Trivial dynamics for a class of analytic homeomorphisms of the plane

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Dedicated to Carles Simó

1 Introduction

A homeomorphism h from $\mathbb{R}^2$ onto $\mathbb{R}^2$ satisfies the *Brown condition* if it is orientation preserving and

$$deg(id - h, \hat{\Gamma}) \neq 1$$

for each Jordan curve $\Gamma \subset \mathbb{R}^2 \setminus \mathrm{Fix}(h)$. Throughout the paper deg refers to the Brouwer degree in the plane, $\hat{\Gamma}$ denotes the bounded component of $\mathbb{R}^2 \setminus \Gamma$ and $\mathrm{Fix}(h)$ is the set of fixed points of h.

Brown observed in [2] that the above condition is sufficient to guarantee that the homeomorphism h produces trivial dynamics. This is a remarkable result showing that, for certain maps in low dimension, it is possible to understand the global dynamics just by computing the degree. In this context trivial dynamics means that the limit set of every bounded orbit is a connected and compact set contained in $\mathrm{Fix}(h)$. When the set of fixed points is totally disconnected this implies the convergence of the orbit but if $\mathrm{Fix}(h)$ contains a non-degenerate continuum then one cannot discard the existence of orbits accumulating around the continuum. Indeed this can happen even for C^∞ diffeomorphisms. The purpose of the present paper is to show that the result by Brown can be sharpened in the analytic setting.

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Theorem 1 *Assume that the homeomorphism h satisfies the Brown's condition and it is a real analytic mapping of $\mathbb{R}^2$. Then, for each $p \in \mathbb{R}^2$, the sequence $\{h^n(p)\}_{n\geq 0}$ is either divergent or convergent to a fixed point. More precisely, either $||h^n(p)|| \to \infty$ or $h^n(p) \to q$ with $q \in \text{Fix}(h)$ as $n \to +\infty$.*

To get some insight on this result we discuss its applicability in simple examples. The Brown's condition is satisfied by the translation $h_1(x, y) = (x+1, y)$ and also by the hyperbolic linear map $h_2(x, y) = (\frac{1}{2}x, 2y)$. For the first map the degree vanishes on any region while for the second this degree can take the values -1 and 0. For the map h_1 all orbits are divergent while the map h_2 shows that convergent and divergent orbits can coexist. The rotation $h_3(x, y) = (y, -x)$ and the contraction $h_4(x, y) = (\frac{1}{2}x, \frac{1}{3}y)$ are not in the conditions of the Theorem. In both cases the degree is 1 whenever the origin belongs to $\hat{\Gamma}$. Despite this fact the map h_4 has trivial dynamics. This shows that the Brown's condition is only a sufficient condition and cannot characterize the trivial dynamics.

The proof of the Theorem will be obtained by combining the theory developed by Brown in [2] with some ideas taken from the paper with Campos and Dancer [3]. After proving the main result we will illustrate its applicability with some examples. First we will consider a class of area preserving maps related to the difference equation

$$x_{n+1} - 2x_n + x_{n-1} = \phi(x_n), \quad n \in \mathbb{Z}, \tag{1}$$

where ϕ is an analytic function. In this family of equations it is possible to give a simple characterization of those with trivial dynamics. This can be obtained as a consequence of Theorem 1 and a criterion for the stability of parabolic fixed points which is due to Simó [12]. The second application will be concerned with the differential equation

$$\ddot{x} = F(t, x, \dot{x}), \tag{2}$$

where the function F is analytic, bounded and periodic in time.

2 Proof of Theorem 1

2.1 Free homeomorphisms on surfaces

Let M be a connected, metric space which is locally homeomorphic to $\mathbb{R}^2$, so that M becomes a two-manifold. A subset D of M is called a disk if it is homeomorphic to $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Following [2] we say that a homeomorphism f from M onto itself is *free* if

$$f(D) \cap D = \emptyset \Rightarrow f^p(D) \cap f^q(D) = \emptyset$$

for each disk D in M and integers $p \neq q$.

Free homeomorphisms are preserved under conjugation and so this is a topological notion. Also it is not hard to prove that the above intersection condition is valid not only for disks but for general continua. More precisely, if we are given a compact and connected set C in M and a free homeomorphism f then

$$f(C) \cap C = \emptyset \Rightarrow f^p(C) \cap f^q(C) = \emptyset, \ \ p \neq q.$$

This is the content of the first Lemma in [2].

Given a point p in M the omega limit set $L_\omega(p, f)$ is defined as the set of points $q \in M$ which are limits of sequences $f^{\sigma(n)}(p)$, where $\sigma(n)$ is a sequence of integers going to $+\infty$. Lemma 3.4 in [2] says that if M is compact then $L_\omega(p, f)$ is contained in $\mathrm{Fix}(f)$. It now follows from the general theory of discrete dynamical systems that $L_\omega(p, f)$ is non-empty, compact and connected. The connectedness is a consequence of Lemma 2.7 in [7] or Proposition 2.5 in [4].

Theorem 5.7 in [2] is a deep result saying that if $M = \mathbb{R}^2$ and $f : \mathbb{R}^2 \cong \mathbb{R}^2$ satisfies the Brown's condition then f is free. In particular this applies to the map $h = f$ of Theorem 1. Consider the Riemann sphere $\mathbb{S}^2 = \mathbb{R} \cup \{\infty\}$ and extend h to $\hat{h} : \mathbb{S}^2 \cong \mathbb{S}^2$ with $\hat{h}(\infty) = \infty$. The point of infinity is fixed and so any disk in the sphere satisfying $\hat{h}(D) \cap D = \emptyset$ is indeed a disk in the plane. This implies that also $\hat{h}$ is free as a homeomorphism of the sphere. Given a point $p \in \mathbb{R}^2$, the set $L_\omega(p, \hat{h})$ is a continuum contained in $\mathrm{Fix}(\hat{h}) = \mathrm{Fix}(h) \cup \{\infty\}$. The proof of Theorem 1 will consist in showing that it is indeed a singleton. At this point the analyticity becomes important, for it is possible to construct examples of C^∞ diffeomorphisms which are free and have non-degenerate continua as limit sets. More details on this type of construction can be found in Examples 3.1 and 3.2 of [4].

We finish this discussion about free homeomorphisms with an observation about Theorem 5.7 in [2]. It can be extended to free embeddings of the plane. This means that the map from $\mathbb{R}^2$ into $\mathbb{R}^2$ is continuous and one-to-one but not necessarily onto. The detailed proofs will appear in [11]. With this result it is possible to extend Theorem 1 to embeddings.

2.2 Connected subsets of analytic sets

A subset A of $\mathbb{R}^2$ is analytic if it can be described by an equation $\Phi(x,y) = 0$ where $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is real analytic. A p-arc will be a vertical segment $\{a\} \times I$ or a set of the type $\{(x, \alpha(x)) : x \in I\}$, where I is a compact non-degenerate interval and $\alpha : I \to \mathbb{R}$ is real analytic.

Lemma 2 *Assume that A is an analytic proper subset of $\mathbb{R}^2$ and C is a connected subset of A which contains at least two points. Then C contains a p-arc.*

Proof. We fix two points $p \neq q$ in C and, for simplicity of the notation, assume that $p = (0,0)$. The real versions of the Weierstrass Preparation Theorem and Puiseux's Theorem (Theorems 6.1.3 and 4.2.8 in [8]) lead to the local description of A around p. Indeed it is possible to find an open neighbourhood N of p and a finite number of subsets $\gamma_1, \ldots, \gamma_r$ with

$$A \cap N = \gamma_1 \cup \ldots \cup \gamma_r, \quad \gamma_i \cap \gamma_j = \{p\} \text{ if } i \neq j.$$

Moreover each set γ_i is either a vertical segment $\{0\} \times J_i$ or a Puiseux branch of the type $y = \varphi_i(|x|^{1/k})$, $x \in J_i, k \geq 1$, with $J_i = [0, \delta[$ or $] - \delta, 0]$ and $\varphi_i : J_i \to \mathbb{R}$ analytic. The neighbourhood N can be chosen arbitrarily small and so we can assume that $q \notin N$.

We claim that one of the branches γ_i must be contained in C. This will be sufficient to prove the Lemma. If we assume by contradiction that none of the subsets γ_i is contained in C then we can find points q_i. $i = 1, \ldots, r$, lying in $\gamma_i \setminus C$. The closed sub-arc of γ_i joining p and q_i will be denoted by $\widehat{pq_i}$. From its definition it is clear that the set $C_1 = C \cap [\widehat{pq_1} \cup \ldots \cup \widehat{pq_r}]$ is simultaneously closed and open with respect to the relative topology of C. Since q is not in C_1 we have arrived at a contradiction with the connectedness of C.■

Let us go back to the proof of Theorem 1. The set of fixed points of h is analytic since it can be expressed as

$$\mathrm{Fix}(h) = \Phi^{-1}(0)$$

with

$$\Phi(x,y) = (x - h_1(x,y))^2 + (y - h_2(x,y))^2.$$

If $\mathrm{Fix}(h) = \mathbb{R}^2$ then h is the identity and the Theorem holds trivially. From now on we assume that $h \neq id$ and so $\mathrm{Fix}(h)$ is a proper subset of the plane. If $L_\omega(p, \hat{h})$ were a non-degenerate continuum then the connected components of $L_\omega(p,\hat{h}) \setminus \{\infty\}$ could not be single points. This would allow us to apply Lemma 2 with $A = \text{Fix}(h)$ and C one of the components of $L_\omega(p,\hat{h}) \setminus \{\infty\}$. The conclusion is that if $L_\omega(p,\hat{h})$ is not a singleton then $L_\omega(p,h) = L_\omega(p,\hat{h}) \setminus \{\infty\}$ must contain a p-arc.

2.3 Free homeomorphisms and limit sets

The proof of Theorem 1 will be complete after proving the following result.

Proposition 3 *Assume that h is an analytic homeomorphism of $\mathbb{R}^2$ and that, for some $p \in \mathbb{R}^2$, the set $L_\omega(p,h)$ contains a p-arc. Then h is not free.*

Proof. It follows along the proof of Theorem 1 in [3]. First of all we notice that it is not restrictive to assume that the p-arc is of the type $y = \alpha(x)$, $x \in I = [a,b]$. Otherwise this arc would be a vertical segment and we could replace h by $r^{-1} \circ h \circ r$, where r is a rotation of 90 degrees. Notice that the Brown's condition is invariant under conjugacy. Next we extend α to a C^∞ function $\hat{\alpha} : \mathbb{R} \to \mathbb{R}$ and consider the C^∞ diffeomorphism of $\mathbb{R}^2$,

$$\psi(x,y) = (x, y - \hat{\alpha}(x)).$$

The strip $[a,b] \times \mathbb{R}$ is invariant under ψ and the arc $y = \alpha(x)$ is mapped onto the horizontal segment $[a,b] \times \{0\}$. The map $g = \psi \circ h \circ \psi^{-1}$ is conjugate to h and so $[a,b] \times \{0\}$ is contained in $L_\omega(p_\star, g)$, where $p_\star = \psi(p)$. From now on we shall work with g instead of h. This new map is not always analytic in the whole plane but it is analytic on $[a,b] \times \mathbb{R}$ and this will be sufficient for the remaining arguments. Any point in $[a,b] \times \{0\}$ is fixed under $g = (g_1, g_2)$ and so

$$g_1(x,0) = x, \quad g_2(x,0) = 0 \text{ if } x \in [a,b].$$

For some $\eta > 0$ we have the identity

$$g_1(x,y) = x + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m g_1}{\partial y^m}(x,0) y^m$$

when $x \in [a,b]$, $|y| \le \eta$. If all the functions $\frac{\partial^m g_1}{\partial y^m}(x,0)$ were identically zero on $[a,b]$ the identity principle would imply that $g_1(x,y) = x$ on $[a,b] \times \mathbb{R}$. This is not possible since the vertical lines $\{x\} \times \mathbb{R}$ would be invariant and so the orbit $\{g^n(p_\star)\}$ could not accumulate along the whole horizontal segment.

Once we know that some of these derivatives is not zero we select the first integer $\mu \ge 1$ such that $\beta(x) = \frac{1}{\mu!} \frac{\partial^\mu g_1}{\partial y^\mu}(x,0)$ is not identically zero on $[a,b]$. We are lead to the expansion

$$g_1(x,y) = x + \beta(x) y^\mu + R(x,y) y^{\mu+1}$$

were R is analytic on $[a,b]\times\mathbb{R}$. By restricting the size of the interval, say $[a_1,b_1]\subset[a,b]$, we can assume that β does not vanish at any x in $[a_1,b_1]$. We can also select $\epsilon>0$ such that if (x,y) is a point in the rectangle $[a_1,b_1]\times[-\epsilon,\epsilon]$ then

$$\frac{\partial g_1}{\partial x}(x,y)>0 \quad\text{and}\quad g_1(x,y)=x \Leftrightarrow y=0. \tag{3}$$

We say that a point $q_\star=(x,0)\in[a_1,b_1]\times\{0\}$ is in $\mathcal{L}^+$ if it can be approached by the orbit $\{g^n(p_\star)\}$ from above. More precisely we define $R^+=[a_1,b_1]\times]0,\epsilon]$ and say that $q_\star$ is in $\mathcal{L}^+$ if there is a sequence of integers $\sigma(n)\to+\infty$ such that the sequence $g^{\sigma(n)}(p_\star)$ remains in R^+ and converges to $q_\star$. The set $\mathcal{L}^-$ is defined in the same way if one replaces R^+ by $R^-=[a_1,b_1]\times[-\epsilon,0[$. The segment $]a_1,b_1[\times\{0\}$ is covered by $\mathcal{L}^+\cup\mathcal{L}^-$ and so we can find a smaller interval $[a_2,b_2]\subset[a_1,b_1]$ such that either $\mathcal{L}^+$ or $\mathcal{L}^-$ is dense in $[a_2,b_2]\times\{0\}$. It is not restrictive to assume that $\mathcal{L}^+$ is the dense set. Otherwise we would replace g by $\tilde{g}=\Psi^{-1}\circ g\circ\Psi$ with $\Psi(x,y)=(x,-y)$. This change does not affect to the condition (3).

From now on we assume that $\mathcal{L}^+$ is dense and, according to the condition (3), we distinguish two cases:

$$\text{Case 1}: \quad g_1(x,y)>x,\ \frac{\partial g_1}{\partial x}(x,y)>0 \ \text{ if } (x,y)\in[a_2,b_2]\times]0,\epsilon].$$

We employ the notation $p_n=g^n(p_\star)$ and find n large enough so that $p_n=g^n(p_\star)$ and $p_{n+1}=g^{n+1}(p_\star)$ lie in $[a_2,b_2]\times]0,\epsilon]$. This is possible because $\mathcal{L}^+$ is dense in $[a_2,b_2]\times\{0\}$ and g satisfies

$$g_1(x,y)=x+O(y^\mu),\quad g_2(x,y)=O(y)$$

around $[a_2,b_2]\times\{0\}$. With the notation $p_n=(x_n,y_n)$, we use that we are in the case 1 and deduce that

$$a_2\le x_n<x_{n+1}\le b_2.$$

The density of $\mathcal{L}^+$ allows us to find $m>n+1$ such that the point $p_m=g^m(p_\star)$ lies in $[a_2,b_2]\times]0,\epsilon]$ and

$$x_n<x_m<x_{n+1}.$$

We consider the arc $\gamma=\gamma_h\cup\gamma_v$ where γ_h is the horizontal segment connecting p_n to (x_m,y_n) and γ_v is the vertical segment from (x_m,y_n) to p_m. Some of these segments can degenerate to a point. We claim that $g(\gamma)\cap\gamma=\emptyset$. Indeed

γ is contained in $[a_2, b_2] \times]0, \epsilon]$ and so the conditions assumed in the case 1 hold along γ. From the positivity of $\frac{\partial g_1}{\partial x}$ we deduce that g_1 is increasing along γ_h and so $g(\gamma_h)$ is contained in $\{x \geq x_{n+1}\}$. From $g_1(x, y) > x$ we deduce that $g(\gamma_v)$ is contained in $\{x > x_m\}$. The arcs γ and $g(\gamma)$ are separated by the line $x = x_m$ and so they cannot intersect.

The point p_m is in $\gamma \cap g^{m-n}(\gamma)$ and this implies that g is not free. The proof is complete for the case 1 because the property of being free is topological and so h cannot be free.

$$\text{Case 2}: \quad g_1(x, y) < x, \quad \frac{\partial g_1}{\partial x}(x, y) > 0 \quad \text{if } (x, y) \in [a_2, b_2] \times]0, \epsilon].$$

The line of reasoning is the same. Now

$$x_{n+1} < x_m < x_n$$

and the horizontal segment γ_h goes from p_n to (x_m, y_n) while the vertical segment γ_v goes from (x_m, y_n) to p_m. The line of separation is again $x = x_m$.■

The dynamics around a segment of fixed points has been considered in [5] and also in [3], Section 2. The results in those papers assume some additional conditions on the derivatives at the fixed points and are not applicable to the general situation considered in the proof of the above Proposition.

3 A family of area preserving maps

We start with the difference equation (1) from the introduction, where the function $\phi : \mathbb{R} \to \mathbb{R}$ is real analytic. After the change $y_n = x_{n+1} - x_n$ we are lead to the study of the analytic area preserving diffeomorphism

$$h : \begin{cases} x_1 = x + y \\ y_1 = y + \phi(x + y). \end{cases}$$

The fixed points of h are the zeros of ϕ or, more precisely,

$$\text{Fix}(f) = \phi^{-1}(0) \times \{0\}.$$

We are going to discuss the Brown's condition for this family. Assume that Γ is a Jordan curve contained in $\mathbb{R}^2 \setminus \text{Fix}(h)$. If $\phi \equiv 0$ then $\text{Fix}(h)$ is the x-axis and Γ lies in one of the half planes $y > 0$ or $y < 0$. This implies that the degree of $id - h$ on $\hat{\Gamma}$ must vanish and so the Brown's condition holds.

Assume now that ϕ is not identically zero and $\xi_1, \ldots, \xi_n$ are the zeros of ϕ with $(\xi_i, 0) \in \hat{\Gamma}$ then

$$deg(id - h, \hat{\Gamma}) = \sum_{i=1}^{n} \sigma_i$$

where $\sigma_i = 1$ [resp. -1] if ϕ is decreasing in a neighbourhood of ξ [resp. increasing] and $\sigma_i = 0$ if ϕ reaches a maximum or minimum at ξ_i. This is a consequence of the general properties of the degree and the reduction of dimension via the Implicit Function Theorem (see [9] or [10], page 223). From here we deduce that the Brown's condition holds if and only if ϕ satisfies one of the following conditions

- There exists $\xi_\star \in \mathbb{R}$ such that $(\xi - \xi_\star)\phi(\xi) \geq 0$ for each $\xi \in \mathbb{R}$
- $\phi(\xi) \geq 0$ for each $\xi \in \mathbb{R}$
- $\phi(\xi) \leq 0$ for each $\xi \in \mathbb{R}$.

In any of these cases the Theorem 1 applies and one has trivial dynamics. It is also possible to understand more details of the local dynamics around the fixed points using the theory of invariant manifolds. The fixed points can be hyperbolic or parabolic. In the first case the Hartman-Grossmann Theorem applies and in the second one can apply the results by Fontich in [6]. It is also interesting to notice that for the family of mappings of this section the Brown's condition is sharp. Indeed, if ϕ has a decreasing zero one can apply Simó's stability criterion for parabolic fixed points (see [12]) and conclude that there is KAM dynamics around this point. This of course excludes the possibility of trivial dynamics in a global sense.

4 Periodically forced Newtonian equations

The Brown's condition is automatically satisfied by orientation preserving homeomorphisms which can be approximated by mappings without fixed points. Indeed if h_n is a sequence of continuous maps from $\mathbb{R}^2$ into $\mathbb{R}^2$ converging to h uniformly on compact sets, then

$$deg(id - h_n, \hat{\Gamma}) = deg(id - h, \hat{\Gamma}),$$

where Γ is a Jordan curve in $\mathbb{R}^2 \setminus \text{Fix}(h)$ and n is large enough. This is a consequence of the invariance of the degree under small perturbations. Now, if we assume that $\text{Fix}(h_n) = \emptyset$ for each n, then this degree always vanishes.

We are going to apply this observation to the study of the dynamics of the differential equation (2) from the introduction. We shall rewrite the equation in the form

$$\ddot{u} = F(t, u, \dot{u}) + s \tag{4}$$

where $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and T-periodic in t and $s \in \mathbb{R}$ acts as a parameter.

We shall assume that F is bounded, say

$$|F(t, u, v)| \leq M \quad \text{for each } (t, u, v) \in \mathbb{R} \times \mathbb{R}^2.$$

In this case it is well known that there exists a non-empty interval $I_F \subset [-M, M]$ such that the existence of a T-periodic solution for (4) is equivalent to

$$s \in I_F.$$

This is a consequence of a result in [1] on upper and lower solutions. The original result was presented for elliptic equations but it adapts easily to periodic problems. More information on this point can be found in the recent paper [13]. The interval I_F can be open, closed or half-open, depending on the nonlinearity F. In some exceptional situations I_F can even degenerate to a point.

To apply the results on analytic homeomorphisms we must assume that F is analytic with respect to (u, v). This means that, given $(u_0, v_0) \in \mathbb{R}^2$, there exists $\delta > 0$ such that if $|u - u_0| + |v - v_0| < \delta$ and $t \in \mathbb{R}$ then F can be expressed as

$$F(t, u, v) = \sum_{m,n=0}^{\infty} \alpha_{n,m}(t)(u - u_0)^n (v - v_0)^m,$$

where $\alpha_{n,m}$ is continuous and T-periodic in t.

The Poincaré map associated to (4) is defined as

$$P_s : (u(0), \dot{u}(0)) \mapsto (u(T), \dot{u}(T))$$

where $u(t)$ is an arbitrary solution. The previous assumptions imply that P_s is an orientation preserving analytic diffeomorphism of the plane. Moreover the dependence with respect to s is continuous and if $s_n \to s$ then P_{s_n} converges to P_s uniformly on compact sets. If $s \notin I_F$ then P_s has no fixed points and so we can deduce that P_s satisfies the Brown's condition as soon as s is not in the interior of I_F. At this point we have in mind the remarks made at the beginning of this Section. After an application of Theorem 1 we conclude that if s is not in the interior of I_F then every solution $u(t)$ of (4) satisfies either

$$|u(nT)| + |\dot{u}(nT)| \to \infty$$

or

$$(u(nT), \dot{u}(nT)) \to \text{fixed point of } P_s$$

as $n \to +\infty$, $n \in \mathbb{Z}$. From the equation we observe that the second derivative of u is bounded by M and we are lead to the following conclusion.

Corollary 4 *Assume that F is in the above conditions. Then if $s \notin I_F$ every solution is divergent and if $s \in \partial I_F$ every solution is divergent or asymptotically T-periodic.*

By a divergent solution we understand that $|u(t)|+|\dot{u}(t)| \to \infty$. The solution $u(t)$ is asymptotically T-periodic if there exists a T-periodic solution $\varphi(t)$ such that $|u(t) - \varphi(t)| + |\dot{u}(t) - \dot{\varphi}(t)| \to 0$ as $t \to +\infty$.

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