Periodic motions of fluid particles induced by a prescribed vortex path in a circular domain

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Abstract

By means of a generalized version of Poincaré-Birkhoff theorem, we prove the existence and multiplicity of periodic solutions for a hamiltonian system modeling the evolution of advected particles in a two-dimensional ideal fluid inside a circular domain and under the action of a point vortex with prescribed periodic trajectory.

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1 Introduction and main result

We consider the motion of a two-dimensional ideal fluid in a circular domain of radius $R > 0$ subjected to the action of a moving point vortex whose position, denoted as $z(t)$, is a prescribed T-periodic function of time. This model plays an important role in Fluid Mechanics as an idealized model of the stirring of a fluid inside a cylindrical

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tank by an agitator. A fundamental reference for this problem is the seminal paper [1], where the concept of *chaotic advection* was coined. Following the classical Lagrangian representation, the mathematical model under consideration is the planar system

$$\dot{\zeta} = \frac{\Gamma}{2\pi i}\left(\frac{|z(t)|^2 - R^2}{(\zeta - z(t))(\zeta\overline{z}(t) - R^2)}\right), \tag{1}$$

where the complex variable ζ represents the particle transport induced by the so-called *stirring protocol* $z(t)$. System (1) is a T-periodically forced planar system with hamiltonian structure, where the stream function

$$\Psi(t,\zeta) = \frac{\Gamma}{2\pi}\ln\left|\frac{\zeta - z(t)}{\overline{z}(t)\zeta - R^2}\right|$$

plays the role of the hamiltonian.

The main contribution of Aref in [1] was to show that the flow may experience regular or chaotic regimes depending on the particular stirring protocol. For instance, system (1) is integrable if $z(t)$ is constant or $z(t) = z_0 \exp(i\Omega t)$ but it is chaotic if $z(t)$ is piecewise constant (blinking protocol in the related literature). A naive way to measure the influence of the ideas presented in [1] is to note the more than a thousand citations of this inspiring paper up to the date. Aref's blinking protocol is piecewise integrable and the theory of linked twist maps permits a good analytical study of the underlying dynamics (see for instance [4, 9]). More recently, other strategies of stirring have been studied, for instance the figure-eight or the epitrochoidal protocol [8], but only from a numerical point of view. Our contribution in this paper is to prove that both regular and chaotic regimes share a common dynamical feature, namely

the existence of an infinite number of periodic solutions labeled by the number of revolutions around the vortex in the course of a period.

To be precise, let us fix $z : \mathbb{R} \to \mathbb{C}$ a T-periodic function such that $|z(t)| < R$ for all t. For a periodic solution ζ of (1) with period kT, the *winding number* of ζ is defined as

$$\mathrm{rot}_{kT}(\zeta) = \frac{1}{2\pi i}\int_0^{kT} \frac{d(\zeta(t) - z(t))}{\zeta(t) - z(t)}$$

and provides the number of revolutions of $\zeta(t)$ around the vortex point $z(t)$ in the time interval $[0, kT]$. We proceed to state our main result.

Theorem 1.1. *Let $z : \mathbb{R} \to \mathbb{C}$ be a T-periodic function of class C^1, such that $|z(t)| < R$ for all t. Then, for every integer $k \geq 1$, system (1) has infinitely many kT-periodic solutions lying in the disk $\mathcal{B}_R(0)$. More precisely, for every integer $k \geq 1$, there exists an integer j_k^* such that, for every integer $j \geq j_k^*$, system (1) has two kT-periodic solutions $\zeta_{k,j}^{(1)}(t)$, $\zeta_{k,j}^{(2)}(t)$ such that, for $i = 1, 2$,*

$$\|\zeta_{k,j}^{(i)}\|_\infty \leq R \quad \text{and} \quad \mathrm{rot}_{kT}(\zeta_{k,j}^{(i)}) = j. \tag{2}$$

Moreover, for every $k \geq 1$, $j \geq j_k^*$ and $i = 1, 2$,

$$\lim_{j \to +\infty} |\zeta_{k,j}^{(i)}(t) - z(t)| = 0, \qquad \textit{uniformly in } t \in [0, kT]. \tag{3}$$

In particular, for $k = 1$, we find that (1) has infinitely many T-periodic solutions. For $k > 1$, we find subharmonic solutions of order k (i.e., kT-periodic solutions which are not lT-periodic for any $l = 1, \dots, k-1$) provided that j and k are relatively prime integers; we remark that in this case it is also possible to show that $\zeta_{k,j}^{(1)}(t)$, $\zeta_{k,j}^{(2)}(t)$ are not in the same periodicity class (namely, $\zeta_{k,j}^{(1)}(\cdot) \not\equiv \zeta_{k,j}^{(2)}(\cdot + lT)$ for every integer $l = 1, \dots, k-1$).

As a final remark, it is worth to point out that the regularity condition on the stirring protocol plays an important role. In fact, Theorem 1.1 is not true for a discontinuous $z(t)$ (e.g. the blinking protocol), because condition (3) would imply unphysical discontinuous particle trajectories. The existence and multiplicity of periodic solutions for a general protocol, as well as their stability properties, remains as an open problem. Intuitively, a vortex induces a singularity on the angular variable, twisting the flux around it, so Poincaré-Birkhoff Theorem becomes a natural tool of potential application in more general contexts like arbitrary boundary domains [6, 10] or the presence of multiple vortices [2, 3]. Such extension will be the subject of future works.

The rest of the paper is divided in two parts. In Section 2 the Poincaré section is defined, whereas Section 3 contains the proof of Theorem 1.1 by an application of a generalized version of Poincaré-Birkhoff Theorem.

2 Definition of the Poincaré section.

For our purposes, it is convenient to write system (1) as

$$\dot{\bar{\zeta}} = \frac{\Gamma}{2\pi i} \left(\frac{1}{\zeta - z(t)} - \frac{1}{\zeta - \frac{R^2}{|z(t)|^2} z(t)} \right). \tag{4}$$

In this form, the first term at the right models the action of the vortex whereas the second term corresponds to the wall influence on the flow. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ and setting $\zeta = (x, y)$, $z(t) = (a(t), b(t))$, we can rewrite system (4) in real notation as

$$\begin{cases} \dot{x} = \dfrac{\Gamma}{2\pi} \left(-\dfrac{y - b(t)}{|\zeta - z(t)|^2} + \dfrac{y - \frac{R^2}{|z(t)|^2} b(t)}{\left| \zeta - \frac{R^2}{|z(t)|^2} z(t) \right|^2} \right) \\[2ex] \dot{y} = \dfrac{\Gamma}{2\pi} \left(\dfrac{x - a(t)}{|\zeta - z(t)|^2} - \dfrac{x - \frac{R^2}{|z(t)|^2} a(t)}{\left| \zeta - \frac{R^2}{|z(t)|^2} z(t) \right|^2} \right), \end{cases} \qquad \zeta = (x, y) \in \mathbb{R}^2. \tag{5}$$

Let $\mathcal{B}_R \subset \mathbb{R}^2$ be the closed disk centered at the origin with radius R. First, we recall a well known property of system (5).

Lemma 2.1. *Let $\zeta : J \to \mathbb{R}^2$ be a solution of* (5)*, with $J \subset \mathbb{R}$ its maximal interval of definition. If $|\zeta(t_0)| \leq R$ for some $t_0 \in J$, then $|\zeta(t)| \leq R$ for every $t \in J$, that is to say, the disk $\mathcal{B}_R$ is invariant for the flow associated to* (5)*.*

Proof. Since $\mathcal{B}_R = \{(x,y) \in \mathbb{R}^2 \mid V(x,y) \leq R^2\}$ for $V(x,y) = x^2 + y^2$, by standard result of flow-invariant sets, it is enough to prove that

$$\langle Z(t,x,y) | \nabla V(x,y) \rangle = 0, \qquad \text{for every } t \in [0,T],\ x^2 + y^2 = R^2,$$

where $Z(t,x,y)$ denotes the vector field of the differential system (5). With simple computations, we find indeed

$$\begin{aligned}
\langle Z(t,x,y) | \nabla V(x,y) \rangle &= \frac{1}{2}\Big(X(t,x,y)x + Y(t,x,y)y\Big) \\
&= \frac{\Gamma}{\pi}\Big(b(t)x - a(t)y\Big)\left(\frac{\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2 - \frac{R^2}{|z(t)|^2}|\zeta - z(t)|^2}{|\zeta - z(t)|^2 \left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\
&= \frac{\Gamma}{\pi}\Big(b(t)x - a(t)y\Big)\left(\frac{\left(1 - \frac{R^2}{|z(t)|^2}\right)\left(|\zeta|^2 - \frac{R^2}{|z(t)|^2}|z(t)|^2\right)}{|\zeta - z(t)|^2 \left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\
&= 0.
\end{aligned}$$

□

From now on, we will study solutions to system (5) belonging to the invariant disk $\mathcal{B}_R$; accordingly, the singularity of the vector field at $\zeta = \frac{R^2}{|z(t)|^2}z(t)$ (for which $|\zeta| > R$) will not play any role. On the contrary, we will take advantage of the singularity at $\zeta = z(t)$. To this aim, it is useful to introduce the change of variable

$$\eta = \zeta - z(t)$$

and set $\eta = (u,v)$, so that system (5) is transformed into

$$\begin{cases}
\dot{u} = \dfrac{\Gamma}{2\pi}\left(-\dfrac{v}{|\eta|^2} + \dfrac{v + b(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|\eta + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) - \dot{a}(t) \\[3ex]
\dot{v} = \dfrac{\Gamma}{2\pi}\left(\dfrac{u}{|\eta|^2} - \dfrac{u + a(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|\eta + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) - \dot{b}(t),
\end{cases} \qquad \eta = (u,v) \in \mathbb{R}^2. \tag{6}$$

In the following, given $\eta_0 \neq 0$, we will denote by $\eta(\cdot;\eta_0)$ the unique solution of (6) satisfying the initial condition $\eta(0) = \eta_0$.

Lemma 2.2. *There exists $r > 0$ such that, if $0 < |\eta_0| \leq r$, then the solution $\eta(\cdot;\eta_0)$ exists on $\mathbb{R}$ and satisfies $|\eta(t;\eta_0) + z(t)| \leq R$, for every $t \in \mathbb{R}$.*

Proof. Define

$$r = R - |z(0)| > 0.$$

Then, for $0 < |\eta_0| \leq r$, the function $\zeta(t) = \eta(t;\eta_0) + z(t)$ solves (5) and

$$|\zeta(0)| \leq |\eta_0| + |z(0)| \leq r + |z(0)| = R.$$

From Lemma 2.1, we have the a priori bound

$$|\eta(t;\eta_0) + z(t)| \leq R, \qquad \text{for every } t \in J, \tag{7}$$

where $J \subset \mathbb{R}$ denotes the maximal interval of definition of $\eta(t;\eta_0)$. Our objective is to show that actually $J = \mathbb{R}$, completing the proof of the lemma. Notice that, in view of the a priori bound (7), we just have to show that $\eta(t;\eta_0)$ cannot reach the singularity $\eta = 0$ in finite time. First, we are going to consider the particular case of $z(t) = a(t), b(t))$ belonging to the C^2 class, then the general case is proved by a standard limiting argument.

Define the function (to simplify the notation, we take advantage here of both real and complex notation)

$$K(t,\eta) = \frac{\Gamma}{2\pi}\Big(\ln|\eta| - \ln\left|\overline{z}(t)(\eta + z(t)) - R^2\right|\Big) + \dot{a}(t)v - \dot{b}(t)u$$

and set $k(t) = K(t, \eta(t;\eta_0))$ for $t \in J$. Since $K(t,\eta)$ is a hamiltonian function for (6), we have

$$\langle \nabla K_\eta(t, \eta(t;\eta_0)) | \eta'(t,\eta_0))\rangle = 0,$$

so that (writing for simplicity $\eta(t;\eta_0) = \eta(t)$),

$$\begin{aligned}
|k'(t)| &= \left|\frac{\partial K}{\partial t}(t, \eta(t;\eta_0))\right| \\
&= \left| -\frac{\Gamma}{2\pi}\frac{\left\langle \overline{z}(t)(\eta + z(t)) - R^2 \middle| \gamma(t)\right\rangle}{\left|\overline{z}(t)(\eta + z(t)) - R^2\right|^2} + \ddot{a}(t)v(t) - \ddot{b}(t)u(t)\right| \\
&\leq \frac{\Gamma}{2\pi}\frac{|\gamma(t)|}{\left|\overline{z}(t)(\eta + z(t)) - R^2\right|} + |\ddot{a}(t)v(t) - \ddot{b}(t)u(t)|,
\end{aligned}$$

being $\gamma(t) = \overline{z}'(t)\eta(t) + 2\langle z(t)|z'(t)\rangle$. From the a priori bound (7) one gets

$$\begin{aligned} \Big|\overline{z}(t)(\eta + z(t)) - R^2\Big| &\geq R^2 - \Big|\overline{z}(t)(\eta(t) + z(t))\Big| \\ &\geq R\Big(R - |\overline{z}(t)|\Big) > 0, \end{aligned} \tag{8}$$

so there exists $M > 0$ (independent on η_0) such that $|k'(t)| \leq M$ for every $t \in J$. Hence,

$$|K(t, \eta(t)) - K(0, \eta_0)| \leq M|t|, \quad \text{for every } t \in J. \tag{9}$$

Since $K(t, \eta)$ is unbounded near $\eta = 0$, this shows that $\eta(t)$ cannot reach the singularity in finite time, thus concluding the proof. For the general C^1 case, one can approach uniformly $z(t)$ by C^1 functions, and the result follows from the continuous dependence of the solutions of the initial value problem with respect to parameters. □

Fix now an integer $k \geq 1$. We can then define the Poincaré map Ψ_k at time kT as

$$\mathcal{B}_r \setminus \{0\} \ni \eta_0 \mapsto \Psi_k(\eta_0) = \eta(kT; \eta_0).$$

By the fundamental theory of ODEs, it turns out that Ψ_k is a global homeomorphism of $\mathcal{B}_r \setminus \{0\}$ onto $\Psi_k(\mathcal{B}_r \setminus \{0\})$, preserving area and orientation; moreover, from (9) we see that Ψ_k can be extended (as an area and orientation preserving homeomorphism) to the whole disc $\mathcal{B}_r$ by setting $\Psi_k(0) = 0$.

3 Proof of the main result.

By Section 2, for any integer $k \geq 1$ there exists a well-defined homeomorphism $\Psi_k : \mathcal{B}_r \to \Psi_k(\mathcal{B}_r)$ preserving area and orientation. Moreover, $\Psi_k(0) = 0$. For the reader's convenience, we recall here the generalized version of Poincaré-Birkhoff theorem which we are going to apply (see [5, 7]).

Generalized Poincaré-Birkhoff theorem. *Let $0 < r_i < r_o$ and set $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \mid r_i^2 \leq x^2 + y^2 \leq r_o^2\}$. Let $\Psi : \mathcal{B}_{r_o} \to \Psi(\mathcal{B}_{r_o})$ be an area-preserving homeomorphism with $\Psi(0) = 0$. Assume that, on the universal covering space $\{(\rho, \theta) \in \mathbb{R}^2 \mid \rho > 0\}$ with covering projection $\Pi(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta)$, $\Psi|_{\mathcal{A}}$ has a lifing of the form*

$$\widetilde{\Psi}(\rho, \theta) = (R(\rho, \theta), \theta + \gamma(\rho, \theta)),$$

being $R(\rho, \theta), \gamma(\rho, \theta)$ continuous functions 2π-periodic in the second variable. Finally, suppose that, for a suitable $j \in \mathbb{Z}$, the twist condition

$$\gamma(r_i, \theta) > 2\pi j \quad \text{and} \quad \gamma(r_o, \theta) < 2\pi j, \quad \text{for every } \theta \in \mathbb{R},$$

is fulfilled. Then there exist two distinct points $(\rho^{(1)}, \theta^{(1)}), (\rho^{(2)}, \theta^{(2)}) \in]r_i, r_e[\times [0, 2\pi[$ such that (for $i = 1, 2$) $\widetilde{\Psi}(\rho^{(i)}, \theta^{(i)}) = (\rho^{(i)}, \theta^{(i)} + 2\pi j)$.

To apply this theorem, we therefore write

$$\eta(t) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t)), \qquad \rho(t) > 0,$$

transforming system (6) into

$$\begin{cases} \dot\rho = I(t,\rho,\theta) \\ \dot\theta = \Theta(t,\rho,\theta), \end{cases} \tag{10}$$

being

$$I(t,\rho,\theta) = \frac{\Gamma}{2\pi}\left(\frac{(b(t)\cos\theta - a(t)\sin\theta)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|(\rho\cos\theta, \rho\sin\theta) + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) - \dot a(t)\cos\theta - \dot b(t)\sin\theta$$

$$\Theta(t,\rho,\theta) = \frac{\Gamma}{2\pi}\left(\frac{1}{\rho^2} - \frac{\rho + (a(t)\cos\theta + b(t)\sin\theta)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\rho\left|(\rho\cos\theta, \rho\sin\theta) + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) + \frac{\dot a(t)\sin\theta - \dot b(t)\cos\theta}{\rho}.$$

We denote by $(\rho(\cdot;\rho_0,\theta_0), \theta(\cdot;\rho,\theta_0))$ the unique solution to (10) satisfying the initial condition $(\rho(0),\theta(0)) = (\rho_0,\theta_0)$. In view of Lemma 2.2, such solutions globally exists (and $\rho(t) \neq 0$) if $\rho_0 \in\,]0,r]$.

Define $j_k^* \geq 1$ as the smallest integer such that

$$\theta(kT; r, \theta_0) - \theta(0; r, \theta_0) < 2\pi j_k^*, \qquad \text{for every } \theta_0 \in [0, 2\pi[\,. \tag{11}$$

Fix now an integer $j \geq j_k^*$; we claim that there exists $r_j \in\,]0,r[$ such that

$$\theta(kT; r_j, \theta_0) - \theta(0; r_j, \theta_0) > 2\pi j, \qquad \text{for every } \theta_0 \in [0, 2\pi[\,. \tag{12}$$

Indeed, arguing similarly as in (8) we see that

$$\left|(\rho\cos\theta, \rho\sin\theta) + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2$$

is bounded away from zero for $\rho \in\,]0,r]$; accordingly, we can find $\hat{r_j} \in\,]0,r[$ such that

$$\Theta(t,\rho,\theta) > \frac{2\pi j}{kT}, \qquad \text{for every } t \in \mathbb{R},\ \rho \in\,]0, \hat{r_j}],\ \theta \in \mathbb{R}. \tag{13}$$

Using a standard compactness argument (usually referred to as “elastic property”) we can find $r_j \in\,]0, \hat{r_j}[$ such that

$$\rho_0 \in\,]0, r_j] \Longrightarrow \rho(t;\rho_0,\theta_0) \leq \hat{r_j}, \qquad \text{for every } t \in [0,kT],\ \theta_0 \in [0,2\pi[\,.$$

Hence (12) follows from (13), after integrating the second equation in (10).

In view of (11) and (12), the Poincaré-Birkhoff fixed point theorem implies the existence of at least two distinct points $(\rho^{(1)}_{k,j}, \theta^{(1)}_{k,j})$, $(\rho^{(2)}_{k,j}, \theta^{(2)}_{k,j}) \in]r_j, r[\, \times [0, 2\pi[$ such that, for $i = 1, 2$,

$$\rho(kT; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j}) = \rho(0; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j}), \quad \theta(kT; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j}) = \theta(0; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j}) + 2\pi j. \tag{14}$$

Accordingly,

$$\zeta^{(i)}_{k,j}(t) = \eta(t; (\rho^{(i)}_{k,j} \cos \theta^{(i)}_{k,j}, \rho^{(i)}_{k,j} \sin \theta^{(i)}_{k,j})) + z(t)$$

is a kT-periodic solution to (5) such that, in view of Lemma 2.2, $\|\zeta^{(i)}_{k,j}\|_\infty \le R$.

The second relation in (2) is just a consequence of (14), using complex notation. Indeed, $\zeta^{(i)}_{k,j}(t) - z(t) = \rho(t; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j}) e^{i\theta(t; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j})}$ so that, with easy computations,

$$\begin{aligned} \mathrm{rot}_{kT}(\zeta^{(i)}_{k,j}) &= \frac{1}{2\pi i} \int_0^{kT} \frac{d(\zeta^{(i)}_{k,j}(t) - z(t))}{\zeta^{(i)}_{k,j}(t) - z(t)} \\ &= \frac{1}{2\pi i} \int_0^{kT} \left(\frac{d}{dt} \Big(\log(\rho(t; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j})) \Big) + i\theta'(t; \rho^{(i)}_{k,j}, \theta^{(i)}_{k,j}) \right) dt = j. \end{aligned}$$

This information finally implies, by using a standard compactness argument, that (3) holds true.

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