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STABILITY OF PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS WITH LOW DIMENSION

Abstract. These notes were prepared for a course on stability of periodic orbits (Torino, November 2015). The contents are structured in six lessons and most of them are taken from previous publications. A new result is presented in the last lesson.

1. Introduction

Consider a Hamiltonian system with two degrees of freedom

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1,2 \tag{1}$$

where $H = H(q,p)$ is a smooth function defined on an open and connected set $\Omega \subset \mathbb{R}^2 \times \mathbb{R}^2$. The study of periodic solutions (existence and stability) is an old problem ($\geq$ 120 years). There are many results but most of the results dealing with stability are either local (small parameters) or use the computer. We are interested in non-local stability results.

The prototype can be the stability of the libration point L_4 in the circular restricted three body problem. The primaries have masses $1-\mu$ and μ with $\mu \in]0, \frac{1}{2}]$. The whole plane is rotating around the vertical axis and the satellite is placed at the vertex of an equilateral triangle composed by the three bodies. Note that in the rotating system L_4 is an equilibrium but in the inertial system it is a periodic solution. In this case the periodic solution is known explicitly and the linearized system is autonomous and can be solved. For linearized stability we find the necessary and sufficient condition

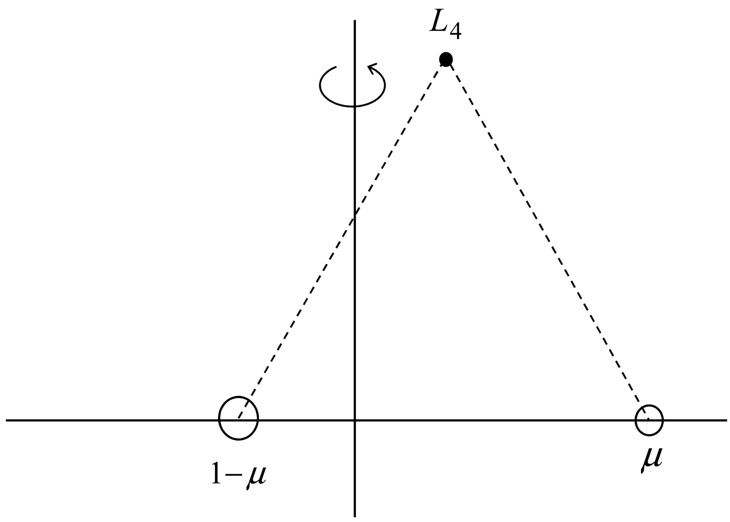
$$0 < \mu < \mu_1 = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right).$$

The eigenvalues lie on the imaginary axis ($\mu < \mu_1$), then collide ($\mu = \mu_1$) and then get out of this axis ($\mu > \mu_1$). For nonlinear stability there exist two numbers $0 < \mu_3 < \mu_2 < \mu_1$ which can be computed and such that L_4 is stable (in the Lyapunov sense) if and only if

$$0 < \mu \leq \mu_1 \quad \text{and} \quad \mu \neq \mu_2, \mu_3.$$

The nonlinear analysis is very delicate and the proof was obtained once KAM theory was available (see [7] for more details).

The purpose of this course is to develop some tools which can be useful to study more general non-local stability problems. Ideally I would like to apply these tools to



the general case (1) but by now I can only deal with a special family: periodic systems with one degree of freedom. That is,

$$H = H(t,q,p), \quad q,p \in \mathbb{R}, \quad H(t+2\pi,q,p) = H(t,q,p)$$

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{2}$$

This system can be immersed in the larger family (1) by introducing new variables

$$Q = t, \quad P = H, \quad \mathcal{H}(q,Q,p,P) = H(Q,q,p) + P.$$

The periodicity of time allows us to interpret Q as an angular variable ($Q \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$),

$$\begin{cases} \dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{\partial H}{\partial p} & \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial H}{\partial q} \\ \dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 1 & \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q} = -\frac{\partial H}{\partial t}. \end{cases}$$

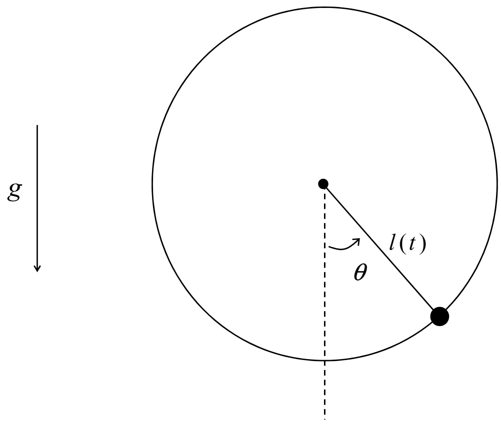
At each energy level $\mathcal{H}$ = constant, the dynamics of (2) is repeated. Let us now see some non-local results for (2).

1.1. The pendulum of variable length

Consider the equation

$$\ddot{\theta} + \alpha(t)\sin\theta = 0 \tag{3}$$

where α is 2π-periodic and positive. After a change of the independent variable this is the equation of a particle moving on a pulsating circle ($\alpha(t) = gl(t)^3$) under the action of gravity. The stability of the equilibrium $\theta = 0$ for small parameters ($\alpha(t) =$



$\omega^2 + \varepsilon p(t)$ with $\omega \neq n, n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$) is a typical illustration of KAM theory (see the book [1]). We are interested in the non-local problem with

$$\alpha \in C(\mathbb{T}), \qquad \min \alpha(t) > 0.$$

Again the periodic solution is explicitly known ($\theta = 0$) but now the linearized equation

$$\ddot{y} + \alpha(t) y = 0 \tag{4}$$

cannot be solved explicitly. As we will see this linear equation contains all the information about the stability of $\theta = 0$ if we exploit the symplectic structure. To explain the result we make our first digression on the symplectic group.

The group $\mathrm{Sp}(\mathbb{R}^2)$ is composed by the 2×2 matrices A satisfying

$$\det A = 1.$$

Associated to this group we have a notion of conjugate matrices, $A \sim B$ if there exists $P \in \mathrm{Sp}(\mathbb{R}^2)$ such that $A = PBP^{-1}$. The conjugacy classes in $\mathrm{Sp}(\mathbb{R}^2)$ are

$$R[\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi[$$

$$H_{\pm}[\theta] = \begin{pmatrix} \pm\mathrm{ch}\theta & \mathrm{sh}\theta \\ \mathrm{sh}\theta & \pm\mathrm{ch}\theta \end{pmatrix}, \quad \theta > 0$$

$$P_{+} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P_{+} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad -P_{+}, \quad -P_{-}.$$

Note that there are matrices which are conjugate in $\mathrm{Gl}(\mathbb{R}^2)$ but not in $\mathrm{Sp}(\mathbb{R}^2)$.

$$R[\theta] \not\sim R[-\theta] \quad \text{in } \mathrm{Sp}(\mathbb{R}^2), \quad \theta \neq 0, \pi$$
$$R[\theta] \sim R[-\theta] \quad \text{in } \mathrm{Gl}(\mathbb{R}^2)$$
$$P_+ \not\sim P_- \quad \text{in } \mathrm{Sp}(\mathbb{R}^2) \quad \text{but} \quad P_+ \sim P_- \quad \text{in } \mathrm{Gl}(\mathbb{R}^2) \ \ldots$$

We can now state the characterization of stability for $\theta = 0$. Let M be the monodromy matrix of (4), then

- $\theta = 0$ is linearly stable if and only if $M \sim R[\theta]$ in $\mathrm{Sp}(\mathbb{R}^2)$ for some θ
- $\theta = 0$ is stable (Lyapunov) if and only if $M^2 \sim R[\theta]$ or $M^2 \sim P_-$ in $\mathrm{Sp}(\mathbb{R}^2)$.

In contrast to L_4, now linearized stability implies stability but there are cases where the linearized equation is unstable and the equilibrium is stable. More details can be found in [11].

Exercise 1.1. Explain why the local result is a consequence of the non-local result.

1.2. A quadratic Newton's equation

Consider the equation

$$\ddot{x} + x^2 = p(t), \qquad \int_0^T p(t)\,dt \leq p_0$$

where p is T-periodic. It was proved in [10] that if

$$T^3 p_0 < 64$$

then there are at most two T-periodic solutions. Assume now that $p(t)$ has been chosen so that there exist exactly two, then if

$$T^3 p_0 < 4$$

one of them is linearly stable. The number 4 is sharp. Moreover, there exists a number σ_*, $\frac{1}{4} < \sigma_* < \frac{64}{81}$, which can be computed such that if

$$T^3 p_0 < \sigma_*$$

then this solution is stable. Then number σ_* was not optimal and it was later improved by Zhang, Chu, Li [20] although the optimal value is not known.

In contrast to the previous results the periodic solution is not explicitly known, the existence is determined via degree theory and some other global techniques. The disadvantage of this type of result is that it depends on very special properties of the nonlinearity x^2. It is my impression that more flexible results should not be valid for all p's but for almost all.

1.3. The forced pendulum equation

Consider the equation

$$\ddot{x} + \beta \sin x = p(t)$$

where $\beta > 0$ and $p \in C(\mathbb{T})$ with $\int_0^{2\pi} p(t)\,dt = 0$. It is well known that there exist at least two 2π-periodic solutions (variational or symplectic methods). If

$$\beta \leq \frac{1}{4}$$

then, for almost all p's, at least one of them is stable (see [12]). Moreover the number $\frac{1}{4}$ is sharp (see [13]). The techniques employed to prove this result are rather flexible. They have been already used by J. Chu, F. Wang in a recent paper [2]. My original intention was to prove this result in the course but later I realized that the proof is too long. Instead I have prepared a toy problem which is easier to prove but employs the same techniques.

For more references on non-local results and different types of extensions I refer to the papers by Liu [6], D. Nuñez [9], M. Zhang and his school [18, 19, 5], P. Torres [17], Hanßmann and Si [4]. This list is not complete.

2. Definition of stability

Let $\phi_t(z)$ be the flow associated to the Hamiltonian flow (1). For each $z \in \Omega$ the solution $\phi_t(z)$ is defined on a maximal (open) interval I_z.

Let $z_* \in \Omega$ be a given point and assume that $\phi_t(z_*)$ is well defined for $t \geq 0$. This solution is called stable (in the Lyapunov sense) if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi_t(z)$ is well defined for $t \geq 0$ and $|\phi_t(z_*) - \phi_t(z)| < \varepsilon$ if $|z - z^*| < \delta$.

This is the notion of stability for the future. If we replace $t \geq 0$ by $t \in \mathbb{R}$ we obtain the notion of perpetual stability.

For periodic solutions this notion is too restrictive. For instance the motion of a planet around the sun is unstable. This is a problem in our framework with $\Omega = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ and $H(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$. The reason for this instability is the third Kepler law: the minimal period of an elliptic orbit is $T = 2\pi a^{3/2}$ where $a = \frac{1}{2} \times$ major axis. If we select an initial condition $z_* = (q_*, p_*)$ with energy $H(q_*, p_*) < 0$ and angular momentum $q_* \wedge p_* \neq 0$ then $\phi_t(z_*)$ is periodic. The same will be true for a small perturbation z, if the major axis of the first orbit is $2a_*$, the major axis of the second, $2a$, will be close but in general it will be different. Then the period T_* and T will be close but different. Let us think of the positions of the two orbits at times $T_*, 2T_*, 3T_* \ldots$ if $a_* < a$

$$q_* = q_*(T_*) = q_*(2T_*) = \cdots = q_*(NT_*).$$

Every year (period T_*) the perturbed planet will not close the orbit and after many years $q_*(NT_*)$ and $q(NT_*)$ can be far away.

The following exercise makes this idea more precise.

Exercise 2.1. Assume that $F : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ is smooth and let $\varphi_n(t)$ and $\varphi(t)$ be periodic solutions of $\dot{z} = F(z)$ with minimal periods T_n and T. In addition,

$$\frac{T_n}{T} \notin \mathbb{Q}, \quad \varphi(t) \text{ is non constant}, \quad \varphi_n(0) \to \varphi(0).$$

Then $\varphi(t)$ is not stable (in the Lyapunov sense).

If we go back to the elliptic orbits of Kepler problem we observe that a small perturbation will change slightly the geometry of the ellipse, hence the orbit

$$\gamma = \{(q(t), p(t)) : t \in \mathbb{R}\}$$

will change slightly.

We are led to the following definition: a closed orbit $\gamma \subset \Omega$ is called *orbitally stable* if given a neighborhood $\mathcal{U} = \mathcal{U}(\gamma)$ there exists another neighborhood $\mathcal{V} = \mathcal{V}(\gamma)$ such that if $z \in \mathcal{V}$ then $\phi_t(z) \in \mathcal{U}$ for each $t \geq 0$.

If γ is a closed orbit (not an equilibrium), $\nabla H(z) \neq 0$ for each $z \in \gamma$ and so the energy level $\{H = c\}$ where $c = H|_\gamma$ is a $3d$-manifold, at least in a neighborhood $\mathcal{U}$ of γ. We can restrict the flow $\phi_t(z)$ to this submanifold and consider the orbital stability only with respect to orbits lying on the same energy level, this leads to the weaker notion of *isoenergetic orbital stability*. In most cases both notions coincide but there are exceptional cases (see the Appendix at the end of this lesson).

Next we are going to introduce an important tool for the study of orbital stability: *transversal sections*.

In the closed orbit γ we fix a point $z_* \in \gamma$. We know that $\nabla H(z_*) \neq 0$ and we assume, for instance,

$$\frac{\partial H}{\partial q_2}(z_*) \neq 0.$$

Then there exists a neighborhood $\mathcal{U}$ of z_* in $\mathbb{R}^4$ such that

$$\Sigma = \{(q,p) \in \Omega \,|\, H(q,p) = 0,\, p_2 = p_2^*\} \cap \mathcal{U}$$

is a surface contained in the energy level. Moreover the flow is transversal to Σ at z_* ($\dot{p}_2 = -\frac{\partial H}{\partial q_2}$).

In a neighborhood $\Sigma_1 \subset \Sigma$ of z_* we can find a return: for each $z \in \Sigma_1$ there exists $\tau = \tau(z) > 0$ such that $\phi_{\tau(z)}(z) \in \Sigma$. Moreover, τ is a smooth function with $\tau(z_*) = \tau_* > 0$ period.

Exercise 2.2. Justify this via the implicit function theorem.

We define the Poincaré map $P : \Sigma_1 \to \Sigma$, $P(z) = \phi_{\tau(z)}(z)$ and it turns out that P is symplectic, that is

$$\omega((dP)_z\zeta, (dP)_z\eta) = \omega(\zeta, \eta) \quad \text{if } z \in \Sigma_1, \zeta, \eta \in T_z(\Sigma).$$

Here ω denotes the restriction to the tangent plane $T_z(\Sigma)$ of the form

$$\omega(\zeta,\eta) = \langle \zeta, J\eta \rangle, \quad \zeta,\eta \in \mathbb{R}^4, \quad J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Note that $\omega|_{T_z(\Sigma)}$ is non-degenerate. In the surface Σ we can express q_2 and p_2 as functions of q_1 and p_1, $p_2 = \text{constant} = p_2^*$ and $q_2 = \varphi(q_1, p_1)$ from $H(q_1,q_2,p_1,p_2) = c$. Then the projection $\Pi(q,p) = (q_1, p_1)$ defines a diffeomorphism between Σ and $\mathbb{R}^2$. We can transport P to $\mathbb{R}^2$ via the commutive diagram

$$\begin{array}{ccc} \Sigma_1 \subset \Sigma & \overset{P}{\to} & \Sigma \\ \downarrow\Pi & & \downarrow\Pi \\ \mathcal{D} \subset \mathbb{R}^2 & \overset{\tilde{P}}{\to} & \mathcal{D}_1 \subset \mathbb{R}^2 \end{array}$$

where $\tilde{P} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is an area-preserving diffeomoprhism from $\mathcal{D}$ onto $\mathcal{D}_1 = \tilde{P}(\mathcal{D})$ with a fixed point at $\Pi(z_*)$. Note that

$$\Pi^*(dq_1 \wedge dp_1) = \omega|_\Sigma.$$

There is a natural notion of stability of fixed points for the discrete system $\zeta_{n+1} = \tilde{P}(\zeta_n)$. Given the fixed point $\zeta_* = \Pi(z_*)$ we say that ζ_* is stable if for each neighborhood $\mathcal{U} = \mathcal{U}(\zeta_*)$ there exists another neighborhood $\mathcal{V} = \mathcal{V}(\zeta_*)$ such that the iterates $\tilde{P}^n(\mathcal{V})$ are well defined for each $n \geq 0$ and contained in $\mathcal{U}$.

By continuous dependence (with some care) it is possible to prove that ζ_* is stable for $\tilde{P}$ if and only if γ is isoenergetically orbitally stable. In the special case of time periodic systems of one degree of freedom this is just stability in the Lyapunov sense for the original system. For more details we refer to [8] and [7]. See also

http://www.ugr.es/~rortega/PDFs/Talca.pdf

Appendix. A closed orbit orbitally unstable but isoenergetically orbitally stable.

We work with symplectic polar coordinates

$$q_1 + ip_1 = \sqrt{2r_1}e^{i\theta_1}, \quad q_2 + ip_2 = \sqrt{2r_2}e^{i\theta_2}$$

and consider the Hamiltonian function

$$H(r_1,r_2) = (r_1 - r_2)\left(1 + (r_1 - r_2)^2 r_1^2\right), \qquad \Omega = \left(\mathbb{R}^2 \setminus \{0\}\right) \times \left(\mathbb{R}^2 \setminus \{0\}\right).$$

Let us concentrate on the energy level $H = 0$, this is equivalent to $r_1 - r_2 = 0$. We compute the derivatives of H

$$\frac{\partial H}{\partial r_1} = 1 + (r_1 - r_2)^2[5r_1^2 - 2r_1 r_2]$$
$$\frac{\partial H}{\partial r_2} = -1 - 3(r_1 - r_2)^2 r_1^2$$

and observe that $\frac{\partial H}{\partial r_1} = 1$, $\frac{\partial H}{\partial r_2} = -1$ on $H = 0$. This implies that on the level $H = 0$ all orbits are 2π-periodic

$$\dot{r}_i = 0, \quad \dot{\theta}_i = -\frac{\partial H}{\partial r_i} \qquad \begin{cases} r_1(t) = \rho_0 = r_2(t) \\ \theta_1(t) = -t + \theta_{10},\, \theta_2(t) = t + \theta_{20}. \end{cases}$$

We select $\gamma : r_1 = r_2 = 1$, $\theta_1 = -\theta_2$ and observe that this closed orbit is isoenergetically stable. Let us prove that γ is orbitally unstable when we include other energy levels. We take $r_1 = 1+\varepsilon$, $r_2 = 1-\varepsilon$, $\theta_{10} = \theta_{20} = 0$ then

$$\dot{\theta}_1 = -\frac{\partial H}{\partial r_1} = -1 - 4\varepsilon^2(1+\varepsilon)(3+7\varepsilon)$$
$$\dot{\theta}_2 = -\frac{\partial H}{\partial r_2} = 1 + 12\varepsilon^2(1+\varepsilon)^2.$$

Now we select ε so that the quotient of the two frequencies is irrational, say $\varepsilon_n = \frac{\sqrt{2}}{n}$. Then the orbit γ_n is dense on the torus $r_1 = 1+\varepsilon_n$, $r_2 = 1-\varepsilon_n$ and so the points of γ_n accumulate on the whole torus $r_1 = r_2 = 1$ which strictly contains the closed orbit γ.

3. Stable fixed points of area-preserving maps

3.1. Stable fixed points

We first recall the notion of *stable fixed point*. Let $\mathcal{D}$ and $\mathcal{D}_1$ be open subsets of $\mathbb{R}^2$ and

$$h : \mathcal{D} \to \mathcal{D}_1, \quad z_1 = h(z)$$

a homeomorphism having a fixed point $z_* \in \mathcal{D}$, $h(z_*) = z_*$. We say that z_* is stable (in the future) if given any neighborhood $\mathcal{U} = \mathcal{U}(z_*)$ there exists another neighborhood $\mathcal{V} = \mathcal{V}(z_*)$ such that the iterate $h^n(\mathcal{V})$ is well defined and

$$h^n(\mathcal{V}) \subset \mathcal{U}$$

for any integer $n \geq 0$. The notion of *perpetual stability* is obtained after replacing $n \geq 0$ by $n \in \mathbb{Z}$.

Exercise 3.1. Discuss the stability properties of $z_* = 0$ if

$$h(z) = \lambda z \ \ (\lambda > 0) \quad \text{or} \quad h(z) = R[\theta]z, \quad R[\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

A rather obvious property of the notion of stability is that it is invariant under changes of variable. By a change of variable we understand a homeomorphism $\Psi : \Delta \to \Delta_1$ between two open sets in $\mathbb{R}^2$ with $\Psi(z_*) = w_*$. Then $\hat{h} = \Psi \circ h \circ \Psi^{-1}$ is well defined on some neighborhood of w_* and w_* is stable under $\hat{h}$ if and only if z_* is stable under h.

Next we present a characterization of stability in terms of invariant neighborhoods.

PROPOSITION 1. *The following statements are equivalent:*

(i) z_* *is stable in the future [resp. perpetually]*

(ii) There exists a sequence $\{\mathcal{U}_n\}$ *of open and bounded sets satisfying*

$$\overline{\mathcal{U}}_{n+1} \subset \mathcal{U}_n, \quad \overline{\mathcal{U}} \subset \mathcal{D}, \quad \bigcap_n \overline{\mathcal{U}}_n = \{z_*\}$$

and

$$h(\mathcal{U}_n) \subset \mathcal{U}_n \quad [\textit{resp.} \quad h(\mathcal{U}_n) = \mathcal{U}_n].$$

Proof. $(ii) \Rightarrow (i)$ The compact sets $\overline{\mathcal{U}}_n$ converge to $\{z_*\}$ in the Hausdorff topology. Given $\mathcal{U}$ we can find n large enough so that $\overline{\mathcal{U}}_n \subset \mathcal{U}$. Then we select $\mathcal{V} = \overline{\mathcal{U}}_n$.

$(i) \Rightarrow (ii)$ Fix a disk D_0 centered at z_* and such that $\overline{D}_0 \subset \mathcal{D}$. We find $\mathcal{V}_0$ such that $h^n(\mathcal{V}_0) \subset D_0$ for each $n \geq 0$. Then $\mathcal{U}_0 = \cup_{n\geq 0} h^n(\mathcal{V}_0)$ is open, contains z_* and satisfies $\mathcal{U}_0 \subset D_0$ and $h(\mathcal{U}_0) \subset \mathcal{U}_0$. We construct $\mathcal{U}_n$ by induction with $D_n \to \{z_*\}$, $\overline{D}_{n+1} \subset \mathcal{U}_n$. □

3.2. Area-preserving maps

The map $h : \mathcal{D} \to \mathcal{D}_1$ is area preserving if

$$\mu(h(B)) = \mu(B)$$

for each Borel set $B \subset \mathcal{D}$. Here μ is the Lebesgue measure in the plane*.

Simple examples are linear maps $h(z) = Az$ with $A \in \mathbb{R}^{2\times 2}$, $|\det A| = 1$.

Exercise 3.2. Prove that the non-linear map $h(z) = R[\theta + \beta|z|^2]z$ with $\beta \neq 0$ is an area-preserving map. Describe the dynamics.

Let $X : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $X = X(t,x)$ be a $C^{0,1}$ vector field with

$$\text{div}_x X = 0.$$

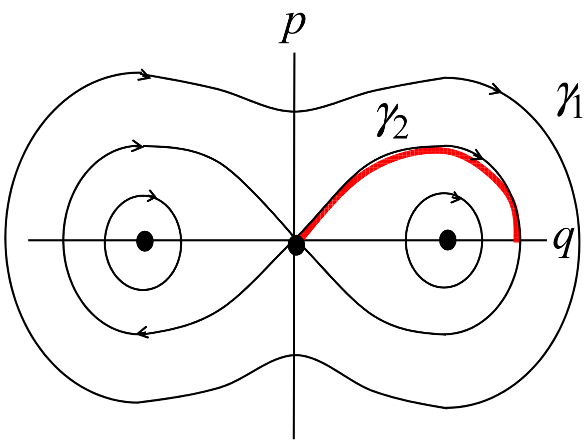
Let $x(t;x_0)$ be the solution of $\dot{x} = X(t,x)$, $x(0) = x_0$. For fixed $\tau \in \mathbb{R}$, the map $h(x_0) = x(\tau;x_0)$ is an area-preserving homeomorphism between appropriate domains. This is a consequence of Liouville theorem. In particular this is the case for Hamiltonian systems $X = J\nabla_x H$.

Area-preserving maps can have positively invariant open sets that are not invariant. We present two examples:

(i) $h(z) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} z, \quad K = \{z = (x,y) \in \mathbb{R}^2 : \ |y| \leq x\}, \quad h(K) \subsetneq K$

(ii) Let $\{\phi_t\}$ be the Hamiltonian flow with phase portrait as below. Let Ω be the region in the interior of γ_1 excepting $\gamma_2 \cap \{p \geq 0\}$ and the origin. Then, for each $t > 0$ $\phi_t(\Omega) \subsetneq \Omega$ because $\phi_t(\Omega)$ contains a smaller piece of γ_2.

*A more restrictive notion appears when Borel sets are replaced by Lebesgue measurable sets (see [14]).



The next result shows that in many cases positive invariance implies invariance.

LEMMA 1. *Assume that Ω is an open set of bounded measure with $\overline{\Omega} \subset \mathcal{D}$ and $\mathrm{int}(\overline{\Omega}) = \Omega$. If $h(\Omega) \subset \Omega$ then $h(\Omega) = \Omega$.*

Proof. If $h(\Omega) \subset \Omega$ we claim that $\Omega \subset h(\overline{\Omega})$ for otherwise $\Omega \setminus h(\overline{\Omega})$ should be a non-empty open subset of Ω, hence a set of positive measure. But

$$\infty > \mu(\Omega) \geq \mu\left(\Omega \setminus h(\overline{\Omega})\right) + \mu\left(h(\Omega)\right) > \mu(\Omega),$$

lead to a contradiction. Once we know that $\Omega \subset h(\overline{\Omega})$ the rest of the argument is purely topological. Indeed,

$$\Omega \subset \mathrm{int}\left(h(\overline{\Omega})\right) = h\left(\mathrm{int}(\overline{\Omega})\right) = h(\Omega).$$

□

Exercise 3.3. Let Ω be an open subset of $\mathbb{R}^2$ and $\omega = \mathrm{int}(\overline{\Omega})$. Then $\Omega \subset \omega \subset \overline{\Omega}$ and $\mathrm{int}(\overline{\omega}) = \omega$.

PROPOSITION 2. *For area-preserving maps, stability in the future and perpetual stability are equivalent notions.*

Proof. If z_* is stable in the future then by Proposition 1 we can find neighborhoods $\{\mathcal{U}_n\}$ in the conditions prescribed by the result. In particular, $h(\mathcal{U}_n) \subset \mathcal{U}_n$. Define $\mathcal{V}_n = \mathrm{int}\left(\overline{\mathcal{U}}_n\right)$, then $\mathcal{U}_n \subset \mathcal{V}_n \subset \overline{\mathcal{U}}_n$ and $h(\mathcal{V}_n) = \mathcal{V}_n$. This is a consequence of the previous exercise and Lemma 1. Again Proposition 1 implies that z_* is perpetually stable. □

3.3. Lyapunov functions and first integrals

Let $V : \mathcal{D} \to \mathbb{R}$ be a continuous function satisfying

$$V(z) > 0 \quad \text{if } z \in \mathcal{D} \setminus \{z_*\}, \qquad V(z_*) = 0.$$

We say that V is a Lyapunov function if

$$V(h(z)) \leq V(z) \quad \text{for each } z \in \mathcal{D} \text{ with } h(z) \in \mathcal{D}.$$

Exercise 3.4. Prove that the existence of a Lyapunov function implies the stability of z_*.

We prove that every Lyapunov function is (locally) a first integral.

PROPOSITION 3. *Assume that there exists a Lyapunov function V. Then there exists a neighborhood $\mathcal{U}$ of z_* such that $h(\mathcal{U}) = \mathcal{U}$ and $V(h(z)) = V(z)$ if $z \in \mathcal{U}$.*

Proof. We know that z_* is stable and so there is a bounded and open invariant set $\mathcal{U}$ with $z_* \in \mathcal{U}$, $\overline{\mathcal{U}} \subset h^{-1}(\mathcal{D})$. Since h is area-preserving and $h(\mathcal{U}) = \mathcal{U}$ we deduce that

$$\int_{\mathcal{U}} V(x)\, dx = \int_{\mathcal{U}} V(h(x))\, dx.$$

Note that $\int_{\mathcal{U}} V = \int_0^\infty F_V(t)\, dt$ and $\int_{\mathcal{U}} V \circ h = \int_0^\infty F_{V\circ h}(t)\, dt$, with

$$F_V(t) = \mu(\{x \in \mathcal{U} : V(x) > t\}) = \mu(\{x \in \mathcal{U} : V(h(x)) > t\}) = F_{V\circ h}(t).$$

From $V - V \circ h \geq 0$ on $\mathcal{U}$ and $\int_{\mathcal{U}}(V - V \circ h) = 0$ we deduce that $V = V \circ h$ almost everywhere, but we are dealing with continuous functions and so it holds everywhere. □

3.4. Invariant "curves" and stability

Up to now all discussions could be adapted to measure-preserving maps in $\mathbb{R}^d$ with $d > 2$. The contents of this section are specific of two dimensions. First we recall some facts of the topology of the plane: given a domain (open + connected) $\Omega \subset \mathbb{R}^2$, we denote by $\widehat{\Omega}$ the smallest simply connected domain containing Ω, that is

i) $\Omega \subset \widehat{\Omega}$, $\widehat{\Omega}$ simply connected domain

ii) if $\Omega \subset \omega$, ω simply connected domain $\Rightarrow \widehat{\Omega} \subset \omega$.

Intuitively we can say that $\widehat{\Omega}$ is obtained by filling in the holes of Ω. Note that $\widehat{\Omega} \cong \mathbb{R}^2$ (this is a consequence of Riemann's theorem on conformal mappings).

Exercise 3.5. Prove that $\widehat{\Omega}$ always exists. Given Ω with $\overline{\Omega} \subset \mathcal{D}$, prove that $h(\widehat{\Omega}) = \widehat{h(\Omega)}$.

Assume that z_* is stable and let $\mathcal{U}_n$ be a sequence of bounded and open sets with

$$\overline{\mathcal{U}}_{n+1} \subset \mathcal{U}_n, \quad \overline{\mathcal{U}} \subset \mathcal{D}, \quad \bigcap_n \overline{\mathcal{U}}_n = \{z_*\}, \quad h(\mathcal{U}_n) = \mathcal{U}_n.$$

We can assume that $\mathcal{U}_n$ is connected since otherwise we would take the connected component of z_*. Next we fill in the holes of $\mathcal{U}_n$, $\mathcal{V}_n = \widehat{\mathcal{U}}_n$. We observe that $\mathcal{V}_n$ is also invariant under h. Moreover $\overline{\mathcal{V}}_n \to \{z_*\}$ in the Hausdorff sense (Take a sequence of disks centered at z_* with $\mathcal{U}_n \subset D_n$ and $D_n \to \{z_*\}$). The sets $\mathcal{V}_n$ are homeomorphic to open disks but the boundaries can be very strange. In any case $h(\partial\mathcal{V}_n) = \partial\mathcal{V}_n$. In some books of topology of the plane the boundary of a simple connected domain is called a "curve". With this terminology we have proved the following result: a fixed point is stable if and only if it is surrounded by invariant "curves". Handel constructed in [3] an example where all invariant "curves" were non locally connected continua (pseudo-circles).

4. Linearization around fixed points

4.1. The symplectic group

We recall that $\mathrm{Sp}(\mathbb{R}^2)$ is composed by the matrices $A \in \mathbb{R}^{2\times 2}$ satisfying $\det A = 1$. They represent linear maps preserving area and orientation.

The Lie group $\mathrm{Sp}(\mathbb{R}^2)$ has a well known topological description

$$\mathrm{Sp}(\mathbb{R}^2) \cong \text{ open solid torus.}$$

To understand this homeomorphism it is convenient to represent the group in complex notation. The linear map induced by A can we written as

$$L_A : \mathbb{C} \to \mathbb{C}, \quad L_A : z \mapsto az + b\bar{z}$$

with $a, b \in \mathbb{C}$ given by

$$a = \alpha + i\beta,\ b = \gamma + i\delta, \quad A = \begin{pmatrix} \alpha+\gamma & \delta-\beta \\ \delta+\beta & \alpha-\gamma \end{pmatrix}.$$

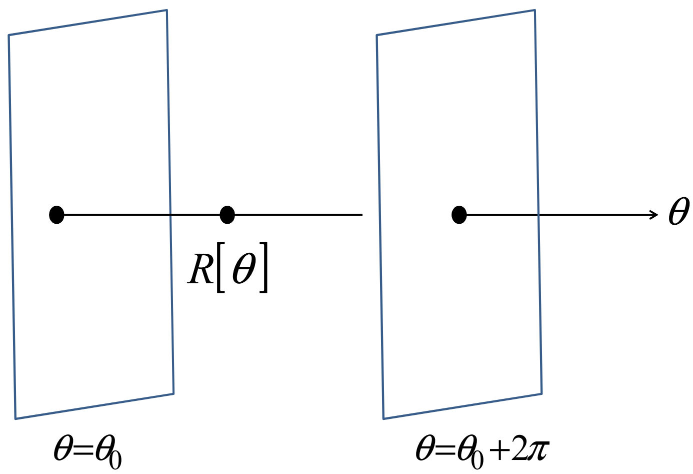
Then $\det A = 1$ is equivalent to

$$|a|^2 - |b|^2 = 1.$$

We construct the homeomorphism

$$\mathbb{C} \times \mathbb{S}^1 \to \mathrm{Sp}(\mathbb{R}^2), \quad (b, e^{i\theta}) \mapsto L_A \quad \text{where } a = \sqrt{1+|b|^2}e^{i\theta}.$$

The θ-axis ($b = 0$) is the circumference of rotations. To understand the stability of the



origin $z_* = 0$ with respect to L_A we employ the trace of A,

$$\Delta := \mathrm{tr} A = 2\Re a = 2\sqrt{1+|b|^2}\cos\theta.$$

If $|\Delta| > 2$ the eigenvalues are real $|\lambda_1| < 1 < |\lambda_2|$. The matrix A is conjugate in $\mathrm{Gl}(\mathbb{R}^2)$ to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and so the origin is unstable.

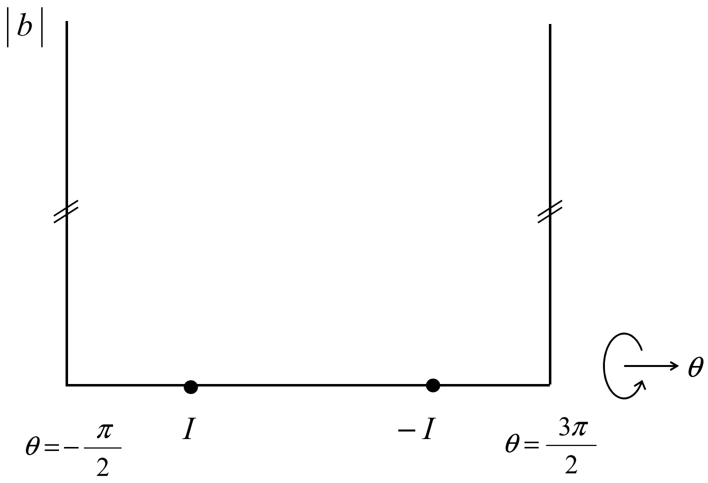
If $|\Delta| < 2$ the eigenvalues are complex and conjugate $\lambda_1 = \bar{\lambda}_2 = e^{i\theta}$, $\theta \in]0,\pi[\cup]\pi, 2\pi[$, and A is conjugate in $\mathrm{Gl}(\mathbb{R}^2)$ to $R[\theta]$, the origin is stable.

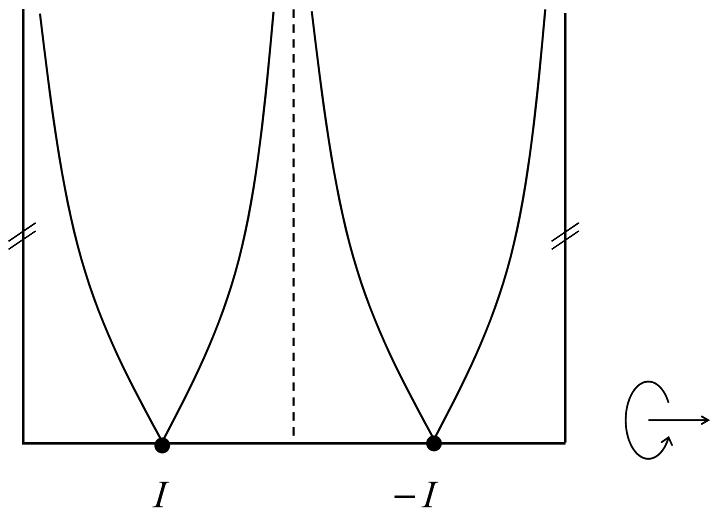
Finally, if $|\Delta| = 2$ there is a double eigenvalue $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$, then either $A = \pm I$ (parabolic stable) or A is conjugate to $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$ in $\mathrm{Gl}(\mathbb{R}^2)$ (parabolic unstable).

Let us now describe this dynamical classification in a topological way. We visualize $\mathrm{Sp}(\mathbb{R}^2)$ as an infinite rectangle with coordinates $|b| \geq 0$ and $\theta \in [\theta_0, \theta_0 + 2\pi]$ that is rotating around the θ-axis and such that θ_0 and $\theta_0 + 2\pi$ are identified. For convenience we chose $\theta_0 = -\frac{\pi}{2}$. We draw the parabolic set $\Delta = \pm 2$, $\sqrt{1+|b|^2} = \frac{\pm 1}{\cos\theta}$. Then $\mathrm{Sp}(\mathbb{R}^2) \setminus \{\Delta = \pm 2\}$ has 4 components, two of them elliptic and two of hyperbolic.

Exercise 4.1. Describe the level sets $\Delta = \text{constant}$.

Exercise 4.2. Describe the stability properties of $z_* = 0$ for linear maps with $\det A = -1$.





4.2. The first approximation

Let $h : \mathcal{D} \to \mathcal{D}_1$ be a C^1-diffeomorphism with

$$\det h'(z) = 1, \quad \text{if } z \in \mathcal{D}.$$

Then h is area-preserving and given a fixed point z_*, the differential at z_*

$$L = h'(z_*)$$

is in $\mathrm{Sp}(\mathbb{R}^2)$. It seems natural to compare the dynamics of $z_{n+1} = h(z_n)$ and $z_{n+1} = Lz_n$ in small neighborhoods of z_* and the origin. We say that z_* is *linearly stable* if the origin is stable for L. This means that either L is elliptic or parabolic-stable ($L = \pm I$). We want to discuss possible connections between stability and linearized stability. When L is hyperbolic, Hartman-Grossman theorem says that the dynamics of h and L are locally conjugate. Hence,

L hyperbolic $\Rightarrow z_*$ unstable for h.

In general there are no further connections between stability and linearized stability. We illustrate it with two examples.

Example I: *z_* is stable but linearly unstable.*

Consider the Hamiltonian function

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4, \qquad z = (x,y)$$

and the associated flow ϕ_t in $\mathbb{R}^2$. The phase-portrait is a center around the origin and the flow is globally defined. Let us fix $T \neq 0$ and consider $h = \phi_T$. Then h is an area-preserving analytic diffeomorphism. Moreover the origin $z_* = 0$ is stable because H is a Lyapunov function ($H \circ h = H$, $H > 0$ if $z \neq 0$, $H(0) = 0$). To compute

$$L = h'(0) = \frac{\partial \phi_t}{\partial z}|_{t=T,z=0}$$

we differentiate with respect to initial conditions. For the Hamiltonian system $\dot{x} = y$, $\dot{y} = -x^3$, the linearized system around a solution $(x(t), y(t))$ is

$$\dot{\xi} = \eta, \qquad \dot{\eta} = -3x(t)^2\xi.$$

For $x = y = 0$ we obtain $\dot{\xi} = \eta$, $\dot{\eta} = 0$. Then

$$L = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$$

because $\frac{\partial \phi_t}{\partial z}(0)$ is the matrix solution $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

Example II: *z_* is linearly stable but unstable.*

We fix an integer $N \geq 3$ and consider the real analytic Hamiltonian function

$$H(z,\bar{z}) = \Im\left(z^N\right).$$

The phase-portrait of the associated flow is a generalized saddle having stable/unstable manifold with N branches. This can be checked via the symplectic change of variable

$z = \sqrt{2r}e^{i\theta}$ leading to the new Hamiltonian function $\mathcal{H}(\theta, r) = (2r)^{N/2}\sin N\theta$. Let ϕ_t be the Hamiltonian flow associated to H and consider $h_1 = \phi_T$. This map is well defined in a neighborhood of the origin and satisfies $h_1(0) = 0$, $h_1'(0) = I$ because the linearized system around the origin is

$$\dot{\xi} = 0, \qquad \dot{\eta} = 0.$$

Note that $H = o(|z|^2)$.

Next we consider a rotation of angle $\frac{2k\pi}{N}$ and define

$$h = R\left[\frac{2k\pi}{N}\right] \circ h_1.$$

From the above computations we know that $h(0) = 0$ and $L = h'(0) = R\left[\frac{2k\pi}{N}\right]$. Moreover $z_* = 0$ is unstable because the stable and unstable manifolds for h_1 are also invariant under h.

Exercise 4.3. Prove that there exists a subset $\mathcal{D} \subset \mathrm{Sp}(\mathbb{R}^2)$ which is dense and such that for each $A \in \mathcal{D}$ there exists an analytic area-preserving map h satisfying $h(0) = 0$, $h'(0) = A$, $z_* = 0$ is unstable.

5. Nonlinear approximation

To simplify the exposition we shall work with a *real analytic* map

$$h : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$$

satisfying $h(0) = 0$ and $\det h'(z) = 1$ if $z \in \mathcal{U}$.

Here $\mathcal{U}$ is an open set containing the origin. We observe that the inverse function theorem implies that h is an area-preserving diffeomorphism between two small neighborhoods of the origin.

5.1. Birkhoff Normal form

Assume that

$$h'(0) \sim R[\theta] \quad \text{in } \mathrm{Sp}(\mathbb{R}^2) \quad \text{with } \theta \neq \frac{2k\pi}{n},\ n = 1,2,3,4.$$

Then there exists a real analytic area-preserving diffeomorphism Ψ (defined on a neighborhood of the origin) and a number $\beta_1 \in \mathbb{R}$ such that

$$\Psi^{-1} \circ h \circ \Psi(z) = R[\theta + \beta_1|z|^2]z + O\left(|z|^4\right)$$

as $z \to 0$.

This result more or less says that if the linear map is elliptic then the third order approximation is a twist map. The proof is purely algebraic and can be found in [16]. See also

http://www.ugr.es/~rortega/PDFs/buenosaires4.pdf

The change of variable Ψ is highly non-unique but the number β_1 is independent of the chosen Ψ. In this sense we can say that β_1 is a symplectic invariant.

When $h'(0) = R[\theta]$ the number β_1 can be computed by the following formula

$$\beta_1 = \Im\left(e^{-i\theta}N\right) - \frac{3\sin\theta}{1-\cos\theta}|A|^2 - \frac{\sin 3\theta}{1-\cos 3\theta}|C|^2 \tag{5}$$

where we are using complex notation and h has the expansion

$$e^{i\theta}z + Az^2 + Bz\bar{z} + C\bar{z}^2 + Mz^3 + Nz^2\bar{z} + Pz\bar{z}^2 + Q\bar{z}^3$$

The excluded angles are called *strong resonances*.

It is possible to obtain normal forms of higher order if some more angles are excluded. In particular if we assume that the angle is not commensurable with 2π,

$$\theta \neq \frac{2k\pi}{n}, \quad n = 1, 2, \ldots, N, \ldots$$

then for each $N \geq 1$ there exists Ψ_N as before and numbers $\beta_1, \ldots, \beta_N$ such that

$$\Psi_N^{-1} \circ h \circ \Psi_N(z) = R[\theta + \beta_1|z|^2 + \cdots + \beta_N|z|^{2N}]z + O\left(|z|^{2N+2}\right).$$

This iterative process suggests a tentative proof for the stability of $z_* = 0$. Assume that we are lucky and the map Ψ_N converges as $N \to \infty$ to some diffeomoprhism Ψ. The numbers $\beta_1, \beta_2, \ldots$ are independent of N and perhaps the power series

$$\theta + \beta_1\zeta + \cdots + \beta_N\zeta^N + \cdots$$

would converge to a function $\Phi(\zeta)$. The we could expect

$$\Psi^{-1} \circ h \circ \Psi(z) = R[\Phi(|z|^2)]z.$$

The origin is stable for $\hat{h}(z) = R[\Phi(|z|^2)]z$ and so the same should be true for h. Typically this program will fail. A typical area preserving analytic map will have a countable set of periodic points but $\hat{h}$ has many continua of periodic points and so they cannot be conjugate. Dynamics of $\hat{h}$: invariant circles, they are composed by periodic points when $\Phi(|z|^2)$ is commensurable with 2π.

5.2. Some consequences of the KAM method

As a corollary of Moser's small twist theorem it can be proved that $z_* = 0$ is stable if some β_N does not vanish. See [16]. Russmann proved, using also the KAM method, that if the number $\frac{\theta}{2\pi}$ satisfies a Diophantine condition and $\beta_N = 0$ for each $N \geq 1$ then h is conjugate to the rotation $R[\theta]$. See [15]. In particular $z_* = 0$ is stable.

We state as corollaries two consequences of there results.

COROLLARY 1. *If* $h'(0) \sim R[\theta]$ *in* $\mathrm{Sp}(\mathbb{R}^2)$ *with*

$$\theta \neq \frac{2k\pi}{n},\ n = 1,2,3,4$$

and

$$\beta_1 \neq 0$$

then $z_* = 0$ *is stable.*

COROLLARY 2. *If* $h'(0) \sim R[\theta]$ *in* $\mathrm{Sp}(\mathbb{R}^2)$ *and* $\frac{\theta}{2\pi}$ *satisfies a Diophantine condition then* $z_* = 0$ *is stable.*

The results on the pendulum of variable length and the quadratic equation (presented in the first lesson) were obtained via Corollary 1 and formula (5). Indeed it was an extension of this corollary which also deals with strong resonances. The result on the forced pendulum equation is based on the second corollary.

5.3. Some remarks on Diophantine numbers

A number $x \in \mathbb{R} \setminus \mathbb{Q}$ is called Diophantine if there exists two positive constants $\gamma > 0$ and $\sigma > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{\gamma}{q^\sigma} \quad \text{for each } \frac{p}{q} \in \mathbb{Q} \quad \text{with } q \geq 1.$$

The class composed by those numbers with fixed constants will be indicated by $DC(\gamma,\sigma)$. Also

$$DC_\sigma = \bigcup_{\gamma>0} DC(\gamma,\sigma).$$

For $\sigma < 2$ the set DC_σ is empty. For $\sigma \geq 2$ the set DC_σ has an interesting structure. From the point of view of category it is a small set but from the point of view of measure is a big set when $\sigma > 2$. Given γ and σ,

$$DC(\gamma,\sigma) = \mathbb{R} \setminus \bigcup_{q=1}^{\infty} \bigcup_{p\in\mathbb{Z}} \left] \frac{p}{q} - \frac{\gamma}{q^\sigma}, \frac{p}{q} + \frac{\gamma}{q^\sigma} \right[.$$

Then $DC(\gamma,\sigma)$ is the complement of an open set in $\mathbb{R}$ and does not contain any rational number. In consequence $DC(\gamma,\sigma)$ is closed and has empty interior. Since

$$DC_\sigma = \bigcup_{n=1}^{\infty} DC\left(\frac{1}{n},\sigma\right)$$

we deduce that DC_σ is of first category.

Let us now assume that $\sigma > 2$. We prove that $DC_\sigma \cap [0,1]$ has measure one. From here it is easy to deduce that DC_σ has full measure in $\mathbb{R}$. From

$$[0,1] \setminus DC(\gamma,\sigma) \subset \bigcup_{q=1}^{\infty} \bigcup_{p=0}^{q} \left] \frac{p}{q} - \frac{\gamma}{q^\sigma}, \frac{p}{q} + \frac{\gamma}{q^\sigma} \right[$$

we deduce that
$$\mu([0,1]\setminus DC(\gamma,\sigma)) \leq \sum_{q=1}^{\infty} \frac{(q+1)2\gamma}{q^{\sigma}} = S_{\sigma}\gamma$$
where $S_{\sigma} = 2\sum_{q=1}^{\infty} \frac{(q+1)}{q^{\sigma}} < \infty$ if $\sigma > 2$. Therefore
$$\mu([0,1]\setminus DC_{\sigma}) \leq \mu([0,1]\setminus DC(\gamma,\sigma)) \leq S_{\sigma}\gamma \to 0$$
as $\gamma \to 0$.

Exercise 5.1. Consider the Haar measure in the symplectic group $\mathrm{Sp}(\mathbb{R}^2)$ and let $\mathcal{E}$ be the open subset of elliptic matrices. Prove that there exists a subset $\widehat{\mathcal{E}} \subset \mathcal{E}$ of full measure in $\mathcal{E}$ such that if $h'(0) \in \widehat{\mathcal{E}}$ then $z_* = 0$ is stable.

6. A result with proof

Consider the equation

$$(\mathcal{E}) \qquad \ddot{x} + \Psi(x) = p(t) + \lambda$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ is analytic and satisfies

(i) For some $n = 0, 1, 2, \ldots$
$$n^2 < \alpha \leq \Psi'(x) \leq \beta < \left(n + \frac{1}{2}\right)^2, \qquad x \in \mathbb{R},$$

(ii) The limits $\Psi'(\pm\infty) = \lim_{x\to\pm\infty} \Psi'(x)$ exist and
$$\Psi'(+\infty) \neq \Psi'(-\infty).$$

The function $p : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic. This function is arbitrary but fixed, in contrast the number $\lambda \in \mathbb{R}$ must be interpreted as a parameter.

THEOREM 1. *In the above conditions, the equation* $(\mathcal{E})$ *has a stable* 2π*-periodic solution for almost every* $\lambda \in \mathbb{R}$.

REMARK 1. 1. Under the less restrictive condition
$$n^2 < \alpha \leq \Psi'(x) \leq \beta < (n+1)^2$$
there exists a unique 2π-periodic solution. This was proved by Loud (1967) and there are also earlier results for Hammerstein's equations which are related.

2. The condition (i) can be replaced by
$$\left(n + \frac{1}{2}\right)^2 < \alpha \leq \Psi'(x) \leq \beta < (n+1)^2.$$

The proof will be divided in three steps:

1. **Existence and uniqueness**. This is very well known and there are many possible proofs (Contraction principle, Global inverse function theorem, Variational methods,...). We choose a proof that will be useful for the following steps.

2. **Linearized stability**. This will be a consequence of a very old stability criterion for Hill's equation (Krein attributes it to Zhukovskii)

3. **Nonlinear stability**. This will be the key step.

6.1. Existence and uniqueness

We start with two results on linear equations. The proofs are in the Appendix. We employ some more or less standard notations on spaces of 2π-periodic functions. Namely $L^\infty(\mathbb{T}), C(\mathbb{T})$,... where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The norm in the Sobolev space $W^{2,\infty}(\mathbb{T})$ is denoted by $\|\cdot\|_{W^{2,\infty}}$.

LEMMA 2. *Assume that $a \in L^\infty(\mathbb{T})$ satisfies*

$$n^2 < \alpha \le a(t) \le \beta < (n+1)^2, \qquad a.e. t \in \mathbb{R}.$$

Then $y = 0$ is the only 2π-periodic solution of

$$\ddot{y} + a(t)y = 0.$$

LEMMA 3. *Assume that a is as before and $b \in L^\infty(\mathbb{T})$. Then there exists a number $K = K(\alpha, \beta) > 0$ such that the only 2π-periodic solution of*

$$\ddot{y} + a(t)y = b(t)$$

satisfies $\|y\|_{W^{2,\infty}} \le K\|b\|_{L^\infty}$.

In this part it is sufficient to assume $\Psi \in C^1(\mathbb{R})$ and

$$n^2 < \alpha \le \Psi'(x) \le \beta < (n+1)^2.$$

Uniqueness. Assume that $x_1 \ne x_2$ are two 2π-periodic solutions of $(\mathcal{E})$ and define

$$a(t) = \begin{cases} \frac{\Psi(x_1(t)) - \Psi(x_2(t))}{x_1(t) - x_2(t)} & \text{if } x_1(t) \ne x_2(t) \\ \Psi'(x_1(t)) & \text{otherwise}. \end{cases}$$

Then $a(t)$ is in the conditions of Lemma 2 and $y = x_1 - x_2$ is a 2π-periodic solution of $\ddot{y} + a(t)y = 0$. This is a contradiction.

Existence. Consider the homotopy

$$(\mathcal{E}_\varepsilon) \qquad \ddot{x} + \Psi(x) = \varepsilon p(t) + \lambda$$

and let us define

$$E = \{\varepsilon \in [0,1] : \text{ there exists a } 2\pi\text{-periodic solution}\}.$$

Since Ψ is a diffeomorphism of $\mathbb{R}$, the constant $x = \Psi^{-1}(\lambda)$ is a solution for $\varepsilon = 0$. Hence $0 \in E$.

<u>E is open in $[0,1]$.</u>

Given $\varepsilon_0 \in E$ let $x_*(t)$ be a 2π-periodic solution for $\varepsilon = \varepsilon_0$. The linearized equation around $x_*(t)$ is

$$\ddot{y} + \Psi'(x_*(t))y = 0.$$

From Lemma 2 we deduce that the number 1 is not a Floquet multiplier. This implies that there exists a 2π-periodic solution $x(t) = x_*(t) + O(\varepsilon)$ for ε close to ε_0.

<u>E is closed.</u>

Given $\varepsilon_n \to \varepsilon_*$ with $\varepsilon_n \in E$, we find a 2π-periodic solution $x_n(t)$ of $(\mathcal{E}_{\varepsilon_n})$. It also satisfies the linear equation

$$\ddot{y} + a_n(t)y = b_n(t)$$

with

$$a_n(t) = \begin{cases} \frac{\Psi(x_n(t)) - \Psi(0)}{x_n(t)} & \text{if } x_n(t) \neq 0 \\ \Psi'(x_n(t)) & \text{otherwise} \end{cases}$$

and $b_n(t) = \varepsilon_n p(t) + \lambda - \Psi(0)$. From Lemma 3 we deduce that

$$\|x_n\|_{W^{2,\infty}} \leq K\|b_n\|_{L^\infty} \leq K\left[\|p\|_{L^\infty} + |\lambda - \Psi(0)|\right].$$

By Ascoli theorem we extract a subsequence x_k converging to a solution of $\mathcal{E}_{\varepsilon_*}$. Hence $\varepsilon_* \in E$.

From now on the unique 2π-periodic solution of $(\mathcal{E})$ will be denoted by $x(t,\lambda)$. We are going to prove that the function

$$\lambda \in \mathbb{R} \mapsto (x(0,\lambda), \dot{x}(0,\lambda)) \in \mathbb{R}^2$$

is analytic.

To this end we introduce the Poincaré map

$$P_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_0, v_0) \mapsto (x(2\pi; x_0, v_0), \dot{x}(2\pi; x_0, v_0))$$

and notice that $P = P_\lambda(x_0, v_0)$ is analytic as a function of the three arguments $(\lambda, x_0, v_0) \in \mathbb{R}^3$. The possible lack of smoothness in t does not play a role because we are freezing the time ($t = 0, t = 2\pi$).

The periodic problem is equivalent to the equation

$$\Phi(x_0, v_0, \lambda) = (\mathrm{id} - P_\lambda)\,(x_0, v_0) = 0.$$

This can be seen as a problem of implicit functions if we seek the solution in the form $x_0 = x_0(\lambda)$, $v_0 = v_0(\lambda)$. Note that, by uniqueness, $x_0 = x(0,\lambda)$, $v_0 = \dot{x}(0,\lambda)$.

To check the transversality condition we observe that

$$\frac{\partial \Phi(x_0, v_0, \lambda)}{\partial (x_0, v_0)} = I - M$$

where the derivative is evaluated at the solution and M is the monodromy matrix associated to

$$\ddot{y} + \Psi'(x(t,\lambda))y = 0.$$

By Lemma 2 we know that 1 is not an eigenvalue of M and so the implicit function theorem (real analytic version) can be applied at each point $(x(0,\lambda), \dot{x}(0,\lambda), \lambda)$.

6.2. Linearized stability

The key is the following *stability criterion for Hill's equations*:

Assume that $a \in C(\mathbb{T})$ satisfies

$$n^2 < \alpha \le a(t) \le \beta < \left(n + \frac{1}{2}\right)^2.$$

Then $\ddot{y} + a(t)y = 0$ is stable.

This is the classical statement but we will see that the proof gives some useful additional information.

Let

$$\Phi(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix}$$

be the matrix solution with solutions $y_1(t)$, $y_2(t)$ satisfying

$$y_1(0) = \dot{y}_2(0) = 1, \quad \dot{y}_1(0) = y_2(0) = 0.$$

The stability of the equation is equivalent to the stability of the monodromy matrix $M = \Phi(2\pi) \in \mathrm{Sp}(\mathbb{R}^2)$. We shall prove that the trace satisfies $|\mathrm{tr} M| < 2$ and so M is *elliptic*.

LEMMA 4. *Assume that $a \in C(\mathbb{T})$ satisfies*

$$n^2 < \alpha \le a(t) \le \beta < \left(n + \frac{1}{2}\right)^2.$$

Then $\ddot{y} + a(t)y = 0$ has no periodic solutions of period 4π different from $y = 0$.

This is a direct consequence of Lemma 2 after changing the time scale.

Consider the homotopy of equations

$$\ddot{y} + (\lambda a(t) + (1-\lambda)\alpha_*)\, y = 0, \qquad \lambda \in [0,1]$$

with α_* a fixed number lying on $[\alpha, \beta]$. Let $\Phi(t,\lambda)$ be the matrix solution as before and

$$\Delta(\lambda) = \operatorname{tr}\Phi(2\pi, \lambda).$$

The discriminant function $\Delta(\lambda)$ is continuous and we can apply Lemma 4 to the equation with λ. This implies $\Delta(\lambda) \neq \pm 2$ for each $\lambda \in [0,1]$. For $\lambda = 0$ the equation $\ddot{y} + \alpha_* y = 0$ can be solved and $|\Delta(0)| < 2$. Then $|\Delta(\lambda)| < 2$ for each λ and, in particular, $|\Delta(1)| < 2$.

6.3. Nonlinear stability

We start with a result on analytic functions.

LEMMA 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant analytic function and let $Z \subset \mathbb{R}$ be a set of zero measure. Then also $f^{-1}(Z)$ has zero measure.*

Proof. The set $\Lambda = \{\lambda \in \mathbb{R} : f'(\lambda) = 0\}$ is discrete and closed. In particular $\mathbb{R} \setminus \Lambda = \cup_i I_i$ where I_i is an open interval and the union is disjoint. Define $f_i = f|_{I_i}$. This is a diffeomorphism from I_i onto $f(I_i)$ and so $f_i^{-1}(Z)$ has zero measure. Finally we observe that

$$f^{-1}(Z) \subset \bigcup_i f_i^{-1}(Z) \cup \Lambda.$$

The set of indexes $I = \{i\}$ is at most countable. Also Λ is countable, implying $\mu(\Lambda) = 0$. □

Let $x(t,\lambda)$ be the 2π-periodic solution of $(\mathcal{E})$. We shall prove that the fixed point $(x(0,\lambda), \dot{x}(0,\lambda))$ is stable with respect to P_λ. The derivative

$$L = P_\lambda'(x(0,\lambda), \dot{x}(0,\lambda))$$

is the monodromy matrix of

$$\ddot{y} + \Psi'(x(t,\lambda))y = 0.$$

From the previous section we know that

$$L \sim R[\theta] \quad \text{in } \operatorname{Sp}(\mathbb{R}^2).$$

The trace is an invariant under conjugacy and so the trace of L is precisely $2\cos\theta$.

Let $\Phi(t,\lambda)$ be the matrix solution of the linearized equation, we define the function

$$\Delta(\lambda) = \operatorname{tr}\Phi(2\pi, \lambda).$$

This is an analytic function and

$$\frac{\theta}{2\pi} \in DC_\sigma \iff \Delta(\lambda) \in \mathcal{D}_\sigma$$

where

$$\mathcal{D}_\sigma = \left\{ 2\cos\theta : \frac{\theta}{2\pi} \in DC_\sigma \right\}.$$

Here we are using that $2\cos\theta$ defines diffeomorphisms $]0,\pi[\approx]-2,2[$ and $]-\pi,\pi[\approx]-2,2[$. If we define $f(x) = 2\cos(2\pi x)$ then $\mathcal{D}_\sigma = f(DC_\sigma)$. Then, if $\sigma > 2$, $\mathcal{D}_\sigma$ has full measure in $]-2,2[$.

Unless Δ were constant we can now say that

$$\Delta^{-1}\left(]-2,2[\setminus \mathcal{D}_\sigma\right)$$

has zero measure. This implies that $\frac{\theta}{2\pi} \in DC_\sigma$ for almost every $\lambda \in \mathbb{R}$ and the conclusion follows from Corollary 2.

We prove that $\Delta(\lambda)$ is not constant in three steps:

1. $x(t,\lambda) = \Psi^{-1}(\lambda) + O(1)$ as $|\lambda| \to \infty$, uniformly in $t \in \mathbb{R}$.

   Let $\mu = \Psi^{-1}(\lambda)$ and $y(t) = x(t) - \mu$. Then $y(t)$ is a 2π-periodic solution of

   $$\ddot{y} + \Psi(y+\mu) - \Psi(\mu) = p(t).$$

   In particular $y(t)$ is also a solution of

   $$\ddot{y} + a_\mu(t)y = p(t) \qquad \text{with } a_\mu(t) = \begin{cases} \frac{\Psi(y(t)+\mu)-\Psi(\mu)}{y(t)} & \text{if } y(t) \neq 0 \\ \Psi'(\mu) & \text{if } y(t) = 0. \end{cases}$$

   We apply again Lemma 3 to deduce that

   $$\|y(\cdot,\lambda)\|_{W^{2,\infty}} \leq K\|p\|_{L^\infty}.$$

2. $\Delta(\lambda) \to 2\cos\left(2\pi\Psi'(\pm\infty)^{1/2}\right)$ as $\lambda \to \pm\infty$.

   From step 1 we deduce that $x(t,\lambda) \to \pm\infty$ as $\lambda \to \pm\infty$ uniformly in t. Then

   $$\Psi'(x(t,\lambda)) \to \Psi'(\pm\infty) \quad \text{as } \lambda \to \pm\infty$$

   uniformly in $t \in \mathbb{R}$. By continuous dependence $\Delta(\lambda) \to \operatorname{tr}(M_\pm)$ where $M_\pm$ is the monodromy matrix of $\ddot{y} + \Psi'(\pm\infty)y = 0$.

3. $\operatorname{tr}(M_+) \neq \operatorname{tr}(M_-)$ because $2\pi\Psi'(+\infty)^{1/2}$ and $2\pi\Psi'(-\infty)^{1/2}$ lie on the interval $]2n\pi,(2n+1)\pi[$.

Appendix.

Proof of Lemma 2. Assume by contradiction that $y(t)$ is a non-trivial 2π-periodic solution. The zeros of this periodic function are simple and so there is an even number of them in any interval of the type $[t_0, t_0+2\pi[$. According to Sturm comparison theory this number must be greater than $2n$ and less than $2(n+1)$. This is absurd.

Proof of Lemma 3. The existence and uniqueness of 2π-periodic solution is a consequence of the previous Lemma. We will prove the estimate by an indirect argument. Assume by contradiction that there exist sequences $a_n \in L^\infty(\mathbb{T})$, $b_n \in L^\infty(\mathbb{T})$ satisfying

$$\alpha \le a_n(t) \le \beta, \; \|b_n\|_{L^\infty} = 1, \; \|y_n\|_{W^{2,\infty}} \to \infty,$$

where $y_n(t)$ is the 2π-periodic solution of

$$\ddot{y} + a_n(t)y = b_n(t).$$

This equation leads to the inequality $\|\ddot{y}_n\|_{L^\infty} \le \beta\|y_n\|_{L^\infty} + 1$. Since $\|\dot{y}_n\|_{L^\infty} \le \pi\|\ddot{y}_n\|_{L^\infty}$ we deduce that $\|y_n\|_{L^\infty} \to \infty$. Define $z_n = y_n/\|y_n\|_{L^\infty}$. From the identities $\|z_n\|_{L^\infty} = 1$ and $\ddot{z}_n + a_n(t)z_n = b_n(t)/\|y_n\|_{L^\infty}$ we deduce that $\|\ddot{z}_n\|_{L^\infty} \le \beta + 1/\|y_n\|_{L^\infty}$. In particular $\|z_n\|_{W^{2,\infty}}$ is bounded and we can extract a subsequence z_k converging to some $z \in C^1(\mathbb{T})$. This convergence means that $\|z_k - z\|_{L^\infty} + \|\dot{z}_k - \dot{z}\|_{L^\infty} \to 0$. In particular, $\|z\|_{L^\infty} = 1$. We extract new subsequences with an additional property: the sequence a_k converges to some $a \in L^\infty(\mathbb{T})$ in the weak* sense. In particular, $\alpha \le a(t) \le \beta$ almost everywhere. Next we employ test functions $\phi \in C^\infty(\mathbb{T})$ and deduce that the identity below holds,

$$\int_{\mathbb{T}} \{-\dot{z}_k\dot{\phi} + a_k z_k \phi - \frac{b_k}{\|y_k\|_{L^\infty}}\phi\} = 0.$$

Letting $k \to \infty$,

$$\int_{\mathbb{T}} \{-\dot{z}\dot{\phi} + az\phi\} = 0.$$

Then $z(t)$ is a 2π-periodic solution of the equation $\ddot{z} + a(t)z = 0$. This is a contradiction with the previous Lemma because we know that z has L^∞-norm one and so it is a non-trivial solution. Note that the previous reasoning really proved that $z(t)$ is a solution in the variational sense, but this implies that it is also a solution in the Caratheodory sense.

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