Resonance tongues in the linear Sitnikov equation *

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Abstract

In this paper it is studied a Hill's equation, depending on two parameters $e \in [0,1)$ and $\Lambda > 0$, that has applications to some problems in Celestial Mechanics of the Sitnikov-type. Due to the nonlinearity of the eccentricity parameter e and the coexistence problem, the stability diagram in the (e,Λ)-plane presents unusual resonance tongues emerging from points $(0,(n/2)^2)$, $n = 1,2,...$ The tongues bounded by curves of eigenvalues corresponding to 2π-periodic solutions collapse into a single curve of coexistence (for which there exist two independent 2π-periodic eigenfunctions), whereas the remaining tongues have no pockets and are very thin. Unlike most of the literature related to resonance tongues and Sitnikov-type problems, the study of the tongues is made from a global point of view in the whole range of $e \in [0,1)$. Indeed, it is found an interesting behavior of the tongues: almost all of them concentrate in a small Λ-interval $[1, 9/8]$ as $e \to 1^-$.

We apply the stability diagram of our equation to determine the regions for which the equilibrium of a Sitnikov $(N+1)$-body problem is stable in the sense of Lyapunov and the regions having symmetric periodic solutions with a given number of zeros. We also study the Lyapunov-stability of the equilibrium in the center of mass of a curved Sitnikov problem.

1 Introduction

In this paper we are going to study the following biparametric equation and its applications

$$\ddot{x} + \frac{\Lambda}{r^3(t,e)}x = 0, \quad \Lambda > 0, \;\; e \in [0,1), \tag{1}$$

hereafter referred as *linear Sitnikov equation*, where $r(t,e)$ is the distance between the focus of a keplerian ellipse of eccentricity e and semi-major axis equals 1 and any of its points, it is defined by

$$r(t,e) = 1 - e\cos u(t,e), \quad t = u(t,e) - e\sin u(t,e), \tag{2}$$

where $u(t,e)$ is the associated eccentric anomaly.

Equation (1) is relevant in several problems in Celestial Mechanics, mostly related to the well known Sitnikov problem. See for example [21], [3], [29]. The classical Sitnikov problem is the elliptic restricted 3-body problem of the lowest dimension, it consists of a massless particle moving under the gravitational influence of two primaries of equal mass that are orbiting around their center of mass. The massless

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particle is confined to the straight line perpendicular to the plane of the primaries through the center of mass. When we linearize this classical problem near the equilibrium we obtain equation (1) with $\Lambda = 8$. Some papers that applied the properties of the linearized equation to the nonlinear problem were [20], [18] and [26]. There are several variations of the Sitnikov problem. We are going to apply our results to two of them. On one hand, in [27] it is formulated a generalization of the Sitnikov problem: $N \geq 2$ primaries of equal masses in elliptic coplanar orbits instead of only two. The circular case has been treated in [4], and the elliptic one in [28], [11] and [30]. The linearization near the equilibrium of this $(N+1)$-Sitnikov problem coincides with equation (1) for certain discrete values of Λ that only depend on N. This allows us to determine the regions of the (e, Λ)-plane for which the equilibrium is stable in the sense of Lyapunov, and, the regions for which there exist symmetric periodic solutions with a given number of zeros. On the other hand, the paper [12] considers a massless particle confined to a circumference instead of a straight line, as in the classical problem. Again, equation (1) is the linear equation near the equilibrium in the center of mass of the two primaries. In this case Λ is proportional to the radius of the circumference, so, here Λ is a continuous parameter. In this paper we find out the points (e, Λ) for which the equilibrium is Lyapunov-stable.

The linear Sitnikov equation is an example of the Hill's equation

$$\ddot{x} + a(t)x = 0, \tag{3}$$

where $a(t)$ is a periodic function. The case of even Hill's equation $a(-t) = a(t)$ is especially considered due to further symmetry properties. We will say that an ODE, like (3), is *stable* if and only if every solution of it is bounded, otherwise it is *unstable*. Our principal theorem gives a full description of the structure of stability/instability regions of equation (1) in the plane of parameters e, Λ. Instability regions of a biparametric Hill's equation are usually called *resonance tongues*.

There is a vast literature that deals with the resonance tongues in the (α, β)-plane of the case $a(t) = \alpha + \beta p(t)$, which is a generalization of the well known Mathieu's equation. The classical Mathieu case, for which $p(t) = \cos t$, presents resonance tongues emanating from the points $((n/2)^2, 0),\ n = 1, 2, ...,$ see [31]. Some papers, such as [1], [5], [6] and [13], deal with alternative versions or perturbations of the Mathieu's equation. It turns out that similar resonance tongues appear with a new feature, the boundaries of a certain tongue cross each other creating, the commonly named, *instability pockets*.

It is remarkable that in our case resonance tongues behave considerably different than what has been found so far: some tongues vanish and the rest have no pockets. Equation (1) is a biparametric even Hill's equation, with Λ as multiplicative parameter, just like β, and e as nonlinear parameter, instead of the additive parameter α. It is important to mention that in the (α, β)-plane standard theory of periodic eigenvalues can be applied for horizontal lines (β constant) and if, furthermore, $p(t) > 0$, standard theory can also be applied for vertical lines (α constant). See [19], [10], or the more recent book [7]. However, in our (e, Λ)-plane, standard theory can only be applied for vertical lines (e constant), since $r(t, e) > 0$, whereas for horizontal lines (Λ constant) a more complicated behavior is shown, as one can see in [20] and [18] for the specific value $\Lambda = 8$.

Under the change of variable $t = t(u) = u - e \sin u$, where u is the eccentric anomaly, the linear Sitnikov equation (1) can be converted into

$$(1 - e\cos u)x'' - e \sin u\ x' + \Lambda x = 0, \tag{4}$$

where $x' = dx/du$. This is an example of the so-called Ince's equation. The most interesting topic related to this equation is the *coexistence* problem, i.e. existence of two independent periodic solutions with the same period. See [19] Chapter 7. Its applications to the linearization of the Sitnikov problem was first studied in [20], also, as it was pointed out in [18], the numerical computation of periodic eigenvalues of (1) is much faster using the form (4). In this paper we extend for all $\Lambda > 0$ the results of [20] concerning coexistence, which turn out to be the key to understand the structure of our particular resonance tongues.

This paper is organized as follows. In Section 2 the stability diagram (Figure 1) of the linear Sitnikov equation and its resonance tongues are described, mostly based on Theorem 1. The proof of it is carried out in Section 3. In Section 4 we apply our results to the $(N+1)$-Sitnikov problem, see [28] and [11], and to the curved Sitnikov problem, see [12].

2 Stability diagram of the linear Sitnikov equation

Theorem 1 *In the (e,Λ)-plane of parameters there exist three families of functions $\Lambda_n^{\downarrow}$, $\Lambda_n^{\uparrow}$ and Λ_n, with $n = 1, 2...$, analytic in $e \in [0,1)$ and such that the equation (1) has a non-trivial 2π-periodic solution if and only if $\Lambda = \Lambda_n(e)$, and, it has a non-trivial 4π-periodic (not 2π-periodic) solution if and only if $\Lambda = \Lambda_n^{\downarrow}(e)$ or $\Lambda = \Lambda_n^{\uparrow}(e)$*

They have the following properties

i) For $e = 0$ they satisfy

$$\Lambda_n^{\uparrow}(0) = \Lambda_n^{\downarrow}(0) = (n - 1/2)^2, \quad \Lambda_n(0) = n^2,$$

whereas for each value $e \in (0,1)$, they are arranged as monotone increasing sequences

$$\Lambda_1^{\downarrow}(e) < \Lambda_1^{\uparrow}(e) < \Lambda_1(e) < ... < \Lambda_n^{\downarrow}(e) < \Lambda_n^{\uparrow}(e) < \Lambda_n(e) < ..., \tag{5}$$

and, furthermore, periodic solutions corresponding to $\Lambda_n^{\downarrow}(e)$ or $\Lambda_n^{\uparrow}(e)$ have $2n-1$ zeros in $[0, 2\pi)$ and the corresponding to $\Lambda_n(e)$ have $2n$ zeros in the same interval.

ii) If $\Lambda = \Lambda_n(e)$ then every solution is 2π-periodic (coexistence), whereas if $\Lambda = \Lambda_n^{\downarrow}(e)$ or $\Lambda = \Lambda_n^{\uparrow}(e)$ then there exists an unbounded solution.

iii) The instability regions of (1) are bounded and defined by

$$R_n = \{(e,\Lambda) \ : \ \Lambda_n^{\downarrow}(e) \le \Lambda \le \Lambda_n^{\uparrow}(e)\}. \tag{6}$$

iv) The function $\Lambda_1(e)$ equals 1 identically, additionally, for each $n \ge 2$

$$1 \le \liminf_{e \to 1^-} \Lambda_n(e) \le \limsup_{e \to 1^-} \Lambda_n(e) \le 9/8,$$

moreover,

$$\lim_{n \to \infty} \left(\liminf_{e \to 1^-} \Lambda_n(e) \right) = \frac{9}{8}.$$

The stability diagram in Figure 1 was first published in [11], in order to study the linear stability of the equilibrium in the $(N+1)$-Sitnikov problem. There, the approach was local: perturbing the associated integrable hamiltonian for $e = 0$ and expanding the functions $\Lambda_n^{\downarrow}$, $\Lambda_n^{\uparrow}$ and Λ_n, $n = 1, 2, 3$, for

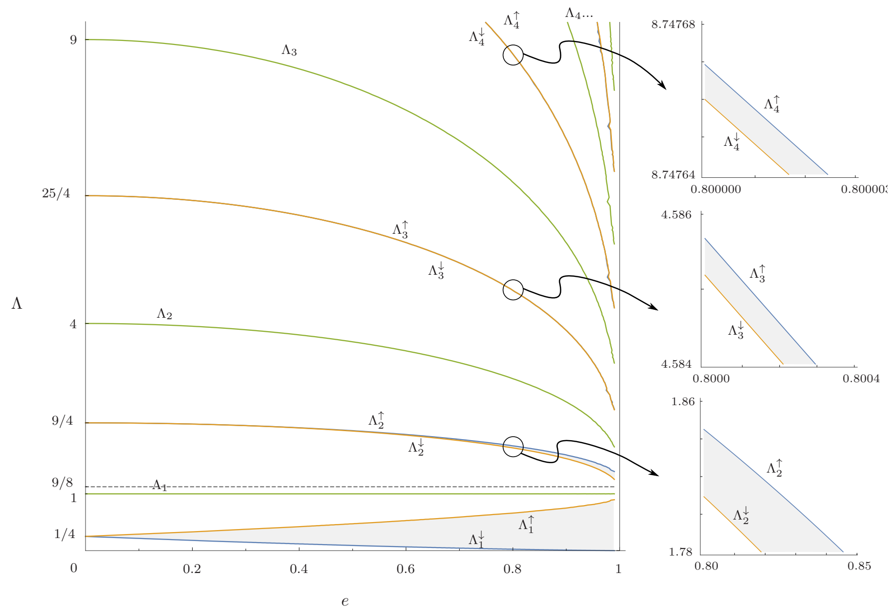

Figure 1: Left: Stability diagram of the linear Sitnikov equation. Unstable regions (shaded regions) are limited by parabolic unstable lines $\Lambda_n^{\downarrow}$ and $\Lambda_n^{\uparrow}$. The blue ones correspond to even 4π-periodic solutions, orange ones to odd 4π-periodic solutions and green ones to 2π-periodic solutions in coexistence. Right: Zoom in of some unstable regions.

small e. Theorem 1 characterizes, from a global approach, the underlying structure of the diagram and shows that there are interesting asymptotic phenomena as $e \to 1$.

The ordering in i) is a sequence of periodic eigenvalues for a fixed value of e. Standard theory of periodic eigenvalues can be applied since $\Lambda/r(t,e)^3 > 0$, see [10]. However, the sequence (5) has additional information that comes from theory of Ince's equation, that is, information about the coexistence problem. As we will see in Lemma 1, there is coexistence for 2π-periodic solutions for every value of e and 4π-periodic solutions cannot be in coexistence for $e \in (0,1)$.

From standard theory of periodic eigenvalues of the even Hill's equation we know that the 4π-periodic solutions associated to the curves $\Lambda_n^{\downarrow}(e)$ or $\Lambda_n^{\uparrow}(e)$ must be either odd or even. Moreover, the symmetry is preserved along each curve and, additionally, for each n, periodic solutions corresponding to $\Lambda_n^{\downarrow}$ has the opposite symmetry than those corresponding to $\Lambda_n^{\uparrow}$. Figure 1 shows that there is no direct correspondence between the type $\uparrow$ or $\downarrow$ and the symmetry of periodic solutions. For instance, for $n = 1$ odd periodic solutions correspond to $\Lambda_1^{\uparrow}$, while for other values of n they correspond to curves of type $\downarrow$.

In the Mathieu-type problems m is the order of the resonance tongue that arises from the point $(\alpha, \beta) = ((m/2)^2, 0),\ m = 1, 2, ...$ Simmilarly, in our case, the resonance tongue of order m is the tongue that emerges from the point $(e, \Lambda) = (0, (m/2)^2)$. Thus, point iii) claims that there only exist resonance tongues of odd order $m = 2n-1$ and are limited by the graphs of $\Lambda_n^{\downarrow}$ or $\Lambda_n^{\uparrow}$, whereas those tongues of even order $m = 2n$ collapse into the curves of coexistence Λ_n. Furthermore, the ordering in i) implies that the remaining resonance tongues have no pockets. This is remarkable because, as far as we know, it has not been studied a Hill's equation with vanishing tongues.

Regarding point iv), it is interesting that the vanishing tongue collapsed in Λ_1 is constant and equals 1 for all e. Actually, it is easy to find the general 2π-periodic solutions explicitly in the variable u, thanks to the form of Ince's equation (4). Note also that the sequence (5) implies that analogous expressions to those of point iv) also hold for $\Lambda_n^{\downarrow}$ and $\Lambda_n^{\uparrow}$.

There are another features for which we did not find an analytical proof and are suggested by Figure 1. For example, it looks like $\Lambda_1^{\downarrow}(e) \to 0$ and $\Lambda_1^{\uparrow}(e) \to 1$ as $e \to 1^-$. It means that under $\Lambda = 1$ there might be only one transition from stability to instability, except for the value $\Lambda = 1/4$, which might be of instability for all $e \in [0,1)$.

Also, it seems that all the curves are strictly decreasing or increasing for $e \neq 0$, this problem is complicated because e is a nonlinear parameter.

If we call $\delta_n(e) = \Lambda_n^{\uparrow}(e) - \Lambda_n^{\downarrow}(e)$ the width of the tongue of order $2n-1$, then, according to [11], as $e \to 0$

$$\delta_1(e) = \frac{3}{4}e + O(e^3), \quad \delta_2(e) = \frac{45}{1024}e^3 + O(e^5), \quad \delta_3(e) = \frac{525}{1048576}e^5 + O(e^7),$$

for $n \geq 4$ it becomes more and more complicated, so, a similar estimate for all n, like that was made in [17], has not been possible to accomplish. However, considering that close to $e = 0$ the solutions of the equation (1) behave as solutions of the Mathieu equation

$$\ddot{x} + \Lambda(1 + 3e\cos t)x = 0,$$

according to [17], the width of the m-resonance tongue is of the order $O(e^m \Lambda^m)$ as $e \to 0$. In our case there are only tongues of odd order $m = 2n-1$, then, $\delta_n(e)$ should be at least of the order $O(e^{2n-1})$ because $\Lambda_n^{\downarrow}(e)$ and $\Lambda_n^{\uparrow}(e)$ are of the order $O(1)$.

It is also remarkable that for $n \geq 2$ the resonance tongues remain very thin even for small values of n and large values of e, actually, almost all the tongues concentrate in $\Lambda \in [1, 9/8]$ as $e \to 1^-$. In contrast, in other stability diagrams with a non-bounded additive parameter α the resonance tongues expand for values of α large enough. In the right panel of Figure 1 we can see that for such a large value of the eccentricity as $e = 0.8$ tongues have to be magnified very much in order to observe clearly the instability region.

3 Proof of Theorem 1

This proof is carried out in four steps.

3.1 Existence of $\Lambda_n^{\downarrow}$, $\Lambda_n^{\uparrow}$ and Λ_n

Let $x = x(t, e, \Lambda)$ be a solution of (1). It will be 2π-periodic in t if and only if it satisfies the periodic boundary conditions

$$x(0, e, \Lambda) = x(2\pi, e, \Lambda), \quad \dot{x}(0, e, \Lambda) = \dot{x}(2\pi, e, \Lambda), \tag{7}$$

respectively, it will be 4π-periodic[1] in t if and only if it satisfies

$$x(0, e, \Lambda) = -x(2\pi, e, \Lambda), \quad \dot{x}(0, e, \Lambda) = -\dot{x}(2\pi, e, \Lambda). \tag{8}$$

Since $\Lambda/r^3(t, e) > 0$ we can apply Theorem 3.1 in [10], Chapter 8. It states that the corresponding periodic eigenvalues of the equation (1), given a value of e, satisfy the following sequence

$$-\infty < \Lambda_0^+(e) < \Lambda_1^-(e) \leq \Lambda_2^-(e) < \Lambda_1^+(e) \leq \Lambda_2^+(e) < \Lambda_3^-(e) \leq \Lambda_4^-(e) < ..., \tag{9}$$

where $\Lambda_i^+(e)$, $i \geq 0$, are eigenvalues corresponding to (7) and $\Lambda_i^-(e)$, $i \geq 1$, are eigenvalues corresponding to (8). One of the strict equalities meets for eigenvalues of type " + " (or " − ") if and only if there exist two independent 2π-periodic solutions (or 4π-periodic solutions). Furthermore, solutions with eigenvalue $\Lambda = \Lambda_0^+(e)$ has no zeros in the interval $[0, 2\pi)$, while solutions with eigenvalue $\Lambda = \Lambda_{2i-1}^+(e)$ or $\Lambda_{2i}^+(e)$, $i \geq 1$, have $2i$ zeros and solutions with eigenvalue $\Lambda = \Lambda_{2i-1}^-(e)$ or $\Lambda_{2i}^-(e)$, $i \geq 1$, have $2i - 1$ zeros.

For $e = 0$ the general solution of (1) is

$$x(t, 0, \Lambda) = A\cos(\sqrt{\Lambda}t) + B\sin(\sqrt{\Lambda}t), \quad A, B \in \mathbb{R}, \tag{10}$$

it will be 2π-periodic if and only if $\Lambda = n^2$, $n = 1, 2...$, consequently,

$$\Lambda_0^+(0) = 0, \quad \Lambda_{2n-1}^+(0) = \Lambda_{2n}^+(0) = n^2,$$

notice that we have taken into account that for $e = 0, \Lambda < 0$, there is no periodic solution. Similarly, $x(t, 0, \Lambda)$ will be 4π-periodic solutions if and only if $\Lambda = (n - 1/2)^2$, $n = 1, 2...$, that is to say,

$$\Lambda_{2n-1}^-(0) = \Lambda_{2n}^-(0) = (n - 1/2)^2.$$

For $\Lambda = 0$ the general solution is

$$x(t, e, 0) = c_1 + c_2 t, \quad c_1, c_2 \in \mathbb{R},$$

[1] From now on with 4π-periodic we mean also not 2π-periodic, unless otherwise indicated.

since for $c_2 = 0$ the periodic conditions (7) are satisfied, then, we can identify $\Lambda_0^+(e) \equiv 0$.

In order to obtain the sequence (5) we need to prove that for the eigenvalues of type + (resp. of type −) the symbols $\leq$ in (9) must be replaced by = (resp. by <). As we mentioned before, this problem is equivalent to the problem of *coexistence*, that is, existence of two independent 2π-periodic solutions (resp. 4π-periodic solutions).

This goal follows directly from the following lemma, that will be proved at the end of this part.

Lemma 1 *The 4π-periodic solutions of (1), with $e \in (0,1)$, are never in coexistence, whereas their 2π-periodic solutions are in coexistence for all $e \in [0,1)$.*

This lemma allows us to rewrite the sequence (9) as

$$0 < \Lambda_1^-(e) < \Lambda_2^-(e) < \Lambda_1^+(e) = \Lambda_2^+(e) < \Lambda_3^-(e) < \Lambda_4^-(e) < \Lambda_3^+(e) = \Lambda_4^+(e) < ...,$$

and define

$$\Lambda_n(e) := \Lambda_{2n-1}^+(e) = \Lambda_{2n}^+(e), \quad \Lambda_n^{\downarrow}(e) := \Lambda_{2n-1}^-(e) \neq \Lambda_n^{\uparrow}(e) := \Lambda_{2n}^-(e),$$

and, as a result, solutions associated with $\Lambda_n(e)$ have $2n$ zeros in $[0, 2\pi)$, whereas solutions associated with $\Lambda_n^{\downarrow}(e)$ and $\Lambda_n^{\uparrow}(e)$ have $2n-1$ zeros in $[0, 2\pi)$.

Proof of Lemma 1.

This proof is based on the fact that, as we mentioned in the Introduction, the linear Sitnikov equation (1) can be converted into an Ince's equation with a change of variable $t = t(u) = u - e \sin u$, where u is the eccentric anomaly, it turns out to be

$$I(x, e, \Lambda) := (1 - e \cos u)x'' - e \sin u \ x' + \Lambda x = 0,$$

where $x' = dx/du$. In our case we are going to make use of Theorems 7.1 and 7.3 in [19] that deal with the coexistence problem of an Ince's equation.

In general, for every Ince's equation

$$4(1 + a \cos u)x'' + 2b \sin u x' + (c + d \cos u)x = 0,$$

there are associated two polynomials

$$Q(\mu) = 2a\mu^2 - b\mu - d/2, \quad Q^*(\mu) = a(2\mu - 1)^2 - b(2\mu - 1) - d,$$

such that (Theorem 7.1 in [19]) if there exist coexistence of 2π-periodic solutions (resp. 4π-periodic solutions) then there exists an integer μ such that $Q(\mu) = 0$ (resp. $Q^*(\mu) = 0$).

The parameters corresponding to our equation are

$$a = -e, \quad b = -2e, \quad c = 4\Lambda, \quad d = 0,$$

so, the Ince's polynomials are in our case

$$Q(\mu) = -2e\mu(\mu - 1), \quad Q^*(\mu) = -e(2\mu - 1)(2\mu - 3),$$

then, for $e \neq 0$, the coexistence is only possible for 2π-periodic solutions.

For the special case $\Lambda = 1$ we easily find the following two independent 2π-periodic solutions in coexistence

$$\cos u - e, \qquad \sin u,$$

but for values $\Lambda \neq 1$ we need the following.

We know that there exists the sequence of periodic eigenvalues (9) with their respective eigenfunctions, we are interested in the 2π-periodic eigenfunctions, whose eigenvalues are $\Lambda_i^+(e)$.

Now, according to Theorem 7.3 in [19], if there exists a 2π-periodic symmetric solution, with Fourier series

$$\sum_{n=0}^{k} A_n \cos nu, \;\; A_k \neq 0, \quad \text{or} \quad \sum_{n=1}^{k} B_n \sin nu, \;\; B_k \neq 0,$$

where k is greater than largest non-negative integer root of $Q(\mu)$, then there is coexistence of 2π-periodic solutions.

In our case the largest non-negative integer root of $Q(\mu)$ is 1. In addition, by direct substitution we check that our 2π-periodic eigenfunctions cannot have any of the following forms

$$\hat{x}_1(u) = A_0 + A_1 \cos u, \;\; A_1 \neq 0, \quad \hat{x}_2(u) = B_1 \sin u, \;\; B_1 \neq 0,$$

since for $e \neq 0$

$$I(\hat{x}_1, e, \Lambda_i^+(e)) = (\Lambda_i^+(e) - 1)A_1 \cos u + eA_1 + \Lambda_i^+(e)A_0 \neq 0,$$

$$I(\hat{x}_2, e, \Lambda_i^+(e)) = (\Lambda_i^+(e) - 1)B_1 \sin u \neq 0,$$

then, our 2π-periodic eigenfunctions must have a Fourier series with more non-vanishing terms, which implies that 2π-periodic solutions are in coexistence, and this is independent of the value of e. ■

3.2 Periodic solutions and stability

Let x_1 and x_2 be the normalized solutions of (1), i.e. solutions generated by the initial conditions

$$x_1(0, e, \Lambda) = 1, \quad \dot{x}_1(0, e, \Lambda) = 0, \quad x_2(0, e, \Lambda) = 0, \quad \dot{x}_2(0, e, \Lambda) = 1,$$

then, x_1 is always an even solution and x_2 an odd solution.

Equation (1) is an even Hill's equation, since $\Lambda/r^3(e, t)$ is an even function in t. Normalized solutions play a special role in this case because they satisfy, not indicating dependence on e and Λ explicitly, the relations

$$\begin{aligned} x_1(2\pi) &= 2x_1(\pi)\dot{x}_2(\pi) - 1 = 2\dot{x}_1(\pi)x_2(\pi) + 1 \\ x_2(2\pi) &= 2x_2(\pi)\dot{x}_2(\pi) \\ \dot{x}_1(2\pi) &= 2x_1(\pi)\dot{x}_1(\pi) \\ \dot{x}_2(2\pi) &= x_1(2\pi), \end{aligned} \tag{11}$$

and

$$
\begin{array}{rcl}
\dot{x}_1(\pi,e,\Lambda)=0 & \Leftrightarrow & x_1 \text{ is } 2\pi\text{-periodic} \\
x_2(\pi,e,\Lambda)=0 & \Leftrightarrow & x_2 \text{ is } 2\pi\text{-periodic} \\
x_1(\pi,e,\Lambda)=0 & \Leftrightarrow & x_1 \text{ is } 4\pi\text{-periodic} \\
\dot{x}_2(\pi,e,\Lambda)=0 & \Leftrightarrow & x_2 \text{ is } 4\pi\text{-periodic.}
\end{array} \qquad (12)
$$

Moreover, if there exists a 2π-periodic (resp. 4π-periodic) solution, then, at least one of the normalized solution is 2π-periodic (resp. 4π-periodic). From this statement follows that the 4π-periodic solutions associated to $\Lambda_n^{\downarrow}(e)$ or $\Lambda_n^{\uparrow}(e)$ are proportional either to x_1 or to x_2, and then, they are symmetric.

All the information about stability of a Hill's equation, such as (1), can be formulated in terms of a monodromy matrix like

$$
\Phi(2\pi,e,\Lambda)=\begin{pmatrix} x_1(2\pi,e,\Lambda) & x_2(2\pi,e,\Lambda) \\ \dot{x}_1(2\pi,e,\Lambda) & \dot{x}_2(2\pi,e,\Lambda)\end{pmatrix},
$$

where $\Phi(t,e,\Lambda)$ is the fundamental matrix of (1) such that $\Phi(0,e,\Lambda)=\mathbb{1}$. Here, $\mathbb{1}$ is the identity matrix. Every fundamental matrix $\Phi(t,e,\Lambda)$ associated to a Hill's equation, such as (1), satisfies that $\det\Phi(t,e,\Lambda)=\det\Phi(0,e,\Lambda)=1$, according to the Abel-Liouville formula. Particularly, it is satisfied by the monodromy matrix $\Phi(2\pi,e,\Lambda)$. This is equivalent to say that $\Phi(2\pi,e,\Lambda)$ belongs to, $Sp(2,\mathbb{R})$, the symplectic group of 2×2 matrices, so it has the form

$$
\Phi(2\pi,e,\Lambda)=\begin{pmatrix} \alpha+\delta & \beta+\gamma \\ -\beta+\gamma & \alpha-\delta\end{pmatrix}, \quad \alpha^2+\beta^2=1+\gamma^2+\delta^2, \qquad (13)
$$

where $\alpha,\beta,\gamma,\delta$ are real-valued functions of (e,Λ). Moreover, for an even Hill's equation, $\Phi(2\pi,e,\Lambda)$ belongs to a three-dimensional subset of $Sp(2,\mathbb{R})$ defined by $\delta=0$. This subset is a subgroup of $Sp(2,\mathbb{R})$ and represents a γ-axis one-sheeted hyperboloid in the (α,β,γ)-space.

The stability criterion for Hill's equation coincides with the standard classification of $Sp(2,\mathbb{R})$ matrices.

Definition 1 *If $\Phi(2\pi,e,\Lambda)$ has the form (13), then it is*

- *Elliptic stable if $|\alpha(e,\Lambda)|<1$*
- *Parabolic if $\alpha(e,\Lambda)=\pm 1$*
  - *Parabolic stable if $\Phi(2\pi,e,\Lambda)=\pm\mathbb{1}$*
  - *Parabolic unstable if $\Phi(2\pi,e,\Lambda)\neq\pm\mathbb{1}$*
- *Hyperbolic unstable if $|\alpha(e,\Lambda)|>1$.*

Considering (11) and (12), $\Phi(2\pi,e,\Lambda)$ is parabolic with $\alpha(e,\Lambda)=+1$ (resp. with $\alpha(e,\Lambda)=-1$) if and only if there exists a non-trivial 2π-periodic (resp. 4π-periodic) solution. It will be parabolic stable if and only if there is coexistence and it will be parabolic unstable if and only if there is not. Then, the points (e,Λ) of the type $(e,\Lambda_n(e))$, $e\in[0,1)$, and $(0,(n-1/2)^2)$ correspond to parabolic stable equations. Similarly, the points of the type $(e,\Lambda_n^{\downarrow}(e))$ and $(e,\Lambda_n^{\uparrow}(e))$ for $e\neq 0$ correspond to parabolic unstable equations.

In addition, from Theorem 2.1 in [19] we know that, for each $e\in[0,1)$, the equation is hyperbolic unstable in the open interval between two consecutive periodic eigenvalues associated to the same period.

According to Lemma 1 the equation is hyperbolic unstable in the intervals of the type $(\Lambda_n^{\downarrow}(e), \Lambda_n^{\uparrow}(e))$. Actually, since $\alpha(e,\Lambda) = \dot{x}_2(2\pi, e, \Lambda) = x_1(2\pi, e, \Lambda)$ varies continuously with Λ it is also true that $\alpha(e,\Lambda) \leq 1$, particularly, the hyperbolic unstable regions satisfy

$$\alpha(e,\Lambda) < -1 \qquad \forall (e,\Lambda) \ : \ \Lambda_n^{\downarrow}(e) < \Lambda < \Lambda_n^{\uparrow}(e), \ e \neq 0, \tag{14}$$

while the elliptic stable regions satisfy that $-1 < \alpha(e,\Lambda) < 1$ for each point (e,Λ) such that

$$0 < \Lambda < \Lambda_1^{\downarrow}(e) \ \text{ or } \ \Lambda_n^{\uparrow}(e) < \Lambda < \Lambda_{n+1}^{\downarrow}(e), \ \Lambda \neq \Lambda_n(e).$$

3.3 Analyticity of functions and associated symmetry

The main tool we will apply repeatedly here is the real analytic version of the implicit function theorem (Theorem 2.4.4 in [16]). According to it, $u(t,e)$, defined in (2), can be seen as a real analytic function defined for $(t,e) \in \mathbb{R} \times (-1,1)$, even though e can be interpreted as eccentricity only for non-negative values. The same is true for $r(t,e)$. In consequence, according to the real analytic version of the theorem of differentiability of solutions respect to parameters, the solutions of equation (1) will be analytic in the whole domain $(t,e,\Lambda) \in \mathbb{R} \times (-1,1) \times \mathbb{R}$. In the following we will take this into account for studying the analyticity of the functions $\Lambda_n^{\downarrow}$, $\Lambda_n^{\uparrow}$ and Λ_n for any fixed value e, including $e = 0$.

It is satisfied that

$$\alpha(e, \Lambda_n(e)) = 1 \qquad \alpha(e, \Lambda_n^{\downarrow}(e)) = \alpha(e, \Lambda_n^{\uparrow}(e)) = -1,$$

this suggests to use the implicit function theorem to prove analyticity of $\Lambda_n^{\downarrow}$, $\Lambda_n^{\uparrow}$ and Λ_n again. However, this will not work for two reasons. First, the identity $\Lambda_n^{\downarrow}(0) = \Lambda_n^{\uparrow}(0)$ implies that the uniqueness of the implicit function theorem is lost at $e = 0$. Second, it can be checked that the coexistence phenomenon for solutions of period 2π implies that the partial derivative $\frac{\partial \alpha}{\partial \Lambda}(e, \Lambda_n(e))$ vanishes. The following lemma shows us that it is more convenient to define the functions $\Lambda_n^{\downarrow}$, $\Lambda_n^{\uparrow}$ and Λ_n implicitly by the expressions (12), that is, in terms of the corresponding symmetric periodic solutions.

Lemma 2 *If $x_1(t,e,\Lambda) = 0$ then*

$$\frac{\partial x_1}{\partial \Lambda}(t,e,\Lambda) \neq 0.$$

This is also true replacing x_1 by any of the functions $x_2, \dot{x}_1, \dot{x}_2$.

Proof of Lemma 2.

We define

$$z_i(t,e,\Lambda) = \frac{\partial}{\partial \Lambda} x_i(t,e,\Lambda), \quad i = 1,2,$$

if we take derivative with respect to Λ in the equation (1) we see that z_1 must satisfy

$$\ddot{z}_1 + \frac{\Lambda}{r^3(t,e)} z_1 = -\frac{x_1(t,e,\Lambda)}{r^3(t,e)},$$

the formula of variation of constantants gives us

$$z_1(t,e,\Lambda) = x_1(t,e,\Lambda)\int_0^t \frac{x_1(s,e,\Lambda)x_2(s,e,\Lambda)}{r^3(s,e)}ds - x_2(t,e,\Lambda)\int_0^t \frac{x_1^2(s,e,\Lambda)}{r^3(s,e)}ds,$$

and if $x_1(t,e,\Lambda) = 0$ we have

$$z_1(t,e,\Lambda) = -x_2(t,e,\Lambda)\int_0^t \frac{x_1^2(s,e,\Lambda)}{r^3(s,e)}ds \neq 0,$$

since the integrand is non-negative and $x_2(t,e,\Lambda) \neq 0$. The last is true because x_1 and x_2 are independent nontrivial solutions and, therefore, neither they nor their derivatives can have a common root.

We proceed analogously in the case of a zero of $\dot{x}_1(t,e,\Lambda), x_2(t,e,\Lambda)$ and $\dot{x}_2(t,e,\Lambda)$. ■

The real analytic version of the implicit function theorem guarantees that, given a fixed point $(\hat{e},\hat{\Lambda})$, $\hat{\Lambda} \neq 0$, such that

$$\dot{x}_1(\pi,\hat{e},\hat{\Lambda}) = 0,$$

there exists an open neighborhood U of $\hat{e}$ and a function $\lambda : U \to \mathbb{R}$, such that

$$\dot{x}_1(\pi,e,\lambda(e)) = 0 \quad \forall e \in U, \tag{15}$$

that is analytic at $\hat{e}$, since Lemma 2 assures that

$$\frac{\partial \dot{x}_1}{\partial \Lambda}(\pi,\hat{e},\hat{\Lambda}) \neq 0.$$

According to (12) and (15), the solutions $x_1(t,e,\lambda(e))$ are 2π-periodic and satisfy the same boundary conditions for all $e \in U$. Besides, from Sturm theory we know that the zeros of the solution $x_1(t,e,\lambda(e))$ are simple. Consequently, the number of zeros in the interval $[0,2\pi)$ of $x_1(t,e,\lambda(e))$ is conserved as long as $e \in U$.

We have proved that if there exists a non-trivial 2π-periodic solution for $(\hat{e},\hat{\Lambda})$, $\hat{\Lambda} \neq 0$, then, there must exists an integer $n \neq 0$ such that $\hat{\Lambda} = \Lambda_n(\hat{e})$. This implies that the solution $x_1(t,e,\lambda(e))$, $e \in U$, has $2n$ zeros in the interval $[0,2\pi)$. The only way for this to happen is that the analytic function λ coincides with the function Λ_n in the whole U. This can be done for each $\hat{e} \in [0,1)$, then, the functions Λ_n are analytic for each $e \in [0,1)$.

Similarly, for a given non-negative integer n, there is coexistence of 4π-periodic solutions with $2n-1$ zeros in $[0,2\pi)$ corresponding to the fixed point $(e,\Lambda) = (0,(n-1/2)^2)$. Then, there must exist two analytic functions $\lambda^{even}, \lambda^{odd} : [0,1) \to \mathbb{R}$, defined respectively by

$$x_1(\pi,e,\lambda^{even}(e)) = 0, \quad \dot{x}_2(\pi,e,\lambda^{odd}(e)) = 0, \tag{16}$$

such that $\lambda^{even}(0) = \lambda^{odd}(0) = (n-1/2)^2$. This functions generate 4π-periodic solutions $x_1(t,e,\lambda^{even}(e))$, $x_2(t,e,\lambda^{odd}(e))$ with $2n-1$ zeros in $[0,2\pi)$. Now we see clearly that, since there is no coexistence for $e \neq 0$, then, one of the functions must be $\Lambda_n^{\downarrow}$ and the other $\Lambda_n^{\uparrow}$.

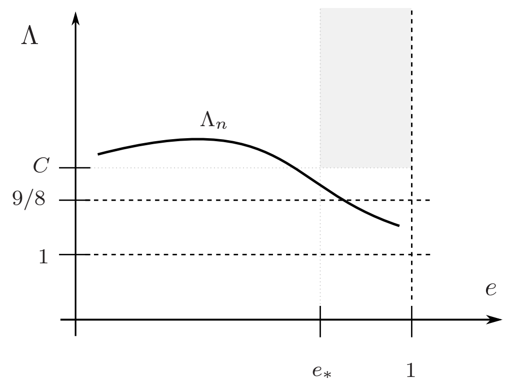

Figure 2: The shaded region is forbidden for $\Lambda_n(e)$.

3.4 Asymptotic behavior as $e \to 1^-$

Recall that we can write equation (1) in the form of Ince's equation as

$$(1 - e\cos u)x'' - e\sin u\; x' + \Lambda x = 0 \tag{17}$$

and if we take $\Lambda = 1$, by direct substitution, we find that

$$\cos u - e, \qquad \sin u, \tag{18}$$

are two symmetric independent 2π-periodic solutions of (17) for each value of $e \in [0, 1)$. These solutions have 2 zeros for $u \in [0, 2\pi)$, the same is true in the variable t. This implies that $\Lambda_1(e) \equiv 1$.

According to the ordering in (5), last result implies that only the graphs of $\Lambda_1^{\downarrow}$ and $\Lambda_1^{\uparrow}$ lie below the horizontal line $\Lambda = 1$, whereas the rest lie above.

The first consequence is that

$$\liminf_{e \to 1^-} \Lambda_n(e) \geq 1, \quad n \geq 2,$$

similarly for $\Lambda_n^{\downarrow}(e)$ and $\Lambda_n^{\uparrow}(e)$. In order to set an upper bound for the upper limit as $e \to 1^-$ we need the following lemma.

Lemma 3 (from [28], Theorem 2) *Given a fixed value of Λ, as $e \to 1^-$ the number of zeros in the interval $[0, 2\pi)$ of a solution $x = x(t, e, \Lambda)$ of the equation (1) tends to infinity if $\Lambda > 9/8$, and it is bounded if $\Lambda < 9/8$.*

Sturm comparison theory guarantees that this Lemma is a direct consequence of Theorem 2 in [28].

Let e and $C > 9/8$ be fixed values and let ν_C be the number of zeros of a solution $x(t, e, C)$ in $[0, 2\pi)$. According to Sturm comparison theory between every two consecutive zeros of $x(t, e, C)$ there must be a zero of any solution $x(t, e, \Lambda), \Lambda > C$, then its number of zeros in $[0, 2\pi)$ is larger or equal to $\nu_C - 1$.

Let n be a fixed positive natural number, then, from Lemma 3 we know that there exists a value e_* such that for any solution $x(t, e, C)$, $e > e_*$ we have that ν_C is larger than some number $2n + 1$.

Then, any solution $x(t,e,\Lambda),\ e > e_*,\ \Lambda > C$, has more than $2n$ zeros in $[0,2\pi)$. Since solutions associated to Λ_n has exactly $2n$ zeros in $[0,2\pi)$, then, the region such that $e > e_*,\ \Lambda > C$, is forbidden for the graph of the function Λ_n (Figure 2), then

$$\limsup_{e\to 1^-} \Lambda_n(e) \le C,$$

and, since we can take C as close as we want to $9/8$, for each $n \ge 2$

$$\limsup_{e\to 1} \Lambda_n(e) \le 9/8.$$

If we now take a horizontal line $\Lambda = C < 9/8$, Lemma 3 implies that there exist a finite number of functions Λ_n whose graph crosses the line $\Lambda = C$. Since the values $\Lambda_n(e),\ e \ne 0$, are organized in an infinite increasing sequence (5), then there exists an integer $n_0 > 1$ such that

$$\liminf_{e\to 1^-} \Lambda_n(e) \ge C, \quad n \ge n_0.$$

But again, we can take C as close as we want to $9/8$, then, it must be satisfied that

$$\lim_{n\to\infty}\left(\liminf_{e\to 1^-} \Lambda_n(e)\right) = \frac{9}{8}.$$

4 Applications

4.1 Stability of the equilibrium of the $(N+1)$-Sitnikov problem

We want to apply our results mainly to the $(N+1)$-Sitnikov problem, treated in [28], [11], [30], and first posed in [27]. Consider a system of N gravitating bodies of equal mass M moving in the same plane. The k-th body is moving according to

$$\ddot{v}_k + GM\sum_{j\ne k}\frac{v_k - v_j}{|v_k - v_j|^3} = 0, \quad j,k = 1,...,N, \tag{19}$$

where $v_k = v_k(t) \in \mathbb{C}$ is its position and G is the gravitational constant. If, at each time, the bodies are distributed in the vertexes of a regular polygon with N edges we can take the polar coordinates $v_k = \rho e^{i(\theta + \frac{2\pi}{N}k)}$, then, equation (19) becomes

$$\ddot{v}_k + \frac{GMf(N)v_k}{\rho(t)^3} = 0, \quad k = 1,...,N, \tag{20}$$

where $f(N)$ must be a real number because $\ddot{v}_k$ must be parallel to v_k. After simplifying we get

$$f(N) = \frac{1}{4}\sum_{m=1}^{N-1}\frac{1}{\sin m\frac{\pi}{N}}. \tag{21}$$

Equation (20) corresponds to the motion of a body in a newtonian potential, consequently, the k-th body's trajectory describes a keplerian ellipse, so, the distance between the focus and the body is of the form

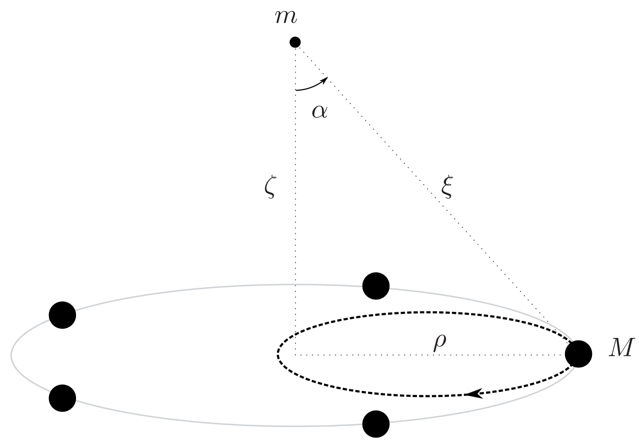

Figure 3: $(N+1)$-Sitnikov problem.

$$\rho(t,e,a) = a(1 - e\cos u(t,e)), \tag{22}$$

where a is the semi-major axis of the ellipse, $e \in [0,1)$ is its eccentricity and $u(t,e)$ is its eccentric anomaly, which satisfies the well known Kepler's equation

$$t = u(t,e) - e\sin u(t,e).$$

By a direct substitution we can say that (22) is a solution of (20) if and only if it is satisfied that

$$GMf(N) = a^3, \tag{23}$$

which is the corresponding Kepler's third law for the body. Note that, comparing to the usual Kepler's third law, $Mf(N)$ plays a role of effective mass that determines the motion of each body. Let us now consider a particle of mass m influenced gravitationally by the N bodies, this particle is confined to the perpendicular line that passes trough the center of the polygon, then, the force exerted on it is

$$F = N\frac{GMm}{\xi^2}\cos\alpha = \frac{NGMm}{\xi^3}\zeta, \quad \xi = \sqrt{\zeta^2+\rho^2},$$

ζ is the distance between the particle and the focus, ξ is the distance between the particle and one of the primaries. See Figure 3. For $M \gg m$ the configuration stays almost unchanged and the particle moves according to the equation

$$\ddot{\zeta} + \frac{GMN\zeta}{(\zeta^2+\rho^2(t,e,a))^{3/2}} = 0, \tag{24}$$

and, since GM can be replaced by $a^3/f(N)$, this equation can be rescaled to

$$\ddot{z} + \frac{\Lambda(N)z}{(z^2+r^2(t,e))^{3/2}} = 0, \quad \Lambda(N) = \frac{N}{f(N)} \tag{25}$$

where $z = \zeta/a$, $r(t,e) = 1 - e\cos u(t,e)$, and $f(N)$ is defined in (21). Note that $\Lambda = 8\lambda$, where λ is the parameter used in [28], whereas for the classical Sitnikov problem $\Lambda(2) = 8$ as we asserted in the Introduction. Note that in the literature it is most common to take the normalization $G = 1$, $M = 1/2$, $a = 1/2$, in equation (24). However, as we have shown here, the problem can be scaled and its qualitative properties do not depend on a or M, which can have any value as long as they satisfy the constraint (23).

In this paper we are going to establish the values of the parameters for which the trivial solution $z = 0$ is stable/unstable in the sense of Lyapunov. Following [23] and [22] we can write equation (25) as

$$\ddot{z} + a(t,e,N)z + c(t,e,N)z^3 + d(t,e,N,z) = 0, \tag{26}$$

where,

$$a(t,e,N) = \frac{\Lambda(N)}{r^3(t,e)}, \quad c(t,e,N) = -\frac{3}{2}\frac{\Lambda(N)}{r^5(t,e)},$$

and the remainder $d(t,e,N,z) = o(z^3)$ as $z \to 0$ and it is 2π-periodic in t. The main results from [23] and [22] relate the stability of the equilibrium for equations in the form (26) and stability of their linearization, in our case (1), when $c(t,e,N)$ meets a requirement of sign. Since our problem satisfies that $c(t,e,N) \le 0$ we can use these results to prove the following theorem. Let

$$R_n^* = \{(e,\Lambda) \;:\; \Lambda_n^{\downarrow}(e) < \Lambda \le \Lambda_n^{\uparrow}(e),\; e \neq 0\},$$

be a set of regions in the stability diagram of parameters (e,Λ) of (1).

Theorem 2 *Given $N \ge 2$, the equilibrium of (25) is stable in the Lyapunov sense if and only if $(e,\Lambda(N)) \notin R_n^*$.*

Figure 4 is the stability diagram for the linear Sitnikov equation incluiding the lines corresponding to the discrete values $\Lambda = N/f(N)$. According to Theorem 1, there are some relevant values of Λ. In the passage from $N = 234$ to 235 the corresponding value of $\Lambda(N)$ becomes smaller than 9/8 (this was already found in [28]), from $N = 472$ to 473 the corresponding $\Lambda(N)$ becomes smaller than 1. As N becomes larger it is more difficult to calculate $f(N)$, even numerically. However, based on the relation $x \ge \sin x \ge 2x/\pi$, $x \in [0,\pi/2]$, we can make the following estimate that is useful for N large

$$\frac{N}{\pi}H_{N/2} < \sum_{m=1}^{N/2}\frac{1}{\sin m\frac{\pi}{N}} < \frac{N}{2}H_{N/2},$$

or equivalently,

$$\frac{2\pi}{H_{N/2}} > \Lambda(N) > \frac{4}{H_{N/2}},$$

where H_n is the n-th harmonic number, i. e., the n-th partial sum of the harmonic series. We have used that $\sin(N-m)\pi/N = \sin m\pi/N$. Since it is well known that, for n large, $H_n = \ln n + \gamma + O(1/n)$, where γ is the Euler–Mascheroni constant, we see clearly that $\Lambda(N) \to 0$ as $N \to \infty$. This also allows us to estimate that $\Lambda(N)$ reaches values close to 1/4 for N such that $9.23 \cdot 10^{10} > N > 9.98 \cdot 10^6$.

In order to introduce the proof of Theorem 2 recall the Definition 1 for $\Lambda = \Lambda(N)$. According to Theorem from [23], if the linear equation associated to (26) is stable, and either $c(t,e,N) \ge 0$ or

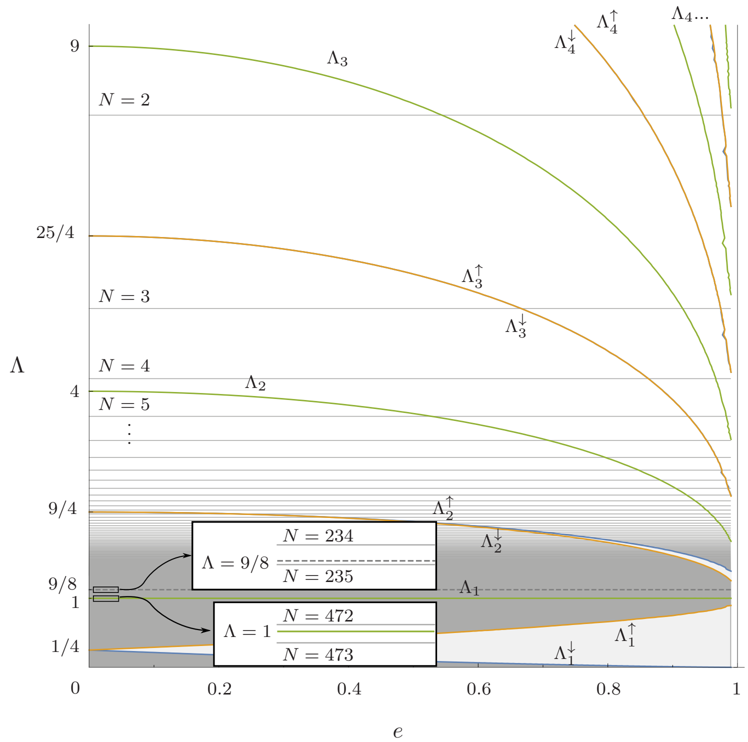

Figure 4: Horizontal grey lines correspond to a $\Lambda = N/f(N)$.

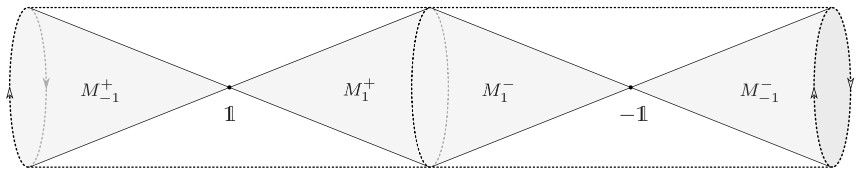

Figure 5: $Sp(2,\mathbb{R})$ as an open solid torus. The oriented curves at the ends of the cylinder are identified in order to get the torus. Each cone, excluded its vertex and its base, represents a connected component of parabolic unstable matrices, it is indicated a representative of the conjugation class.

$c(t,e,N) \leq 0$ for all $t \in \mathbb{R}$, then its equilibrium is Lyapunov-stable. This result guarantees that, if the monodromy matrix associated to (1) is elliptic or parabolic stable, then, the equilibrium of the nonlinear equation (25) is Lyapunov-stable. However, as it was pointed out in [22], if the monodromy matrix is parabolic unstable the equilibrium can still be stable or unstable in the sense of Lyapunov, depending on its conjugation class in $Sp(2,\mathbb{R})$.

Let us provide some definitions needed to apply Theorem 5.1 in [22]. We say that two matrices $A, B \in Sp(2,\mathbb{R})$ are *conjugate* if there exists another matrix $G \in Sp(2,\mathbb{R})$ such that $B = GAG^{-1}$; we denote this relation by $A \sim B$.

It is well known that every parabolic matrix $A \in Sp(2,\mathbb{R})$ is conjugate to a matrix of the form

$$M^{\pm}_{\mu} = \begin{pmatrix} \pm 1 & \mu \\ 0 & \pm 1 \end{pmatrix}, \quad \mu \in \mathbb{R},$$

furthermore, if A is a monodromy matrix associated to an even Hill's equation, then, A has either the form $M^{\pm}_{\mu}$ or $(M^{\pm}_{\mu})^T$, where $(\cdot)^T$ denote transposition.

Lemma 4 *It is satisfied that $M^{\pm}_{\mu} \sim (M^{\pm}_{-\mu})^T$. Furthermore, assume $\mu \neq 0$, then, $M^{\pm}_{\mu} \sim M^{\pm}_{k\mu}$ if and only if $k > 0$.*

This algebraic lemma, whose proof is left to the reader, shows us that there are four conjugation classes of parabolic unstable matrices in $Sp(2,\mathbb{R})$ that, moreover, correspond to four connected components, see Figure 5.

Lemma 5 *Let the monodromy matrices of (1) for $e \neq 0$, then, for each n*

$$\Phi(2\pi, e, \Lambda^{\downarrow}_n(e)) \sim M^-_1, \quad \Phi(2\pi, e, \Lambda^{\uparrow}_n(e)) \sim M^-_{-1}.$$

Proof of Lemma 5.

First, it is easy to check that, given n, the monodromy matrices associated to the values $\Lambda = \Lambda^{\downarrow}_n(e)$ (or $\Lambda = \Lambda^{\uparrow}_n(e)$) belong to the same conjugation class for all $e \neq 0$. Note that the map $e \mapsto \Phi(2\pi, e, \Lambda^{\downarrow}_n(e))$ is a continuous curve that starts at $-\mathbb{1}$ and moves along one of the cones labeled by M^-_1 or M^-_{-1} shown in Figure 5. It must stay in one of that cones because there is no coexistence for $e \neq 0$.

In order to know what is the corresponding cone let us first consider the case when the 4π-periodic eigenfunctions associated to $\Lambda = \Lambda^{\downarrow}_n(e)$ are even, which implies, according to relations (11) and (12), that

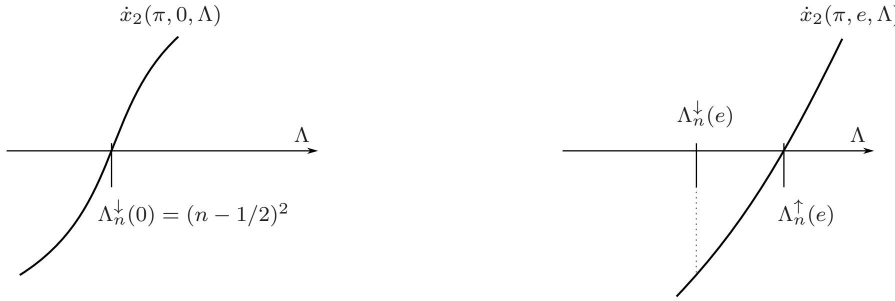

Figure 6: Left: For $e=0$, the graph of $\dot x_2(\pi,0,\Lambda)$. Right: For a fixed $e\neq 0$ we obtain the graph of $\dot x_2(\pi,e,\Lambda)$ deforming the previous one.

$$x_1(\pi,e,\Lambda_n^{\downarrow}(e))=0,\quad \Phi(2\pi,e,\Lambda_n^{\downarrow}(e))=M_\mu^-,$$

where

$$\mu=x_2(2\pi,e,\Lambda_n^{\downarrow}(e))=2x_2(\pi,e,\Lambda_n^{\downarrow}(e))\dot x_2(\pi,e,\Lambda_n^{\downarrow}(e))\neq 0.$$

In order to find out the sign of μ, recall (10) for $e=0$ and $t=\pi$

$$x_2(\pi,0,\Lambda)=\frac{1}{\sqrt{\Lambda}}\sin\sqrt{\Lambda}\pi,\quad \dot x_2(\pi,0,\Lambda)=\cos\sqrt{\Lambda}\pi. \tag{27}$$

From analiticity of solutions with respect to the parameters we know that, for a small enough $e>0$,

$$\operatorname{sgn} x_2(\pi,e,\Lambda_n^{\downarrow}(e))=\operatorname{sgn} x_2(\pi,0,\Lambda_n^{\downarrow}(0))=(-1)^{n+1}.$$

since $\Lambda_n^{\downarrow}(0)=(n-1/2)^2$. We cannot find out the sign of $\dot x_2(\pi,e,\Lambda_n^{\downarrow}(e))$ exactly in the same way, since $\dot x_2(\pi,0,\Lambda_n^{\downarrow}(0))=0$. However, note that the zeros of the graph of the function $\Lambda\mapsto\dot x_2(\pi,0,\Lambda)$ are simple. Then, due to analiticity of solutions with respect to the parameters, we can smoothly deform the previous graph, keeping its zeros simple, in order to obtain the graph of the map $\Lambda\mapsto\dot x_2(\pi,e,\Lambda)$, for a fixed and small enough $e>0$. See Figure 6. Cosidering that for $e\neq 0$, $\dot x_2(\pi,e,\Lambda)=0$ if and only if $\Lambda=\Lambda_n^{\uparrow}(e)$, and it is satisfied that $\Lambda_n^{\downarrow}(e)<\Lambda_n^{\uparrow}(e)$, then, the sign of $\dot x_2(\pi,e,\Lambda_n^{\downarrow}(e))$ must be the same as the sign of $\dot x_2(\pi,0,\Lambda)$ just to the left of the point $\Lambda=\Lambda_n^{\downarrow}(0)$, that is to say,

$$\operatorname{sgn}\dot x_2(\pi,e,\Lambda_n^{\downarrow}(e))=(-1)^{n+1},$$

then,

$$\operatorname{sgn}\mu=\operatorname{sgn} x_2(\pi,e,\Lambda_n^{\downarrow}(e))\operatorname{sgn}\dot x_2(\pi,e,\Lambda_n^{\downarrow}(e))=(-1)^{2n+2}=+1.$$

If now we assume that the 4π-periodic eigenfunctions associated to $\Lambda_n^{\downarrow}(e)$ are odd, i.e.

$$\dot x_2(\pi,e,\Lambda_n^{\downarrow}(e))=0,\quad \Phi(2\pi,e,\Lambda_n^{\downarrow}(e))=(M_{\mu'}^-)^T,$$

an analogous procedure gives us that $\operatorname{sgn}\mu' = -1$. Finally, according to Lemma 4 we have that $\Phi(2\pi, e, \Lambda_n^{\downarrow}(e)) \sim M_1^-$. Analogously, it can be checked that $\Phi(2\pi, e, \Lambda_n^{\uparrow}(e)) \sim M_{-1}^-$ as we claimed. ■

Now we can introduce the main result of [22]. Let us first define the invariants of the conjugation class within the subgroup of parabolic matrices.

Definition 2 *Let A be a parabolic matrix such that $A \sim M_\mu^\pm$, then*

$$\sigma(A) = \pm 1, \quad \nu(A) = \operatorname{sgn}\mu.$$

Here it must be understood that if $\mu = 0$ then $\operatorname{sgn}\mu = 0$. Thus, we associate a pair of numbers σ, ν to every equation in the form (1), whose monodromy matrix is parabolic. With these two numbers we are able to state Theorem 5.1 in [22]. According to it, if the linear equation associated to (26) is parabolic unstable, then the equilibrium is stable if

$$\sigma\nu c(t, e, N) \geq 0 \quad \forall t \in \mathbb{R},$$

and unstable if

$$\sigma\nu c(t, e, N) \leq 0 \quad \forall t \in \mathbb{R}.$$

Proof of Theorem 2.

From the First Lyapunov Method we see that the trivial solution $z = 0$ of (25) is Lyapunov-unstable within the interior of the regions R_n for $\Lambda = \Lambda(N)$, where R_n are defined in (6), because the linear equation (1) is hyperbolic unstable, as we observe in (14).

Since equation (1) is stable for any point in the (e, Λ)-diagram, with $\Lambda = \Lambda(N)$, that does not belong to R_n, and $c(t, e, N) \leq 0$, from the main theorem of [23] the equilibrium of (25) is Lyapunov-stable in that points.

Thus, the only points that we have left are those that belong to the curves $\Lambda = \Lambda_n^{\downarrow}(e)$ and $\Lambda = \Lambda_n^{\uparrow}(e)$. From Lemma 5

$$\sigma(\Phi(2\pi, e, \Lambda_n^{\downarrow}(e))) = -1, \quad \nu(\Phi(2\pi, e, \Lambda_n^{\downarrow}(e))) = 1,$$
$$\sigma(\Phi(2\pi, e, \Lambda_n^{\uparrow}(e))) = -1, \quad \nu(\Phi(2\pi, e, \Lambda_n^{\uparrow}(e))) = -1.$$

Finally, from Theorem 5.1 in [22] we conclude that the equilibrium of (25) is Lyapunov-stable for points $(e, \Lambda(N))$, $\Lambda(N) = \Lambda_n^{\downarrow}(e)$, since $\sigma\nu c \geq 0$, and unstable for points $(e, \Lambda(N))$, $\Lambda(N) = \Lambda_n^{\uparrow}(e)$, since $\sigma\nu c \leq 0$. ■

4.2 Existence of symmetric periodic solutions of the $(N+1)$-Sitnikov problem

Next application of Theorem 1 to the problem (25) deals with the existence of symmetric 2π-periodic solutions with a prescribed number of zeros.

Even and odd 2π-periodic solutions of equation (1) are eigenfunctions in coexistence associated to the eigenvalues $\Lambda = \Lambda_n(e)$ for a given e. From relations (12) it is satisfied that

$$x_2(\pi, e, \Lambda_n(e)) = 0, \quad \dot{x}_1(\pi, e, \Lambda_n(e)) = 0.$$

For $e = 0$ the symmetric solutions corresponding to $\Lambda = \Lambda_n(0)$ have n zeros in $[0, \pi)$, see (10). Besides, the number of zeros of the symmetric solutions is constant as we move along the graph of the function $e \mapsto \Lambda_n(e)$, since the boundary conditions are the same for all e and the zeros of the solutions are simple.

Theorem 3 *For a given value $e \in [0, 1)$, there exist an even and an odd 2π-periodic solutions of (25) with n zeros in $[0, \pi)$ if and only if $\Lambda(N) > \Lambda_n(e)$.*

To prove this theorem we need the following lemma, which is an extension of Theorem 2 in [25], because [25] deals with odd periodic solutions and our lemma deals with both types of symmetric periodic solutions.

Lemma 6 (Extension of Theorem 2 in [25]) *Given two integers $m \geq 1$ and $n \geq 0$, the following statements are equivalent*

i) There exists an even $2m\pi$-periodic[2] non-trivial solution of (25) with n critical points ($\dot{z}(t, e, \Lambda) = 0$) in $[0, m\pi)$.

ii) The even normalized solution $x_1(t, e, \Lambda)$ of the linear equation (1) has more than n critical points ($\dot{x}_1(t, e, \Lambda) = 0$) in $[0, m\pi)$.

The same is true replacing "even" by "odd", "x_1" by "x_2" and "critical points" by "zeros".

Remark. Note that symmetric periodic solutions of (25) must have the same number of zeros than of critical points in $[0, \pi)$ because it has the form

$$\ddot{z} + D(t, z)z = 0, \quad D(t, z) > 0,$$

this is why in Theorem 3 we do not mention the number of critical points, but only the number of zeros. According to this, we can express one of the assertions of Lemma 6 as: *There exists an even 2π-periodic solution of (25) with n zeros in $[0, \pi)$ if and only if $x_1(t, e, \Lambda)$ has more than n critical points in the same interval.*

Proof of Lemma 6.

Theorem 2 in [25] treats only the case of the zeros of odd solutions. Our goal is to prove that it has an extension for the case of even solutions. But now, instead of paying attention to the zeros of the odd solutions, we pay attention to the critical points of even solutions. Anyhow, our proof is entirely analogous to that of [25].

First, let us consider a more general equation

$$\ddot{z} + D(t, z)z = 0, \tag{28}$$

where $D : [0, L] \times \mathbb{R} \to \mathbb{R}$ is a continuous function in the first argument and of class C^1 in the second argument, that also satisfies,

$$D(t, 0) > D(t, z) > 0, \quad z \neq 0,$$

and

$$|D(t, z)| \leq \frac{C}{1 + |z|}, \quad (t, z) \in [0, L] \times \mathbb{R}.$$

[2] In this subsection we do not require the period to be minimum.

Let us consider the solution generated by the initial conditions

$$z(0) = d \in \mathbb{R}, \quad \dot{z}(0) = 0,$$

we denote this solution by $z(t,d)$. We are looking for even $2L$-periodic solutions, that is to say, we want to know if there exists a value of d such $z(t,d)$ satisfies the Neumann boundary conditions

$$\dot{z}(0,d) = \dot{z}(L,d) = 0.$$

Now, note that $z(t,0) \equiv 0$, then, the uniqueness of the Cauchy problem of (28) implies that the zeros of $z(t,d)$, $d \neq 0$, must be simple, then, between two consecutive zeros of $z(t,d)$ there must be a zero of $\dot{z}(t,d)$, but considering the strict inequality $D(t,z) > 0$, there must be only one zero. Consequently, the zeros of $\dot{z}(t,d)$ must be also simple. We denote the number of zeros of $\dot{z}(t,d)$ in $[0,L)$ by $\mu(d)$, which is a finite non-negative integer.

Instead, in [25] it is defined $\nu(v)$, the number of zeros in $(0,L)$ of the solution with initial conditions $z(0) = 0$, $\dot{z}(0) = v \in \mathbb{R}$. We want to prove that $\mu(d)$ satisfies the equivalent four properties than $\nu(v)$.

Let μ_0 the number of zeros in $[0,L)$ of $\dot{\chi}(t)$, where χ satisfies

$$\ddot{\chi} + D(t,0)\chi = 0, \quad \chi(0) = 1, \ \dot{\chi}(0) = 0. \tag{29}$$

Property I: $\mu(d) \leq \mu_0$ for each $d \neq 0$.

In order to prove this property we need Theorem V in [15], which is a generalization of the Sturm comparison theory. We can see equation (28) as the system

$$\dot{y}_1 = D(t,z_1)z_1, \quad \dot{z}_1 = -y_1,$$

where $z_1(t) = z(t,d)$, for a given $d \neq 0$, and equation (29) as the system

$$\dot{y}_2 = D(t,0)z_2, \quad \dot{z}_2 = -y_2,$$

where $z_2(t) = \chi(t)$. We can see that the coefficientes, particularly $D(t,z(t,d))$ and $D(t,0)$, are continuous functions in $[0,L]$ such, at least for $t = 0$, $D(t,0) > D(t,z(t,d)) > 0$, and $y_1(0) = -\dot{z}(0,d) = 0$. Then we have two systems fulfilling the hypotheses of Theorem V in [15]. Consequently, if t_n and t_n^* are the n-th zeros of $y_1(t) = -\dot{z}(t,d)$ and $y_2(t) = -\dot{\chi}(t)$, respectively, then $t_n > t_n^*$. Thus, $\dot{z}(t,d)$ cannot have more zeros than $\dot{\chi}(t)$ in $[0,L)$.

Property II: Given $h \neq 0$, there exists $\delta > 0$ such that $\mu(h) \leq \mu(d) \leq \mu(h) + 1$ if $|d-h| \leq \delta$. Assuming in addition that $\dot{z}(L,h) \neq 0$, the identity $\mu(h) = \mu(d)$ holds if $|d-h| \leq \delta$.

This property is consequence of Lemma 3 in [25], let us check the hypotheses. If we take a sequence $\{d_k\}$ converging to d, then, by continuous dependence on initial conditions, there is uniform convergence of functions

$$z(\cdot,d_k) \to z(\cdot,d), \quad \dot{z}(\cdot,d_k) \to \dot{z}(\cdot,d),$$

but there is also uniform convergence of

$$\ddot{z}(\cdot,d_k) \to \ddot{z}(\cdot,d)$$

since $\ddot{z} = -D(t,z)z$, and D is a continuous function such $D(t,z) > 0$. This let us assert that $\dot{z}(\cdot, d_k)$ converges uniformly to $\dot{z}(\cdot, d)$ in $C^1[0,L]$, whose zeros are simple and $\dot{z}(0,d) = 0$. Finally, we can apply Lemma 3 in [25], which gives us directly Property II.

Property III: There exists $d_* > 0$ such that $\mu(d) = \mu_0$ if $|d| \leq d_*$.

In an analogous way that in [25] we see that for a sequence $\{d_k\}$ converging to 0 there is uniform convergence of

$$\frac{z(t,d_k)}{d_k} \to \chi(t), \quad \frac{\dot{z}(t,d_k)}{d_k} \to \dot{\chi}(t),$$

then, by Lemma 3 in [25], for some k large enough $\mu_0 \leq \mu(d_k) \leq \mu_0 + 1$. But, moreover, Property I says that $\mu(d_k) \leq \mu_0$, which implies finally that $\mu(d_k) = \mu_0$.

Property IV: There exists $d^* > 0$ such that $\mu(d) = 0$ if $|d| \geq d^*$.

We can write

$$\dot{z}(t,d) = -\int_0^t D(s, z(s,d))z(s,d)ds,$$

and since there exists a $C > 0$ such $|D(t,z)| \leq C/(1+|z|)$, then

$$|\dot{z}(t,d)| \leq CL.$$

Since for $d \to \infty$ the functions $f_d(t) = \frac{1}{d}\dot{z}(t,d)$ converge uniformly to $f(t) = 1$ in $C^1[0,L]$, which has no zeros, then we can apply Lemma 3 in [25], so, for d large enough $\mu(d) = 0$.

Following [25], this four properties are enough to guarantee that for any $N < \mu_0$ there exists a $d_N > 0$, such that $\dot{z}(t, d_N)$ has exactly N zeros and so $z(t, d_N)$ satisfies the Neumann boundary conditions for it to be the even $2L$-periodic solution we were looking for. Actually, the condition $N < \mu_0$ is necessary for existence of an even $2L$-periodic solution with N critical points, due to the generalization of Sturm comparison theorem in [15]. ■

Now it is easy to prove Theorem 3.

Proof of Theorem 3. Let us start with the case of odd solutions. Remember that $x_2(\pi, e, \Lambda) = 0$ if and only if there exists n such that $\Lambda = \Lambda_n(e)$, and consequently, x_2 is 2π-periodic with n zeros in $[0,\pi)$. Note that given $e \in [0,1)$ the first zero of $x_2(t, e, \Lambda_n(e))$ arise at $t = 0$, while the equation $x_2(\pi, e, \Lambda_n(e)) = 0$ implies that its $(n+1)$-th zero is not included in the interval. With this in mind, it is straightforward from the Sturm comparison theory that x_2 has exactly n zeros in $[0,\pi)$ if and only if the point (e, Λ) belongs to the region

$$\{(e,\Lambda) \ : \ \Lambda_{n-1}(e) < \Lambda \leq \Lambda_n(e)\}.$$

Then, x_2 has more than n zeros in $[0,\pi)$ if and only if $\Lambda > \Lambda_n(e)$. We finish the proof applying Lemma 6, for $m = 1$, in the case of odd 2π-periodic solutions.

For even solutions we proceed exactly in the same way regarding the number of critical points instead of the number of zeros. ■

In fact, Lemma 6 allows us to extend Theorem 3 to symmetric $2m\pi$-periodic solutions of (25), for any $m = 1, 2, ...$ Here we sketch the method to accomplish it. We only need to define analogous functions to Λ_n.

It is well known that if a Hill's equation like (1) has a non-trivial $2m\pi$-periodic solution, with $m = 3, 4...$, then, there is always coexistence, consequently, there should exists a family of functions $\Lambda_{n,m}$ satisfying

$$x_2(m\pi, e, \Lambda_{n,m}(e)) = 0, \quad \dot{x}_1(m\pi, e, \Lambda_{n,m}(e)) = 0,$$

such that the corresponding symmetric solutions to $\Lambda_{n,m}(e)$ are $2m\pi$-periodic with n zeros and n critical points in the interval $[0, m\pi)$ and such that

$$\Lambda_{n,m}(0) = \left(\frac{n}{m}\right)^2.$$

Then, the extension of Theorem 3 would be the following: *Given a positive integer $m \geq 3$, there exist an even and an odd $2m\pi$-periodic non-trivial solutions of (25) with n zeros in $[0, m\pi)$ if and only if $\Lambda(N) > \Lambda_{n,m}(e)$.*

Note that in the case $m = 2$ we should keep in mind that there is no coexistence of non-trivial periodic solutions of (1) with minimum period 4π for $e \neq 0$ (Lemma 1). Thus, we need to distinguish two families of functions $\Lambda_{n,2}^{even}$ and $\Lambda_{n,2}^{odd}$ satisfying

$$x_2(2\pi, e, \Lambda_{n,2}^{odd}(e)) = 0, \quad \dot{x}_1(2\pi, e, \Lambda_{n,2}^{even}(e)) = 0,$$

such that, for $\Lambda = \Lambda_{n,2}^{odd}(e)$ (resp. for $\Lambda = \Lambda_{n,2}^{even}(e)$) there exists an odd (resp. even) 4π-periodic non-trivial solution with n zeros (resp. n critical points) in the interval $[0, m\pi)$ and such that

$$\Lambda_{n,2}^{odd}(0) = \Lambda_{n,2}^{even}(0) = \left(\frac{n}{2}\right)^2.$$

Note that $\Lambda_{2n,2}^{odd} = \Lambda_{2n,2}^{even} = \Lambda_n$ and that $\Lambda_{2n-1,2}^{odd}$ is one of the functions $\Lambda_n^{\downarrow}, \Lambda_n^{\uparrow}$ and $\Lambda_{2n-1,2}^{even}$ is the other one.

Then, the extension of Theorem 3 would be in this case: *There exists an even (resp. odd) 4π-periodic non-trivial solution of (25) with n zeros in the interval $[0, 2\pi)$ if and only if $\Lambda(N) > \Lambda_{n,2}^{even}(e)$ (resp. $\Lambda(N) > \Lambda_{n,2}^{odd}(e)$).*

4.3 Stability of the center of mass in the Curved Sitnikov problem

In [12] it is considered a variant of Sitnikov problem: two primaries orbit around their center of mass in keplerian ellipses, we deal with the motion of a third massless particle confined to a circumference of radius R under the influence of the primaries. See Figure 7. The plane of the circumference and the plane of primaries' orbits intersect each other through the common line of major axes. Also, the circumference itself intersect the primaries' plane through the center of mass. Both this point and the other intersection point are equilibria for the dynamics because the forces exerted by the primaries lie in their plane.

If the distance between the center of mass and each of the primaries is $\frac{1}{2}r(t, e)$, then the massless particle moves according to

$$\ddot{q} + \left(\frac{R + \frac{1}{2}r(t,e)\cos t}{d_1(t,e,R,q)^3} + \frac{R - \frac{1}{2}r(t,e)\cos t}{d_2(t,e,R,q)^3} \right) \sin q = 0, \tag{30}$$

where q is the particle's angular position in the circumference (taking the center of mass as $q = 0$) and $d_i(t, e, R, q),\ i = 1, 2$, are the distances from the massless particle to each primary.

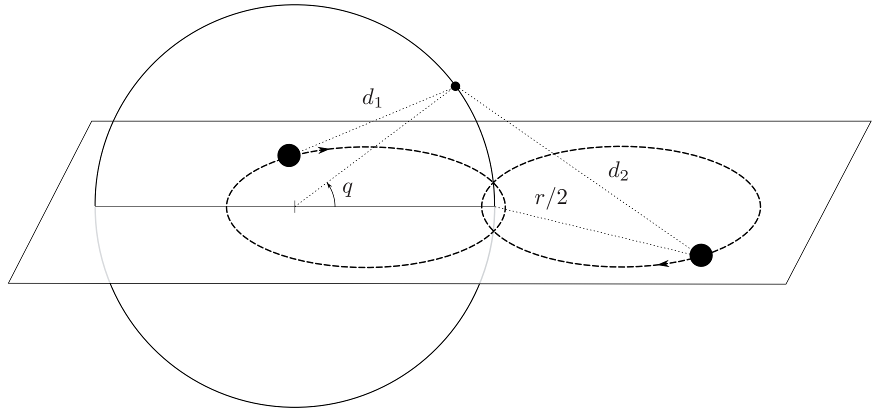

Figure 7: Curved Sitnikov problem.

The equilibria of (30) correspond to $q = 0$ and $q = \pi$. In [12] it is studied mainly the Lyapunov-stability of the equilibrium $q = \pi$ as an application of a more general theorem that regards the proximity of an equilibrium to one of the orbits of the primaries. The equilibrium $q = 0$ is only studied in the case of $e = 0$, whereas for $e \neq 0$ they mention some works, as [20], that suggest alternation of Lyapunov-stability/instability intervals as we change e.

Theorem 1 allows us to state a complete and precise result for the Lyapunov-stability of the equilibrium $q = 0$ for any $e \in [0, 1)$. First let us recall the following regions of the stability diagram of equation (1) defined in page 15

$$R_n^* = \{(e, \Lambda) \ : \ \Lambda_n^{\downarrow}(e) < \Lambda \leq \Lambda_n^{\uparrow}(e), \ e \neq 0\}.$$

Theorem 4 *The equilibrium $q = 0$ of (30) is Lyapunov-stable if and only if $(e, 16R) \notin R_n^*$.*

This theorem's proof is equivalent to that of Theorem 2, so we will not make it explicitly. We only mention that everything works because it is possible to write equation (30) for q close to 0 as

$$\ddot{q} + \frac{16R}{r^3(t,e)} q - \frac{8R}{3} \frac{(1 + 9\cos^2 t) r^2(t,e) + 36R}{r^5(t,e)} q^3 + O(q^5) = 0$$

since $d_i(t, e, R, 0) = \frac{1}{2} r(t, e), \ i = 1, 2$.

Note that, as we see in [12], it is important for the equilibrium $q = \pi$ to avoid the possibility of contact of the circumference with one of the elliptic orbits of the primaries. However, in this case it is irrelevant because we are considering only motions close enough to $q = 0$, so we consider all the possible values of R and e.

We conclude from Theorems 1 and 4 that for a fixed $R > 9/128$ there is alternation of infinite intervals (in the range of the eccentricity e of the primaries' orbits) of Liapunov-stability/instability of the equilibrium of (30). For $R \in (1/16, 9/128)$ the number of alternating intervals is finite. For $R = 1/16$ the equilibrium is Lyapunov-stable for each value $e \in [0, 1)$. For $R \in (0, 1/64) \cup (1/64, 1/16)$ it looks that there is only one transition from stability to instability. Finally, for $R = 1/64$, it seems that the equilibrium is unstable for each $e \in [0, 1)$.

5 Conclusions and open questions

The linear Sitnikov equation (1) has been shown to be relevant for some Sitnikov-like problems, for which there are several primaries in an elliptic motion determined by $r(t, e)$. Here we have focused on the $(N+1)$-Sitnikov problem and on the curved Sitnikov problem, but we can easily imagine a $(N+1)$-curved Sitnikov problem or other problems for which equation (1) is useful. With this paper we point out two possible applications: in one hand, Lyapunov stability of the equilibrium in the center of mass of the nonlinear Sitnikov-like problem, and on the other hand, existence of symmetric periodic solutions for the problem.

Despite the foregoing, keplerian elliptic motion is one of the most common considered in Celestial Mechanics. Then, it makes sense to look for other models related to the linear Sitnikov equation. In the search for it we have found the following interesting relation.

The following biparametric equation

$$\ddot{\theta} + \frac{\epsilon}{r^3(t,e)} \sin 2(\theta - f(t,e)) = 0, \quad e \in [0,1),\ \epsilon > 0, \tag{31}$$

determine the motion of a triaxial rigid satellite of ellipticity ϵ in a elliptic orbit determined by $r(t, e)$, with eccentricity e and associated true anomaly $f(t, e)$. This is the so-called spin-orbit problem, it has been widely studied, see for example [8] for the conservative case or [9] and [14] for the dissipative case. Consider

$$\Theta(t) = \theta(t) - t, \quad \phi(t,e) = f(t,e) - t,$$

then equation (31) turns into

$$\ddot{\Theta} + \frac{\epsilon}{r^3(t,e)} \sin 2(\Theta - \phi(t,e)) = 0, \tag{32}$$

whose solutions are denoted by $\Theta = \Theta(t, e, \epsilon)$. Note that a 2π-periodic solution of equation (32) corresponds to a spin-orbit resonance $1:1$ of equation (31), that is to say, solutions such that $\theta(t+2\pi) = \theta(t) + 2\pi$. Since the nonlinear term is bounded, there exists a solution for the Dirichlet boundary conditions

$$\Theta(0, e, \epsilon) = \Theta(\pi, e, \epsilon) = 0, \tag{33}$$

moreover, since $\phi(-t, e) = -\phi(t, e)$, then equation (32) is of the form

$$\ddot{\Theta} + F(t, \Theta) = 0, \quad F(-t, -\Theta) = -F(t, \Theta)$$

consequently, a solution of (32) with boundary conditions (33) is odd and 2π-periodic. Consider the Dirichlet problem for $e = 0$

$$\begin{cases} \ddot{\Theta} + \epsilon \sin 2\Theta = 0 \\ \Theta(0, 0, \epsilon) = \Theta(\pi, 0, \epsilon) = 0, \end{cases} \tag{34}$$

which has a solution $\Theta(t, 0, \epsilon) \equiv 0$. This solution has a local analytic continuation $\Theta(t, e, \epsilon)$, solution of (32) with boundary conditions (33), for a small enough $e \neq 0$, if and only if $2\epsilon \neq n^2$, for $n \in \mathbb{Z}$, because in that points the linearized equation of (34), i.e. $\ddot{y} + 2\epsilon y = 0$, has Floquet multiplier 1.

Since for e small $\phi(t,e)$ is close to zero and the local continuation $\Theta(t,e,\epsilon)$ is close to $\Theta(t,0,\epsilon) \equiv 0$, then, this solution $\Theta(t,e,\epsilon)$ might be seen as solution of

$$\ddot{\Theta} + \frac{2\epsilon}{r^3(t,e)}(\Theta - \phi(t,e)) = 0, \tag{35}$$

which has the form of a non-homogeneous linear Sitnikov equation (1). This suggests that the stability diagram in Figure 1 for e close to zero must be also significant for the capture into the resonance 1:1 of the spin-orbit problem.

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