A new relativistic extension of the harmonic oscillator satisfying an isochronicity principle

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Abstract

A new relativistic extension of the one-dimensional simple harmonic oscillator is presented. The equations of motion express the proportionality between the relativistic force acting on a particle and the distance to the oscillation center measured by a comoving observer with the particle, instead of the distance measured by a certain inertial observer. The resulting trajectory is really harmonic, i.e., the period of the oscillations does not depend on the amplitude, against the anharmonicity of the standard relativistic harmonic oscillator.

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1 Introduction

In Classical Mechanics, a linear restoring force acting over a mass produces isochronous oscillations, that is, the period of the oscillations is constant and does not depend on the amplitude. The classical harmonic oscillator is a key piece in the modelling of many physical phenomena related to mechanical devices, electric circuits and Quantum Mechanics. For a Newtonian oscillator, a classical result [12] states that the quadratic potential is the unique symmetric isochronous potential; in this way isochronicity characterizes the harmonic oscillator. Then, it is natural to ask if this property of isochronicity is preserved in the framework of Special Relativity.

Let us consider the one-dimensional motion of a particle of mass m subjected to a linear restoring force (Hooke's law) with stiffness constant k. In the Newtonian ambient, the equation of motion is as simple as

$$mx'' + kx = 0.$$

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The standard relativistic version of the harmonic oscillator simply replaces the Newtonian momentum mx' by the *relativistic momentum*, giving rise to

$$\left(\frac{mx'}{\sqrt{1 - \frac{x'^2}{c^2}}} \right)' + kx = 0. \tag{1}$$

Here, m is the mass of the particle at rest, c is the speed of light in the vacuum, and it is understood that $|x'| < c$ at any time. This equation is generally recognized as the natural relativistic extension of the harmonic oscillator and has been studied by many authors [5, 6, 7, 8, 9, 10]. One of the main features of the model is that the period of the oscillations depends on the amplitude, i.e., the relativistic harmonic oscillator is in fact anharmonic (see for instance [8]).

In the present article, we propose a different model for the relativistic extension of the harmonic oscillator. The standard and the alternative equations are derived from the same scheme: the 'proper force' acting on a relativistic particle is proportional to the 'distance to a fixed point'. Of course, in the non relativistic setting, this property is tantamount to Hooke's law leading to the simple harmonic oscillator. However, in Relativity each term written between commas must be clarified. First, the 'proper force' acting on a particle is the intrinsic force which a comoving observer is able to measure by using an accelerometer. An accelerometer may be intuitively thought of as a sphere in whose center there is a small round object which is supported on elastic radii of the sphere surface. So, if a free falling observer carries such an accelerometer, then it will notice that the small round object remains right at the center. Yet it will be displaced if the observer obeys an accelerated motion. Therefore, the proper force may be computed as the product of the proper acceleration (measured by the accelerometer) and the rest mass of the particle. It is important to remark that this proper force is in general different from the force measured by another observer, even though this observer is inertial (obviously, the last claim does not happen in Classical Mechanics). On the other hand, we fix an inertial observer to be considered as a 'fixed point'. The distance between the moving particle and this inertial observer depends on *who* measures it. In the standard relativistic harmonic oscillator (1), this distance is measured by the fixed inertial observer, situated in the 'center' of the oscillations. In the new proposed model, the distance is measured by the observer comoving with the particle.

Summarizing, we pretend to describe the harmonic oscillator motion as the oscillating particle would observe it. The solutions of this new equation turn out to be periodic and really harmonic, i.e., the period of the oscillations does not depend on the amplitude. It is important to remark that this period is measured in the proper time of the moving particle. In the time of the fixed inertial observer, it is possible to design an isochronous restoring force [3], but this force is singular and not defined on the whole space, so the linearity of the restoring force is lost. Our model tries to conceal isochronicity and linearity of the restoring force.

The rest of the paper is organized in three sections. In Section 2 we expound the notation and some preliminaries of Relativistic Dynamics. Section 3 provides the details of the new model in comparison with the standard model stated in [9]. Finally, we present some conclusions in Section 4.

2 'Newtonian' and Lagrangian approach to the relativistic equations of motion

We consider the motion of a particle of rest mass m in the (n+1)-dimensional Minkowski spacetime $\mathbb{A}$. Throughout the paper, the signature of the metric will be $(- + \cdots +)$. Its world line is represented by a timelike future pointing curve $\sigma : I \subseteq \mathbb{R} \longrightarrow \mathbb{A}$ satisfying the normalization condition $|\sigma'(\tau)|^2 = \langle \sigma'(\tau) \,, \sigma'(\tau) \rangle = -c^2$ (see for instance [11]). Here, the parameter τ is the proper time of σ and the tangent vector σ' is known as the relativistic velocity of σ. Also, we denote by $P = m\,\sigma'$ the relativistic momentum of the particle σ.

Now, we fix an arbitrary inertial observer in $\mathbb{A}$ whose world line is described by the unit, future timelike curve $O : \mathbb{R} \longrightarrow \mathbb{A}$. Its proper time will be denoted by t. Associated with O there exists a family of inertial observers, at rest relative to O and containing O, which are synchronized with their proper time (also denoted by t). This family defines a reference frame for the spacetime and a global chart $\varphi = (t, x_1, \cdots x_n)$. Coordinate functions $x_1, \cdots x_n$ will be the usual Cartesian coordinates that determine the position in each hyperplane of the foliation $\{t = \text{constant}\}$. The coordinate t of an event $p \in \mathbb{A}$ precisely corresponds to the proper time elapsed until p from an initial hyperplane of reference, relative to the unique observer of the family passing through p. Therefore, these inertial observers are given by the integral curves of the vector field ∂_t. By simplicity, we suppose that the observer O has the expression $O(t) = (t, 0, \cdots, 0)$. The metric of the spacetime is given by

$$\langle \,,\rangle = -c^2\, dt^2 + dx_1^2 + \cdots + dx_n^2,$$

where c is the speed of light in the vacuum.

The classical way to derive the equation of motion of a relativistic particle comes from considering the analogue of the classical Newton law,

$$m\frac{D\sigma'}{d\tau} = \frac{DP}{d\tau} = F, \tag{2}$$

where $\frac{D}{d\tau}$ is the covariant derivative along σ induced by the Levi-Civita connection in $\mathbb{A}$, and F is a spacelike vector field along σ, named relativistic force. We recall that F is orthogonal to $\sigma'(\tau)$ at each event $\sigma(\tau)$. As a consequence, F necessarily depends on the relativistic velocity because, at each event, it must be simultaneously orthogonal to all possible velocities of the particle in that event [2]. We may appreciate this dependence by expressing F in the coordinate system associated to the observer O. The relativistic velocity and force are expressed as follows,

$$\sigma'(\tau) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\big(\,\partial_t + \mathbf{v}\,\big), \qquad F = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Big(\,\frac{\langle \mathbf{f}, \mathbf{v}\rangle}{c^2}\,\partial_t + \mathbf{f}\,\Big),$$

where $\mathbf{v} = \frac{d}{dt}(x_1 \circ \sigma)\partial_1 + \cdots + \frac{d}{dt}(x_n \circ \sigma)\partial_n$, with σ parametrized by the coordinate time t, and $v^2 = \langle \mathbf{v} \,, \mathbf{v} \rangle < c^2$. The vector $\mathbf{v}$ is the velocity of the particle relative to the observer O and $\mathbf{f}$ is interpreted as the force acting on the particle measured by O. Then, equation (2) provides the following relativistic equations of motion, obtained first by Poincaré,

$$\frac{d}{dt}\left(\frac{m\,c^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \langle \mathbf{f}, \mathbf{v}\rangle, \tag{3}$$

$$\frac{d}{dt}\left(\frac{m\,\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\mathbf{f}. \tag{4}$$

It may be checked that Equation 4 is deduced from 3, which is the better known relativistic equation for the motion of a particle, from the point of view of an inertial reference system. Observe that $\frac{m\,\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}$ is the relativistic momentum of the particle measured by O.

The Lagrangian setting has been extensively explored in the related literature, and its study is still underway. The free motion in Special Relativity is an extremal of the proper time of the particle. However, when there exists some interaction, the Lagrangian is more complicated and its choice depends on the nature of the physical phenomenon under study. If we are interested in describing a motion through covariant equations, we must impose that the Lagrangian is a (Lorentz) scalar. For instance, this is the case of the Lorentz force equation, which describes the motion of a charge in the presence of an electromagnetic field. However, relativistic harmonic oscillator equations are not Lorentz invariant; rather, they come from the following sort of Lagrangian (not Lorentz invariant in general). Let us consider a differentiable function $V:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$, and let $\varphi:\mathbb{A}\longrightarrow\mathbb{R}^{n+1}$, $\varphi=(t,x_1\cdots,x_n)$ be the chart of the inertial observer O. For every λ-parametrized curve $\xi:I\subseteq\mathbb{R}\longrightarrow\mathbb{A}$, we define the Lagrangian

$$L[\xi]=-mc\,|\xi'|+(V\circ\varphi)(\xi)\,(t\circ\xi)'. \tag{5}$$

The prime means the derivative with respect to λ and $|\xi'|=\sqrt{\langle\xi',\xi'\rangle}$. The function $V\circ\varphi$ represents the potential *measured by the observer* O. Note that the second added term is not a Lorentz invariant in general, because we are using a special frame of reference ($V\circ\varphi$ is not a Lorentz scalar). By imposing the restrictions $t'=1$ or $t'=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ on the Euler Lagrange equations of L (see [9]), we arrive at the relativistic equations of motion (3)-(4). Another alternative Lagrangian formalism of relativistic mechanics is more recently exposed in [5].

3 Two models for the relativistic harmonic oscillator

In this section we intend to model a relativistic oscillator attending to the *proper force* that the particle measures through an accelerometer. By analogy with the classical harmonic oscillator, we consider an oscillator in which the proper force is proportional to the distance to a 'fixed point'. However, distances in Relativity depend on which observer does the measurement. Depending on the distance we take, we will obtain two models of relativistic harmonic oscillators. The first one will coincide with the standard relativistic harmonic oscillator, but the second one will be different. Moreover, as we have commented in the Introduction, the second model is more suitable to be named *harmonic* oscillator because the period of the oscillations observed by the moving particle does not depend on the amplitude. The key difference between the two models is illustrated by Figure 1.

For simplicity's sake, from now on we will consider the 2-dimensional Minkowski spacetime $\mathbb{A}$. From [1], we know that in a 2-dimensional spacetime, each observer obeys an unchanged direction motion, i.e., it measures a 'constant' proper acceleration (if we consider an n-dimensional spacetime, it would be necessary to impose that the particle is in addition an unchanged direction observer).

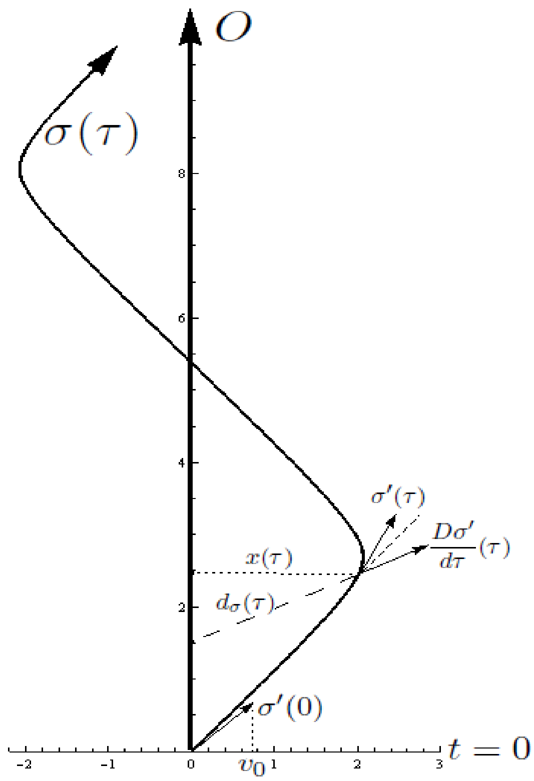

Figure 1: For a trajectory $\sigma(\tau) = (t(\tau), x(\tau))$, this picture illustrates the idea behind the two models. In the first model, which corresponds to the standard relativistic harmonic oscillator, the proper force is proportional to $x(\tau)$, while in the new model it is proportional to $d_\sigma(\tau)$, defined by (14).

We continue considering the coordinate system associated to the fixed observer O. This inertial observer will be considered as the center of the oscillations. The coordinates will be denoted by $(t,x) : \mathbb{A} \longrightarrow \mathbb{R}^2$. To simplify notation we will respectively write t and x instead of $t \circ \sigma$ and $x \circ \sigma$ whenever there is no confusion. In this way, we write $\sigma(\tau) = (t(\tau), x(\tau))$.

With respect to the initial conditions, we suppose that when the clocks of the particle and the observer O mark zero, the particle passes through the spacial origin of O at a velocity equal to $\mathbf{v}_0 = v_0\, \partial_x$. Written in coordinates,

$$t(0) = 0, \quad x(0) = 0, \quad t'(0) = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}, \quad x'(0) = \frac{v_0}{\sqrt{1 - \frac{v_0^2}{c^2}}}, \tag{6}$$

where we have taken into account that $\mathbf{v} = x'/t'\, \partial_x$ and the normalization condition

$$\langle \sigma'(\tau)\, , \sigma'(\tau) \rangle = -c^2 t'^2 + x'^2 = -c^2. \tag{7}$$

3.1 The classical relativistic an-harmonic oscillator

In this first model, the proper force is proportional to the distance between the particle and the observer O, *measured by O*. This means that

$$\left| m\, \frac{D\sigma'}{d\tau} \right|^2 = k^2\, x^2,$$

k being a positive constant with units of $[M][T]^{-2}$, the usually named elastic or stiffness constant. Therefore, taking into account that $\sigma' \perp \frac{D\sigma'}{d\tau}$, the equation of this oscillator may be expressed as

$$m\, \frac{D\sigma'}{d\tau}(\tau) = k\, x(\tau)\, N(\tau), \tag{8}$$

where N is the unit normal vector along σ pointing to the observer O (the center of the oscillations). The coordinates of this vector may be easily computed as

$$N = -\frac{x'}{c^2} \partial_t - t' \partial_x.$$

Thus, equation (8) results in

$$m\, t'' = -\frac{k}{c^2}\, x\, x', \qquad m\, x'' = -k\, x\, t', \tag{9}$$

plus the initial conditions (6). Since $t' \geq 1$, we may parametrize σ through the coordinated time t. Denoting by $\mathbf{v} = \frac{dx}{dt}\, \partial_x = \frac{x'}{t'}\, \partial_x$ and $v = |\mathbf{v}|$, from (7) we deduce that

$$\frac{d}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\, \frac{d}{dt}.$$

Therefore, according to the second equation of (9), we obtain

$$m\, \frac{d}{dt} \left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = -k\, x. \tag{10}$$

This equation is recognized as the standard relativistic equation (1). Let us remark that it also may be obtained from equation (4) when the force observed by O is

$$\mathbf{f} = -k\, x\, \partial_x,$$

The relativistic force associated with this $\mathbf{f}$ is

$$F = k\, x \left[\frac{1}{c^2}\, x'\, \partial_t - t'\, \partial_x \right],$$

which is explicitly non Lorentz covariant.

Equations (9) can also be obtained from the Lagrangian formalism, as is done in [9]. These equations can be manipulated to obtain a suitable representation of the solutions. Integrating the first equation of (9) and taking into account the initial conditions (6), we have

$$t'(\tau) = -\frac{k}{2m\, c^2}\, x^2(\tau) + \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2}. \tag{11}$$

Inserting this identity into the second equation of (9), it turns out that $x(\tau)$, that is, the motion of the particle in its proper time, obeys the Duffing equation

$$x'' + \frac{k}{m\sqrt{1 - \frac{v_0^2}{c^2}}}\, x - \frac{k^2}{2m^2\, c^2}\, x^3 = 0. \tag{12}$$

A first integral of this equation is

$$\phi(x, x') = \frac{m}{2} x'^2 + \frac{k}{2\sqrt{1 - \frac{v_0^2}{c^2}}}\, x^2 - \frac{k^2}{8m\, c^2}\, x^4.$$

By imposing the initial conditions, the energy level of our solution is

$$\phi(x, x') = E(v_0) := \frac{m\, v_0^2}{2\left(1 - \frac{v_0^2}{c^2}\right)} > 0. \tag{13}$$

Besides, the normalization condition (7) implies that $t' \geq 1$, and making use of (11), we deduce the boundedness of the orbit, in fact,

$$x^2(\tau) \leq \frac{2mc^2}{k} \left(\frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} - 1 \right).$$

In view of the phase plane of the Duffing equation, the orbit is closed and it is inside the separatrix (see Fig. 2), which corresponds to the energy level $E_0 = \frac{m\, c^2}{2\left(1 - \frac{v_0^2}{c^2}\right)}$. In conclusion, the trajectory of the particle is periodic and implicitly given by (13). As a matter of fact, an explicit expression of the solution can be given by using Jacobian elliptic functions (see [4, Section 4.2] or [9, Eq. 41]).

Summing up, it is shown that $x(\tau)$ is periodic, although the period is not constant and depends on the initial condition v_0. Indeed, it tends to $+\infty$ as $v_0 \to c$. On the other hand, if the oscillations are measured with the proper time of the inertial observer O, from (11) and the periodicity of $x(\tau)$, we conclude that the particle σ describes a periodic motion as well. Nevertheless, the period still depends on v_0, and so it is not constant. In conclusion, this motion can not really be considered *harmonic*.

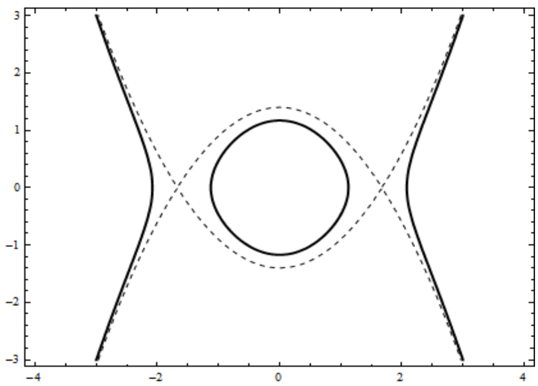

Figure 2: The energy level $\phi(x, x') = E(v_0)$ (thick curve) and the separatrix (dashed curve) of the Duffing equation (12).

3.2 A new model for the relativistic harmonic oscillator

For the second model, we assume that the proper force is proportional to the distance between the particle and the inertial observer O *measured by the particle*, i.e., by the inertial observer which is instantaneously at rest with O. From now on, we will refer to this observer as the *comoving* inertial observer with O. If the observer O measures a distance $x(\tau)$ when its clock marks $t = t(\tau)$, then the distance measured from the particle when its clock marks τ is equal to

$$d_\sigma(\tau) := x(\tau)\sqrt{1 - \frac{v^2(\tau)}{c^2}}, \tag{14}$$

where $v(\tau) = \frac{x'}{t'}(\tau)$ is the velocity of the particle measured by O. In this way, the equation for this model is expressed as

$$m\,\frac{D\sigma'}{d\tau}(\tau) = k\,x(\tau)\sqrt{1 - \frac{v^2(\tau)}{c^2}}\,N(\tau), \tag{15}$$

$k > 0$ being the elastic constant and N the unit normal to σ'. Developing this equation, we arrive at

$$m\,t'' = -\frac{k}{c^2}\,x\,x'\sqrt{1 - \frac{\left(x'/t'\right)^2}{c^2}}, \qquad m\,x'' = -k\,x\,t'\sqrt{1 - \frac{\left(x'/t'\right)^2}{c^2}}, \tag{16}$$

with the initial conditions (6). Surprisingly, by using the normalization condition (7), the second one simplifies to the familiar equation of a Newtonian harmonic oscillator

$$m\,x'' + k\,x = 0, \tag{17}$$

whose solution is

$$x(\tau) = \frac{v_0}{w\sqrt{1 - \frac{v_0^2}{c^2}}}\,\sin\left(w\,\tau\right), \tag{18}$$

where $w = \sqrt{\frac{k}{m}}$.

As in the standard model presented in Subsection 3.1, the solution $x(\tau)$ is periodic and, taking into account that

$$t'^2(\tau) = 1 + \frac{x'^2(\tau)}{c^2}, \tag{19}$$

we conclude that the proper acceleration $\frac{D\sigma'}{d\tau}$ is periodic. Therefore, in its proper time the particle obeys a periodic motion with period $T = 2w\pi$. But we highlight that now this period does not depend on the amplitude of the oscillations (or the initial velocity v_0), i.e., a sort of 'isochronicity principle' holds for this model. This is a justified reason to claim that the particle σ is *harmonically* oscillating, at least from the perspective of the moving particle.

From the viewpoint of the inertial observer O, the particle also describes a periodic trajectory. In order to check this claim, let us write (17) in terms of the variable t, which is the proper time of O, giving

$$\frac{m\,x''}{\left(1 - \frac{x'^2}{c^2}\right)^2} + k\,x = 0. \tag{20}$$

This equation presents the conserved quantity,

$$\psi(x, x') = \frac{mc^2}{1 - x'^2/c^2} + k\,x^2,$$

so in consequence its solutions may be represented in the phase plane as the level curves $\psi(x, y) = E$. Such level curves are closed and therefore all the solutions are periodic. The quantitative difference between the solutions of equations (10) and (20) is illustrated in Fig. 3.

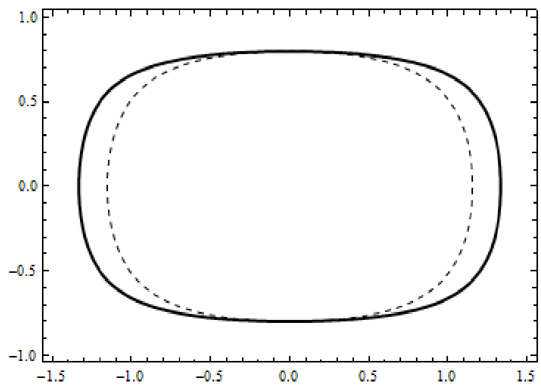

Figure 3: For $k = m$, $c = 1$ and initial velocity $v_0 = 0.8$, this figure depicts the orbit (x, x') of eq. (10) (dashed curve) and eq. (20) (thick curve) respectively.

Despite the fact that the inertial observer O also observes a periodic trajectory of the particle σ, it will not measure isochronous oscillations. The relation between the time t of the inertial observer O and the proper time τ of the particle is directly obtained from (19) as

$$t(\tau) = \int_0^\tau \sqrt{1 + \frac{v_0^2}{c^2 - v_0^2}\cos^2(ws)}\; ds. \tag{21}$$

Hence, the relation between the two periods is

$$T^O = \int_0^T \sqrt{1 + \frac{v_0^2}{c^2 - v_0^2}\cos^2(ws)}\; ds.$$

Therefore, the period measured by O, T^O, is variable and depends on initial velocity, tending to $+\infty$ when $v_0 \to c^-$.

As a last remark, note that, when comparing (4) and (20), equation (15) may be interpreted as the relativistic equation of motion of a particle under the action of a force which, for the observer O, takes the value of $\mathbf{f} = -k\,x\,\sqrt{1 - \frac{v^2}{c^2}}\,\partial_x$, v being the velocity of σ relative to O. Since $\mathbf{f}$ depends explicitly on the velocity, equation (15) does not derive from a Lagrangian of the form (5). Moreover, once again (15) is not Lorentz covariant because of the explicit use of a special reference frame in its formulation.

4 Conclusions

In this note, a new model for the relativistic harmonic oscillator is presented. In the standard model, the proper force acting on the moving particle is proportional to the distance measured by an inertial fixed observer. In contrast, in our model the proper force is proportional to the distance measured from the point of view of the moving particle. With this modification, we have proven that the oscillations observed by the particle are isochronous. Also, it is shown that the two models are not Lorentz covariant.

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References

[1] D. de la Fuente, A. Romero, P.J. Torres, Unchanged direction motion in General Rela tivity: the problems of prescribing acceleration and the extensibility of trajectories, J. Math. Phys.,**56**, DOI:10.1063/1.4935854 (2015).

[2] Y. Friedman, T. Scarr, Making the relativistic dynamics equation covariant: explicit solutions for motion under a constant force, *Physica Scripta*, **86**, 065008 (2012).

[3] J.-H. Kim, S.-W. Lee, H. Maassen, H.-W. Lee, Relativistic oscillator of constant period, *Phys. Rev. A*, **53**, 2991–2997 (1996).

[4] I. Kovacic, M.J. Brenan, *The Duffing Equation*, John Wiley & Sons, London, 2011.

[5] Y.S. Huang, An alternative derivation of the equations of motion of the relativistic (an)harmonic oscillator, *Am. J. Phys.*, **67**, 142–145 (1999).

[6] E.H. Hutten, Relativistic (Non-Linear) Oscillator, *Nature*, **205**, p.892 (1965).

[7] L. A. MacColl, Theory of the Relativistic Oscillator, *Am. J. Phys.* **25**, 535–538 (1957).

[8] R.E. Mickens, Periodic solutions of the relativistic harmonic oscillator, *Journal of Sound and Vibration*, **212**, 905–908 (1998).

[9] W. Moreau, R. Easther, R. Neutze, Relativistic (an)harmonic oscillator, *Am. J. Phys.*, **62**, 531–535 (1994).

[10] R. Penfield, H. Zatzkis, The relativistic linear harmonic oscillator, Journal of Franklin Institute, **262**, 121–125 (1956).

[11] R.K. Sachs, H. Wu, *General Relativity for Mathematicians*, Grad. Texts in Math., **48**, Springer-Verlag, 1977.

[12] M. Urabe, Potential forces which yield periodic motion of a fixed period, Journal of Mathematics and Mechanics **10**, 569–578 (1961).