Twist Mappings, Invariant Curves and Periodic Differential Equations

Rafael Ortega

1 Introduction

Let us consider the periodic differential system in the plane,

$$\dot{x} = X(t,x), \quad x \in \mathbb{R}^2,$$

where X satisfies

$$X(t+2\pi) = X(t,x), \quad \forall (t,x) \in \mathbb{R}^2.$$

In this course we will study the quasi-periodic solutions of this equation and we will show that these solutions play an important role in the study of the dynamics when the equation has a hamiltonian structure.

Before trying to do any theory, we discuss a simple case at an intuitive level.

Example.

$$\ddot{x} + 2x = \sin t. \tag{1.1}$$

This equation has the unique 2π-periodic solution

$$x^\star(t) = \sin t.$$

The other solutions are not periodic, in fact they are of the form

$$x(t) = \sin t + c_1 \sin \sqrt{2}t + c_2 \cos \sqrt{2}t, \quad c_1, c_2 \in \mathbb{R}.$$

These functions are each sums of two periodic functions with nonconmensurable periods ($T_1 = 2\pi$, $T_2 = \sqrt{2}\pi$). As we shall see, they are quasi-periodic solutions with frequencies $\omega_1 = \frac{2\pi}{T_1} = 1$ and $\omega_2 = \frac{2\pi}{T_2} = \sqrt{2}$.

From the mechanical point of view these solutions are very natural. In the absence of external force the oscillator

$$\ddot{x} + 2x = 0$$

has periodic solutions with frequency $\omega_2 = \sqrt{2}$. We now add an external force with frequency $\omega_1 = 1$ and the typical solution of the forced equation will combine both frequencies.

Next we shall look at the equation (1.1) from a geometrical point of view. Let us consider the space of three dimensions with coordinates $(x, \dot{x}, t)$. First we draw in this space the curve produced by the periodic solution $x^\star$. This is the helix with parametric equation $(\sin t, \cos t, t)$. The other solutions of (1.1) satisfy

$$\frac{1}{2}(\dot{x}(t) - \cos t)^2 + (x(t) - \sin t)^2 = constant, \quad \forall t \in \mathbb{R}. \tag{1.2}$$

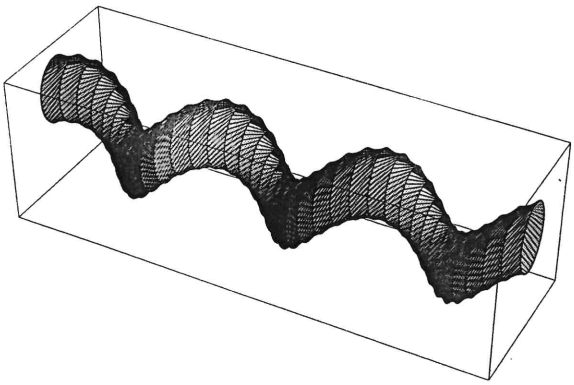
This identity follows from the conservation of energy for the autonomous equation. The solutions corresponding to the level *constant* $= 0.25$ are

$$x_c(t) = \sin t + 0.5 \sin(\sqrt{2}t + c), \quad c \in \mathbb{R}.$$

The corresponding initial conditions at $t = 0$ define an ellipse in the plane $(x, \dot{x})$, namely

$$\mathcal{E}_0 : \quad \frac{1}{2}(\dot{x} - 1)^2 + x^2 = 0.25.$$

This ellipse is transported by the flow according to (1.2). It becomes an ellipse $\mathcal{E}(t)$ with moving center at the helix generated by $x^\star$. In this way we produce a cylinder that is invariant by the flow. See the figure below.



The intersection of the cylinder with $t = 0, \pm 2\pi, \pm 4\pi, \ldots$ is always the initial ellipse. However, the corresponding solutions are not periodic and will not arrive at the same point of the ellipse. To be more precise let us define the Poincaré map

$$\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x(0), \dot{x}(0)) \mapsto (x(2\pi), \dot{x}(2\pi)),$$

where $x(t)$ is an arbitrary solution of (1.1). The ellipse $\mathcal{E}_0$ is invariant by $\mathcal{P}$, that is

$$\mathcal{P}(\mathcal{E}_0) = \mathcal{E}_0.$$

The following exercise gives more details on this curve.

Exercise 1. Define $\xi = (x_0, \dot{x}_0)^t \in \mathbb{R}^2$ and prove that the Poincaré map can be expressed in the form

$$\mathcal{P}(\xi) = A\xi + b.$$

Compute the matrix A and the vector b and prove that, for each $\xi \in \mathcal{E}_0$, the orbit $\{\mathcal{P}^n(\xi)\}_{n\in\mathbb{Z}}$ is dense in $\mathcal{E}_0$. □

The periodic solution $x^\star$ corresponds to the fixed point $(0,1)^t$ of $\mathcal{P}$. In a similar way, the family of quasi-periodic solutions $\{x_c\}_{c\in\mathbb{R}}$ is associated to the invariant curve $\mathcal{E}_0$. As we shall see later, this correspondence between invariant curves and quasi-periodic solutions is typical in the class of periodic equations.

We have been able to understand so well the previous equation because it is linear and the solutions have been explicitly computed. We can now ask what happens in the case of nonlinear equations. As an example consider the forced oscillator

$$\ddot{x} + g(x) = f(t), \tag{1.3}$$

where g is nonlinear, $g(x) \to \pm\infty$ as $x \to \pm\infty$ and f is 2π-periodic. Now the dynamics can be complicated but still many things can be said. In 1976 Morris studied this equation when $g(x) = 2x^3$ (see [14]). He proved the existence of many invariant curves for the Poincaré map of this equation. Each of these curves produces a family of quasi-periodic solutions with frequencies ω_1 and ω_2. The frequency $\omega_1 = 1$ is due to the period of f but ω_2 changes with the invariant curve. This is an important difference with respect to the linear case. These families of quasi-periodic solutions can be visualized as invariant cylinders in the space $(t, x, \dot{x})$. These cylinders are 2π-periodic in time and they become the so-called invariant tori after the identification $t \equiv t + 2\pi$. By uniqueness, the solution starting inside a cylinder will never escape. This implies that all the solutions are bounded. In contrast to the linear case the invariant curves probably do not foliate the plane. The work of Morris was motivated by a problem posed by Littlewood, and several authors have extended it to more general equations of the type (1.3). See [11] for a description of the results obtained in the superlinear case. In all these works the existence of invariant curves is proved using the Twist Theorem. This result was obtained by Moser in the sixties in order to solve a problem in the theory of stability. By now it has become a crucial tool for the understanding of the dynamics of periodic hamiltonian systems in the plane.

As we mentioned already, this course is devoted to the study of quasi-periodic solutions. They are interesting by themselves, especially from a mechanical point of view, but they are also useful to obtain interesting properties of the equation. Due to the associated invariant cylinders they can be employed to prove boundedness (invariant curves around infinity) or stability (invariant curves around the origin). The plan of this course is the following. First we shall study some generalities of quasi-periodic functions and quasi-periodic solutions. This includes the precise connection between invariant curves and quasi-periodic solutions. After this we shall state a simple version of the Twist Theorem that is ready for applications. It follows from the works of Moser in [15] and Herman in [4, 5]. Finally we shall apply all the previous theory to a concrete equation of the type (1.3): the asymmetric oscillator. This section combines [16, 3] with ideas taken from [13].

To conclude this preliminary chapter we introduce some notation. The circle will be identified with the quotient group

$$\mathbf{T}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

Given a real number $\theta \in \mathbb{R}$, the corresponding equivalence class is denoted by

$$\overline{\theta} = \theta + 2\pi\mathbb{Z}.$$

The rotation of angle α is defined as

$$\mathcal{R}_\alpha : \mathbf{T}^1 \to \mathbf{T}^1, \quad \mathcal{R}_\alpha(\overline{\theta}) = \overline{\theta} + \overline{\alpha}.$$

Periodic functions $f = f(\theta)$ will be identified with functions defined on the circle.

The torus is defined as

$$\mathbf{T}^2 = \mathbf{T}^1 \times \mathbf{T}^1$$

and, in an analogous way, doubly periodic functions $f = f(\theta_1, \theta_2)$ will be identified with functions defined on the torus.

2 Quasi-periodic functions with two frequencies

Let $\omega_1, \omega_2 \in \mathbb{R} - \{0\}$ be two real numbers which are not commensurable; that is,

$$\frac{\omega_1}{\omega_2} \notin \mathbb{Q}. \tag{2.1}$$

A function $x : \mathbb{R} \to \mathbb{R}^N$, $t \mapsto x(t)$, is said to be quasi-periodic with frequencies (ω_1, ω_2) if there exists another function $F \in C(\mathbf{T}^2, \mathbb{R}^N)$, $F = F(\theta_1, \theta_2)$, such that

$$x(t) = F(\omega_1 t, \omega_2 t), \quad \forall t \in \mathbb{R}.$$

The condition (2.1) says that the set $\{(\overline{\omega_1 t}, \overline{\omega_2 t}) : t \in \mathbb{R}\}$ is dense in $\mathbf{T}^2$. In consequence, F is uniquely determined by x.

Example. The function $x(t) = \sin t + 0.5 \sin \sqrt{2}t$ is quasi-periodic with frequencies $\omega_1 = 1, \omega_2 = \sqrt{2}$. Notice that $\sup x(t) = 1.5$ and $\inf x(t) = -1.5$ are not reached and therefore x is not periodic of any period.

The class of quasi-periodic functions with frequencies (ω_1, ω_2) will be denoted by $QP(\omega_1, \omega_2)$. The target space $\mathbb{R}^N$ will not be specified in this notation because it will be fixed. We have in mind the case $N = 2$.

Exercise 2. Prove that every function in $QP(\omega_1, \omega_2)$ is almost periodic. □

Exercise 3. Given $x \in QP(\omega_1, \omega_2)$, the limit below exists,

$$\overline{x} := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt.$$

Moreover,

$$\overline{x} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} F(\theta_1, \theta_2) d\theta_1 d\theta_2.$$

[Hint: study first the case $x(t) = e^{i(n\omega_1 + m\omega_2)t}$]. □

A function x in $QP(\omega_1, \omega_2)$ can also belong to other spaces $QP(\omega_1^\star, \omega_2^\star)$. In such a case, the function F representing x will change when we change the frequencies. As an example let us go back to the function $x(t) = \sin t + 0.5 \sin \sqrt{2}t$. It also belongs to $QP(3 + \sqrt{2}, 5 + 2\sqrt{2})$ and the corresponding function $F^\star$ representing x is now $F^\star(\theta_1, \theta_2) = \sin(2\theta_1 - \theta_2) + 0.5 \sin(-5\theta_1 + 3\theta_2)$. The next result clarifies this.

Proposition 2.1. *Let ω_1, ω_2 and $\omega_1^\star, \omega_2^\star$ be two couples of frequencies satisfying* (2.1). *Then the following statements are equivalent:*

(*i*) $QP(\omega_1, \omega_2) = QP(\omega_1^\star, \omega_2^\star)$.

(*ii*) *There exists a 2×2 matrix A with entries in $\mathbb{Z}$ and such that*

$$\det A = \pm 1, \quad (\omega_1^\star, \omega_2^\star)^t = A(\omega_1, \omega_2)^t.$$

To prove this result we need an algebraic result on subgroups of $(\mathbb{R}, +)$ with two generators. Given ω_1 and ω_2 satisfying (2.1), the subgroup generated by these two numbers is denoted by

$$\langle \omega_1, \omega_2 \rangle = \{n\omega_1 + m\omega_2 : \; n, m \in \mathbb{Z}\}.$$

The reader is invited to prove the following result.

Lemma 2.2. *The condition* (*ii*) *is equivalent to*

(*iii*) $\langle \omega_1, \omega_2 \rangle = \langle \omega_1^\star, \omega_2^\star \rangle$.

Proof of Proposition 2.1. The implication (*ii*) $\Rightarrow$ (*i*) is easy once we notice that the matrix A^{-1} has also its entries in $\mathbb{Z}$. For instance, given

$x \in QP(\omega_1, \omega_2)$ with $x(t) = F(\omega_1 t, \omega_2 t)$, we can also represent this function as $x(t) = F^\star(\omega_1^\star t, \omega_2^\star t)$ where $F^\star = F \circ A^{-1}$. This function is doubly periodic precisely because A^{-1} has integer coefficients.

In view of the previous Lemma we shall now prove $(i) \Rightarrow (iii)$. Since we can interchange the roles of the frequencies ω_i and $\omega_i^\star$, it is sufficient to prove the implication

$$QP(\omega_1, \omega_2) \subset QP(\omega_1^\star, \omega_2^\star) \implies \langle \omega_1, \omega_2 \rangle \subset \langle \omega_1^\star, \omega_2^\star \rangle.$$

By assumption, the real and imaginary parts of the function $e^{i\omega_1 t}$ belong to $QP(\omega_1^\star, \omega_2^\star)$. Let F be a function in $C(\mathbf{T}^2, \mathbb{C}^N)$ with Fourier series

$$F(\theta_1, \theta_2) \sim \sum_{n,m} F_{n,m} e^{i(n\theta_1 + m\theta_2)}$$

and such that

$$e^{i\omega_1 t} = F(\omega_1^\star t, \omega_2^\star t).$$

Since the system $e^{i(n\theta_1 + m\theta_2)}$ is complete in $L^2(\mathbf{T}^2)$ it is possible to find some integers n and m for which $F_{n,m} \neq 0$. We now apply Exercise 3 to the function $e^{i\omega_1 t} e^{-i(n\omega_1^\star + m\omega_2^\star)t}$ to deduce that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{i(\omega_1 - n\omega_1^\star - m\omega_2^\star)t} dt \neq 0.$$

This implies that $\omega_1 = n\omega_1^\star + m\omega_2^\star$. In the same way one can prove that also ω_2 is a $\mathbb{Z}$-linear combination of $\omega_1^\star$ and $\omega_2^\star$. This shows that the group generated by the omegas is included in the group generated by the omegas*.

For the applications to 2π-periodic differential equations, it is convenient to freeze the first frequency $\omega_1 = 1$ while the second will be simply denoted by ω, where $\omega \notin \mathbb{Q}$. The previous proposition implies that the identity

$$QP(1, \omega) = QP(1, \omega^\star)$$

occurs only in the cases

$$\omega + \omega^\star \in \mathbb{Z} \quad \text{or} \quad \omega - \omega^\star \in \mathbb{Z}.$$

The minimal period (or maximal frequency) is a very useful concept in the theory of periodic functions. We now define a related concept in the class $QP(1, \omega)$. A function $x \in QP(1, \omega)$ belongs to the class $\mathcal{M}(\omega)$ if 2π is the minimal period of F with respect to the second variable θ_2; that is,

$$F(\theta_1, \theta_2 + P) = F(\theta_1, \theta_2) \;\; \forall (\theta_1, \theta_2) \in \mathbb{R}^2 \implies P \in 2\pi\mathbb{Z}.$$

As an example, consider the functions $x_1(t) = \sin t + 0.5 \sin \sqrt{2} t$ and $x_2(t) = \sin t + 0.5 \sin 3\sqrt{2} t$. The function x_1 belongs to $\mathcal{M}(\sqrt{2})$ while x_2 is in $QP(1, \sqrt{2}) - \mathcal{M}(\sqrt{2})$.

Exercise 4. Let $x : \mathbb{R} \to \mathbb{R}^N$ be a continuous function with minimal period P and assume that $\omega = \frac{2\pi}{P} \notin \mathbb{Q}$. Prove $x \in \mathcal{M}(\omega)$. □

Exercise 5. Prove the equivalence below for a function $x \in QP(1,\omega)$,

$$x \in \mathcal{M}(\omega) \iff x \notin QP(1, n\omega),\ n = 2, 3, \ldots . \qquad \square$$

There are several nonequivalent definitions of quasi-periodic function, depending on the smoothness imposed on the function F. In the book of Siegel and Moser [21, Section 36] the function F is real analytic. Nevertheless, the previous discussions were inspired by this book.

3 Periodic differential equations. The Poincaré map

Let us consider the differential equation

$$\dot{x} = X(t, x), \tag{3.1}$$

where $X : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $(t,x) \mapsto X(t,x)$ is continuous and 2π-periodic in t. We also assume that there is uniqueness for the associated initial value problem. Given $p \in \mathbb{R}^2$, $\phi_t(p)$ denotes the solution of (3.1) satisfying $x(0) = p$. It is defined in the maximal interval I_p. For each $t \in \mathbb{R}$ we can define the map

$$\phi_t : \mathcal{D}_t \subset \mathbb{R}^2 \to \mathbb{R}^2,\ \ p \mapsto \phi_t(p),$$

where $\mathcal{D}_t = \{p \in \mathbb{R}^2 : t \in I_p\}$.

The theorem on continuous dependence implies that $\mathcal{D}_t$ is open and ϕ_t is a homeomorphism from $\mathcal{D}_t$ onto $\phi_t(\mathcal{D}_t)$. The family $\{\phi_t\}$ satisfies the property

$$\phi_t \circ \phi_{2\pi} = \phi_{2\pi} \circ \phi_t = \phi_{t+2\pi} \quad \forall t \in \mathbb{R}.$$

The map $\mathcal{P} = \phi_{2\pi}$ will be called the Poincaré map of (3.1). It satisfies

$$\mathcal{P}^n = \phi_{2\pi n},\ \ n \in \mathbb{Z}.$$

Exercise 6. Construct examples where the open set $\mathcal{D}_{2\pi}$ is disconnected. Construct another example with $\mathcal{D}_{2\pi} = \emptyset$. $\square$

Exercise 7. Construct an example of a differential equation (3.1) such that the field X is not locally Lipschitz continuous but there is uniqueness for the initial value problem. $\square$

The dynamics of (3.1) can be studied through $\mathcal{P}$. Usually the properties of the differential equation are translated to the language of discrete dynamics via $\mathcal{P}$. For example, a 2π-periodic solution corresponds to a fixed point, a subharmonic solution with minimal period $2\pi n$ corresponds to an n-periodic point. Let us now assume that $x(t)$ is a solution of (3.1) in the class $QP(1,\omega)$. How is this property reflected on the Poincaré map? This will be the problem of the next section.

4 Quasi-periodic solutions and invariant curves

4.1 Families of quasi-periodic solutions

Given an autonomous system $\dot{x} = X(x)$ and a nontrivial periodic solution $\gamma(t)$, we know that also $\gamma(t+c)$ is a periodic solution for each $c \in \mathbb{R}$. In a similar way we shall show that quasi-periodic solutions of a periodic system appear in families depending on one parameter.

Lemma 4.1. *Let $x(t)$ be a solution of* (3.1) *that belongs to the class $QP(1,\omega)$ and let $F \in C(\mathbf{T}^2, \mathbb{R}^2)$ be the associated function on the torus. For each $c \in \mathbb{R}$ define*

$$x_c(t) = F(t, \omega t + c).$$

Then $x_c(t)$ is also a solution of (3.1).

Proof. The periodicity of (3.1) implies that the translates

$$x(t + 2\pi n), \quad n \in \mathbb{Z}$$

are also solutions. From the definition of the family $\{x_c\}$ we deduce that

$$x_{2\pi n\omega}(t) = F(t, \omega t + 2\pi n\omega) = F(t + 2\pi n, \omega(t + 2\pi n)) = x(t + 2\pi n).$$

This implies that the functions $x_{2\pi n\omega}$ are solutions of the equation. Since ω is not in $\mathbb{Q}$, the sequence $\{\overline{2\pi\omega n}\}_{n\in\mathbb{Z}}$ is dense in $\mathbf{T}^1$. Using this density and the theorem on continuous dependence we deduce that all the functions x_c are solutions. $\square$

The next result studies the properties of this family of solutions when we are in the class $\mathcal{M}(\omega)$. The proof is left as an exercise.

Lemma 4.2. *Assume that $x \in \mathcal{M}(\omega)$ is a solution of* (3.1) *and let $\{x_c\}$ be the associated family. Then,*

(i) $x_{c_1} = x_{c_2} \iff c_1 - c_2 \in 2\pi\mathbb{Z}$.

(ii) $x_c(t + 2\pi) = x_{c+2\pi\omega}(t) \quad \forall t \in \mathbb{R}$.

An interesting fact about the family $\{x_c\}$ is that it allows us to reconstruct the function F from it. In fact, if we consider the definition

$$x_c(t) = F(t, \omega t + c)$$

and define the new variables $\theta_1 = t, \theta_2 = \omega t + c$ we are lead to the formula

$$F(\theta_1, \theta_2) = x_{\theta_2 - \omega\theta_1}(\theta_1). \tag{4.1}$$

The next result shows that quasi-periodicity can be characterized in terms of properties *(i)* and *(ii)*.

Proposition 4.3. *Let* $\{x_c\}_{c\in\mathbb{R}}$ *be a family of solutions of* (3.1)*, defined in* $(-\infty, +\infty)$ *and depending continuously on the parameter; that is, the function*

$$\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2,\quad (t,c)\mapsto x_c(t)$$

is continuous. In addition, properties (i) *and* (ii) *of the previous Lemma hold. Then* x_c *is in* $\mathcal{M}(\omega)$.

Proof. Define F as in (4.1). This function belongs to $C(\mathbb{R}^2,\mathbb{R}^2)$ and satisfies

$$x_c(t) = F(t, \omega t + c).$$

It remains to prove that it is doubly periodic and that 2π is the minimal second period. The 2π-periodicity in θ_1 follows from property (ii). The rest is a consequence of (i). □

We are now ready to understand the effect of a quasi-periodic solution on the Poincaré map.

4.2 From quasi-periodic solutions to invariant curves

Given a solution $x(t)$ of (3.1) that belongs to $\mathcal{M}(\omega)$ and the associated family $\{x_c\}$, we define the parametrized curve

$$\psi:\mathbb{R}\to\mathbb{R}^2,\quad \psi(c) = x_c(0) = F(0,c).$$

The property (i) and the uniqueness for (3.1) imply that ψ is 2π-periodic and one-to-one on $[0,2\pi)$. Thus, $\Gamma = \psi(\mathbb{R})$ is a Jordan curve in the plane. Moreover,

$$\mathcal{P}\psi(c) = \mathcal{P}x_c(0) = x_c(2\pi) = x_{c+2\pi\omega}(0) = \psi(c+2\pi\omega).$$

This identity implies that Γ is invariant under $\mathcal{P}$, that is

$$\mathcal{P}(\Gamma) = \Gamma.$$

This is not the only information given by the previous identity. In fact, if we parametrize Γ with respect to a circle, the mapping

$$\overline{\psi}:\mathbf{T}^1\to\Gamma,\quad \overline{c}\mapsto\psi(c)$$

becomes a homeomorphism and the following diagram is commutative.

$$\begin{array}{ccccc} & & \mathcal{P} & & \\ & \Gamma & \longrightarrow & \Gamma & \\ \overline{\psi} & \uparrow & & \uparrow & \overline{\psi} \\ & \mathbf{T}^1 & \longrightarrow & \mathbf{T}^1 & \\ & & \mathcal{R}_{2\pi\omega} & & \end{array}$$

In other words, the restriction of the Poincaré map to the invariant curve is conjugate to a rotation of the circle. Moreover, the frequency of the quasi-periodic solutions can be recovered from the rotation number.

Exercise 8. Let Γ be a Jordan curve such that $\mathcal{P}(\Gamma) \subset \Gamma$. Then Γ is invariant under $\mathcal{P}$. [Hint: $\mathbf{T}^1$ is not homeomorphic to any subset of $\mathbb{R}$]. □

4.3 From invariant curves to quasi-periodic solutions

Let us now see the converse. We start with a Jordan curve in the plane, $\Gamma \subset \mathbb{R}^2$, which is invariant under $\mathcal{P}$ and such that the restriction of the Poincaré map to Γ, denoted by $\mathcal{P}_\Gamma$, is conjugate to a rotation $\mathcal{R}_{2\pi\omega}$, for some $\omega \notin \mathbb{Q}$. We shall show that quasi-periodic solutions can be produced from Γ. First we allow the flow to evolve from Γ and consider the family of solutions starting at this curve. The invariance of Γ implies that these solutions are defined in $(-\infty, +\infty)$ and we can define

$$x_c(t) = \phi_t(\psi(c)), \quad c, t \in \mathbb{R},$$

where $\psi : \mathbb{R} \to \Gamma$ is the 2π-periodic parametrization such that

$$\mathcal{P}_\Gamma \circ \overline{\psi} = \overline{\psi} \circ \mathcal{R}_{2\pi\omega}.$$

We shall show that this family satisfies the conditions (i) and (ii) of Lemma 4.2 and so Proposition 4.3 can be applied to deduce that $x_c \in \mathcal{M}(\omega)$. The property (i) is immediate because, by assumption, ψ is one-to-one in $[0, 2\pi)$. To prove (ii) we use the commutativity of $\{\phi_t\}$ with $\mathcal{P}$ and obtain

$$x_c(t + 2\pi) = \phi_{t+2\pi}(\psi(c)) = \phi_t \circ \phi_{2\pi}(\psi(c)) = \phi_t(\mathcal{P} \circ \psi(c))$$
$$= \phi_t(\psi(c + 2\pi\omega)) = x_{c+2\pi\omega}(t).$$

To sum up, we can say that finding a solution in $\mathcal{M}(\omega)$ is equivalent to finding a Jordan curve Γ which is invariant under $\mathcal{P}$ and such that $\mathcal{P}_\Gamma$ is conjugate to $\mathcal{R}_{2\pi\omega}$.

Exercise 9. Given $\omega, \omega^\star \notin \mathbb{Q}$, prove the equivalence

$$\mathcal{M}(\omega) = \mathcal{M}(\omega^\star) \iff \mathcal{R}_{2\pi\omega} \text{ is conjugate to } \mathcal{R}_{2\pi\omega^\star}\ .$$ □

4.4 Invariant cylinders

Given a Jordan curve Γ in the plane, the bounded component of $\mathbb{R}^2 - \Gamma$ will be denoted by $R_i(\Gamma)$.

Exercise 10. Let Γ be a Jordan curve included in $\mathcal{D}_t$. Prove that $R_i(\Gamma) \subset \mathcal{D}_t$ and $\phi_t(R_i(\Gamma)) = R_i(\phi_t(\Gamma))$. □

Let us now assume that Γ is an invariant curve for $\mathcal{P}$. Since ϕ_t is a homeomorphism also $\Gamma_t = \phi_t(\Gamma)$ is a Jordan curve. Moreover,

$$\Gamma_{t+2\pi} = \Gamma_t \quad \forall t \in \mathbb{R}.$$

Define

$$C = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : \ t \in \mathbb{R}, \ x \in R_i(\Gamma_t)\}.$$

This set is invariant with respect to the differential equation (3.1). That is, given a solution $x(t)$ of (3.1), such that $(x(t_0), t_0) \in C$ for some t_0, then it is defined in $(-\infty, \infty)$ and

$$(x(t), t) \in C \quad \forall t \in \mathbb{R}.$$

This is a consequence of Exercise 10.

4.5 Some examples

We shall now analyze the quasi-periodic solutions of three equations.

Example 1.

$$\ddot{x} + \omega^2 x = f(t), \quad f \in C(\mathbf{T}^1), \ \omega > 0, \ \omega \notin \mathbb{Q}.$$

This is a continuation of the starting example, where $\omega = \sqrt{2}$ and $f(t) = \sin t$. There is a unique 2π-periodic solution and the other solutions are in the class $\mathcal{M}(\omega)$. The Poincaré map has a fixed point that is the center of a family of concentric ellipses which are invariant under $\mathcal{P}$. Moreover, the restriction of the Poincaré map to each of these ellipses is conjugate to the rotation of angle $2\pi\omega$.

Exercise 11. Discuss the case $\omega \in \mathbb{Q}$. □

Example 2. Let us assume now that ω is a smooth function from $[0, \infty)$ into $\mathbb{R}$, with

$$\omega'(\rho) > 0 \quad \forall \rho > 0.$$

We consider the system

$$\dot{x}_1 = -\omega(\rho) x_2, \quad \dot{x}_2 = \omega(\rho) x_1,$$

where $\rho = x_1^2 + x_2^2$. This system is autonomous but we shall look at it as a 2π-periodic system. The nontrivial solutions are

$$x_1(t) = \rho_0^{1/2} \cos(\omega(\rho_0)t + c), \quad x_2(t) = \rho_0^{1/2} \sin(\omega(\rho_0)t + c)$$

with $\rho_0 > 0$ and $c \in \mathbb{R}$. This system is nonlinear but easy to integrate because the function ρ is a first integral. When $\omega(\rho_0) \notin \mathbb{Q}$ the solution belongs to $\mathcal{M}(\omega(\rho_0))$. The origin is a fixed point of the Poincaré map and the circles around this point are invariant. The difference with the previous example is that the rotation number of each of these curves depends on ρ. If we perturb this system and introduce periodic coefficients, in most cases the first integral will disappear. Also, the foliation of the plane by invariant curves will be destroyed but still some curves will remain if the perturbation is not too large. These statements are not rigorous or precise but the reader can be convinced by herself (or himself?) via numerical experiments.

Example 3.

$$\ddot{x} + c\dot{x} = f(t, x), \quad c > 0.$$

This is the general equation of motion in the presence of friction. The force f will be smooth and 2π-periodic in t. We are going to prove that this equation cannot have quasi-periodic solutions in $\mathcal{M}(\omega)$ for any $\omega \notin \mathbb{Q}$. We do it by contradiction. We know that such a solution would produce an invariant curve Γ for the Poincaré map. The region $R_i(\Gamma)$ would also be invariant. Due to the friction, the mapping $\mathcal{P}$ is area contracting. Thus,

$$meas(\mathcal{P}(R_i(\Gamma))) < meas(R_i(\Gamma)),$$

and this is not compatible with the invariance of $R_i(\Gamma)$.

5 Invariant curves of mappings of the annulus

We shall consider a system of polar coordinates in the plane. Every point in $\mathbb{R}^2 - \{0\}$ has coordinates $(\overline{\theta}, r)$ where $\theta \in \mathbb{R}$ and $r > 0$. Given $b > a > 0$, A is the annulus defined by

$$A = \{(\overline{\theta}, r) : \overline{\theta} \in \mathbf{T}^1, \ a \leq r \leq b\}.$$

A universal cover of this annulus is the strip

$$\mathcal{A} = \{(\theta, r) : \theta \in \mathbb{R}, \ a \leq r \leq b\}.$$

Given a mapping of the annulus

$$M : A \to \mathbb{R}^2, \quad (\overline{\theta}, r) \mapsto (\overline{\theta}_1, r_1)$$

we can find a lift to $\mathcal{A}$, given by

$$M : \mathcal{A} \to \mathbb{R}^2, \quad (\theta, r) \mapsto (\theta_1, r_1).$$

Both, the mapping and the lift, are denoted by M.

An invariant curve of M is a Jordan curve $\Gamma \subset A$, which is homotopic in A to the circle $r = a$ and such that

$$M(\Gamma) = \Gamma.$$

Intuitively speaking, to assume that the curve is homotopic to $r = a$ just means that the curve goes around the hole. It is easy to construct examples of mappings M and curves Γ such that $M(\Gamma) = \Gamma$ but Γ can be deformed to a point in A. They are not considered as invariant curves in the previous definition. Let us now study some simple mappings in the annulus.

Example 1. Rotations.

$$\theta_1 = \theta + \beta, \quad r_1 = r.$$

Here $\beta \in \mathbb{R}$ is the angle of rotation. All circles $r =$ *constant* are invariant curves. Moreover, the restriction of M to each of these circles is always conjugate to $\mathcal{R}_\beta$.

Example 2. Twist mappings.

$$\theta_1 = \theta + \beta + \alpha(r), \quad r_1 = r.$$

Here $\alpha : [a, b] \to \mathbb{R}$ is a smooth function with

$$\alpha'(r) > 0 \qquad \forall r \in [a, b].$$

Again the circles $r =$ *constant* are invariant but now the restriction of M to each of them is conjugate to a different rotation $\mathcal{R}_{\beta+\alpha(r)}$. The monotonicity of α says that the rotation number is monotone with respect to the radius r. The name "twist mapping" can be justified by the following geometrical observation: given a radial segment $\theta =$ *constant* the map M transforms it into a twisted arc.

Our goal is to obtain a theorem on the existence of invariant curves for small perturbations of the twist mapping. However, the next example will show that we shall have to impose some additional condition.

Example 3. Twisted contractions.

$$\theta_1 = \theta + \beta + \alpha(r), \quad r_1 = (1 - \epsilon)r.$$

If $\epsilon > 0$ is small this mapping becomes a small perturbation of the twist mapping. However, it is clear that it has no invariant curves because all orbits will escape from the annulus in a finite number of iterations.

To exclude mappings like those in Example 3 we shall introduce the following definition. M has the *intersection property in* A if

$$M(\Gamma) \cap \Gamma \neq \emptyset,$$

for any Jordan curve Γ in A which is homotopic to $r = a$.

The mapping of Example 3 does not satisfy the intersection property. In fact, circles $r =$ *constant* are transformed in circles of smaller radius. On the other hand, rotations and twist mappings have this property in any annulus centered at the origin. To prove this we notice that in these cases M can be seen as an area-preserving homeomorphism of the whole plane with $M(0) = 0$. Given a Jordan curve Γ with $0 \in R_i(\Gamma)$, the curve $\Gamma_1 = M(\Gamma)$ also satisfies $0 \in R_i(\Gamma_1)$. If Γ and Γ_1 do not intersect then one of the regions $R_i(\Gamma)$, $R_i(\Gamma_1)$ should be strictly included in the other. This is not compatible with the area-preserving character of the mapping because

$$meas R_i(\Gamma) = meas R_i(\Gamma_1).$$

Exercise 12. Let M be the twist mapping of Example 2. Find a Jordan curve Γ in the plane such that $M(\Gamma) \cap \Gamma = \emptyset$. □

Exercise 13. Assume that M is a homeomorphism from A onto $M(A)$. For each $k = 0, 1, \ldots, \infty, \omega$, we say that M has the C^k-intersection property if the previous definition is restricted to Jordan curves of class C^k. Prove that all these properties are equivalent. [Hint: an elegant proof using Riemann's theorem on conformal mappings can be seen in [6]]. □

We are now going to state the Twist Theorem. It guarantees the existence of invariant curves for small perturbations of the twist mapping having the intersection property. The perturbation will be small in class C^4.

Theorem 5.1. *Let $\alpha \in C^4[a, b]$ be a function satisfying*

$$\alpha'(r) > 0 \quad \forall r \in [a, b].$$

Then there exists $\epsilon > 0$, depending on $b - a$ and α, such that a mapping $M : A \to \mathbb{R}^2$ has invariant curves if it satisfies the conditions below,

- *M has the intersection property,*
- *the lift of M can be expressed in the form*

$$\theta_1 = \theta + \beta + \alpha(r) + \varphi_1(\theta, r), \quad r_1 = r + \varphi_2(\theta, r),$$

  with $\varphi_1, \varphi_2 \in C^4(A)$,

- $||\varphi_1||_{C^4(A)} + ||\varphi_2||_{C^4(A)} < \epsilon.$

Remarks. 1. This theorem was proved by Moser in [15] assuming that M was of class 333. An analytic version of the theorem was presented in [21] and a version in class C^5 can be seen in Russmann [19]. The version we have just stated can be proved using the techniques developed in the two works by Herman [4, 5].
2. The proof of this theorem gives additional information and implies the existence of many invariant curves in A. In fact, there exist infinitely many numbers ν in the interval $[\beta + \alpha(a), \beta + \alpha(b)]$ for which it is possible to find an invariant curve Γ such that M_Γ is conjugate to $\mathcal{R}_\nu$. That is, the

diagram below is commutative,

$$\begin{array}{ccccc} & \Gamma & \xrightarrow{M_\Gamma} & \Gamma & \\ \overline{\psi} & \uparrow & & \uparrow & \overline{\psi} \\ & \mathbf{T}^1 & \xrightarrow[\mathcal{R}_\nu]{} & \mathbf{T}^1 & \end{array}$$

(As before, M_Γ denotes the restriction of M to Γ. The admissible numbers ν are not in $2\pi\mathbb{Q}$ and must satisfy some additional properties. See [4].) Here $\overline{\psi}$ is some homeomorphism.

Exercise 14. Consider the annulus $A = \{1 \leq r \leq 2\}$ and the mapping

$$M : A \to \mathbb{R}^2, \quad \theta_1 = \theta + \pi, \; r_1 = r + \epsilon \sin 2\theta.$$

i) Prove that, for small ϵ, this mapping has the intersection property in A. [Hint: apply Stokes Theorem with the differential form $\omega = r d\theta$].
ii) Prove that all orbits with $\theta_0 \in [0, 2\pi) - \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ escape from the annulus.
iii) Deduce that M has no invariant curves for small ϵ. This proves that there is no analogue of the Twist Theorem when $\alpha = 0$. □

In many applications the mapping M is a perturbation of the small twist mapping defined by

$$\theta_1 = \theta + \beta + \delta\alpha(r), \quad r_1 = r,$$

where δ is a small parameter. In these cases the previous theorem is not applicable but one can use the so-called Small Twist Theorem.

Theorem 5.2. *Let $\alpha \in C^4[a,b]$ be a function satisfying*

$$\alpha'(r) > 0 \quad \forall r \in [a,b].$$

Then there exists $\epsilon > 0$, depending on $b - a$ and α, such that a mapping $M : A \to \mathbb{R}^2$ has invariant curves if it satisfies the conditions below,

- *M has the intersection property,*
- *the lift of M can be expressed in the form*

$$\theta_1 = \theta + \beta + \delta\alpha(r) + \delta\varphi_1(\theta, r), \quad r_1 = r + \delta\varphi_2(\theta, r),$$

  for some $\delta \in (0,1)$ and $\varphi_1, \varphi_2 \in C^4(A)$,
- $||\varphi_1||_{C^4(A)} + ||\varphi_2||_{C^4(A)} < \epsilon$.

Remarks. 1. The proof of this result is similar. Notice that it does not follow directly from the Twist Theorem because the number ϵ is independent of δ.
2. As δ tends to 0 the rotation numbers of the invariant curves will tend to β. Again it is possible to find many invariant curves.

6 The asymmetric oscillator

Let us consider the differential equation

$$\ddot{x} + ax^+ - bx^- = 1 + \epsilon p(t), \tag{6.1}$$

where a and b are positive constants, ϵ is a parameter and the function p is 2π-periodic. This equation was proposed by Lazer and McKenna as a simplified version of their model of a nonlinear suspension bridge (see [10]).

When $a = b$ we go back to the linear oscillator and we can have the classical phenomenon of resonance. As an example consider the equation

$$\ddot{x} + x = 1 + \epsilon \sin t.$$

In this case it is easy to prove that, for any $\epsilon \neq 0$, all solutions are unbounded. The next result shows that the situation is different for the case $a \neq b$.

Theorem 6.1. *Assume that a and b are positive constants with $a \neq b$. In addition, the function p is of class $C^5(\mathbf{T}^1)$. Then there exists $\epsilon^\star > 0$ such that all the solutions of* (6.1) *are bounded if* $|\epsilon| < \epsilon^\star$.

Remarks. 1. This theorem was proved in [16] assuming less regularity for p, namely $p \in C^4$. The extra regularity C^5 will allow us to obtain a simpler proof. I do not know of an example showing that unbounded solutions can exist when ϵ is still small but p is not smooth.

2. In [3], J.M. Alonso and I constructed many examples showing that unbounded solutions can exist if ϵ is not small and the parameters a, b satisfy

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q}. \tag{6.2}$$

I do not know of an example showing that unbounded solutions can exist when ϵ is large and (6.2) does not hold.

In a recent paper [13], Liu Bin has given a new proof of the theorem. In the next pages I shall present a proof that combines ideas from [13] and from [16, 3]. However, the most technical part of the proof (the estimates) will be obtained by a new procedure. I hope that the reader will find this procedure rather simple. The proof will follow after several steps.

Step 1. The asymmetric polar coordinates

We start with the autonomous ("homogeneous") equation

$$\ddot{x} + ax^+ - bx^- = 0. \tag{6.3}$$

The nontrivial solutions of this equation oscillate and satisfy alternatively the linear equations $\ddot{x} + ax = 0$ if $x > 0$ and $\ddot{x} + bx = 0$ if $x < 0$. They are periodic, with minimal period

$$T = \frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}},$$

and they look more or less like a sinusoidal function with two asymmetric bumps. It will be convenient to select one particular solution that plays the role of the "asymmetric cosine". Let $C(t)$ be the solution of (6.3) satisfying $C(0) = 1, \dot{C}(0) = 0$. Since the equation is autonomous and positively homogeneous, all solutions can be expressed in the form

$$x(t) = \alpha C(t+\beta), \quad \alpha \geq 0,\ \beta \in \mathbb{R}.$$

Exercise 15. Prove

$$\int_0^T C(t)dt = 2\sqrt{a}(\frac{1}{a} - \frac{1}{b}). \qquad \square$$

Next we define the "asymmetric sine" as $S(t) = \dot{C}(t)$. The conservation of energy for (6.3) leads to

$$S(t)^2 + aC^+(t)^2 + bC^-(t)^2 = a \quad \forall t \in \mathbb{R}. \tag{6.4}$$

It is convenient to notice that, for $a = b = 1$, the functions C and S are $C(t) = \cos t$ and $S(t) = -\sin t$. In such a case the identity (6.4) becomes the classical trigonometric identity.

We shall now analyze (6.3) from a geometric perspective. If we look at the phase portrait in the plane $(x, \dot{x})$, we find that the nontrivial orbits are closed curves obtained by gluing two ellipses. Namely, $\frac{1}{2}\dot{x}^2 + \frac{a}{2}x^2 = c_1$, if $x \geq 0$ and $\frac{1}{2}\dot{x}^2 + \frac{b}{2}x^2 = c_2$, if $x \leq 0$, where c_1 and c_2 are appropriate constants. The energy

$$E = \frac{1}{2}\dot{x}^2 + \frac{a}{2}(x^+)^2 + \frac{b}{2}(x^-)^2$$

is preserved along these orbits. Since the minimal period is always T, the origin is an isochronous center. We shall use this phase portrait to define a system of coordinates.

Define

$$x = \gamma I^{1/2} C(\frac{\theta}{\Omega}), \quad y = \gamma I^{1/2} S(\frac{\theta}{\Omega}), \quad I > 0,\ \theta \in \mathbb{R},$$

where $\gamma > 0$ is a parameter that will be determined later and $\Omega = \frac{2\pi}{T}$. The mapping

$$\Psi : \mathbf{T}^1 \times (0, \infty) \to \mathbb{R}^2 - \{0\},\ (\bar{\theta}, I) \mapsto (x, y)$$

is one-to-one and onto. This can be proved using (6.4). Since C is C^2 and S is C^1, we can say that Ψ is C^1. We shall now prove that, for an appropriate value of γ, Ψ will transport the symplectic forms. This means

$$dx \wedge dy = d\theta \wedge dI,$$

or, in the language of jacobians,

$$\det \Psi'(\theta, I) = 1.$$

This property and the local inverse function theorem imply that Ψ is a C^1 symplectic diffeomorphism. A computation and the identity (6.4) lead to

$$dx \wedge dy = \frac{\gamma^2}{2\Omega}[S(\frac{\theta}{\Omega})^2 - C(\frac{\theta}{\Omega})\dot{S}(\frac{\theta}{\Omega})]d\theta \wedge dI = \frac{\gamma^2}{2\Omega} a d\theta \wedge dI.$$

We define $\gamma = \sqrt{\frac{2\Omega}{a}}$.

Exercise 16. Prove that the equations

$$x = \gamma I^\alpha C(\frac{\theta}{\Omega}), \quad y = \gamma I^\alpha S(\frac{\theta}{\Omega}), \quad I > 0, \ \theta \in \mathbb{R},$$

define a C^1-diffeomorphism for any $\alpha \neq 0$. For which values of α and γ is it symplectic?. □

A mechanical interpretation. The variables (θ, I) are the so-called "action-angle" variables for the oscillator (6.3). In this case the action is, up to a constant, the energy. In fact, using again (6.4),

$$E = \frac{1}{2}y^2 + \frac{a}{2}(x^+)^2 + \frac{b}{2}(x^-)^2 = \frac{\gamma^2 a I}{2} = \Omega I.$$

The angle θ can be interpreted by the formula $\theta = \frac{2\pi}{T}\tau(x, y)$, where τ is the time employed by a particle to travel from the point $(\gamma I^{1/2}, 0)$ to (x, y). Of course this motion follows the law (6.3).

The coordinates that we have constructed are important because they reduce (6.3) to its simplest possible form. To change variables in this equation we use that Ψ is symplectic and so the hamiltonian structure is preserved. The equation (6.3) is equivalent to

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \quad H(x, y) = \frac{1}{2}y^2 + \frac{a}{2}(x^+)^2 + \frac{b}{2}(x^-)^2.$$

In the new variables,

$$\dot{\theta} = h_I = \Omega, \quad \dot{I} = -h_\theta = 0, \quad h(\theta, I) = H(\Psi(\theta, I)) = \Omega I.$$

Let us now consider the nonautonomous equation

$$\ddot{x} + ax^+ - bx^- = f(t) \tag{6.5}$$

with $f \in C(\mathbf{T}^1)$. It is reasonable to expect that the previous change of variables will simplify it. The hamiltonian

$$H(t,x,y) = \frac{1}{2}y^2 + \frac{a}{2}(x^+)^2 + \frac{b}{2}(x^-)^2 - xf(t)$$

is transformed into

$$h(t,\theta,I) = \Omega I - \gamma I^{1/2} C(\frac{\theta}{\Omega}) f(t)$$

and we are lead to the system

$$\dot{\theta} = \Omega - \frac{\gamma}{2I^{1/2}} C(\frac{\theta}{\Omega}) f(t), \quad \dot{I} = \frac{\gamma}{\Omega} I^{1/2} S(\frac{\theta}{\Omega}) f(t). \tag{6.6}$$

It is convenient to notice that this system is not completely equivalent to (6.5). The reason is that Ψ introduces a singularity at the origin and so some of the solutions of (6.6), $(\theta(t), I(t))$, have a maximal interval of definition smaller than $(-\infty, +\infty)$. They correspond to the solutions $x(t)$ of (6.5) passing through $x = \dot{x} = 0$ at some time t.

Finally we define $\rho = \sqrt{I}$ and (6.6) becomes

$$\dot{\theta} = \Omega - \frac{\gamma}{2\rho} C(\frac{\theta}{\Omega}) f(t), \quad \dot{\rho} = \frac{\gamma}{2\Omega} S(\frac{\theta}{\Omega}) f(t). \tag{6.7}$$

This system is not hamiltonian. This is not surprising because $(\theta, I) \mapsto (\theta, \rho)$ is not symplectic.

Let $\mathcal{P}$ be the Poincaré map associated to (6.7). It is easy to prove that there is a disk such that $\mathcal{P}$ is well defined outside this disk. Let us assume for a moment that we could prove the existence of a family of Jordan curves that were invariant under $\mathcal{P}$ and surround infinity. They would produce invariant cylinders in the space $(t; x, \dot{x})$ and the proof of the Theorem would be essentially complete. In view of this optimistic argument one could try to apply the Twist Theorem to $\mathcal{P}$. However we shall not follow this idea, due to the lack of regularity of the equation. Since (6.7) is only C^1 in θ we cannot guarantee that $\mathcal{P}$ is of class C^4 as required in the Twist Theorem. To overcome this difficulty we notice that if f is smooth, then (6.7) is smooth with respect to t and ρ. This fact will motivate us to interchange the roles of θ and t. The new independent variable will be θ while the new unknowns will be $t = t(\theta)$ and $\rho = \rho(\theta)$. In this way we shall obtain a new (and smooth) Poincaré map. This trick was employed by M. Levi in [11] to prove boundedness in a superlinear oscillator. The idea of applying it to the asymmetric oscillator is due to Liu Bin (see [13]).

Step 2. The successor mapping

Consider the system

$$\frac{dt}{d\theta} = F(\theta, t, \rho), \quad \frac{d\rho}{d\theta} = G(\theta, t, \rho), \tag{6.8}$$

with

$$F(\theta,t,\rho) = [\Omega - \frac{\gamma}{2\rho}C(\frac{\theta}{\Omega})f(t)]^{-1},$$
$$G(\theta,t,\rho) = \frac{\gamma}{2\Omega}S(\frac{\theta}{\Omega})f(t)[\Omega - \frac{\gamma}{2\rho}C(\frac{\theta}{\Omega})f(t)]^{-1}.$$

Let $\rho_\star > 0$ be a positive number such that

$$\Omega - \frac{\gamma}{2\rho_\star}||C||_\infty ||f||_\infty > 0.$$

The functions F and G are well defined for $\rho \geq \rho_\star$ and, if f is of class $C^n(\mathbf{T}^1)$, they belong to $C^{0,n}(\mathbf{T}^1 \times \mathcal{E}_\star)$, where

$$\mathcal{E}_\star = \{(t,\rho) \in \mathbb{R}^2 : \rho \geq \rho_\star\}.$$

These functions are also 2π-periodic with respect to t and so we can interpret (t,ρ) as a system of polar coordinates in the plane. Then we can consider that the equation (6.8) is defined in the exterior of the disk $\rho < \rho_\star$ if $\mathcal{E}_\star$ is interpreted as the universal covering of

$$E_\star = \{(\bar{t},\rho) \in \mathbf{T}^1 \times \mathbb{R} : \rho \geq \rho_\star\}.$$

Let $(t(\theta),\rho(\theta))$ be a solution of (6.8) defined in a certain interval $I = [\theta_0,\theta_1]$ and such that $\rho(\theta) > \rho_\star$ for all θ in I. The derivative $\frac{dt}{d\theta}$ is positive and so the function t is a diffeomorphism from I onto $J = [t_0,t_1]$, where $t(\theta_0) = t_0$ and $t(\theta_1) = t_1$. The inverse function will be denoted by $\theta = \theta(t)$. It maps J onto I. Let us define

$$x(t) = \gamma\rho(\theta(t))C(\frac{\theta(t)}{\Omega}), \quad t \in J.$$

It is easy to verify that this function is a solution of the original equation (6.5). Of course this is not surprising in view of the way we constructed (6.8). There are some interesting aspects of this solution that we have constructed. The derivative can be expressed in the form

$$\dot{x}(t) = \gamma\rho(\theta(t))S(\frac{\theta(t)}{\Omega}), \quad t \in J$$

and the zeros of x correspond to the values of θ such that $C(\frac{\theta}{\Omega}) = 0$. These zeros are nondegenerate because S and C do not vanish simultaneously. The zeros of $\dot{x}$ correspond to the values of θ such that $S(\frac{\theta}{\Omega}) = 0$. In particular, if $x(t)$ reaches a local maximum at $t_2 \in (t_0,t_1)$, then $\theta(t_2) \in 2\pi\mathbb{Z}$ and $x(t_2) = \gamma\rho(t_2)$.

Let $(t(\theta),\rho(\theta))$ be the solution of (6.8) satisfying the initial conditions $t(\theta_0) = t_0, \rho(\theta_0) = \rho_0$. It is not hard to show the existence of a number $\rho^\star > \rho_\star$ such that if $\rho_0 \geq \rho^\star$, then $(t(\theta),\rho(\theta))$ is well defined in $[\theta_0,\theta_0 + 2\pi]$

and remains in $\mathcal{E}_\star$. The Poincaré map associated to (6.8) will be denoted by $\mathcal{S}$. The previous property implies that $\mathcal{S}$ is well defined on the set

$$E^\star = \{(\bar{t}, \rho) : \rho \geq \rho^\star\}$$

and satisfies $\mathcal{S}(E^\star) \subset E_\star$. The smoothness of $\mathcal{S}$ will not be a big problem because $\mathcal{S}$ is of class C^n in $E^\star$ if f belongs to $C^n(\mathbf{T}^1)$.

Understanding $\mathcal{S}$ geometrically. Let us use the notation $\mathcal{S} : (t_0, \rho_0) \mapsto (t_1, \rho_1)$. The mapping $\mathcal{S}$ can be defined directly from the original equation. We consider the solution of (6.5) satisfying

$$x(t_0) = \gamma\rho_0, \quad \dot{x}(t_0) = 0.$$

If ρ_0 is large enough the function x reaches a local maximum at this instant. Then $t_1 > t_0$ is the next instant where x reaches a maximum and $x(t_1) = \gamma\rho_1$. This observation justifies the name of successor mapping for $\mathcal{S}$. See [18, 1, 7, 16] where variants of the successor mapping were employed in the study of second order scalar differential equations.

A strategy for the proof. To prove the Theorem we will find a family of invariant curves of $\mathcal{S}$ that surround infinity. To be precise, we shall look for a sequence of numbers $\{R_n\}$ and a sequence of Jordan curves in $E^\star$, denoted by $\{\Gamma_n\}$, satisfying the conditions

- $\rho^\star < R_0 < R_1 < \ldots < R_n \ldots, \quad R_n \to \infty,$
- $\Gamma_n \subset A_n := \{(\overline{\theta}, \rho) : R_n < \rho < R_{n+1}\},$
- Γ_n is homotopic to $\rho = \mathit{constant}$ in A_n,
- $\mathcal{S}(\Gamma_n) = \Gamma_n$.

Lemma 6.2. *Assume that $f \in C(\mathbf{T}^1)$ and we can find sequences R_n and Γ_n in the previous conditions. Then the solutions of* (6.5) *are bounded.*

Proof. The previous assumptions imply that $R_i(\Gamma_n) \subset R_i(\Gamma_{n+1})$ and $\mathbb{R}^2 = \bigcup_{n\geq 0} R_i(\Gamma_n)$. The invariant curves of $\mathcal{S}$ produce invariant cylinders for (6.8) in the space $(\theta; t, \rho)$. The solutions lying between two of these cylinders are bounded. From these facts and from the equation itself we can deduce the existence of a new number $\rho^{\star\star} > \rho^\star$, such that if $(t(\theta), \rho(\theta))$ is a solution of (6.8) with $\rho(\theta_0) > \rho^{\star\star}$ for some θ_0, then the solution is defined in the whole line, bounded and such that $\rho(\theta) > \rho^\star$, $\forall\theta \in \mathbb{R}$. From this solution we can construct a bounded solution of (6.5).

Now we prove the Lemma. Given a solution $x(t)$ of (6.5) we distinguish two cases. If

$$\rho(t) := \frac{1}{\Omega^{1/2}}\{\frac{1}{2}\dot{x}(t)^2 + \frac{a}{2}x^+(t)^2 + \frac{b}{2}x^-(t)^2\}$$

remains always below the number ρ^{**}, then the solution is obviously bounded. On the other hand, if for some t the function $\rho(t)$ is above ρ^{**}, then x will correspond to a solution of (6.8) living between two invariant cylinders and so ρ will be uniformly bounded in $\mathbb{R}$. □

Exercise 17. Prove that, in the conditions of the previous lemma, the solutions of (6.5) are equi-bounded; that is, given a constant $\gamma_\star > 0$ we can find another constant $\gamma^\star > \gamma_\star$ such that

$$\inf_{t\in\mathbb{R}} |x(t)| + |\dot{x}(t)| < \gamma_\star \Rightarrow \sup_{t\in\mathbb{R}} |x(t)| + |\dot{x}(t)| < \gamma^\star,$$

for any solution $x(t)$. □

Exercise 18. Consider the equation (6.5) where f is an arbitrary function in $C(\mathbf{T}^1)$. Let $x(t)$ be a solution and let M_x denote the set of instants $t_0 \in \mathbb{R}$ where x reaches a local maximum. Prove that $x(t)$ is bounded if and only if one of the conditions below holds:

(i) $\sup\{x(t_0) : t_0 \in M_x\} < \infty$,
(ii) $M_x = \emptyset$.
[Compare with proposition 4.2 in [16]]. □

Exercise 19. Let $\Gamma \subset E^\star$ be a Jordan curve such that $\mathcal{S}(\Gamma) = \Gamma$ and such that the following diagram is commutative,

$$\begin{array}{ccccc} & & \mathcal{S} & & \\ & \Gamma & \longrightarrow & \Gamma & \\ \bar{\psi} & \uparrow & & \uparrow & \bar{\psi} \\ & \mathbf{T}^1 & \longrightarrow & \mathbf{T}^1 & \\ & & \mathcal{R}_{2\pi\alpha} & & \end{array}$$

where $\alpha \notin \mathbb{Q}$. We want to produce a family of quasi-periodic solutions of (6.5).

Let $\tau = \tau(s), \rho = \rho(s)$ be a lift of $\bar{\psi}$. We can assume

$$\tau(s + 2\pi) = \tau(s) + 2\pi, \quad \rho(s + 2\pi) = \rho(s).$$

(If $\tau(s + 2\pi) = \tau(s) - 2\pi$ we replace α by $-\alpha$).
Let $x(t; \tau, \rho)$ denote the solution of (6.5) satisfying $x(\tau) = \gamma\rho,\ \dot{x}(\tau) = 0$. Prove

(i) $x(t; \tau(s), \rho(s)) = x(t; \tau(s + 2\pi\alpha), \rho(s + 2\pi\alpha))$,

(ii) $x(t; \tau(s + 2\pi), \rho(s + 2\pi)) = x(t - 2\pi; \tau(s), \rho(s))$,

(iii) $x_c(t) = x(t; \tau(\alpha c), \rho(\alpha c))$ is a family of quasi-periodic solutions in $\mathcal{M}(\frac{1}{\alpha})$. □

Step 3. Applying the Twist Theorem

Intersection property

Lemma 6.3. *Let Γ be a Jordan curve in $E^\star$ that is homotopic to $\rho = \rho^\star$. Then*

$$\mathcal{S}(\Gamma) \cap \Gamma \neq \emptyset.$$

Proof. In $\mathbb{R}^3$ we consider a system of cylindrical coordinates defined by

$$\bar{t} \in \mathbf{T}^1, \quad \rho > 0, \quad \theta \in \mathbb{R},$$

where the associated cartesian coordinates are

$$X = \rho \cos t, \quad Y = \rho \sin t, \quad Z = \theta.$$

Let A be the vector field in $\mathbb{R}^3$ described, in cylindrical coordinates, by the equations

$$A_t = 1, \quad A_\rho = \frac{\gamma^2}{2\Omega} S(\frac{\theta}{\Omega}) f(t), \quad A_\theta = \Omega - \frac{\gamma^2}{2\rho} C(\frac{\theta}{\Omega}) f(t).$$

Using the standard formula for cylindrical coordinates,

$$\operatorname{div} A = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_t}{\partial t} + \frac{\partial A_\theta}{\partial \theta},$$

we conclude that $\operatorname{div} A = 0$ for $\rho \neq 0$. For $\rho = 0$ the field has a singularity but it is rather weak. Around the singularity A satisfies

$$A_X = O(1), \quad A_Y = O(1), \quad A_Z = O(\frac{1}{\rho}). \tag{6.9}$$

Define $\Gamma_1 = \mathcal{S}(\Gamma)$. Since $\mathcal{S}$ is a topological mapping, Γ_1 is a Jordan curve in $E_\star$ that is homotopic to $\rho = \rho_\star$. Let us consider the flow in $\mathbb{R}^3$ given by $\dot{t} = A_t$, $\dot{\rho} = A_\rho$, $\dot{\theta} = A_\theta$. If we start with the curve $\Gamma \times \{0\}$, lying in the plane $Z = 0$, and allow the flow to evolve up to $Z = 2\pi$, then we arrive at $\Gamma_1 \times \{2\pi\}$ via a smooth cylinder. The domain enclosed by this cylinder in $\{0 < Z < 2\pi\}$ will be denoted by D. The boundary of D is composed by the cylinder itself and the two faces $R_i(\Gamma) \times \{0\}$ and $R_i(\Gamma_1) \times \{2\pi\}$. The outward normal vector to ∂D satisfies $A \cdot n = 0$ on the cylinder. Also, $n = (0,0,-1)$ [resp. $n = (0,0,1)$] at $R_i(\Gamma) \times \{0\}$ [resp. $R_i(\Gamma_1) \times \{2\pi\}$]. Given a small $\epsilon > 0$ we apply the Divergence Theorem to the vector field A on the domain

$$D_\epsilon = \{(X,Y,Z) \in D : \ X^2 + Y^2 > \epsilon, \ 0 < Z < 2\pi\}.$$

Letting ϵ go to 0 and using (6.9) one obtains

$$0 = \int_D \operatorname{div} A = \int_{\partial D} A \cdot n = \int_{R_i(\Gamma_1)\times\{2\pi\}} A_\theta - \int_{R_i(\Gamma)\times\{0\}} A_\theta. \tag{6.10}$$

If we assume for a moment that the conclusion of the Lemma does not hold and Γ and Γ_1 do not intersect, since they are both homotopic to $\rho = \rho^\star$

in $E_\star$, either $R_i(\Gamma_1) \subset R_i(\Gamma)$ or $R_i(\Gamma) \subset R_i(\Gamma_1)$. Assume for example that the first inclusion holds. We reach a contradiction with the identity (6.10) because

$$\int_{R_i(\Gamma)\times\{0\}} A_\theta - \int_{R_i(\Gamma_1)\times\{2\pi\}} A_\theta = \int_{R_i(\Gamma)-R_i(\Gamma_1)} \{\Omega - \frac{\gamma^2}{2\rho} f(t)\}\rho d\rho dt > 0. \ \square$$

Inversion and change of scale. To apply the Twist Theorem we need a fixed annulus and, for this reason, we perform the change of variables

$$r = \frac{1}{\delta\rho}, \quad r \in [0.5, 2.5],$$

where $\delta > 0$ is a parameter. As $\delta \to 0$ the annulus $r \in [1,2]$ is mapped onto the annulus around infinity $\rho \in [\frac{1}{2\delta}, \frac{1}{\delta}]$. In this way we introduce a small parameter in the system (6.8). It becomes

$$\begin{cases} \frac{dt}{d\theta} = [\Omega - \delta\frac{\gamma r}{2} f(t) C(\frac{\theta}{\Omega})]^{-1}, \\ \frac{dr}{d\theta} = -\delta\frac{\gamma r^2}{2\Omega} f(t) S(\frac{\theta}{\Omega})[\Omega - \delta\frac{\gamma r}{2} f(t) C(\frac{\theta}{\Omega})]^{-1}. \end{cases} \tag{6.11}$$

The Poincaré map M_δ is well defined in the annulus $r \in [1,2]$ for small δ. Moreover, the previous Lemma implies that M_δ satisfies the intersection property. In order to apply the Twist Theorem we also need some estimates. To obtain these estimates we are going to consider that (6.11) is a differential equation depending on a parameter and we shall apply the following consequence of Peano's Theorem.

Differentiability with respect to parameters. For the moment we shall consider a general differential equation depending on one parameter of the type

$$\frac{dz}{d\theta} = F(\theta, z, \delta), \tag{6.12}$$

where the function

$$F : [0, 2\pi] \times \mathcal{D} \times [0, \Delta] \to \mathbb{R}^N, \quad (\theta, z, \delta) \mapsto F(\theta, z, \delta)$$

is of class $C^{0,\nu+1,\nu+1}$ for some $\nu \geq 0$. Here $\mathcal{D}$ is an open and connected subset of $\mathbb{R}^N$ and $\Delta > 0$. The solution of (6.12) satisfying $z(0) = z_0$ will be denoted by $z(\theta; z_0, \delta)$. The general theory of differential equations says that z is of class $C^{0,\nu+1,\nu+1}$ in its three arguments whenever it is defined. The following result is a consequence of this fact.

Proposition 6.4. *Let K be a compact subset of $\mathcal{D}$ such that for every $z_0 \in K$ and $\delta \in [0, \Delta]$ the solution $z(\theta; z_0, \delta)$ is well defined in $[0, 2\pi]$. Then, for each $(\theta; z_0, \delta) \in [0, 2\pi] \times K \times [0, \Delta]$, the expansion below holds,*

$$z(\theta; z_0, \delta) = z(\theta; z_0, 0) + \frac{\partial z}{\partial \delta}(\theta; z_0, 0)\delta + R(\theta; z_0, \delta)\delta$$

where the remainder R satisfies

$$||R(\theta;\ \cdot\ ,\delta)||_{C^\nu(K)} \to 0, \quad \delta \to 0,$$

uniformly in $\theta \in [0, 2\pi]$.

This result is a consequence of the regularity of the solution together with the following Lemma.

Lemma 6.5. *Let φ be a function in $C^{0,\nu+1,\nu+1}([0,2\pi]\times K\times[0,\Delta])$. Then*

$$\varphi(\theta; z, \delta) = \varphi(\theta; z, 0) + \frac{\partial\varphi}{\partial\delta}(\theta; z, 0)\delta + \mathcal{R}(\theta; z, \delta)\delta$$

with $\mathcal{R}$ satisfying the same conditions as R in Proposition 6.4.

Proof. It is a consequence of the identity

$$\varphi(\theta; z, \delta) = \varphi(\theta; z, 0) + \frac{\partial\varphi}{\partial\delta}(\theta; z, 0)\delta + \delta\int_0^1 \{\frac{\partial\varphi}{\partial\delta}(\theta; z, \delta s) - \frac{\partial\varphi}{\partial\delta}(\theta; z, 0)\}ds. \ \square$$

Exercise 20. Let $z(\theta; z_0, \delta, \epsilon)$ be the solution of

$$\frac{dz}{d\theta} = F(\theta, z, \delta, \epsilon), \quad z(0) = z_0,$$

where

$$F : [0, 2\pi]\times\mathcal{D}\times[0,\Delta]\times[-1,1] \to \mathbb{R}^N, \quad (\theta, z, \delta, \epsilon) \longmapsto F(\theta, z, \delta, \epsilon)$$

is of class $C^{0,\nu+1,\nu+1,0}$ for some $\nu \geq 0$. Assuming that $z(\theta; z_0, \delta, \epsilon)$ is well defined in $[0, 2\pi]$ if $z_0 \in K$, $\delta \in [0, \Delta]$, $\epsilon \in [-1, 1]$, then

$$z(\theta; z_0, \delta, \epsilon) = z(\theta; z_0, 0, \epsilon) + \frac{\partial z}{\partial\delta}(\theta; z_0, 0, \epsilon)\delta + R(\theta; z_0, \delta, \epsilon)\delta$$

where the remainder R satisfies

$$||R(\theta;\ \cdot\ , \delta, \epsilon)||_{C^\nu(K)} \to 0, \quad \delta \to 0,$$

uniformly in $\theta \in [0, 2\pi], \epsilon \in [-1, 1]$. □

The estimates. We are going to apply the previous Proposition to the system (6.11). For $\delta = 0$ this system becomes $\frac{dt}{d\theta} = \frac{1}{\Omega}$, $\frac{dr}{d\theta} = 0$. Then

$$t(\theta; t_0, r_0, 0) = t_0 + \frac{\theta}{\Omega}, \quad r(\theta; t_0, r_0, 0) = r_0.$$

The derivatives with respect to the parameter will be denoted by $\xi(\theta) = \frac{\partial t}{\partial\delta}(\theta;\ t_0, r_0, 0)$ and $\eta(\theta) = \frac{\partial r}{\partial\delta}(\theta; t_0, r_0, 0)$. They satisfy

$$\begin{cases} \frac{d\xi}{d\theta} = \frac{\gamma}{2\Omega^2} r_0 f(t_0 + \frac{\theta}{\Omega})C(\frac{\theta}{\Omega}), & \xi(0) = 0, \\ \frac{d\eta}{d\theta} = -\frac{\gamma}{2\Omega^2} r_0^2 f(t_0 + \frac{\theta}{\Omega})S(\frac{\theta}{\Omega}), & \eta(0) = 0. \end{cases}$$

In consequence,

$$\xi(\theta) = \frac{\gamma}{2\Omega^2} r_0 \int_0^\theta f(t_0 + \frac{\Theta}{\Omega}) C(\frac{\Theta}{\Omega}) d\Theta, \quad \eta(\theta) = -\frac{\gamma}{2\Omega^2} r_0^2 \int_0^\theta f(t_0 + \frac{\Theta}{\Omega}) S(\frac{\Theta}{\Omega}) d\Theta.$$

If we apply Proposition 6.4 to the system (6.11) with Δ sufficiently small and $K = \{(t,r) \in \mathbb{R}^2 : 0 \leq t \leq 2\pi,\ 1 \leq r \leq 2\}$, then we are lead to the following result.

Proposition 6.6. *Assume that $f \in C^{n+1}(\mathbf{T}^1)$. Then the Poincaré map M_δ of (6.11) satisfies the expansion*

$$\begin{cases} t_1 = t_0 + T + \delta\xi(2\pi) + o(\delta), \\ r_1 = r_0 + \delta\eta(2\pi) + o(\delta), \end{cases}$$

and the remainders $o(\delta)$ are understood in the C^n sense.

Remark. This proposition is also true if one only assumes $f \in C^n(\mathbf{T}^1)$. See Proposition 6.1 in [17]. It is possible to prove this result using a refinement of Proposition 6.4.

Proof of Theorem 6.1. Let us now assume that $f(t) = 1 + \epsilon p(t)$ with $p \in C^5(\mathbf{T}^1)$ and $|\epsilon| \leq 1$. The numbers $\rho_\star$ and $\rho^\star$ employed in the construction of the Successor mapping can be chosen independent of ϵ. Let M_δ^ϵ be the Poincaré map of (6.11) when $f = 1 + \epsilon p$. We can also find another number independent of ϵ, $\Delta > 0$, such that if $|\epsilon| \leq 1$ and $\delta \in [0, \Delta]$, then M_δ^ϵ is well defined in the annulus $A: \ 1 \leq r \leq 2$. We want to prove the existence of $\epsilon^\star > 0$ such that if $|\epsilon| < \epsilon^\star$, then M_δ^ϵ has invariant curves in A for small δ. Once we have proved this, the Theorem will follow from Lemma 6.2.

To find the invariant curves of M_δ^ϵ we are going to apply the Small Twist Theorem. The intersection property is an easy consequence of Lemma 6.3. To obtain the estimates we can employ Exercise 20 to obtain a variant of Proposition 6.6 that is valid for $f = 1 + \epsilon p$. More precisely, M_δ^ϵ has the expansion

$$t_1 = t_0 + T + \delta\alpha(r_0) + \delta\{\epsilon\frac{\gamma r_0}{2\Omega^2}\int_0^{2\pi} p(t_0 + \frac{\Theta}{\Omega}) C(\frac{\Theta}{\Omega}) d\Theta + R_1(t_0, r_0; \delta, \epsilon)\},$$

$$r_1 = r_0 + \delta\{-\epsilon\frac{\gamma r_0^2}{2\Omega^2}\int_0^{2\pi} p(t_0 + \frac{\Theta}{\Omega}) S(\frac{\Theta}{\Omega}) d\Theta + R_2(t_0, r_0; \delta, \epsilon)\}$$

with

$$\alpha(r_0) = \frac{\gamma}{2\Omega}(\int_0^T C(t)dt) r_0$$

and

$$||R_i(\cdot, \cdot; \delta, \epsilon)||_{C^4(K)} \to 0, \quad \delta \to 0, \ i = 1, 2,$$

uniformly with respect to $\epsilon \in [-1, 1]$. Exercise 15 implies that α has a non-vanishing derivative and therefore the Small Twist Theorem is applicable whenever $\delta \in [0, \Delta]$ and ϵ is small enough. □

Remark. More results on the boundedness problem for asymmetric oscillators can be found in [16, 3, 13]. Other nonlinear oscillators with linear growth have been studied in [20, 2, 9, 17, 12]. The paper of Markus Kunze in these lectures notes [8] also deals with semilinear oscillators and contains a new application of Proposition 6.4.

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Rafael Ortega
Departamento de Matemática Aplicada
Facultad de Ciencias, Universidad de Granada
18071 Granada
Spain
e-mail: `rortega@goliat.ugr.es`