Non-degeneracy and Uniqueness of Periodic Solutions For $2n$-order Differential Equations *

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Abstract— We analyze the non-degeneracy of the linear $2n$-order differential equation $u^{(2n)}+\sum\limits_{m=1}^{2n-1}a_m u^{(m)}=q(t)u$ with potential $q(t)\in L^p(\mathbb{R}/T\mathbb{Z})$, by means of new forms of the optimal Sobolev and Wirtinger inequalities. The results is applied to obtain existence and uniqueness of periodic solution for the prescribed nonlinear problem in the semilinear and superlinear case.

Keyword—Non-degeneracy; Uniqueness; Superlinear; Semilinear; $2n$-order differential equation.

MSC2000—34C25.

1 Introduction

Given $q(t)\in L^p(\mathbb{S}_{\mathbb{T}})$, $\mathbb{S}_T=\mathbb{R}/T\mathbb{Z}$, $1\le p\le\infty$, $a_m\in\mathbb{R}$, it is said that the linear periodic boundary value problem

$$u^{(2n)}+\sum_{m=1}^{2n-1}a_m u^{(m)}=q(t)u,\qquad t\in\mathbb{R},\quad u\in\mathbb{R},\tag{1.1}$$

$$u^{(i)}(0)=u^{(i)}(T),\quad i=0,1,\cdots,2n-1,\tag{1.2}$$

is non-degenerate, if problem (1.1)-(1.2) has only the trivial solution $u(t)=0$. In this case, we also say that $q(t)$ is a non-degenerate potential of problem (1.1) and (1.2).

The periodic solution problem for the high-order differential equations has attracted much attention (see for instance [1]-[3], [10]-[14]), however, the study on non-degenerate problems for

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high-order differential equation is not adequately covered in the related literature. The main objective of this paper is to contribute to the literature with a new criterium of non-degeneracy in the general case.

The interest of a good understanding of the non-degeneracy problem is twofold. Besides the intrinsic theoretical interest, generally speaking a concrete non-degeneracy result can be applied to obtain existence and uniqueness results for a nonlinear problem. For the second order equation, such techniques have been widely developed for the semilinear case. This line of research can be traced back at least to the seminal paper of Lasota and Opial [6] a present a number of variants, see for instance [4, 8, 15] and the references therein. The superlinear case has been considered in [9]. The analysis of higher-order problems with this technique is more rare. Just recently, Li and Zhang [7] have used some Sobolev constants to explicitly characterize a class of potentials $q(t) \in L^p(0,T)$ for which the beam equation with periodic boundary conditions

$$\begin{cases} u^{(4)}(t) = q(t)u(t), & t \in (0,T), \\ u^{(i)}(0) = u^{(i)}(T), & 0 \le i \le 3, \end{cases} \tag{1.3}$$

admits only the trivial solution. As an application of non-degeneracy, they obtain the uniqueness of periodic solutions of a certain class of superlinear beam equations.

In this paper, we develop a novel non-degeneracy criterium for problem (1.1)-(1.2). Later, inspired in the cited papers [7, 9, 15], such criterium is applied to the existence and uniqueness of periodic solutions of the related nonlinear differential equation. In section 2, we present new forms of optimal Sobolev and Wirtinger inequalities recently developed in [5]. In section 3, by using the previous optimal Sobolev and Wirtinger inequalities, we get sufficient conditions for a potential to be non-degenerate for (1.1)-(1.2). Section 4 and 5 are devoted to applications of the main result for non-degenerate potentials to the nonlinear problem. Section 4 deals with the semilinear case and applies the technique developed in [15]. In section 5, firstly, the classes $\mathcal{C}(\sigma; A, B)$ of nonlinearities to be considered are given in Definition 5.1. These nonlinearities $f(x)$ can grow superlinearly as $x \to \infty$. Besides the existence for equations of Landesman-Lazer type [14] where the nonlinearities are monotone, by mimicking the technique employed in [7] it is shown in Theorem 5.1 that, for those classes of nonlinear equations, the periodic solution is unique.

We fix some notations. For a function $h(t)$ in the Lebesgue space $L^1(\mathbb{S}_T)$ of T-periodic function, $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$, the mean value of $h(t)$ is $\bar{h}(t) = \frac{1}{T}\int_0^T h(t)dt$. Then $L^1(\mathbb{S}_T)$ can be decomposed as $L^1(\mathbb{S}_T) = \mathbb{R} \oplus \tilde{L}^1(\mathbb{S}_T)$, where $\tilde{L}^1(\mathbb{S}_T) = \{h \in L^1(\mathbb{S}_T) : \bar{h} = 0\}$ and $\mathbb{R}$ is identified as the set of constant functions of $L^1(\mathbb{S}_T)$. Analogously, the Hilbert space $H^n(\mathbb{S}_T)$ can be decomposed as $H^n(\mathbb{S}_T) = \mathbb{R} \oplus \tilde{H}^n(\mathbb{S}_T)$, where $\tilde{H}^n(\mathbb{S}_T) = H^n(\mathbb{S}_T) \cap \tilde{L}^1(\mathbb{S}_T)$. The uniform norm is as usual $||x||_\infty = \max |x(t)|$. Finally, the positive and negative part of a function $q(t)$ are given by $q_+(t) =$

$\max\{q(t),0\}, q_-(t) = \max\{-q(t),0\}$.

2 Optimal Sobolev and Wirtinger inequalities.

In this section, we recall some novel Sobolev and Wirtinger inequalities recently proved in [5].

As a preparation, we explain briefly about Riemann zeta function, Bernoulli polynomial and Bernoulli number. Riemann zeta function is a meromorphic function defined by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \qquad (\mathrm{Re}\ z > 1).$$

Bernoulli polynomial $b_n(x)$ is defined by the following recurrence relation.

$$b_0(x) = 1, \quad b_n'(x) = b_{n-1}(x), \qquad \int_0^1 b_n(x)dx = 0 \quad (n = 1, 2, 3, \cdots).$$

Bernoulli number is defined by

$$B_M = (2M)!(-1)^{M-1} b_{2M}(0) \qquad (M = 1, 2, 3, \cdots).$$

It can be obtained by the following recurrence relate

$$\begin{cases} \sum_{j=0}^{n-1} (-1)^j \begin{pmatrix} 2n \\ 2j \end{pmatrix} B_j = -n \qquad (n = 1, 2, 3, \cdots) \\ B_0 = -1 \end{cases}.$$

Bernoulli numbers are positive rational numbers.

Next lemmas have been proved in [5].

Lemma 2.1. (Sobolev) For each fixed $M = 1, 2, 3, \cdots$ and for every function $u(x) \in \tilde{H}^M(\mathbb{S}_1)$, we have a suitable positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds

$$\left(\sup_{0 \le y \le 1} |u(y)| \right)^2 \le C \int_0^1 \left| u^{(M)}(x) \right|^2 dx.$$

Among such C the best constant $C_M := \frac{2\zeta(2M)}{(2\pi)^{2M}} = \frac{B_M}{(2M)!}$.

Lemma 2.2. (Wirtinger) For each fixed $M = 1, 2, 3, \cdots$ and for every function $u(x) \in \tilde{H}^M(\mathbb{S}_1)$, we have a suitable positive constant $\hat{C}$ which is independent of $u(x)$ such that the following Wirtinger inequality holds

$$\int_0^1 |u(x)|^2 dt \le \widehat{C} \int_0^1 \left| u^{(M)}(x) \right|^2 dx.$$

Among such $\hat{C}$ the best constant $\widehat{C}_M := \frac{1}{(2\pi)^{2M}}$.

Such inequalities are directly generalized to T-periodic functions through a time rescalling. If $\phi(t) \in \tilde{H}^n(\mathbb{S}_T)$, we know that $\psi(t) := \phi(Tt) \in \tilde{H}^n(\mathbb{S}_1)$. Since

$$||\psi||^2_{L^2(\mathbb{S}_1)} = T^{-1}||\phi||^2_{L^2(\mathbb{S}_T)}, \qquad ||\psi^{(n)}||^2_{L^2(\mathbb{S}_1)} = T^{2n-1}||\phi^{(n)}||^2_{L^2(\mathbb{S}_T)},$$

the previous inequalities are readily generalized as follows.

Lemma 2.3. (Sobolev inequality) Let $x \in \tilde{H}^M(\mathbb{S}_T)$. Then we have

$$||x||^2_\infty \le C_M \int_0^T \left|x^{(M)}(t)\right|^2 dt, \tag{2.1}$$

where $C_M := \frac{T^{2M-1}B_M}{(2M)!}$ is the best constant for this inequality.

Lemma 2.4. (Wirtinger inequality) Let $x \in \tilde{H}^M(\mathbb{S}_T)$. Then we have

$$\int_0^T |x(t)|^2 dt \le \widehat{C}_M \int_0^T \left|x^{(M)}(t)\right|^2 dt, \tag{2.2}$$

where $\widehat{C}_M := \left(\frac{T}{2\pi}\right)^{2M}$ is the best constant for this inequality.

3 Sufficient conditions for a potential to be non-degenerate.

In this section the main result is stated and proved. To this purpose, let us define $\sigma = \{1, 2, \ldots, n-1\}$ and the subsets

$$\sigma_1 = \{k \in \sigma \,:\, (-1)^k a_{2k} < 0\}, \qquad \sigma_2 = \{k \in \sigma \,:\, (-1)^k a_{2k} > 0\}.$$

Of course, one (or both) of these subsets can be empty. In this case, the usual convention $\sum_{\emptyset} = 0$ is used.

Theorem 3.1. Given $q(t) \in L^\alpha(\mathbb{S}_T)$ for some $\alpha \in [1, \infty]$, let us assume that one of the following conditions holds

(1) n is even, $\bar{q} > 0$ and

$$C_n T^{\frac{1}{\alpha^*}} \left\|q_+\right\|_\alpha < 1 + C_n \sum_{k\in\sigma_2} |a_{2k}|\, C_k^{-1} - \sum_{k\in\sigma_1} |a_{2k}|\, \widehat{C}_{n-k}. \tag{3.1}$$

where $\alpha^* = \frac{\alpha}{\alpha-1}$.

(2) n is odd, $\bar{q} < 0$ and

$$C_n T^{\frac{1}{\alpha^*}} \left\|q_-\right\|_1 < 1 + C_n \sum_{k\in\sigma_1} |a_{2k}|\, C_k^{-1} - \sum_{k\in\sigma_2} |a_{2k}|\, \widehat{C}_{n-k}. \tag{3.2}$$

Then (1.1)-(1.2) is non-degenerate.

Proof. We argue by contradiction. Assume that (1.1)-(1.2) has a non-trivial solution $x \in H(\mathbb{S}_T)$. Let us write $x = \bar{x} + \tilde{x}$, where $\tilde{x} := x - \bar{x} \in \tilde{H}^n(\mathbb{S}_T)$. Now (1.1) for $\tilde{x}$ is

$$\tilde{x}^{(2n)}(t) + \sum_{m=1}^{2n-1} a_m \tilde{x}^{(m)} = q(t)\bar{x} + q(t)\tilde{x}(t). \tag{3.3}$$

Integrating this equation over one period, we have, by the T-periodicity of $\tilde{x}$, $\int_0^T q(t)\bar{x}dt + \int_0^T q(t)\tilde{x}(t)dt = 0$. Since $\bar{q} \neq 0$, one has $\bar{x} = -\left(\int_0^T q(t)\tilde{x}(t)dt\right)/(T\bar{q})$. Multiplying (3.3) by $\bar{x} - \tilde{x}(t)$, we have

$$\bar{x}\tilde{x}^{(2n)}(t) - \tilde{x}(t)\tilde{x}^{(2n)}(t) + \bar{x}\sum_{m=1}^{2n-1} a_m \tilde{x}^{(m)} - \tilde{x}(t)\sum_{m=1}^{2n-1} a_m \tilde{x}^{(m)} = q(t)\bar{x}^2 - q(t)\tilde{x}^2(t).$$

Integrating this equation over one period and making use of the T-periodicity of $\tilde{x}(t)$, we get

$$\begin{aligned} &-\int_0^T \tilde{x}(t)\tilde{x}^{(2n)}(t)dt - \sum_{m=1}^{2n-1} a_m \int_0^T \tilde{x}(t)\tilde{x}^{(m)}dt \\ =& -(-1)^n \int_0^T (\tilde{x}^{(n)}(t))^2 dt - \sum_{m=1}^{2n-1} a_m \int_0^T \tilde{x}(t)\tilde{x}^{(m)}(t)dt \\ =& T\bar{q}\bar{x}^2 - \int_0^T q(t)\tilde{x}^2(t)dt. \end{aligned} \tag{3.4}$$

Note that integrating by parts one gets $\int_0^T \tilde{x}(t)\tilde{x}^{(m)}dt = 0$ for every odd m. Then, by reindexing $m = 2k$, (3.4) reads

$$-(-1)^n \int_0^T (\tilde{x}^{(n)}(t))^2 dt - \sum_{k=1}^{n-1} a_{2k}(-1)^k \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt = T\bar{q}\bar{x}^2 - \int_0^T q(t)\tilde{x}^2(t)dt. \tag{3.5}$$

First, let us assume that (1) holds. Since n is even, we have

$$\int_0^T (\tilde{x}^{(n)}(t))^2 dt + \left(\sum_{k\in\sigma_1} (-1)^k a_{2k} \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt + \sum_{k\in\sigma_2} (-1)^k a_{2k} \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt\right) = \int_0^T q(t)\tilde{x}^2(t)dt - T\bar{q}\bar{x}^2,$$

i.e.,

$$\int_0^T (\tilde{x}^{(n)}(t))^2 dt - \sum_{k\in\sigma_1} |a_{2k}| \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt = \int_0^T q(t)\tilde{x}^2(t)dt - T\bar{q}\bar{x}^2 - \sum_{k\in\sigma_2} |a_{2k}| \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt. \tag{3.6}$$

Using Wirtinger inequality in left-hand side of (3.6), we have

$$\begin{aligned} \int_0^T (\tilde{x}^{(n)}(t))^2 dt - \sum_{k\in\sigma_1} |a_{2k}| \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt &\geq \left\|\tilde{x}^{(n)}\right\|_2^2 - \sum_{k\in\sigma_1} |a_{2k}|\, \widehat{C}_{n-k} \left\|\tilde{x}^{(n)}\right\|_2^2 \\ &= \left(1 - \sum_{k\in\sigma_1} |a_{2k}|\, \widehat{C}_{n-k}\right) \left\|\tilde{x}^{(n)}\right\|_2^2, \end{aligned} \tag{3.7}$$

where $\widehat{C}_k$ are the optimal constants defined in Lemma 2.4.

On the other hand, by using now Sobolev inequality and $\bar{q} > 0$, the right-hand side of (3.6) can be bounded above as follows

$$\begin{aligned}
&\int_0^T q(t)\tilde{x}^2(t)dt - T\bar{q}\bar{x}^2 - \sum_{k\in\sigma_2} |a_{2k}| \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt \\
\leq &\int_0^T q_+(t)\tilde{x}^2(t)dt - \|\tilde{x}\|_\infty^2 \sum_{k\in\sigma_2} |a_{2k}|\, C_k^{-1} \\
\leq &\|\tilde{x}\|_\infty^2 \left(\|q_+\|_1 - \sum_{k\in\sigma_2} |a_{2k}|\, C_k^{-1}\right) \\
\leq &C_n \left(T^{\frac{1}{\alpha^*}} \|q_+\|_\alpha - \sum_{k\in\sigma_2} |a_{2k}|\, C_k^{-1}\right) \left\|\tilde{x}^{(n)}\right\|_2^2 .
\end{aligned} \tag{3.8}$$

Therefore,

$$\left(1 - \sum_{k\in\sigma_1} |a_{2k}|\, \widehat{C}_{n-k}\right) \left\|\tilde{x}^{(n)}\right\|_2^2 \leq C_n \left(T^{\frac{1}{\alpha^*}} \|q_+\|_\alpha - \sum_{k\in\sigma_2} |a_{2k}|\, C_k^{-1}\right) \left\|\tilde{x}^{(n)}\right\|_2^2 .$$

Under assumption (4.3), it is necessary that $||\tilde{x}^{(n)}||_2 = 0$. Thus $\tilde{x}^{(n-1)}$ is constant. Since $\tilde{x} \in \tilde{H}^n(\mathbb{S}_T)$, one has $\tilde{x}(t) \equiv 0$. Now $\bar{x} = -\left(\int_0^T q(t)\tilde{x}(t)dt\right)/(T\bar{q}) = 0$. Thus $x = 0$, which contradicts the assumption $x \neq 0$.

Under assumption (2), an analogous argument can be done. As n is odd, then (3.6) reads

$$\int_0^T (\tilde{x}^{(n)}(t))^2 dt - \sum_{k\in\sigma_2} |a_{2k}| \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt = T\bar{q}\bar{x}^2 - \int^T q(t)\tilde{x}^2(t)dt - \sum_{k\in\sigma_1} |a_{2k}| \int_0^T \left(\tilde{x}^{(k)}(t)\right)^2 dt,$$

and the proof follows the same steps as before.

□

4 Semilinear case

As a direct application of general non-degenerate potentials, one can obtain reasonable existence results for periodic solutions of nonlinear beam equation

$$u^{(2n)} + \sum_{m=1}^{2n-1} a_m u^{(m)} = pu + h(t,u), \tag{4.1}$$

here $h(t,u)$ grows semilinearly when $|u| \to \infty$. Denote

$$\varphi(t) = \lim \sup_{|u|\to\infty} \frac{|h(t,u)|}{|u|}$$

exist in the sense that for any given $\varepsilon > 0$, there is $\psi_\varepsilon(t) \in L^1(\mathbb{S}_T)$ such that

$$|h(t,u)| \leq (\varphi(t)+\varepsilon)|u| + \psi_\varepsilon(t), \qquad \text{for all} \quad x \in \mathbb{R}, \quad \text{a.e.} \ \ t \in [0,T],$$

and $\varphi \in L^1(\mathbb{S}_T)$.

The proof of the main result of this section follows the strategy adopted by [15] for the second-order equation. Let us consider an m-th order systems of the form

$$\begin{cases} x^{(m)} = g\left(x, x', \dots, x^{(m-1)}\right) + h\left(t, x, x', \dots, x^{(m-1)}\right), \quad t \in [0,T], \\ x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, \dots, m-1, \end{cases} \tag{4.2}$$

where

$$g\left(kx, kx', \dots, kx^{(m-1)}\right) = kg\left(x, x', \dots, x^{(m-1)}\right)$$

for all $k > 0$, $\left(x, x', \dots, x^{(m-1)}\right) \in \mathbb{R}^{mn}$, and suppose that

$$\varphi^*(t) = \lim_{|x|+|x'|+\cdots+|x^{(m-1)}|\to\infty} \frac{\left|h\left(t, x, x', \dots, x^{(m-1)}\right)\right|}{|x| + |x'| + \cdots \left|x^{(m-1)}\right|}$$

exists and $\varphi^* \in L^p(\mathbb{S}_T)$.

Lemma 4.1. ([15]) Assume that

(H_1) The problem

$$\begin{cases} x^{(m)} = g\left(x, x', \dots, x^{(m-1)}\right), \quad t \in [0,T], \\ x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, \dots, m-1, \end{cases}$$

has no T-periodic solution other than $x = 0$; and

(H_2) $\deg(\tilde{g}, B(0,r), 0) \neq 0$ for some $r > 0$, where $\tilde{g}(x) = g(x, 0, \dots, 0)$, deg means the Brouwer degree and $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

Then there is a constant $c_0 > 0$ such that if

$$||\varphi^*|| < c_0,$$

the problem (4.2) has at least one T-periodic solution.

The main result of this section is as follows.

Theorem 4.1. Let us assume that one of the following conditions holds

(1) n is even, $p > 0$ and

$$C_n T|p| < 1 + C_n \sum_{k\in\sigma_2} |a_{2k}|\, C_k^{-1} - \sum_{k\in\sigma_1} |a_{2k}|\, \widehat{C}_{n-k}. \tag{4.3}$$

(2) n is odd, $p < 0$ and

$$C_n T|p| < 1 + C_n \sum_{k\in\sigma_1} |a_{2k}|\, C_k^{-1} - \sum_{k\in\sigma_2} |a_{2k}|\, \widehat{C}_{n-k}. \tag{4.4}$$

Then there is a constant $c_0 > 0$ such that if

$$||\varphi|| < c_0,$$

the problem (4.1) has at least one T-periodic solution.

Proof. Comparing (4.1) to (4.2), we have

$$g\left(u, u', \ldots, u^{(2n-1)}\right) = -\sum_{m=1}^{2n-1} a_m u^{(m)} + pu, \qquad h\left(t, u, u', \ldots, u^{(2n-1)}\right) = h(t, u).$$

Obviously, it is easy to see that

$$g\left(ku, ku', \ldots, ku^{(2n-1)}\right) = k\left(-\sum_{m=1}^{2n-1} a_m u^{(m)} + pu\right) = kg\left(u, u', \ldots, u^{(2n-1)}\right).$$

Besides,

$$\varphi^*(t) = \varphi(t) = \lim_{|u|\to\infty} \frac{|h(t, u)|}{|u|}.$$

Firstly, let us consider the linear problem

$$\begin{cases} u^{(2n)} + \sum\limits_{m=1}^{2n-1} a_m u^{(m)} = pu, \\ u^{(i)}(0) = u^{(i)}(T), \qquad i = 0, 1, \ldots, m-1, \end{cases} \tag{4.5}$$

From Theorem 3.1, we know that if n is even, $p > 0$ and

$$C_n T|p| < 1 + C_n \sum_{k\in\sigma_2} |a_{2k}| \, C_k^{-1} - \sum_{k\in\sigma_1} |a_{2k}| \, \widehat{C}_{n-k},$$

or alternatively if n is odd, $p < 0$ and

$$C_n T|p| < 1 + C_n \sum_{k\in\sigma_1} |a_{2k}| \, C_k^{-1} - \sum_{k\in\sigma_2} |a_{2k}| \, \widehat{C}_{n-k},$$

then (4.5) is non-degenerate, therefore condition (H_1) holds.

On the other hand, $\tilde{g}(u) = g(u, 0, \ldots, 0) = pu$. Therefore, we have trivially $\deg(\tilde{g}(u), B(0, r), 0) \neq 0$. Then, condition (H_2) holds and the result is a direct consequence of Lemma 4.1. $\square$

5 Superlinear case

In this section, we will give an application of the class of non-degenerate potentials constructed above to the study of existence and uniqueness of T-periodic solution for equations with superlinear term. We will combine techniques from [14] and [7, 9]. Let us consider the nonlinear differential equation

$$u^{(2n)} + \sum_{m=1}^{2n-1} a_m u^{(m)} = f(u) - s + \tilde{h}(t), \tag{5.1}$$

where $s \in \mathbb{R}$, $\tilde{h} \in \tilde{L}^1(\mathbb{S}_T)$, and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous and monotone function. The parameter s is the mean value of the external term $-s + \tilde{h}(t)$.

It is easy to find a necessary condition for existence of T-periodic solutions. In fact, integrating (5.1) on $[0, T]$, we have

$$s = T^{-1} \int_0^T f(u(t))dt = f(u(t_*)) \in \mathcal{R}(f) := \{f(x) : x \in \mathbb{R}\}. \tag{5.2}$$

The proof of the existence of periodic solution of (5.1) follows the strategy adopted by [14]. Let us consider an m-th order equation of the form

$$y^{(m)} + a_{m-1}y^{(m-1)} + \cdots + a_1 y' + g(t, y) = p(t) \quad (m > 1). \tag{5.3}$$

where $a_1, \ \cdots, \ a_{m-1}$ is real constants. $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic in its first variable; i.e., $g(t + T, y) = g(t, y)$ for all $t, \ y$. We define two measurable functions $\mu_+, \ \mu_- : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ by

$$\mu_+(t) = \lim \inf_{y \to \infty} g(t, y), \qquad t \in \mathbb{R};$$

$$\mu_-(t) = \lim \inf_{y \to -\infty} g(t, y), \qquad t \in \mathbb{R}.$$

Let us denote

$$Ly \equiv y^{(m)} + a_{m-1}y^{(m-1)} + \cdots + a_1 y'.$$

The following lemma is the main result of [14].

Lemma 5.1 ([14])**.** Assume that $g(t, y)$ is bounded below for $y \geq 0$ and bounded above for $y \leq 0$, and the following conditions hold

(c_1) The only T-periodic solutions to the equation $Ly = 0$ are the constants.

(c_2) There are numbers α_1 and β_1 such that for all $(t, y) \in \mathbb{R} \times \mathbb{R}$, $|g(t, y)| \leq g(t, y) + \alpha_1|y| + \beta_1$.

(c_3) $\int_0^T \mu_-(t)dt < \int_0^T p(t)dt < \int_0^T \mu_+(t)dt$.

Then there is a number $\varepsilon > 0$ such that (5.3) has a T-periodic solution provided $\alpha_1 \leq \varepsilon$.

Our existence result is the following one.

Proposition 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded below for $u \geq 0$ and bounded above for $u \leq 0$, $s \in \text{int}\mathcal{R}(f)$, and there are two non-negative constants α and β such that

$$|f(u)| \leq f(u) + \alpha|x| + \beta.$$

Assume that one of the following conditions holds

(1) n is even, and $\sum\limits_{k \in \sigma_1} |a_{2k}|\widehat{C}_{n-k} < 1$.

(2) n is odd, and $\sum\limits_{k \in \sigma_2} |a_{2k}|\widehat{C}_{n-k} < 1$,

Then there exists a positive constant α_0 such that (5.1) has at least one T-periodic solution provided $\alpha \leq \alpha_0$.

Proof. Comparing (5.1) to (5.3), we have

$$g(t,y)=f(u),\quad p(t)=-s+\tilde{h}(t).$$

It is evident to see that (c_2) and (c_3) hold. It remains to prove that condition (c_1) is satisfied.

Assume first that n is even. Let $\phi(t)$ be a T-periodic solution of the equation

$$\phi^{(2n)}(t)+\sum_{m=1}^{2n-1}a_m\phi^{(m)}(t)=0. \tag{5.4}$$

Multiplying both sides of (5.4) by $\phi(t)$ and integrating over $[0,T]$, we have

$$(-1)^n\int_0^T|\phi^{(n)}(t)|^2dt+\int_0^T\sum_{m=1}^{2n-1}a_m\phi^{(m)}(t)\phi(t)dt=0.$$

Note that $\int_0^T\phi^{(m)}(t)\phi(t)dt=0$ for every odd m. By reindexing $m=2k$ and using the definition of σ_1,σ_2 from Section 3, we have

$$(-1)^n\int_0^T|\phi^{(n)}(t)|^2dt+\sum_{m=1}^{n-1}a_{2k}(-1)^k\int_0^T|\phi^{(k)}(t)|^2dt=0.$$

Since n is even and $\sum\limits_{k\in\sigma_1}(-1)^ka_{2k}<0,\ \sum\limits_{k\in\sigma_2}(-1)^ka_{2k}>0$ by definition, we get

$$\begin{aligned}\int_0^T|\phi^{(n)}(t)|^2dt&=-\sum_{k\in\sigma_1}(-1)^ka_{2k}\int_0^T|\phi^{(k)}(t)|^2dt-\sum_{k\in\sigma_2}(-1)^ka_{2k}\int_0^T|\phi^{(k)}(t)|^2dt\\&\le-\sum_{k\in\sigma_1}(-1)^ka_{2k}\int_0^T|\phi^{(k)}(t)|^2dt.\end{aligned}$$

Therefore, from Lemma 2.4, we have

$$\begin{aligned}\int_0^T|\phi^{(n)}(t)|^2dt&\le\sum_{k\in\sigma_1}|a_{2k}|\int_0^T|\phi^{(k)}(t)|^2dt\\&\le\sum_{k\in\sigma_1}|a_{2k}|\widehat{C}_{n-k}\int_0^T|\phi^{(n)}(t)|^2dt.\end{aligned}$$

Since $\sum\limits_{k\in\sigma_1}|a_{2k}|\widehat{C}_{n-k}<1$, we get $||\phi^{(n)}||_2^2=0$. From $||\phi^{(n-1)}||_2\le\left(\frac{T}{2\pi}\right)||\phi^{(n)}||_2$, we know $||\phi^{(n-1)}||_2=0$. As $\phi^{(n-1)}(t)$ is continuous, we get $\phi^{(n-1)}(t)\equiv0$. Hence, we have $\phi(t)\equiv c$, here c is a constant. Therefore, (c_1) holds. From Lemma 5.1, we know that there exists a positive constant α_0 such that if $\alpha_0\ge\alpha$, (5.1) has at least one T-periodic solution.

On the other hand, if n is odd, the proof follows the similar steps as before.

□

In the following, we will consider the uniqueness problem. Let us introduce the following definition from [9].

Definition 5.1. Given $\sigma \in [1,\infty)$ and $A,\ B \in [0,\infty)$.

We say that f satisfies the condition $\mathcal{C}(\sigma; A, B)$ if

$$\left(\frac{f(x_1)-f(x_2)}{x_1-x_2}\right)_+^{\sigma} \le A\left(\frac{f(x_1)+f(x_2)}{2}\right)+B \tag{5.5}$$

for every $x_1,\ x_2 \in \mathbb{R}$, and $x_1 \neq x_2$. Here $\varphi_+ = (\varphi)_+ = \max(\varphi, 0)$ for $y \in \mathbb{R}$.

Or we say that f satisfies the condition $\mathcal{C}^*(\sigma; A, B)$ if

$$\left|\left(\frac{f(x_1)-f(x_2)}{x_1-x_2}\right)_-\right|^{\sigma} \le A\left(\frac{f(x_1)+f(x_2)}{2}\right)+B \tag{5.6}$$

for every $x_1,\ x_2 \in \mathbb{R}$, and $x_1 \neq x_2$. Here $\varphi_- = (\varphi)_- = \min(\varphi, 0)$ for $y \in \mathbb{R}$.

The main result for uniqueness is as follows.

Proposition 2. Assume that one of the following conditions holds

(1) n is odd, $f \in \mathcal{C}^*(\sigma; A, B)$ is non-increasing. Suppose that $s \in \mathcal{R}(f)$ satisfies

$$As + B < \frac{(M'(\sigma^*, n))^{\sigma}}{T}, \quad \text{and} \quad \sum_{k\in\sigma_2} |a_{2k}|\widehat{C}_{n-k} < 1, \tag{5.7}$$

where $M'(\sigma^*, n) := \frac{1+C_n \sum\limits_{k\in\sigma_1} |a_{2k}|C_k^{-1} - \sum\limits_{k\in\sigma_2} |a_{2k}|\widehat{C}_{n-k}}{C_n T^{\frac{1}{\sigma^*}}}$.

(2) n is even, $f \in \mathcal{C}(\sigma; A, B)$ is non-decreasing. Suppose that $s \in \mathcal{R}(f)$ satisfies

$$As + B < \frac{(M(\sigma^*, n))^{\sigma}}{T}, \quad \text{and} \quad \sum_{k\in\sigma_1} |a_{2k}|\widehat{C}_{n-k} < 1, \tag{5.8}$$

where $M(\sigma^*, n) := \frac{1+C_n \sum\limits_{k\in\sigma_2} |a_{2k}|C_k^{-1} - \sum\limits_{k\in\sigma_1} |a_{2k}|\widehat{C}_{n-k}}{C_n T^{\frac{1}{\sigma^*}}}$.

Then (5.1) has at most one T-periodic solution.

Proof. Firstly, assume that n is odd. Let $x_1(t)$ and $x_2(t)$ be two different T-solutions of (5.1), we have

$$x_i^{(2n)}(t) + \sum_{m=1}^{2n-1} a_m x_i^{(m)} = f(x_i(t)) - s + \tilde{h}(t), \quad \text{a.e. } t, \quad i = 1, 2. \tag{5.9}$$

Integrating (5.9) on $[0, T]$, we get

$$\int_0^T f(x_i(t))dt = Ts, \quad i = 1, 2. \tag{5.10}$$

The difference $x(t) := x_1(t) - x_2(t)$ is a non-trivial T-periodic solution of the equation (1.1) with

$$q(t) = \frac{f(x_1(t)) - f(x_2(t))}{x_1(t) - x_2(t)},$$

which is well defined for all $t \in I := \{t \in \mathbb{R} : x(t) \neq 0\}$, which is a non-empty open subset of $\mathbb{R}$. It is easy to see that $q(t) \in C(I)$, and $q(t) = 0$ on $J := \mathbb{R} \setminus I$. Obviously, $q(t)$ is measurable. As $f(x)$ is non-increasing in x, one has $q(t) \le 0$ for all t. Moreover, for all $t \in I$, we have from (5.6) that

$$|q(t)|^{\sigma} \le A(f(x_1(t)) + f(x_2(t)))/2 + B \le C \tag{5.11}$$

where C is a constant and $C \geq 0$, since $f(x)$ is continuous and the $x_i(t)$ are T-periodic. Therefore, $q(t) \leq 0$ for all t and $q \in L^\infty(\mathbb{S}_T)$. From (5.11), we have

$$\begin{aligned}
||q||_\sigma^\sigma &= \int_{I\cap[0,T]} |q(t)|^\sigma dt \leq \int_{I\cap[0,T]} (A(f(x_1(t)) + f(x_2(t)))/2 + B) \\
&\leq \int_{I\cap[0,T]} (A(f(x_1(t)) + f(x_2(t)))/2 + B) + \int_{J\cap[0,T]} (A(f(x_1(t)) + f(x_2(t)))/2 + B) \\
&= \frac{A}{2}\left(\int_0^T f(x_1(t))dt + \int_0^T f(x_2(t))dt\right) + BT \\
&= (As + B)T,
\end{aligned}$$

and then $||q||_\sigma \leq ((As+B)T)^{\frac{1}{\sigma}}$. From (5.7), we get $||q||_\sigma < M'(\sigma^*, n)$.

Under assumption (5.7), if we have $\bar{q} < 0$, by Theorem 3.1, we have $x(t) \equiv 0$, contradicting with the assumption $x_1 \neq x_2$. Then $\bar{q} = 0$. As $q(t) \leq 0$, we know that $q(t) \equiv 0$. Therefore,

$$x^{(2n)}(t) + \sum_{m=1}^{2n-1} a_m x^{(m)}(t) = 0. \tag{5.12}$$

Multiplying both sides of (5.12) by $x(t)$ and integrating over $[0, T]$, we have

$$(-1)^n \int_0^T |x^{(n)}(t)|^2 dt + \int_0^T \sum_{m=1}^{2n-1} a_m x^{(m)}(t)x(t)dt = 0.$$

Note that $\int_0^T x^{(m)}(t)x(t)dt = 0$ for every odd m. By reindexing $m = 2k$ and using the definition of σ_1, σ_2 from Section 3, we have

$$(-1)^n \int_0^T |x^{(n)}(t)|^2 dt + \sum_{m=1}^{n-1} a_{2k}(-1)^k \int_0^T |x^{(k)}(t)|^2 dt = 0.$$

From n is odd and $\sum\limits_{k\in\sigma_1} (-1)^k a_{2k} < 0$, $\sum\limits_{k\in\sigma_2} (-1)^k a_{2k} > 0$, we have

$$\begin{aligned}
\int_0^T |x^{(n)}(t)|^2 dt &= \sum_{k\in\sigma_1} (-1)^k a_{2k} \int_0^T |x^{(k)}(t)|^2 dt + \sum_{k\in\sigma_2} (-1)^k a_{2k} \int_0^T |x^{(k)}(t)|^2 dt \\
&\leq \sum_{k\in\sigma_2} (-1)^k a_{2k} \int_0^T |x^{(k)}(t)|^2 dt.
\end{aligned}$$

So, from Lemma 2.4, we have

$$\begin{aligned}
\int_0^T |x^{(n)}(t)|^2 dt &\leq \sum_{k\in\sigma_2} |a_{2k}| \int_0^T |x^{(k)}(t)|^2 dt \\
&\leq \sum_{k\in\sigma_2} |a_{2k}| \widehat{C}_{n-k} \int_0^T |x^{(n)})(t)|^2 dt.
\end{aligned}$$

Since $\sum_{k\in\sigma_2} |a_{2k}|\widehat{C}_{n-k} < 1$, we get $||x^{(n)}||_2^2 = 0$. From $||x||_2 \leq \left(\frac{T}{2\pi}\right)^n ||x^{(n)}||_2$, we know $||x||_2 = 0$. As $x(t)$ is continuous, $x(t) \equiv 0$ and the proof is done.

On the other hand, if n is even, the proof follows the similar steps as before. $\square$

In the following we consider equations of the Landesman-Lazer type.

Theorem 5.1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded below for $u \geq 0$ and bounded above for $u \leq 0$, and there are two non-negative constants α and β such that

$$|f(u)| \leq f(u) + \alpha|x| + \beta.$$

Assume that one of the following conditions holds

(1) n is odd, $f \in \mathcal{C}^*(\sigma; A, B)$ is strictly decreasing and $s \in \mathcal{R}(f)$ satisfies (5.7).

(2) n is even, $f \in \mathcal{C}(\sigma; A, B)$ is strictly increasing and $s \in \mathcal{R}(f)$ satisfies (5.8).

Then there exists a positive constant α_0 such that (5.1) has exactly one T-periodic solution provided that $\alpha \leq \alpha_0$.

Proof. It follows directly from Propositions 1 and 2. □

We conclude the paper with some illustrative examples.

Example 5.1. Theorem 5.1 can be applied to the example $f(x) = \exp(x) \in \mathcal{C}(1; 1, 0)$ in a direct way. In this case, one has $\mathcal{R}(f) = (0, \infty)$. Hence the equation

$$x^{(8)} + \sum_{m=1}^{7} \left(\frac{1}{2}\right)^m x^{(m)} = \exp(x) - s + \sin t \tag{5.13}$$

has at least one 2π-periodic solution for each $s > 0$. Obviously, $n = 4$ is even, $T = 2\pi$ and $a_m = \left(\frac{1}{2}\right)^m$, $m = 2k$, $\sigma_1 = 1, 3,\ \sigma_2 = 2$. Besides $|e^x| \leq e^x + 5$, here $\alpha = 0,\ \beta = 5$. Then,

$$C_4 = \frac{T^{2n-1} B_n}{(2n)!} = \frac{(2\pi)^7 \cdot \frac{1}{30}}{(8)!} = \frac{\pi^7}{9450},$$

$$\sum_{k=\sigma_2} |a_{2k}| C_k^{-1} = \left(\frac{1}{2}\right)^4 \times C_2^{-1} = \frac{1}{16} \times \left(\frac{\pi^3}{90}\right)^{-1} = \frac{45}{8\pi^3}.$$

and

$$\sum_{k \in \sigma_1} |a_{2k}| \widehat{C}_{n-k} = \left(\frac{1}{2}\right)^2 \times \widehat{C}_3 + \left(\frac{1}{2}\right)^6 \times \widehat{C}_1 = \frac{1}{4} + \frac{1}{64} = \frac{17}{64} < 1.$$

Hence, condition (5.8) is

$$s < \frac{M(\infty, 4)}{T} = \frac{1 + \frac{\pi^4}{1680} - \frac{17}{64}}{\frac{\pi^7}{9450} \times 2\pi} = \frac{315 \times (4935 + 4\pi^4)}{448 \times \pi^8}. \tag{5.14}$$

Theorem 5.1 asserts that for $s > 0$ satisfying (5.14), eq. (5.13) has exactly one T-periodic solution.

Example 5.2. Let $p \in (1, \infty)$. The function $f(x) = x_+^p \in \mathcal{C}(p^*; p^{p^*}, 0)$ is non-decreasing, but is not strictly increasing. Theorem 5.1 can be applied to the following superlinear equation:

$$x^{(2n)} + \sum_{m=1}^{2n-1} a_m x^{(m)} = x_+^p - s + \tilde{h}(t) \tag{5.15}$$

in an indirect way, here $a_m \in \mathbb{R}$. For this case, one have $\mathcal{R}(f) = [0, \infty)$, $|x_+^p| \leq x_+^p + 4$, here $\alpha = 0,\ \beta = 4,\ \alpha_0 \geq 0$. Then (5.15) has at least one T-periodic solution for each $s > 0$ and each $\tilde{h} \in \tilde{L}^1(\mathbb{S}_T)$. Note that the function $f(x) = x_+^p$ is strictly increasing in $x \in (0, \infty)$, $C_n = \frac{T^{2n-1}B_n}{(2n)!}$, $\widehat{C}_{n-k} = \left(\frac{T}{2\pi}\right)^{2(n-k)}$. After a modification of the proof of Theorem 5.1, we conclude that if n is even,

$$0 < s < \frac{(M(p,n))^{p^*}}{p^{p^*} \cdot T}, \qquad \text{and} \qquad \sum_{k \in \sigma_1} |a_{2k}| \widehat{C}_{n-k} < 1. \tag{5.16}$$

If n is odd

$$0 < s < \frac{(M'(p,n))^{p^*}}{p^{p^*} \cdot T}, \qquad \text{and} \qquad \sum_{k \in \sigma_2} |a_{2k}| \widehat{C}_{n-k} < 1 \tag{5.17}$$

then for each $\tilde{h} \in \tilde{L}^1(\mathbb{S}_T)$, (5.15) has exactly one T-periodic solution. The reasons are as follows. Note that the second inequality of (5.16) (or (5.17)) corresponds to (5.8) (or (5.7)) for $f(x) = x_+^p$.

Example 5.3. Consider the following superlinear equation:

$$x^{(4)} + \sum_{m=1}^{3} (-1)^m x^{(m)} = x_+ + x_+^2 - s + \tilde{h}(t). \tag{5.18}$$

Here $x_+ + x_+^2 \in \mathcal{C}(2; 4, 1)$, $T < 2\pi$, $n = 2$ is even, $a_m = (-1)^m$, $m = 2k$, $\sigma_1 = 1$, $\sigma_2 = 0$, $|x_+ + x_+^2| \leq x_+ + x_+^2 2|x| + 3$, here $\alpha = 2,\ \beta = 3,\ \alpha_0 > 2$. $C_2 = \frac{T^{2\times 2-1}B_2}{4!} = \frac{T^7 \times \frac{1}{30}}{4!} = \frac{T^3}{720}$, $\sum\limits_{k=\sigma_2} |a_{2k}| C_k^{-1} = 0$ and $\sum\limits_{k=\sigma_1} |a_{2k}| C_k^{-1} = \widehat{C}_1 = \frac{T}{2\pi} < 1$. Then,

$$M(2,2) = \frac{1 - \frac{T}{2\pi}}{\frac{T^3}{720} \cdot T^{\frac{1}{2}}} = \frac{360 \times (2\pi - T)}{\pi \times T^{\frac{7}{2}}}$$

Condition (5.8) are now $s > 0$ and

$$4s + 1 < \frac{(M(2,2))^2}{T} = \frac{129600 \times (2\pi - T)^2}{\pi^2 \times T^8}.$$

In order to obtain reasonable conditions, T should be satisfy $129600 \times (2\pi - T)^2 > \pi^2 \times T^8$. We conclude that when

$$129600 \times (2\pi - T)^2 > \pi^2 \times T^8, \qquad 0 < s < \frac{129600 \times (2\pi - T)^2 - \pi^2 \times T^8}{4\pi^2 \times T^8}. \tag{5.19}$$

Then (5.18) has exactly one T-periodic solution for each $\tilde{h} \in \tilde{L}^1(\mathbb{S}_T)$. Different from the case for (5.14) and (5.16), we have now a restriction on the period T in (5.19).

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