Existence and extendibility of rotationally symmetric graphs with a prescribed higher mean curvature function in Euclidean and Minkowski spaces

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Abstract

In this paper we investigate the existence of rotationally symmetric entire graphs (resp. entire spacelike graphs) with prescribed k-th mean curvature function in Euclidean space $\mathbb{R}^{n+1}$ (resp. Minkowski spacetime $\mathbb{L}^{n+1}$). As a previous step, we analyze the associated homogeneous Dirichlet problem on a ball, which is not elliptic for $k > 1$, and then we prove that it is possible to extend the solutions. Moreover, a sufficient condition for uniqueness is given in both cases.

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1 Introduction

Along this paper, $\mathbb{R}_a^{n+1}$, $a = 0, 1$, will denote, for $a = 0$, the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ endowed with its standard Riemannian metric $\langle \, , \, \rangle = \sum_{i=1}^{n+1} dx_i^2$ and, for $a = 1$, the $(n+1)$-dimensional Lorentzian spacetime $\mathbb{L}^{n+1}$ endowed with its standard Lorentzian metric $\langle \, , \, \rangle = -dx_1^2 + \sum_{j=2}^{n+1} dx_j^2$ and with the time orientation defined by $\partial/\partial x_1$. For a two sided hypersurface ($a = 0$) or spacelike hypersurface ($a = 1$) in $\mathbb{R}_a^{n+1}$, the k-th mean curvatures are geometric invariants which encode the geometry of the hypersurface. From an algebraic point of view, each one of these functions corresponds to a coefficient of the characteristic polynomial of the shape operator corresponding to a unit normal vector field

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($a = 0$) or to a unit timelike vector field pointing to future ($a = 1$). In fact, each k-th mean curvature is described as a certain type of average measure of the principal curvatures of the hypersurface (see Section 2 for details). In particular, the 1-th mean curvature corresponds with the usual mean of the principal curvatures if $a = 0$ or its opposite if $a = 1$, the 2-th mean curvature is, up to a constant factor, the scalar curvature, and the n-th mean curvature is the Gauss-Kronecker curvature if $a = 0$ and $(-1)^{n+1}$ times the Gauss-Kronecker curvature if $a = 1$. Each k-th mean curvature has a variational nature [21], and in Riemannian geometry the constant k-th mean curvature case has been extensively studied ([16], [22] for instance). From a physical perspective, the k-th mean curvatures have a relevant role in General Relativity. A spacelike hypersurface is a suitable subset of the spacetime where the initial value problems for the different field equations are naturally stated. Roughly speaking, a spacelike hypersurface represents the physical space at one time instant. Each k-th mean curvature function intuitively measures the time evolution towards the future or the past of the spatial universe (see Remark 2.1).

In the Euclidean context, the pioneer work on the prescribed mean curvature Dirichlet problem was given by Serrin [23], who found necessary and sufficient conditions for its solvability. In Minkowski spacetime, Cheng and Yau proved in [10] the Bernstein's property for entire solutions of the maximal (i.e., zero mean curvature) hypersurface equation and, later, Treibergs [25] classified the entire solutions of the constant mean curvature spacelike hypersurface equation. An important universal existence result was proved by Bartnik and Simon [5], and Bartnik proved the existence of prescribed mean curvature spacelike hypersurfaces under certain asymptotic assumptions [2]. The Dirichlet problem in a more general spacetime was solved by Gerhardt [14]. More recently, there are more contributions (see for instance, [1]) and the interest is many times focused on the existence of positive solutions, by using a combination of variational techniques, critical point theory, sub-supersolutions and topological degree (see for instance [6, 7, 9, 11, 12] and the references therein). Respect to the scalar curvature, we refer to [8] in the Euclidean context. On the other hand, Bayard proved the existence of prescribed scalar entire spacelike hypersurfaces in Minkowski spacetime [3], by using other previous works on the Dirichlet problem ([4] and [26] and references therein) and Gerhardt [15] obtained important results on the case of more general ambient spacetimes. Finally, the Gauss-Kronecker curvature has been also quite well studied in both settings. In Euclidean space, Wang [28] prescribed the Gauss-Kronecker curvature of a convex hypersurface. In Minkowski spacetime, we highlight the work of Li [19] on constant Gauss curvature and Delanoè [13], in which the existence of entire spacelike hypersurfaces asymptotic to a lightcone with prescribed Gauss-Kronecker curvature function is proved.

Up to the last decade, little attention has been paid to hypersurfaces with prescribed k-th mean curvature when $3 \leq k < n$. One of the first works in this direction was done by Ivochkina (see [18] and references therein). More recently, several contributions (for instance, [27], [17]) especially on the Dirichlet problem has been done. However, the general question is still open in both settings. The study has usually been focused in the search of some *a priori* bounds on the length of the shape operator, assuming that solutions are k-stable to ensure the ellipticity of some involved differential operators (see [27] for more details). Then, some special dependence in the prescription function are imposed in order to obtain partial results. In this paper, we provide several existence and uniqueness results on this open problem, assuming that the prescription function is rotationally symmetric respect to a unit parametrized line or an inertial observer γ (i.e., a unit timelike parametrized line pointing to the future) in $\mathbb{R}_a^{n+1}$ with $a = 0$ or $a = 1$, respectively. Especially, we prove the existence of rotationally symmetric entire graphs with prescribed k-th mean curvature of the associated Dirichlet problem when the domain in a n-dimensional ball, by using a suitable fixed point operator. Besides, we prove that such graph can be extended to the whole space, providing some information about uniqueness as well.

Along the paper, we will use the usual cylindrical coordinates (t, r, Θ) in $\mathbb{R}_a^{n+1}$ associated to γ, namely, $t \in \mathbb{R}$ is the parameter of γ, $r \in \mathbb{R}^+$ is the radial distance to γ and $\Theta = (\theta_1, ..., \theta_{n-1})$ are the standard spherical coordinates of the $(n-1)$-dimensional unit round sphere $\mathbb{S}^{n-1}$. The prescription functions H_k will be assumed to be radially symmetric with respect to γ. Therefore, it is natural to consider $H_k(t, x) = H_k(t, r)$ where r denotes the distance of $x \in \mathbb{R}^n$ to γ.

Hypersurfaces in Euclidean space and spacelike hypersurface in Lorentz-Minkowski spacetime have a different geometry. Therefore, in the related literature, one can make a clear distinction between two large groups of papers, depending if they consider the Euclidean or the Lorentzian ambient. However, we have decided to present the results of both contexts in a single paper because, even if the results are different, the mathematical treatment is similar. It is interesting to mention that solving the problem in the last case is easier than the in first one. This is the reason to present in this order the content of the paper.

Below, we summarize the main results on entire graphs in Lorentzian ambient. The first theorem is a kind of "universal existence result" when k is odd.

Theorem A *Let $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, with k an odd positive integer, be a continuous function which is rotationally symmetric with respect to an inertial observer γ of $\mathbb{L}^{n+1}$. Then, for each $R > 0$, there exists at least an entire spacelike graph, rotationally symmetric respect to γ, whose k-th mean curvature equals to H_k and such that it intersects the hyperplane orthogonal to γ at $\gamma(0)$ in an $(n-1)$-sphere with radius R centered at $\gamma(0)$. In addition, if H_k is non decreasing with respect to the proper time of γ, then the spacelike graph is unique.*

For k even, we have to introduce a natural restriction on the curvature, as it is shown in the next result.

Theorem B *Let $H_k : \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, with k an even positive integer, be a continuous function such that*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0 \quad \text{for all} \quad r \in \mathbb{R}^+, \quad \text{and} \quad v \in C^1, \ |v'| < 1, \tag{1}$$

and which is rotationally symmetric with respect to an inertial observer γ of $\mathbb{L}^{n+1}$. If $H_k(0, \cdot) \not\equiv 0$, for each $R > 0$, then there exists at least two different entire spacelike graphs and rotationally symmetric whose k-th mean curvature equals to H_k and such that it intersects the hyperplane orthogonal to γ at $\gamma(0)$ in an $(n-1)$-sphere with radius R centered in $\gamma(0)$. Moreover, the radial profile curve of one of them is increasing and the other one is decreasing. Besides, condition (1) is necessary for the existence of such graphs.

These results are obtained from the local existence results provided by Proposition 3.1 and Proposition 3.3, and the extendibility result given by Lemma 6.1.

On the other hand, in the case of entire graphs in $\mathbb{R}^{n+1}$ we can not expect a kind of universal result like Theorem A, for if we want to obtain a rotationally symmetric graph with prescribed k-curvature that intersects the orthogonal hyperplane in a n-sphere with radius R, it is necessary to introduce an additional limit in the size of the prescribed curvature, as in the following result.

Theorem C *Let $H_k : \mathbb{R}^{n+1} = \mathbb{R} \times \Pi \longrightarrow \mathbb{R}$, with k an odd positive integer, be a continuous function which is rotationally symmetric respect to an oriented line γ, orthogonal to Π. Given a fixed $R > 0$, assume there is some $\alpha \in (0, R^{-k})$, satisfying*

$$|H_k(t,r)| \leq \alpha \qquad \text{for all} \qquad r \in [0,R]\,, \quad t \in [-R\,\beta, R\,\beta], \tag{2}$$

where $\beta := \dfrac{R\alpha^{1/k}}{\sqrt{1 - R^2\alpha^{2/k}}}$, and, for each $r > R$,

$$\big|H_k(t,r)\big| < 1/r^k, \qquad \text{for all} \qquad t \in \mathbb{R}. \tag{3}$$

Then, there exists at least an entire graph, rotationally symmetric respect to γ, whose k-th mean curvature equals to H_k and such that it intersects the hyperplane Π in an $(n-1)$-sphere with radius R centered in $\gamma(0)$. In addition, if H_k is non decreasing along the line γ, then the graph is unique.

Remark 1.1 In fact, hypothesis (3) may be weakened to the more artificial condition,

$$\Big| \int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \Big| < \frac{r^{n-k}}{n}, \qquad \text{for all} \qquad r > R \quad \text{and} \quad v \in C^1.$$

Later, we will see that the condition (2) is quite natural. In particular, it is necessary when the prescription function H_k is constant (see the example given in (14)). The last result considers the case of k even in the Euclidean space.

Theorem D *Let $H_k : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, with k an even positive integer, be a continuous function which is rotationally symmetric respect to a line γ. Given a fixed $R > 0$, assume there is some $0 < \alpha < R^{-k}$, satisfying (2), (3) and*

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0, \qquad \text{for all} \qquad r \in [0,R] \quad \text{and} \quad v \in B_{R\beta,\beta}, \tag{4}$$

being $B_{R\beta,\beta} = \{v \in C^1 : \|v\|_\infty < R\beta,\ \|v'\|_\infty < \beta\}$. Then, if $H_k(0,\cdot) \not\equiv 0$, there exists at least two different entire graphs, rotationally symmetric, whose k-th mean curvatures equal to H_k and such that they intersect the hyperplane orthogonal to γ in $\gamma(0)$ in an $(n-1)$-sphere with radius R centered in $\gamma(0)$. Moreover, the radial profile curve of one of them is increasing and the other one is decreasing.

The proof of Theorems C and D follows from the local existence results given in Theorem 4.1 and Theorem 4.2, taking into account Proposition 5.1 and Proposition 5.2, respectively, and Lemma 6.1 for the extendibility.

The remaining content of the paper is structured as follows. Section 2 is devoted to expose several definitions and preliminary results. Sections 3 and 4 are respectively focused on the study of the Dirichlet problem in Minkowski spacetime $\mathbb{R}^{n+1}_1$ and in Euclidean space $\mathbb{R}^{n+1}$. In Section 5 several uniqueness results are obtained. Finally, we analyse in Section 6 the extendibility of the solutions of the previous Dirichlet problems.

2 Preliminaries

Let us consider a smooth immersion $\varphi : \Sigma \longrightarrow \mathbb{R}^{n+1}_a$ of an n-dimensional manifold Σ in $\mathbb{R}^{n+1}_a$, which is two sided if $a = 0$ and spacelike (i.e., the induced metric via φ is Riemannian) if $a = 1$. Assume N is a unit normal vector field along φ, which we choose pointing to the future if $a = 1$. The shape operator of Σ relative to N, is defined by

$$A(X) = -\nabla_X N,$$

where $X \in T_p\Sigma$, $p \in \Sigma$, and ∇ denote the Levi-Civita connection of $\mathbb{R}^{n+1}_a$ which is given by

$$\nabla_X N = (X(N_1), ..., X(N_{n+1})),$$

where $N = (N_1, ..., N_{n+1})$, contemplated as a map from Σ to $\mathbb{R}^{n+1}_a$. The linear operator A of $T_p\Sigma$, $p \in \Sigma$, is self-adjoint with respect to the induced metric. Its eigenvalues $\kappa_1(p), \kappa_2(p), ..., \kappa_n(p)$ are called the principal curvatures of the hypersurface. Consider the the characteristic polynomial of A,

$$\det(tI - A) = \sum_{k=0}^{n} c_k\, t^{n-k} = \prod_{i=1}^{n} (t - \kappa_i),$$

where we put $c_0 = 1$. It is not difficult to see that

$$\begin{aligned} c_1 &= -\sum_{i=1}^{n} \kappa_i \\ c_k &= (-1)^k \sum_{i_1<...<i_k} \kappa_{i_1} \cdots \kappa_{i_k}, \quad 2 \le k \le n. \end{aligned}$$

The k-th mean curvature S_k of Σ is defined as follows,

$$S_k = \frac{(-1)^{k(a+1)}}{\binom{n}{k}}\, c_k,$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. For instance, when $k = 1$, we get $S_1 = \frac{(-1)^{a+1}}{n}\, c_1 = \frac{(-1)^a}{n}\, \mathrm{trace}(A)$, the usual mean curvature of Σ. Moreover, S_2 is, up to a constant, the scalar curvature of Σ and, when $k = n$, we recover the Gauss-Kronecker curvature $S_n = (-1)^{an} \det(A)$ of Σ. It is interesting to note that k-th mean curvatures are in fact intrinsic geometric invariants of the hypersurfaces when k is even. Precisely, the parity of k plays an important role in the treatment of the equations, as it will be shown in next sections.

Remark 2.1 For the case of a spacelike hypersurface Σ in a (general) spacetime M, the geometric information contained in a k-th mean curvature S_k can be locally propagated to the future and the past in M and, then, physically interpreted. In fact, for each $p_0 \in \Sigma$ there exist an open neighbourhood U of p_0 in M and a reference frame Q on U such that $Q_p = N_p$ for all $p \in \Sigma \cap U$. The operator field $X \mapsto -\nabla_X Q$, $X \in T_qM$, $q \in U$, may be restricted to $Q^\perp$ providing with $A_Q : Q^\perp \to Q^\perp$ on U. Note that A_Q equals to the shape operator corresponding to N on $\Sigma \cap U$. On U, consider the $n + 1$ smooth functions $\overline{S}_0$, $\overline{S}_1$,...,$\overline{S}_n$ defined by

$$\overline{S}_k = \frac{1}{\binom{n}{k}}\, c_k,$$

where the function c_k is defined as previously from

$$\det(tI - A_Q) = \sum_{k=0}^{n} c_k\, t^{n-k}.$$

Each $\overline{S}_k$ is a *relative quantity* for the observers in Q and it equals to the k-th mean curvature S_k on $\Sigma \cap U$. Assume $S_k > 0$ (resp. $S_k < 0$) at some $p \in \Sigma \cap U$. Then $\overline{S}_k > 0$ (resp. $\overline{S}_k < 0$) near p in U. In particular, if the mean curvature H satisfies $H > 0$ (resp. $H < 0$) at p then $\operatorname{div}(Q) > 0$ (resp. $\operatorname{div}(Q) < 0$) near p in U, i.e., for the observers in Q the universe is expanding (resp. contracting) near p in U.

The prescribed k-th mean curvature problem in $\mathbb{R}^{n+1}_a$ consists in finding, for a given prescription function H_k, a (embedded) hypersurface if $a = 0$ or a spacelike hypersurface if $a = 1$, Σ in $\mathbb{R}^{n+1}_a$ which satisfies

$$S_k(p) = H_k(p) \qquad \text{for all} \quad p \in \Sigma. \tag{5}$$

We will focus here the problem as follows. Consider a line γ in $\mathbb{R}^{n+1}_a$ (as defined before) and put Π the hyperplane through $p = \gamma(0)$ and orthogonal to $\mathbb{R}^{n+1}_a$. We will look for Σ as a graph if $a = 0$ or a spacelike graph if $a = 1$ for a suitable function v defined on Π, i.e., $\Sigma = \{(v(x), x) : x \in \Pi\} \subset \mathbb{R} \times \mathbb{R}^n$. If the prescription function H_k were assumed rotationally symmetric with respect to γ, then it would be natural to assume v also has the same symmetry, i.e., $v(x) = v(r)$ where $r = r(x)$ is the distance in Π from x to $\gamma(0)$.. On the other hand, using a cylindrical coordinate system (t, r, Θ), $\Theta = (\theta_1, ..., \theta_{n-1})$, as before of $\mathbb{R}^{n+1}_a$, the metric of $\mathbb{R}^{n+1}_a$ may be expressed as

$$\langle\, , \rangle = \epsilon\, dt^2 + dr^2 + r^{n-1} d\Theta^2, \qquad \Theta = (\theta_1, \cdots, \theta_{n-1}),$$

where $\epsilon = (-1)^a$ and $d\Theta^2$ the standard Riemannian metric on the unit round sphere $\mathbb{S}^{n-1}$. With respect to the coordinate frame $\{\partial_t, \partial_r, \partial_{\theta_1}, ..., \partial_{\theta_{n-1}}\}$, the unit normal vector field N along Σ in $\mathbb{R}^{n+1}_a$ is given by

$$N = \frac{\partial_t - \epsilon v' \partial_r}{\sqrt{1 + \epsilon\, v'^2}}.$$

where v', the derivative of $v = v(r)$, satisfies $|v'| < 1$ if $a = 1$.

The value of the principal curvatures is given by the following result.

Lemma 2.2 *At each $(v(r), r) \in \Sigma$, the vectors ∂_{θ_i} and $v'\partial_t + \partial_r$ are eigenvectors of the shape operator A, with eigenvalues*

$$\kappa_1(v, r) = \cdots = \kappa_{n-1}(v, r) = \frac{\epsilon\, v'}{r\sqrt{1 + \epsilon\, v'^2}}, \qquad \text{and} \qquad \kappa_n(v, r) = \frac{\epsilon\, v''}{(1 + \epsilon\, v'^2)^{3/2}}.$$

Proof. In order to check that ∂_{θ_i} are $n - 1$ eigenvectors of A, we compute

$$A(\partial_{\theta_i}) = -\frac{1}{\sqrt{1 + \epsilon\, v'^2}} \left[\nabla_{\partial_{\theta_i}} \partial_t - \epsilon\, v'\, \nabla_{\partial_{\theta_i}} \partial_r\right] = \frac{\epsilon\, v'}{\sqrt{1 + \epsilon\, v'^2}}\, \Gamma^{\theta_i}_{\theta_i r}\, \partial_{\theta_i}.$$

In the last step, we have taken into account that the only non-zero Christoffel symbols involving the angles θ_i are $\Gamma^{\theta_i}_{\theta_i r} = \Gamma^{\theta_i}_{r\theta_i} = 1/r$, and $\Gamma^{r}_{\theta_i\theta_i} = -r$, and the result follows directly. The last eigenvector is obtained by imposing the orthogonality respect to N and each eigenvector ∂_{θ_i}, $1 \le i \le n-1$. $\square$

The differential operators S_k, $1 \le k \le n$, associated to the k-curvature of rotationally symmetric graphs in $\mathbb{R}^{n+1}_a$, can be written as follows,

$$S_k^+ \,:\, \{v \in C^2(\mathbb{R}^+) \,:\, v'(0) = 0\} \longrightarrow \mathbb{R},$$

$$S_k^+[v](r) = \begin{cases} \dfrac{1}{n\, r^{n-1}} \left(r^{n-k}\psi^k(v')\right)' & \text{in} \quad (0,\infty), \\ 0 & \text{in} \quad r = 0, \end{cases}$$

where $\psi(s) := \dfrac{s}{\sqrt{1+s^2}}$ in the case $a = 0$, and

$$S_k^- \,:\, \{v \in C^2(\mathbb{R}^+) \,:\, v'(0) = 0,\, |v'| < 1\} \longrightarrow \mathbb{R},$$

$$S_k^-[v](r) = \begin{cases} \dfrac{1}{n\, r^{n-1}} \left(r^{n-k}\phi^k(v')\right)' & \text{in} \quad (0,\infty), \\ 0 & \text{in} \quad r = 0, \end{cases}$$

where $\phi(s) := \dfrac{s}{\sqrt{1-s^2}}$ in the case $a = 1$.

Then, our first aim is to prove the existence of solutions of the equations

$$S_k^{\pm}[v](r) = H_k(v(r), r) \qquad r \in \mathbb{R}^+, \tag{6}$$

for a given prescription function $H_k : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$.

Note that, in general, these differential operators are not elliptic. Although we are interested in the existence of entire graphs, we also will deal with graphs defined over a ball $B_{\gamma(0)}(R) \subset \Pi$, with Dirichlet boundary conditions. This problem is not only a previous step to face the main purpose, but it has its own interest. Next two sections are devoted to this aim.

3 Existence results of the Dirichlet problem in Minkowski spacetime

Associated to equation (6), in the Minkowski spacetime $\mathbb{R}^{n+1}_1$, we can consider the corresponding Dirichlet problem on a ball with radius R contained in Π. Passing to polar coordinates, we get the following boundary value problem,

$$\begin{aligned} \left(r^{n-k}\phi^k(v')\right)' &= n\, r^{n-1}\, H_k(v(r), r) && \text{in} \quad (0,R), \\ |v'| &< 1 && \text{in} \quad (0,R), \\ v'(0) &= 0 = v(R), && \end{aligned} \tag{7}$$

where $\phi(s) := \dfrac{s}{\sqrt{1-s^2}}$ and $1 \le k \le n$.

It is easy to compute the profile of the rotationally symmetric graphs with constant k-th mean curvature. If H_k is constant (non negative if k is even), we can integrate directly equation (7) in order to obtain an hyperboloid,

$$v(r) = \sqrt{R^2 + H_k^{-2/k}} - \sqrt{r^2 + H_k^{-2/k}},$$

for each $1 \leq k \leq n$. On the other hand, if k is even and $H_k < 0$, it is easy to realize that (7) has no solution. By this reason, for a more general prescription of the curvature we need to distinguish two cases, depending if k is an even or odd natural number.

We fix some notation which will be used in the rest of the section. Let C be the Banach space of the real continuous functions in $[0, R]$, with the maximum norm, and C^1 the space of continuously differentiable functions with its usual norm $\|v\| = \|v\|_\infty + \|v'\|_\infty$. We write $B_{R,1} = \{v \in C^1 : \|v\|_\infty < R, \|v'\|_\infty < 1\}$.

3.1 Case 1: k odd

In this case, the existence problem is a straightforward application of the results exposed in [6] (see Proposition 2.4 therein), taking into account that $\phi^k : (-1, 1) \longrightarrow \mathbb{R}$ is also an increasing homeomorphism such that $\phi(0) = 0$. The result is enunciated as follows.

Proposition 3.1 *Let be k odd. Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a spacelike hyperplane $\Pi \subset \mathbb{L}^{n+1}$ orthogonal to a inertial observers vector field. For every rotationally symmetric (in the second argument) and continuous function $H_k : [-R, R] \times B_0(R) \subset \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, there exists at least one rotationally symmetric spacelike graph with k-curvature equal to H_k such that its boundary is in the hyperplane Π.*

3.2 Case 2: k even

When k is even, ϕ^k is not a homeomorphisms between $(-1, 1)$ and $\mathbb{R}$ and then, the results of [6] cannot be applied.

First of all, from equation (7) we have that

$$\big[\phi(v')\big]^k(r) = \frac{n}{r^{n-k}} \int_0^r s^{n-1} H_k\big(v(s), s\big) ds. \tag{8}$$

Hence, since k is an even number, we have that the previous integral term is non negative. Then, it is quite natural to impose the following condition on the mean k-curvature prescription function,

$$\int_0^r s^{n-1} H_k\big(v(s), s\big)\, ds \geq 0 \qquad \text{for all} \qquad r \in [0, R]\,, \quad v \in B_{R,1}. \tag{9}$$

Note that condition (1) implies in particular condition (9) for any $R > 0$. Our first step is to construct a *fixed point operator* $\mathcal{A}$ such that its fixed points are solutions of to problem (7). We start by defining

$$K : C^1 \longrightarrow C^1,$$
$$K(v)(r) = \int_r^R v(t) dt,$$

$$S : C \longrightarrow C^1,$$
$$S(v)(r) = \frac{n}{r^{n-k}} \int_0^r t^{n-1} v(t)dt \quad (r \in (0, R]), \quad S(v)(0) = 0.$$

Besides, consider the Nemytskii operator associated to H_k,

$$N_{H_k} : B_{R,1} \subset C^1 \longrightarrow C, \quad N_{H_k}(v) = H_k(\cdot, v).$$

Obviously, N_{H_k} is continuous and $N_F(\overline{B}_{R,1})$ is a bounded subset of C. Finally, we define the operator

$$\mathcal{A} : \overline{B}_{R,1} \subset C^1 \longrightarrow C^1, \quad \mathcal{A} = K \circ (\phi^{-1})^{1/k} \circ S \circ N_F, \tag{10}$$

where $(\phi^{-1})^{1/k} : \mathbb{R}^+ \longrightarrow [0, 1)$ means the (positive) k-root composed with the inverse of ϕ, i.e., $(\phi^{-1})^{1/k}(s) = \phi^{-1}(s^{1/k})$. Note that $\mathcal{A}$ is well-defined thanks to condition (9).

Note that $\mathcal{A}$ is a composition of continuous operators, hence it is continuous. Moreover, from the compactness of K, $\mathcal{A}$ is a compact and continuous operator. Note that the image of the operator $\mathcal{A}$ is contained in $C^2[0, R]$, so the fixed points (solutions of the equation (7)) will be of class C^2.

Fixed points of $\mathcal{A}$ always verify the restrictions $v'(0) = v(R) = 0$, in consequence we can consider the Banach subspace $\widehat{C}^1 \subset C^1$ of the functions that satisfy these boundary conditions. Let us define the set

$$\widehat{B}_{R,1} = \{v \in \overline{B}_{R,1} : \, v'(0) = 0 = v(R)\}.$$

A straightforward checking shows that if a function $v \in \widehat{C}^1$ is a fixed point of the nonlinear compact operator (10), then v is a solution of equation (7).

More explicitly, operator $\mathcal{A}$ can be written as

$$\mathcal{A}(v)(r) = -\int_r^R \phi^{-1} \left[\left(\frac{n}{s^{n-k}} \int_0^s \tau^{n-1} H_k(\tau, v(\tau)) d\tau \right)^{1/k} \right] ds,$$

and its derivative is

$$\big(\mathcal{A}(v)\big)'(r) = \phi^{-1} \left[\left(\frac{n}{r^{n-k}} \int_0^r \tau^{n-1} H_k(\tau, v(\tau)) d\tau \right)^{1/k} \right].$$

By using that $\phi^{-1}(\mathbb{R}^+) = [0, 1)$, one gets

$$\|\big(\mathcal{A}(v)\big)'\|_\infty < 1 \quad \text{and} \quad \|\mathcal{A}(v)\|_\infty < R \quad \text{for all} \quad v \in B_{R,1}. \tag{11}$$

Such inequalities imply that $\mathcal{A}(\widehat{B}_{R,1}) \subset \widehat{B}_{R,1}$. Since $\widehat{B}_{R,1}$ is closed and contractible to a point, and $\mathcal{A}$ (restricted to $\widehat{B}_{R,1}$) is a continuous and compact operator, the Schauder Point Fixed theorem applies, leading to the following result.

Proposition 3.2 *Assume condition* (9) *over the prescription function* H_k. *Then, problem* (7) *has at least one radially symmetric solution.*

Note that the solution given in previous result satisfies

$$v'(r) = \phi^{-1}\left[\left(\frac{n}{r^{n-k}}\int_0^r \tau^{n-1}H_k(\tau, v(\tau))d\tau\right)^{1/k}\right] \geq 0,$$

then, v is increasing and negative.

Nevertheless, it is possible to obtain a second solution of (7) by taking the negative k-root in equality (8) and proceeding in the same way. In this second case, the solution is decreasing and positive. Moreover, one solution is not the symmetric respect to the hyperplane Π of the other one, except when $H_k(t, r) = H_k(-t, r)$ for all $t \in [-R, R]$ and $r \in [0, R]$.

Summarizing, we have proved the following result.

Proposition 3.3 *Let be k even. Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a spacelike hyperplane $\Pi \subset \mathbb{L}^{n+1}$ orthogonal to a inertial observers vector field. For every rotationally symmetric and continuous function $H_k : [-R, R] \times B_0(R) \subset \mathbb{L}^{n+1} \longrightarrow \mathbb{R}$, satisfying (9) such that $H_k(0, \cdot) \not\equiv 0$, there exist at least two different rotationally symmetric spacelike graphs with k-curvature equal to H_k such that its boundary is in the hyperplane Π. One is above and the other one below the hyperplane Π.*

4 Existence results of the Dirichlet problem in Euclidean space

In the Euclidean ambient, the prescribed k-th mean curvature equation for a rotationally symmetric graph $\Sigma_v \subset \mathbb{R}^n$ with Dirichlet boundary conditions is written as

$$\begin{aligned} \left(r^{n-k}\psi^k(v')\right)' &= n\, r^{n-1}\, H_k(v(r), r) \qquad \text{in} \quad (0, R), \\ v'(0) &= 0 = v(R), \end{aligned} \tag{12}$$

where $\psi(s) := \dfrac{s}{\sqrt{1+s^2}}$ and $1 \leq k \leq n$.

From $\psi(\mathbb{R}) = (-1, 1)$, we immediately note that k-th mean curvature function along the graph must satisfy the inequality

$$\left|\int_0^r s^{n-1}H_k\big(v(s), s\big)\, ds\right| < \frac{r^{n-k}}{n} \qquad \text{for all} \qquad r \in [0, R]. \tag{13}$$

Analogously to the Minkowski case, if H_k is a constant (non negative if k is even) satisfying $H_k \leq R^{-k}$, a straight integration of (12) gives

$$v(r) = \sqrt{H_k^{-2/k} - R^2} - \sqrt{H_k^{-2/k} - r^2}, \tag{14}$$

for each $1 \leq k \leq n$. On the other hand,for $H_k > R^{-k}$ the inequality (13) means that (12) has no solution. This fact suggests that, contrarily to the Minkovski case, in order to find solutions one has to impose a restriction on the size of the prescribed curvature function. This is a common feature of the Euclidean ambient. The following results are based on the analysis performed in [6].

4.1 Case 1: k odd

The following result for k odd is proved by adapting the proof of [6, Proposition 2.5] applied to (12), due to the fact that $\psi^k : \mathbb{R} \longrightarrow (-1,1)$ is an increasing homeomorphism and $\psi(0)=0$. The result is picked up in the following theorem.

Theorem 4.1 *Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$, and let $H_k : \mathbb{R}\times B_0(R) \subset \mathbb{R}^{n+1} = \mathbb{R}\times\Pi \longrightarrow \mathbb{R}$ (k odd) be a rotationally symmetric and continuous function such that, for some $0<\alpha<R^{-k}$, satisfies*

$$|H_k(t,r)| \le \alpha \qquad \text{for all} \qquad r\in[0,R]\,, \quad t\in[-R\,\beta\,, R\,\beta],$$

where $\beta := \psi^{-1}(R\alpha^{1/k})$. Then, there exists at least one rotationally symmetric graph with k-curvature equal to H_k such that its boundary is in the hyperplane Π.

Proof. Denote by $\Omega_\alpha := [-R\,\beta\,, R\,\beta]$. We show that

$$\mathcal{B}(\Omega_\alpha) \subset \Omega_\alpha, \tag{15}$$

where $\mathcal{B}$ is the same operator than $\mathcal{A}$ but replacing ϕ by ψ. Let $u\in\Omega_\alpha$ and $v=\mathcal{B}(u)$. By using the assumption $|H_k|\le\alpha$, we have

$$\big|(\psi(v'(r)))^k\big| = \big|\frac{n}{r^{n-k}}\int_0^r t^{n-1}H_k(u(t),t)dt\big| \le \alpha R^k$$

for all $r\in(0,R]$, and the hypothesis $\alpha<R^{-k}$ ensures that the image is less than 1.
Since $\psi^k(v'(0))=0$, and $\psi^k : \mathbb{R}\longrightarrow(-1,1)$ is an homeomorphism, it follows that

$$v'(r) \in [-\psi^{-1}(R\alpha^{1/k}), \psi^{-1}(R\alpha^{1/k})].$$

Therefore, $v(r)\in\Omega_\alpha$, and (15) is proved. Now, using the fact that Ω_α is a closed convex set in $\widehat{C}^1$ invariant by the compact operator $\mathcal{B}$, from the Schauder fixed point theorem, we conclude that there exists $u\in\Omega_\alpha$ such that $\mathcal{B}(u)=u$, which is a solution of the initial Dirichlet problem. □

4.2 Case 2: k even

Our aim here consist in to construct a fixed point operator associated to the equation (12). In order to do this, we take $0<\alpha<R^{-k}$ and we name $\beta := \psi^{-1}(R\alpha^{1/k})$.

As in Minkowski setting, we need to impose some restriction on the prescription function H_k,

$$\int_0^r s^{n-1}H_k\big(v(s),s\big)\,ds \ge 0 \qquad \text{for all} \qquad r\in[0,R]\,, \quad v\in B_{R\beta,\beta}. \tag{16}$$

In addition, as in the case with k odd, we introduce a boundedness assumption on H_k,

$$|H_k(t,r)| \le \alpha \qquad \text{for all} \qquad r\in[0,R]\,, \quad t\in[-R\,\beta, R\,\beta]. \tag{17}$$

Now, we can proceed in the same way that in the Minkowski setting, using the same notation, and define the operator

$$\mathcal{B} : \overline{B}_{R\beta,\beta} \subset C^1 \longrightarrow C^1, \quad \mathcal{B} = K\circ(\psi^{-1})^{1/k}\circ S\circ N_F, \tag{18}$$

where $(\psi^{-1})^{1/k} : [0,1) \longrightarrow \mathbb{R}^+$ is defined by $(\psi^{-1})^{1/k}(s) := \psi^{-1}(+s^{1/k})$.

Explicitly,

$$\mathcal{B}(v)(r) = -\int_r^R \psi^{-1}\left[\left(\frac{n}{s^{n-k}}\int_0^s \tau^{n-1}H_k(\tau, v(\tau))d\tau\right)^{1/k}\right] ds,$$

Note that $\mathcal{B}$ is well defined due to (16) and (2). Now, restricting $\mathcal{B}$ to the subset $\widehat{B}_{R\beta,\beta} := \{v \in B_{R\beta,\beta} \;:\; v'(0) = v(R) = 0\}$.

Again, if a function $v \in \widehat{C}^1$ is a fixed point of the nonlinear compact operator (18), then v is a solution of equation (12). Following the arguments of the Minkowski setting, with k even, we conclude that there exist one increasing solution and other decreasing solution. We may enunciate this result,

Proposition 4.2 *Let $B_0(R)$ be an Euclidean ball centered at 0 with radius R contained in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$. For every rotationally symmetric and continuous function $H_k : \mathbb{R} \times B_0(R) \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, ($k$ even), satisfying (16) and (2) such that $H_k(0,\cdot) \not\equiv 0$, there exists at least two different rotationally symmetric graph with k-curvature equal to H_k such that its boundary is in the hyperplane Π. One is above and the other one below the hyperplane Π.*

5 Uniqueness results

It is possible to ensure the uniqueness of certain rotationally symmetric solutions of equation (7) under some hypothesis on the prescription function. As before, these results depend again of the parity of k, but the treatment will be the same in the Minkowski and the Euclidean cases. Therefore, we will denote both, $\dfrac{s}{\sqrt{1-s^2}}$ and $\dfrac{s}{\sqrt{1+s^2}}$, by $\chi(s)$.

Proposition 5.1 (*Case k odd*) *If $H_k(\cdot, r)$ is a non decreasing prescription function for each fixed $r \in [0,R]$, then equation*

$$\begin{aligned} \left(r^{n-k}\chi^k(v')\right)' &= n\, r^{n-1}\, H_k(v(r), r) \qquad \text{in} \quad (0,R), \\ v'(0) &= 0 = v(R), \end{aligned} \tag{19}$$

has at most one solution.

Proof. Suppose that u and v are different solutions of equation (7). Since $u(R) = v(R) = 0$, the set $F = \{r \in [0,R] : u'(r) \neq v'(r)\}$ has positive measure. Multiplying by $(u - v)$ the identity

$$\left[r^{n-k}\left(\chi^k(u') - \chi^k(v')\right)\right]' = nr^{n-1}\left[H_k(u(r), r) - H_k(v(r), r)\right]$$

and integrating over $[0,R]$, and using the boundary conditions we have

$$\begin{aligned} &-\int_F \left[\chi^k(u'(r)) - \chi^k(v'(r))\right]\left[u'(r) - v'(r)\right] r^{n-k} dr \\ &= n\int_0^R r^{n-1}\left[H_k(u(r), r) - H_k(v(r), r)\right]\left[u(r) - v(r)\right] dr. \end{aligned} \tag{20}$$

From the increasing character of χ^k, the first term is strictly negative, while the second one is non negative due to the increasing assumption over H_k. This is a contradiction and the result follows. $\square$

Now we deal with the case with k even. The proof of the following proposition is similar to the previous one, but taking into account that χ^k is only increasing on the positive real numbers in which it is defined.

Proposition 5.2 (*Case k even*)

- *If $H_k(\cdot, r)$ is a non decreasing prescription function for each fixed $r \in [0, R]$, then equation* (19) *has at most one increasing solution.*
- *If $H_k(\cdot, r)$ is a non increasing prescription function for each fixed $r \in [0, R]$, then equation* (19) *has at most one decreasing solution.*

6 Extendibility of the solutions as entire graphs

In order to end the proof of Theorems A,B,C and D, it suffices to guarantee that every solution v, given by Theorems 3.1, 3.3, 4.1, 4.2, once R is fixed, can be continued until $+\infty$ as a solution of equations (6). We need the following lemma.

Lemma 6.1 *Every solution $v \in C^2[0, \varrho]$ of* (7) *verifies that $|v'| < 1$ on $[0, \varrho]$. Analogously, each solution $v \in C^2[0, \varrho]$ of* (12) *satisfies that $|v'| < +\infty$ on $[0, \varrho]$.*

Proof. From (19), we have

$$v'(r) = \chi^{-1}\left[\left(\frac{n}{r^{n-k}}\int_0^r \tau^{n-1} H_k(\tau, v(\tau))d\tau\right)^{1/k}\right],$$

and, taking into account (3) in the Euclidean case, the result follows immediately. $\square$

Remark 6.2 Graphs defined by the solution of Equation (7) are spacelike on the open ball. However, there could exist solutions which are of lightlike on the boundary, ∂B. The previous lemma ensures a priori that each possible solution v of (7) is spacelike on the boundary too.

The rest of the proof does not essentially depend on the ambient space (Euclidean or Minkowski), thus we follow with the notation of the previous section. However, we have to distinguish odd and even cases again.

First, assume that k is odd. Let v be a solution of equation (19), and let $[0, b[$ be the maximal interval of definition of v. Suppose that $b < +\infty$. We can rewrite equation (19) as a system of two ordinary differential equations of first order

$$v'(r) = \chi^{-1}\big[\big(\frac{z(r)}{r^{n-k}}\big)^{1/k}\big]$$
$$z'(r) = n\, r^{n-1} H(v(r), r),$$

which we can abbreviate

$$\begin{bmatrix} v' \\ z' \end{bmatrix} = \mathcal{F}\big(r, v, z\big),$$

where $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R} \times J \longrightarrow \mathbb{R}^2$, and J is $\mathbb{R}$ or $(-b^{n-k}, b^{n-k})$ if the ambient is Minkowski or Euclidean space respectively.

By the standard prolongability theorem of ordinary differential equations (see for instance [24, Section 2.5]), we have that the graph $\{(r, v(r), z(r)) \;:\; r \in [R/2, b[\}$ goes out of any compact subset of $\mathbb{R}^+ \times \mathbb{R} \times J$. However, by Lemma 6.1, $|v'(r)| < \rho$ (of course, ρ depends on the chosen solution v), then $|v(r)| < b\rho$. Therefore, the graph can not go out of the compact subset $[R/2\,, b] \times [-b\rho, b\rho] \times [-b^{n-k}\chi^k(\rho), b^{n-k}\chi^k(\rho)]$ contained in the domain of $\mathcal{F}$. This is a contradiction, then $b = +\infty$.

If k is even, we know that at least there exist one increasing and one decreasing solutions of equation (19). For instance, let v be a increasing solution (the argument of the proof is similar for a decreasing solution), and let $b < +\infty$ its maximal interval of definition. In this way, $v'(r) > 0$ for all $r \in (0, R]$. Moreover, if condition (1) (or (3) in the Euclidean setting) holds, then extension of v will be also increasing on $(0, b)$. Hence, v and z, where $z(r) := r^{n-k}\chi^k(v'(r))$, verify the ODE system (21), taking the positive k-root in the first equation. From this point, the proof continues being the same that case k odd, and we deduce that v can be extended to $+\infty$ as an increasing solution of (6).

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