On the open problems connected to the results of Lazer and Solimini

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Abstract

A well-known theorem proved by A. C. Lazer and S. Solimini claims that the singular equation

$$u'' + \frac{1}{u^{\lambda}} = h(t), \qquad \lambda > 0,$$

has a periodic solution if and only if the mean value of the continuous external force is positive. In this paper, we show that this result cannot be extended to the case when h is an integrable function, unless the additional assumptions are introduced. In addition, for each $p \geq 1$ and h integrable function in the p-th power, we give a sharp condition guaranteeing the existence of periodic solutions to the above-mentioned equation, showing that there is a close relation between p and the order of singularity, λ.

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1 Introduction

Boundary value problems for ordinary differential equations with singularities (both in the time and the phase variable) arise in applications, especially in physics; therefore this topic has been of substantial interest of scientists and engineers for decades. An interesting overview of history and current state of the matter can be found in [6] (see also references therein).

In this paper, we focus on the existence of positive periodic solutions to the second-order ordinary differential equation with singularity in the phase variable, in particular

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on the problem

$$u''(t) + g(u(t)) = h(t) \qquad \text{for a. e. } t \in \mathbb{R}, \tag{1.1}$$
$$u(t) = u(t+\omega) \qquad \text{for } t \in \mathbb{R}, \tag{1.2}$$

where $h \in L^p_\omega(\mathbb{R})$, $p \geq 1$, and $g \in C(\mathbb{R}_+;\mathbb{R}_+)$ satisfies the conditions

$$\lim_{x\to 0+} g(x) = +\infty, \qquad \lim_{x\to+\infty} g(x) = 0. \tag{1.3}$$

The pioneer paper about this topic was written by A. C. Lazer and S. Solimini and published in 1987 (see [4]). They proved, among others, that if h is continuous and g satisfies (1.3), then there exists a positive solution to (1.1), (1.2) in the space $C^2(\mathbb{R};\mathbb{R})$ if and only if $\overline{h} > 0$. The way of the proof of Lazer and Solimini allows us to formulate their result as follows:

Theorem 1.1 (Lazer and Solimini [4]). *Let $g \in C(\mathbb{R}_+;\mathbb{R}_+)$ satisfy* (1.3) *and let*

$$h \in L^\infty_\omega(\mathbb{R}). \tag{1.4}$$

Then there exists a positive solution $u \in AC^1_\omega(\mathbb{R}_+)$ to (1.1), (1.2) *if and only if $\overline{h} > 0$.*

The proof of this result is a simple and elegant application of the method of lower and upper functions. There are other papers in the literature where the problem of the type (1.1), (1.2) is studied in the framework of the Carathéodory theory, i.e., if $h \in L_\omega(\mathbb{R})$ and a positive function $u \in AC^1_\omega(\mathbb{R}_+)$ is understood as a solution to (1.1) (see, e.g., [1–3, 6, 7] and references therein). However, also in the works [1, 7], the boundedness of the function h is needed. In [2, 3, 6], the condition (1.4) is replaced by another condition dealing with the oscillation of the primitive of h.

On the other hand, the major part of the results dealing with the continuous input functions can be formulated also in the framework of the Carathéodory theory without any essential changes. This fact may encourage one's expectation that also Theorem 1.1 can be extended to the case when $h \in L_\omega(\mathbb{R})$ without any other additional conditions. Such a question was formally posted in [2, Open Problem 4.1] and it runs as follows: *Does Theorem 1.1 remain still valid if the condition* (1.4) *is violated?* Despite of all expectations, the answer is negative. The condition (1.4) is essential and, as shown in this paper (see Example 5.1), Theorem 1.1 is not valid anymore if the condition (1.4) is withdrawn unless the additional assumptions are involved—such an additional condition is, e.g., the relation (4.1) in Theorem 4.1 formulated below. Moreover, we prove that the condition (4.1) is optimal in a certain sense. More precisely, if we consider the equation

$$u''(t) + \frac{1}{u^\lambda(t)} = h(t) \qquad \text{for a. e. } t \in \mathbb{R}, \tag{1.5}$$

from our results, for every $p \in [1, +\infty)$, we obtain

- If $\lambda \geq 1/(2p-1)$ and $h \in L^p_\omega(\mathbb{R})$ then (1.5), (1.2) has a positive solution if and only if $\overline{h} > 0$, and such a solution is unique.
- If $0 < \lambda < 1/(2p-1)$ then there exists a function $h \in L^p_\omega(\mathbb{R})$ with $\overline{h} > 0$ such that (1.5), (1.2) has no positive solution.

At this point we would like to emphasize the following: for $h \in L^p_\omega(\mathbb{R})$, there exists a relation between p and the order of singularity, λ. In other words, there exists a critical value depending on p such that if the power of the singularity λ is greater than or equal to this value then there exists a positive periodic solution. Moreover, if $h \in L^\infty_\omega(\mathbb{R})$, then also $h \in L^p_\omega(\mathbb{R})$ for every $p \in [1, +\infty)$, and so applying our results for p sufficiently large we obtain that (1.5), (1.2) has a positive solution for every $\lambda > 0$ (provided $h \in L^\infty_\omega(\mathbb{R})$). Thus Theorem 1.1 can be understood as a limit case of Theorem 4.1 formulated below.

Theorem 4.1 deals also with the uniqueness of a solution in the case when g is strictly decreasing function. Such a result can be also found in [5], however this fact is worth mentioning here because in the original paper of Lazer and Solimini, the question of the uniqueness was not discussed.

The paper is organized as follows: in Section 2 we introduce some basic notation together with definitions and a preliminary result. In Section 3, some auxiliary results are proven. Section 4 is devoted to the formulation and the proof of the main result. Finally, in Section 5, we introduce an example showing the optimality of the obtained result.

2 Preliminaries

For convenience, we are going to introduce a list of notation which is used throughout the paper:
$\mathbb{N}$ is the set of all natural numbers;
$\mathbb{Z}$ is the set of all integer numbers;
$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = (0, +\infty)$;
$C_\omega(\mathbb{R})$ is the Banach space of ω-periodic continuous functions $u : \mathbb{R} \to \mathbb{R}$, endowed with the norm

$$\|u\|_C = \max\big\{|u(t)| : t \in [0, \omega]\big\};$$

$C(\mathbb{R}_+; \mathbb{R}_+)$ is the set of continuous functions $u : \mathbb{R}_+ \to \mathbb{R}_+$;
$AC^1_\omega(D)$, where $D \subseteq \mathbb{R}$, is the set of ω-periodic functions $u : \mathbb{R} \to D$ which are absolutely continuous together with their first derivative;
$L^p_\omega(\mathbb{R})$, where $1 \leq p < +\infty$, is the Banach space of ω-periodic functions $u : \mathbb{R} \to \mathbb{R}$ which are Lebesgue integrable on $[0, \omega]$ in the p-th power, endowed with the norm

$$\|u\|_p = \left(\int_0^\omega |u(t)|^p dt \right)^{1/p};$$

$L_\omega(\mathbb{R}) = L^1_\omega(\mathbb{R})$;

$L^\infty_\omega\big(\mathbb{R}\big)$ is the Banach space of ω-periodic essentially bounded functions $u:\mathbb{R}\to\mathbb{R}$, endowed with the norm

$$\|u\|_\infty = \text{ess sup}\,\big\{|u(t)| : t\in[0,\omega]\big\};$$

if $h\in L_\omega\big(\mathbb{R}\big)$ then

$$\overline{h} = \frac{1}{\omega}\int_0^\omega h(s)ds;$$

$I(s,t) = (\min\{s,t\},\max\{s,t\})$.

In what follows, we study the problem (1.1), (1.2) with $h\in L^p_\omega\big(\mathbb{R}\big)$, $p\in[1,+\infty)$, and $g\in C\big(\mathbb{R}_+;\mathbb{R}_+\big)$. By a solution to (1.1), (1.2) we understand a function $u\in AC^1_\omega\big(\mathbb{R}_+\big)$ satisfying (1.1) almost everywhere on $\mathbb{R}$ and verifying (1.2).

We recall the notion of upper and lower functions in a form suitable for us.

Definition 2.1. A function $\alpha\in AC^1_\omega\big(\mathbb{R}_+\big)$ is said to be a *lower function* to the problem (1.1), (1.2) if

$$\alpha''(t)+g(\alpha(t))\ge h(t) \qquad \text{for a. e. } t\in\mathbb{R}.$$

Definition 2.2. A function $\beta\in AC^1_\omega\big(\mathbb{R}_+\big)$ is said to be an *upper function* to the problem (1.1), (1.2) if

$$\beta''(t)+g(\beta(t))\le h(t) \qquad \text{for a. e. } t\in\mathbb{R}.$$

The following assertion is well-known in the theory of boundary value problems. Its proof can be found, e.g., in [6].

Proposition 2.1. *Let α and β be lower and upper functions, respectively, to the problem* (1.1), (1.2) *such that*

$$\alpha(t)\le\beta(t) \qquad \textit{for } t\in\mathbb{R}.$$

Then there exists a solution u to the problem (1.1), (1.2) *satisfying*

$$\alpha(t)\le u(t)\le\beta(t) \qquad \textit{for } t\in\mathbb{R}.$$

3 Auxiliary results

The following results are important to prove our main results

Lemma 3.1. *Let $h\in L^p_\omega\big(\mathbb{R}\big)$ and let $u\in AC^1_\omega\big(\mathbb{R}_+\big)$ satisfy*

$$u''(t)\le h(t) \qquad \textit{for a. e. } t\in\mathbb{R}. \tag{3.1}$$

Then

$$|u'(t)|\le\left(\frac{(2p-1)}{p}\|h\|_p\right)^{p/(2p-1)}\big(u(t)\big)^{(p-1)/(2p-1)} \qquad \textit{for } t\in\mathbb{R}. \tag{3.2}$$

Proof. Let $t_0 \in \mathbb{R}$ be arbitrary but such that $u'(t_0) \neq 0$, and let $\sigma = \operatorname{sgn} u'(t_0)$. Then there exists $s_0 \in I(t_0, t_0 - \sigma\omega)$ such that

$$\sigma u'(t) > 0 \qquad \text{for } t \in I(s_0, t_0), \qquad u'(s_0) = 0.$$

Multiplying both sides of (3.1) by $|u'(t)|^{(p-1)/p}$ and integrating over $I(s_0, t_0)$ we get

$$\frac{p}{2p-1}|u'(t_0)|^{(2p-1)/p} = \sigma \int_{s_0}^{t_0} u''(t)|u'(t)|^{(p-1)/p} dt \leq \sigma \int_{s_0}^{t_0} h(t)|u'(t)|^{(p-1)/p} dt \leq$$
$$\|h\|_p \left(\sigma \int_{s_0}^{t_0} |u'(t)| dt\right)^{(p-1)/p} = \|h\|_p \big(u(t_0) - u(s_0)\big)^{(p-1)/p} \leq \|h\|_p \big(u(t_0)\big)^{(p-1)/p}. \quad (3.3)$$

Thus (3.2) follows from (3.3). □

Lemma 3.2. *Let $u \in AC^1_\omega(\mathbb{R}_+)$, $g \in C(\mathbb{R}_+; \mathbb{R}_+)$, and let*

$$g_*(x) \overset{def}{=} \inf\{g(s) : s \in (0, x]\}. \quad (3.4)$$

Let, moreover, $s_0, t_0 \in \mathbb{R}$ be such that $s_0 < t_0$ and

$$u'(s_0) = 0, \qquad u'(t_0) = 0. \quad (3.5)$$

Then

$$\int_{s_0}^{t_0} u''(t) g_*^{p-1}(u(t)) dt \geq 0. \quad (3.6)$$

Proof. Let $g_n \in C(\mathbb{R}_+; \mathbb{R}_+)$ be a sequence of non-increasing functions which are continuous together with their derivatives and such that

$$\lim_{n \to +\infty} \|g_n^{p-1} \circ u - g_*^{p-1} \circ u\|_C = 0. \quad (3.7)$$

Obviously, since g_n are non-increasing, in view of (3.5) we have

$$\int_{s_0}^{t_0} u''(t) g_n^{p-1}(u(t)) dt = -(p-1) \int_{s_0}^{t_0} {u'}^2(t) g_n^{p-2}(u(t)) g_n'(u(t)) dt \geq 0. \quad (3.8)$$

Now, (3.6) follows from (3.7) and (3.8), because

$$\left| \int_{s_0}^{t_0} u''(t) g_n^{p-1}(u(t)) dt - \int_{s_0}^{t_0} u''(t) g_*^{p-1}(u(t)) dt \right| \leq \|g_n^{p-1} \circ u - g_*^{p-1} \circ u\|_C \int_{s_0}^{t_0} |u''(t)| dt.$$

□

Lemma 3.3. *Let $h \in L^p_\omega(\mathbb{R})$, $g \in C(\mathbb{R}_+; \mathbb{R}_+)$, and let $u \in AC^1_\omega(\mathbb{R}_+)$ be such that*

$$u''(t) + g(u(t)) \leq h(t) \qquad \text{for a. e. } t \in \mathbb{R}. \quad (3.9)$$

Let, moreover, $s_0, t_0 \in \mathbb{R}$ be such that $s_0 < t_0$ and (3.5) is fulfilled. Then

$$\int_{s_0}^{t_0} g(u(t)) g_*^{p-1}(u(t)) dt \leq \|h\|_p^p \quad (3.10)$$

where g_* is given by (3.4).

Proof. Multiplying both sides of (3.9) by $g_*^{p-1}(u(t))$ and integrating from s_0 to t_0, in view of (3.5) and according to Lemma 3.2, we arrive at

$$\int_{s_0}^{t_0} g(u(t))g_*^{p-1}(u(t))dt \le \int_{s_0}^{t_0} h(t)g_*^{p-1}(u(t))dt \le \|h\|_p \left(\int_{s_0}^{t_0} g_*^p(u(t))dt\right)^{(p-1)/p},$$

whence in view of (3.4) we get (3.10). □

Lemma 3.4. *Let* $u_i \in AC^1_\omega\big([0,+\infty)\big)$ $(i=1,2)$ *be such that*

$$\text{meas}\big\{t \in [0,\omega] : u_i(t) = 0\big\} = 0 \qquad (i=1,2), \tag{3.11}$$

$$u_i''(t) + \widetilde{g}(u_i(t)) = h(t) \qquad \text{for a. e. } t \in \mathbb{R} \quad (i=1,2), \tag{3.12}$$

where

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \end{cases} \tag{3.13}$$

$g \in C\big(\mathbb{R}_+;\mathbb{R}_+\big)$ *is a decreasing function, and* $h \in L_\omega\big(\mathbb{R}\big)$. *Then* $u_1 \equiv u_2$.

Proof. First assume that

$$u_1(t) \ge u_2(t) \qquad \text{for } t \in \mathbb{R}. \tag{3.14}$$

Put

$$z(t) = u_1(t) - u_2(t) \qquad \text{for } t \in \mathbb{R}. \tag{3.15}$$

Then, in view of (3.11)–(3.14) we have

$$z''(t) \ge 0 \qquad \text{for a. e. } t \in \mathbb{R}, \qquad z \in AC^1_\omega\big([0,+\infty)\big). \tag{3.16}$$

However, (3.16) implies that z is a constant function, i.e., with respect to (3.11), (3.12), (3.13), and (3.15), we have

$$0 = u_1''(t) - u_2''(t) = -g(u_1(t)) + g(u_2(t)) \qquad \text{for a. e. } t \in \mathbb{R}. \tag{3.17}$$

Now from (3.17), according to the fact that g is assumed to be decreasing, it follows that $u_1 \equiv u_2$.

Further suppose that (3.14) is not valid. Then there exist $t_0, t_1 \in \mathbb{R}$ such that $t_0 < t_1$ and

$$u_1(t) > u_2(t) \qquad \text{for } t \in (t_0,t_1), \qquad u_1(t_0) = u_2(t_0), \qquad u_1(t_1) = u_2(t_1). \tag{3.18}$$

Define z by (3.15). Then, in view of (3.11)–(3.13), and (3.18) we have

$$z''(t) \ge 0 \qquad \text{for a. e. } t \in [t_0,t_1], \qquad z(t_0) = 0, \qquad z(t_1) = 0. \tag{3.19}$$

However, (3.19) implies $z(t) \le 0$ for $t \in [t_0,t_1]$, which, on account of (3.15) contradicts (3.18). □

4 Main Result

Theorem 4.1. *Let* $h \in L^p_\omega(\mathbb{R})$ *and let* $g \in C(\mathbb{R}_+; \mathbb{R}_+)$ *verify* (1.3)*. Let, moreover,*

$$\lim_{x \to 0+} \int_x^1 g\left(w(s)\right) g_*^{p-1}\big(w(s)\big) ds = +\infty, \tag{4.1}$$

where

$$w(s) \overset{def}{=} s^{(2p-1)/p} \qquad \text{for } s \in \mathbb{R}^+. \tag{4.2}$$

Then the problem (1.1), (1.2) *has a positive solution if and only if* $\overline{h} > 0$*. If, in addition, g is a decreasing function, then such a solution is unique.*

Proof. If u is a positive solution to (1.1), (1.2) then the integration of (1.1) from 0 to ω results in

$$\int_0^\omega g(u(s))ds = \int_0^\omega h(s)ds,$$

whence it follows that $\overline{h} > 0$ as the function g is positive.

Now suppose that $\overline{h} > 0$. Together with (1.1), for every $k \in \mathbb{N}$, consider the auxiliary equation

$$u''(t) + g(u(t)) = h_k(t) \qquad \text{for a. e. } t \in \mathbb{R}, \tag{4.3}$$

where

$$h_k(t) = \begin{cases} k & \text{if } h(t) > k, \\ h(t) & \text{if } h(t) \le k, \end{cases} \qquad \text{for a. e. } t \in \mathbb{R}, \quad k \in \mathbb{N}. \tag{4.4}$$

Obviously,

$$h_k(t) \le h_m(t) \le h(t) \qquad \text{for a. e. } t \in \mathbb{R}, \quad k \le m, \tag{4.5}$$

$$|h_k(t)| \le |h(t)| \qquad \text{for a. e. } t \in \mathbb{R}, \quad k \in \mathbb{N}, \tag{4.6}$$

$$\lim_{k \to +\infty} \overline{h}_k = \overline{h}. \tag{4.7}$$

According to (1.3), (4.7), and $\overline{h} > 0$, there exist $x_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$g(x) \le \overline{h}_{k_0} \qquad \text{for } x \ge x_0. \tag{4.8}$$

Let z be a solution to the Dirichlet problem

$$z''(t) = h_{k_0}(t) - \overline{h}_{k_0}, \qquad z(0) = 0, \qquad z(\omega) = 0 \tag{4.9}$$

and put

$$\beta(t) = \widetilde{z}(t) + r \qquad \text{for } t \in \mathbb{R}, \tag{4.10}$$

where $\widetilde{z}$ is an ω-periodic prolongation of z to the real axis and $r > 0$ is large enough such that

$$x_0 \le \beta(t) \qquad \text{for } t \in \mathbb{R}. \tag{4.11}$$

Obviously, $\beta \in AC^1_\omega(\mathbb{R}_+)$ and in view of (4.5) and (4.8)–(4.11),

$$\beta''(t) + g(\beta(t)) \le h_k(t) \qquad \text{for a. e. } t \in \mathbb{R}, \quad k \ge k_0. \tag{4.12}$$

On the other hand, on account of (1.3) and (4.4), for every $k \ge k_0$ there exists $x_k \in (0, x_0)$ such that

$$g(x_k) \ge h_k(t) \qquad \text{for a. e. } t \in \mathbb{R}. \tag{4.13}$$

If we put $\alpha_k(t) = x_k$ for $t \in \mathbb{R}$ then, in view of (4.11) and (4.13), we have

$$\alpha_k''(t) + g(\alpha_k(t)) \ge h_k(t) \qquad \text{for a. e. } t \in \mathbb{R}, \quad k \ge k_0, \tag{4.14}$$

$$\alpha_k(t) \le \beta(t) \qquad \text{for } t \in \mathbb{R}, \quad k \ge k_0. \tag{4.15}$$

Thus, for every $k \ge k_0$, there exists a pair of well-ordered upper and lower functions. According to Proposition 2.1, there exists a sequence of solutions $\{u_k\}_{k=k_0}^{+\infty}$ to (4.3), (1.2) such that

$$0 < \alpha_k(t) \le u_k(t) \le \beta(t) \qquad \text{for } t \in \mathbb{R}. \tag{4.16}$$

From Lemma 3.1 and (4.16) it follows that

$$\|u_k'\|_C \le \left(\frac{(2p-1)}{p}\|h\|_p\right)^{p/(2p-1)} \|\beta\|_C^{(p-1)/(2p-1)} \qquad \text{for } k \ge k_0. \tag{4.17}$$

Further, we show that the set of functions $\{u_k\}_{k=k_0}^{+\infty}$ is bounded from below. The integration of (4.3) from 0 to ω, in view of (4.5), yields

$$\int_0^\omega g(u_k(s))ds \le \omega \overline{h}. \tag{4.18}$$

On the other hand, (1.3) implies the existence of $y_0 > 0$ (which does not depend on k) such that

$$g(x) > \overline{h} \qquad \text{for } x \in (0, y_0). \tag{4.19}$$

From (4.18) and (4.19) it follows that for every $k \ge k_0$ we have

$$\|u_k\|_C \ge y_0. \tag{4.20}$$

Let $r_k \in [0, \omega]$ and $\xi_k \in [r_k - \omega, r_k)$ be such that

$$u_k(r_k) = \max\{u_k(t) : t \in [0, \omega]\}, \qquad u_k(\xi_k) = \min\{u_k(t) : t \in [0, \omega]\}. \tag{4.21}$$

Obviously, $u_k'(\xi_k) = 0$, $u_k'(r_k) = 0$, and in view of (4.20), (4.21), and Lemmas 3.1 and 3.3 we have

$$\frac{2p-1}{p} \int_{(u_k(\xi_k))^{p/(2p-1)}}^{y_0^{p/(2p-1)}} g\left(w(s)\right) g_*^{p-1}\big(w(s)\big) ds \le \int_{\xi_k}^{r_k} \frac{u_k'(t) g\left(u_k(t)\right) g_*^{p-1}\big(u_k(t)\big)}{\big(u_k(t)\big)^{(p-1)/(2p-1)}} dt \le$$
$$\left(\frac{(2p-1)}{p}\|h\|_p\right)^{p/(2p-1)} \int_{\xi_k}^{r_k} g\left(u_k(t)\right) g_*^{p-1}\big(u_k(t)\big) dt \le \left(\frac{2p-1}{p}\right)^{p/(2p-1)} \|h\|_p^{2p^2/(2p-1)},$$

where w is given by (4.2).

Thus the assumption (4.1) implies the existence of an $\varepsilon > 0$ such that

$$\varepsilon \le u_k(t) \qquad \text{for } t \in \mathbb{R}, \quad k \ge k_0. \tag{4.22}$$

Finally, using (4.6), (4.16), and (4.22), from (4.3) we obtain

$$|u_k''(t)| \le g^* + |h(t)| \qquad \text{for a. e. } t \in \mathbb{R}, \quad k \ge k_0$$

where

$$g^* = \max\big\{g(x) : x \in [\varepsilon, \|\beta\|_C]\big\}.$$

Thus, the sequences $\{u_k\}_{k=k_0}^{+\infty}$ and $\{u_k'\}_{k=k_0}^{+\infty}$ are uniformly bounded and equicontinuous. Therefore, according to Arzelà–Ascoli Theorem, we can assume without loss of generality that there exist $u_0, v_0 \in C_\omega\big(\mathbb{R}\big)$ such that

$$\lim_{k\to+\infty} \|u_k - u_0\|_C = 0, \qquad \lim_{k\to+\infty} \|u_k' - v_0\|_C = 0. \tag{4.23}$$

Moreover, since u_k are solutions to (4.3), (1.2), in view of (4.4), (4.22), and (4.23) we have $u_0 \in AC^1_\omega\big(\mathbb{R}_+\big)$, $u_0' \equiv v_0$, and u_0 is a positive solution to (1.1), (1.2).

The uniqueness of a solution in the case when g is a decreasing function follows from Lemma 3.4. □

Remark 4.1. Note that assumption $h \in L^p_\omega\big(\mathbb{R}\big)$ in Theorem 4.1 can be weakened to $h \in L_\omega\big(\mathbb{R}\big)$, $[h]_+ \in L^p_\omega\big(\mathbb{R}\big)$, where $[h]_+$ is a non-negative part of the function h, i.e.,

$$[h]_+(t) \overset{def}{=} \frac{|h(t)| + h(t)}{2} \qquad \text{for a. e. } t \in \mathbb{R}.$$

5 Optimality and Counter-Example

A particular case of the equation (1.1) is the equation (1.5) where $\lambda > 0$. For this equation, Theorems 1.1 and 4.1 yield the following assertions:

Corollary 5.1. *Let*

$$h \in L^\infty_\omega\big(\mathbb{R}\big), \qquad \lambda > 0. \tag{5.1}$$

Then the problem (1.5), (1.2) *has a positive solution if and only if* $\overline{h} > 0$, *and such a solution is unique.*

Corollary 5.2. *Let* $p \in [1, +\infty)$ *and*

$$h \in L^p_\omega\big(\mathbb{R}\big), \qquad \lambda \ge 1/(2p-1). \tag{5.2}$$

Then the problem (1.5), (1.2) *has a positive solution if and only if* $\overline{h} > 0$, *and such a solution is unique.*

Remark 5.1. In spite of the fact that nor in Theorem 1.1 nor in the original result of Lazer and Solimini, the uniqueness of a positive solution to (1.1), (1.2) is discussed, Corollary 5.1 is valid—the uniqueness of a solution follows from Lemma 3.4.

Before we formulate our theorem, we introduce an example:

Example 5.1. Let $p \geq 1$ and $\lambda \in \left(0, \frac{1}{2p-1}\right)$. Choose $\mu \in \left(2 - \frac{1}{p\lambda}, \frac{1}{p}\right)$, $\varepsilon \in \left(0, \frac{\omega}{4}\right)$, and put

$$\varphi(t) = \begin{cases} -t^{-\mu} & \text{for } t \in (0, \varepsilon], \\ 0 & \text{for } t \in \left(\varepsilon, \frac{\omega}{2} - \varepsilon\right), \\ \left(\frac{\omega}{2} - t\right)^{-\mu} & \text{for } t \in \left[\frac{\omega}{2} - \varepsilon, \frac{\omega}{2}\right), \end{cases} \qquad \varphi(t) = \varphi(\omega - t) \quad \text{for } t \in \left(\frac{\omega}{2}, \omega\right),$$

$$v(t) = \int_t^{\frac{\omega}{2}} \int_s^{\frac{\omega}{2}} \varphi(\xi) d\xi ds \qquad \text{for } t \in [0, \omega).$$

If we periodically extend the functions φ and v to the whole real axis by setting

$$\varphi(t) \stackrel{def}{=} \varphi(t - k\omega) \qquad \text{for a. e. } t \in (k\omega, (k+1)\omega), \qquad k \in \mathbb{Z} \setminus \{0\},$$
$$v(t) \stackrel{def}{=} v(t - k\omega) \qquad \text{for } t \in [k\omega, (k+1)\omega), \qquad k \in \mathbb{Z} \setminus \{0\},$$

we obviously obtain

$$\varphi \in L^p_\omega(\mathbb{R}), \quad v \in AC^1_\omega([0, +\infty)), \quad v''(t) = \varphi(t) \quad \text{for a. e. } t \in \mathbb{R}, \tag{5.3}$$

and by a direct calculation, the following relations can be verified:

$$v(t) > 0 \qquad \text{for } t \in [0, \omega/2) \cup (\omega/2, \omega], \qquad v(\omega/2) = 0, \tag{5.4}$$
$$v(t) = \frac{|\omega/2 - t|^{2-\mu}}{(2-\mu)(1-\mu)} \qquad \text{for } t \in \left(\frac{\omega}{2} - \varepsilon, \frac{\omega}{2} + \varepsilon\right).$$

Now it can be easily seen that

$$\frac{1}{v^\lambda} \in L^p_\omega(\mathbb{R}). \tag{5.5}$$

Put

$$h(t) \stackrel{def}{=} \varphi(t) + \frac{1}{v^\lambda(t)} \qquad \text{for a. e. } t \in \mathbb{R}. \tag{5.6}$$

Obviously, in view of (5.3)–(5.5) we have $h \in L^p_\omega(\mathbb{R})$ and $\overline{h} > 0$. Consider the problem (1.5), (1.2) and suppose that there exists a positive solution u to (1.5), (1.2). According to (5.3), (5.4), (5.6), and Lemma 3.4, it follows that $u(t) = v(t)$ for $t \in \mathbb{R}$. However, that is impossible, because $v(\omega/2) = 0$. Thus, (1.5), (1.2) has no positive solution with h defined by (5.6).

Example 5.1 proves the following assertion:

Theorem 5.1. *Let* $p \in [1, +\infty)$, $0 < \lambda < 1/(2p-1)$. *Then there exists* $h \in L^p_\omega(\mathbb{R})$ *with* $\overline{h} > 0$ *such that the problem* (1.5), (1.2) *has no positive solution.*

According to Theorem 5.1, it can be seen that the condition (4.1) in Theorem 4.1 is essential and cannot be omitted. Moreover, Theorem 5.1 shows that the condition (5.2) in Corollary 5.2 is unimprovable.

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