T]} \varphi(t)$ and $\varphi_M := \max_{t\in[0,T]} \varphi(t)$ for a T-periodic function, $\varphi(t)$.

Proof. The result of existence and uniqueness follows as [5] using the theorem of the previous section.
Concerning the estimates, let us multiply the first equation of system (1) by u^{p-2} and the second one by v^{p-2}. After integrating both equations over a period T and using the T-periodicity of $(u(t), v(t))$ together with the following inequality for $f, g \in L^p(\mathbb{T}_T)$, $p \in [1, \infty[$, $\|f^{p-1}g\|_{L^1(\mathbb{T}_T)} \leq \|f\|^{p-1}_{L^p(\mathbb{T}_T)}\|g\|_{L^p(\mathbb{T}_T)}$, we get:

$$b_L\,\|u\|^p_{L^p(\mathbb{T}_T)} + \|c\,u^{p-1}v\|_{L^1(\mathbb{T}_T)} \leq \|a\|_{L^p(\mathbb{T}_T)}\|u\|^{p-1}_{L^p(\mathbb{T}_T)} \implies \|u\|_{L^p(\mathbb{T}_T)} < \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L},$$

for the first equation and

$$f_L\,\|v\|^p_{L^p(\mathbb{T}_T)} \leq \|(d + e\,u)\,v^{p-1}\|_{L^1(\mathbb{T}_T)} \leq \|(d + e\,u)\|_{L^p(\mathbb{T}_T)}\,\|v\|^{p-1}_{L^p(\mathbb{T}_T)} \implies \|v\|_{L^p(\mathbb{T}_T)} < \frac{\|d\|_{L^p(\mathbb{T}_T)} + e_M \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}}{f_L}$$

for the second one. □

Remark 3.1. By the previous proof, we conclude that the previous theorem holds if (22) is verified for all u and v such that

$$\|u\|_{L^p(\mathbb{T}_T)} < \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}, \quad \|v\|_{L^p(\mathbb{T}_T)} < \frac{\|d\|_{L^p(\mathbb{T}_T)} + e_M \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}}{f_L}.$$

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