Maps of the plane with a finite number of fixed points

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Abstract

In this paper, we find two families of planar maps with a finite number of fixed points. Further we apply our results to study the number of periodic solutions of some forced second order differential equations.

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1 Introduction

Given a fixed number $T > 0$, we consider a system of differential equations in the plane

$$\mathbf{x}' = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{x} \in \mathcal{D} \subseteq \mathbb{R}^2, \tag{1.1}$$

where the vector field $\mathbf{X} : \mathbb{R} \times \mathcal{D} \to \mathbb{R}^2$ is T-periodic in time, i.e.,

$$\mathbf{X}(t + T, \mathbf{x}) = \mathbf{X}(t, \mathbf{x}), \qquad \text{for } (t, \mathbf{x}) \in \mathbb{R} \times \mathcal{D}.$$

We will say that this system has the *NS* property if the set of T-periodic solutions is finite. Nakajima and Seifert found in [8] a class of real analytic systems with this property. The systems in [8] were defined in the whole plane, $\mathcal{D} = \mathbb{R}^2$, and they include as a remarkable example the system associated to the forced Duffing equation

$$u'' + cu' + \alpha u + \beta u^3 = B \sin t, \tag{1.2}$$

where the parameters c and β satisfy

(C1) $c > 0, \ \beta > 0.$

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In this case, $\mathbf{x} = (u, u')$ and the period can have the values $T = 2n\pi$ for each $n = 1, 2, \cdots$.

The goal of the present paper is to introduce some variants in the ideas developed in [8] that allow to enlarge the class of known systems with the *NS* property. In particular, for the Duffing equation we will prove that this property is satisfied for the set of parameters described by any of the two conditions below:

(C2) $c \neq 0,\ \beta \neq 0$;

(C3) $c = 0,\ \beta < 0,\ \alpha \leq \frac{1}{n^2}$ if $T = 2n\pi$.

In [8], the condition $\beta > 0$ was imposed to guarantee that the system associated to the Duffing equation was dissipative in the sense of Levinson (see [6] and [8]). In a dissipative system all solutions will eventually enter into a compact set and will remain there for larger times. This implies in particular that there exists a common bound for all T-periodic solutions. Indeed, it is the existence of this bound what plays a role in the proof of *NS* property, rather than the more restrictive dissipativity assumption. This observation will allow to replace the condition (C1) by (C2). Note that the cases $c > 0$ and $c < 0$ are equivalent via the change of independent variable $t \mapsto -t$.

Another condition imposed to the systems in [8] was the negative divergence of the vector field $\mathbf{X}(t, \cdot)$. This excludes all Hamiltonian systems and, in particular, the Duffing equation with $c = 0$. Some modifications in the geometric arguments of the proof in [8] have allowed us to obtain the condition (C3). Note that in some aspect that condition is optimal because there exists infinitely many 2π-periodic solutions of (1.2) when $c = 0$ and $\beta > 0$ (see [7] and [2]). An elementary phase portrait argument shows that this is also the case if $c = 0,\ \beta < 0,\ B = 0$ and $\alpha > \frac{1}{n^2}$.

As it is well known the search for T-periodic solutions for the system (1.1) can be reduced to the study of the fixed points of a certain planar map, sometimes called the Poincaré map (see [10]). Most of the proofs will be connected with this map rather than with the differential equation. For this reason we present our results in the language of maps. We will consider a real analytic map

$$\mathbf{T} : \mathfrak{D} \subseteq \mathbb{R}^2 \to \mathbb{R}^2,$$

and we will look for conditions allowing to conclude that the set of fixed points is finite. An important assumption will be the compactness of the set of fixed points. In the applications this assumption will be linked to the existence of a priori bounds for T-periodic solutions. Another important feature of the results is that the domain of the map $\mathfrak{D}$ can be a proper subset of the plane. This gives more flexibility in the applications and we will deal with systems having solutions that can blow up in finite time or even with systems having singularities, $\mathcal{D} \neq \mathbb{R}^2$.

In the paper [11], Smith obtained an extension of Nakajima-Seifert theorem to higher dimensions. In $\mathbb{R}^d$ with $d \geq 3$, the condition of negative divergence is replaced by a condition on the second truncated trace of the vector field. We will only work on the plane, but some extensions to higher dimensions in the line of [11] seem plausible.

The rest of the paper is organized in three sections. In Section 2, we state and prove two results for analytic maps in the plane. In Section 3, we present an extension of Nakajima-Seifert's result. Finally, in Section 4 some applications to Hamiltonian systems are given.

2 Two theorems on analytic maps

Let $\mathfrak{D}$ be an open and simply connected subset set of $\mathbb{R}^2$ and let

$$\mathbf{T} : \mathfrak{D} \to \mathbb{R}^2$$

be a real analytic map. We will be concerned with the set of fixed points

$$\mathrm{Fix}(\mathbf{T}) := \{\mathbf{x} \in \mathfrak{D} : T(\mathbf{x}) = \mathbf{x}\}. \tag{2.1}$$

Our first result can be seen as a discrete version of the main theorem in [8]. The novelty is that the map is not necessarily defined in the whole plane.

Theorem 2.1. *Assume that* $\mathbf{T}$ *is one-to-one and two conditions below hold:*

(A1) $\mathrm{Fix}(\mathbf{T})$ *is compact;*

(A2) $0 < \det\, \mathbf{T}'(\mathbf{x}) < 1$ *for each* $\mathbf{x} \in \mathfrak{D}$.

Then $\mathrm{Fix}(\mathbf{T})$ *is finite.*

From now on, the Jacobian matrix of the map $\mathbf{T}$ is denoted by $\mathbf{T}'(\mathbf{x})$.

Remark 2.1. The previous result is not valid when the domain $\mathfrak{D}$ is an open set with holes. To illustrate this, we consider the case of an open set $\mathfrak{D}$ such that

$$S^1 \subseteq \mathfrak{D} \subseteq \left\{ \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| > \frac{1}{3} \right\}.$$

and the map

$$\mathbf{T}(\mathbf{x}) = \left(\frac{1}{2\|\mathbf{x}\|} + \frac{1}{2} \right) \mathbf{x}.$$

Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$ and

$$S^1 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}.$$

A direct computation shows that

$$0 < \det\, \mathbf{T}'(\mathbf{x}) = \frac{1}{4} \left(\frac{1}{\|\mathbf{x}\|} + 1 \right) < 1, \qquad \text{for } \mathbf{x} \in \mathfrak{D}.$$

The map $\mathbf{T}$ is one-to-one and $\mathrm{Fix}(\mathbf{T}) = S^1$. In this case the condition (A1) and (A2) hold but the set of fixed points is a continuum.

To prepare the proof of Theorem 2.1, we need two preliminary results. The first of them is concerned with the set of zeros of an analytic function and the second has a more topological flavor.

Lemma 2.1. *Let* $\mathbf{F} : \mathfrak{D} \to \mathbb{R}^2$ *be a real analytic map with set of zeros*

$$\mathbf{O}(\mathbf{F}) := \{\mathbf{x} \in \mathfrak{D} : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}. \tag{2.2}$$

Assume that

$$\mathbf{F}'(\mathbf{x}) \neq \mathbf{O} \qquad \text{for each } \mathbf{x} \in \mathbf{O}(\mathbf{F}), \tag{2.3}$$

where $\mathbf{O}$ *denotes the* 2×2 *zero matrix. In addition, assume that* $\mathbf{O}(\mathbf{F})$ *is compact and infinite. Then at least one of the connected components of this set is a Jordan curve.*

In the case $\mathfrak{D} = \mathbb{R}^2$, this result is exactly Lemma 5 in [8]. The proof in [8] is easily adapted to the case $\mathfrak{D} \neq \mathbb{R}^2$.

Given a Jordan curve $\Gamma \subseteq \mathbb{R}^2$, the bounded connected component of $\mathbb{R}^2 \setminus \Gamma$ will be denoted by $R_i(\Gamma)$. Since $\mathfrak{D}$ is simply connected, $\Gamma \subseteq \mathfrak{D}$ implies that $R_i(\Gamma) \subseteq \mathfrak{D}$.

Lemma 2.2. *Assume that* $\Gamma \subseteq \mathfrak{D}$ *is a Jordan curve. Then* $\mathbf{T}(R_i(\Gamma)) = R_i(\mathbf{T}(\Gamma))$.

Proof. Since $\mathbf{T}$ is one-to-one, we know that also $\mathbf{T}(\Gamma)$ is a Jordan curve. In the case $\mathfrak{D} = \mathbb{R}^2$, this result is a consequence of Lemma 6, Chapter 3 of [10], where this result is proved for topological embeddings of the plane. In the case $\mathfrak{D} \neq \mathbb{R}^2$, we first select a homeomorphism $\varphi : \mathbb{R}^2 \to \mathfrak{D}$. This is possible because $\mathfrak{D}$ is simply connected (see [9]). The map φ can be seen as a topological embedding and therefore

$$\varphi(R_i(\gamma)) = R_i(\varphi(\gamma)),$$

where $\gamma := \varphi^{-1}(\Gamma)$.
Define $\widehat{\mathbf{T}} = \mathbf{T} \circ \varphi$. Again we have an embedding of the plane and we know that

$$\widehat{\mathbf{T}}(R_i(\gamma)) = R_i(\widehat{\mathbf{T}}(\gamma)).$$

Combining both identities, we obtain the same conclusion for the curve Γ and the map $\mathbf{T}$. $\square$

Proof of Theorem 2.1. By a contradiction argument assume that $\mathrm{Fix}(\mathbf{T})$ is infinite. The map $\mathbf{F}(\mathbf{x}) = \mathbf{x} - \mathbf{T}(\mathbf{x})$ is real analytic in $\mathfrak{D}$ and

$$\mathbf{O}(\mathbf{F}) = \mathrm{Fix}(\mathbf{T}).$$

We know that $\mathbf{O}(\mathbf{F})$ is compact by assumption (A1). Also, from (A2) it is easy to deduce that $\mathbf{T}'(\mathbf{x})$ cannot be the identity matrix and so the non-degeneracy condition (2.3) will hold. We employ now Lemma 2.1 to deduce that $\mathrm{Fix}(\mathbf{T})$ contains a Jordan curve, say $\Gamma \subseteq \mathrm{Fix}(\mathbf{T}) \subseteq \mathfrak{D}$. In particular, this curve is invariant under $\mathbf{T}$,

$$\mathbf{T}(\Gamma) = \Gamma.$$

From Lemma 2.2, we deduce that also $R_i(\Gamma)$ is invariant,

$$\mathbf{T}(R_i(\Gamma)) = R_i(\Gamma).$$

This is not compatible with (A2). In fact, this condition implies that $\mathbf{T}$ is area contracting. From the theorem of change of variable in Lebesgue integral it is easy to deduce that

$$\mu(\mathbf{T}(A)) < \mu(A),$$

where A is any non-empty, open and bounded subset of $\mathfrak{D}$, and μ is the Lebesgue measure in the plane. The choice $A = R_i(\Gamma)$ leads to a contradiction. $\square$

The previous argument also works for mappings contracting other measures in the plane. These measures must have the property: the measure of bounded and open sets is positive and finite. However, Theorem 2.1 cannot be applied to area-preserving maps. For that case, we need a different principle.

Theorem 2.2. *Assume that the two conditions below hold,*

(H1) $\mathrm{Fix}(\mathbf{T})$ *is homeomorphic to a compact subset of the real line;*

(H2) $\mathbf{T}'(\mathbf{x}) \neq \mathbf{I}$ *for each* $\mathbf{x} \in \mathrm{Fix}(\mathbf{T})$.

Then $\mathrm{Fix}(\mathbf{T})$ *is finite.*

The 2×2 identity matrix is denoted by $\mathbf{I}$.

Proof. We proceed as in the proof of the previous theorem to obtain a Jordan curve $\Gamma \subseteq \mathrm{Fix}(\mathbf{T}) \subseteq \mathfrak{D}$. The condition (H2) plays a role in the argument. Then we find a contradiction with (H1) because S^1 is not homeomorphic to any subset of $\mathbb{R}$.
This argument is valid for any open set $\mathfrak{D}$, it is not necessary to assume that $\mathfrak{D}$ is simply connected. $\square$

3 A variant on Nakajima-Seifert theorem

In this section, we consider the system (1.1) where $\mathcal{D}$ is an open and connected subset of $\mathbb{R}^2$, and the vector field $\mathbf{X}$ is real analytic in $\mathbf{x}$, and continuous and T-periodic in t. This assumption can be understood in different ways and we refer to [10, p. 121] for a more detailed formulation. In particular, it implies that the initial value problem associated to (1.1) is unique.

We present an extension of the main result in [8].

Theorem 3.1. *In addition to the above conditions on the vector field, assume that the set $\mathcal{D}$ is simply connected and the following conditions hold,*

(α1) There exists a number $C > 0$ such that

$$\|\mathbf{x}(t)\| \leq C, \qquad \operatorname{dist}(\mathbf{x}(t), \partial\mathcal{D}) \geq \frac{1}{C}, \qquad \text{for } t \in \mathbb{R},$$

for each $\mathbf{x}(t)$, T-periodic solution of (1.1)*.*

(α2) $\operatorname{div}_{\mathbf{x}} \mathbf{X}(t, \mathbf{x}) := \frac{\partial X_1}{\partial x_1}(t, \mathbf{x}) + \frac{\partial X_2}{\partial x_2}(t, \mathbf{x}) < 0$ *for* $(t, \mathbf{x}) \in \mathbb{R} \times \mathcal{D}$.

Then the set of T-periodic solutions is finite.

There are two differences with respect to the original result in [8]. The domain can be a proper subset of $\mathbb{R}^2$ and the dissipativity condition has been replaced by the existence of a priori bounds. Note that we must control the distance to the boundary of $\mathcal{D}$ when $\mathcal{D} \neq \mathbb{R}^2$. In contrast to previous papers such as [1], we do not need to assume that the solutions of the initial value problem are globally defined. This is crucial for the applicability of the theorem to the Duffing equation.

Example 3.1. The equation (1.2) has the NS property if $c > 0$ and $\beta < 0$.

Proof. In this case, $\mathbf{x} = (u, u')$ and $\mathcal{D} = \mathbb{R}^2$. The condition ($\alpha$2) is satisfied because $\operatorname{div}_{\mathbf{x}} \mathbf{X}(t, \mathbf{x}) \equiv -c$. To prove that there are a priori bounds, we take a periodic solution $u(t)$ and assume that it reaches its maximum at some instant t_M. The $u'(t_M) = 0$ and $u''(t_M) \leq 0$. From the equation (1.2), we obtain the inequality

$$-B \leq B \sin t_M \leq \alpha u(t_M) + \beta u^3(t_M).$$

Since β is negative, it is not hard to get an upper estimate of $u(t_M)$ depending only on B, α and β. Similarly, we can obtain a lower bound for the minimum. Once we have obtained a uniform bound for $|u(t)|$, we observe that $v(t) := u'(t)$ satisfies a first order equation of the type $v' + cv = b(t)$ with $|b(t)| \leq C_1$ and $C_1 = C_1(B, \alpha, \beta)$. Then, given $t_* \in \mathbb{R}$ with $v'(t_*) = 0$,

$$|v(t_*)| \leq \frac{1}{c} C_1.$$

We have obtained a bound of $\max_{t \in \mathbb{R}}[|u(t)| + |u'(t)|]$ depending only on B, α and β. □

Example 3.2. Consider the singular differential equation

$$u'' + cu' + g(u) = p(t),$$

where $c > 0, g :]0, +\infty[\to \mathbb{R}$ is real analytic and

$$\lim_{u \to 0^+} g(u) = \lim_{x \to +\infty} g(u) = +\infty.$$

The function $p(t)$ is continuous and T-periodic. This equation has been considered in several papers. See [3] and the reference therein. As in the previous example we can prove that the NS property is valid.

Proof. Again $\mathbf{x} = (u, u')$ but now the domain is a half-plane. More precisely, $\mathcal{D} =]0, +\infty[\times \mathbb{R}, \partial\mathcal{D} = \{0\} \times \mathbb{R}$ and

$$\operatorname{dist}((x_1, x_2), \partial\mathcal{D}) = x_1 \quad \text{if } (x_1, x_2) \in \mathcal{D}.$$

To check (α1), we must take an arbitrary T-periodic solution $u(t)$ and obtain an upper bound of $|u(t)| + |u'(t)|$ and a positive lower bound of $u(t)$. For those estimates, we refer to [3, Lemma 2.2]. □

After these examples, we are going to prepare the proof of Theorem 3.1 with several auxiliary results.

Given $\mathbf{p} \in \mathcal{D}$, the solution of (1.1) satisfying $\mathbf{x}(0) = \mathbf{p}$ will be denoted by $\mathbf{x}(t;\mathbf{p})$. This solution is defined on a forward maximal interval $[0, \omega(\mathbf{p})[$. Further, according to [4], $\omega = \omega(\mathbf{p})$ is a lower semi-continuous function. Define

$$\mathfrak{D} := \{\mathbf{p} \in \mathcal{D} : \mathbf{x}(t;\mathbf{p}) \text{ is defined on } [0,T]\}. \tag{3.1}$$

This set can be also described by $\omega(\mathbf{p}) > T$, and so it is open. We want to prove the following result.

Proposition 3.1. *Assume that $\mathcal{D} \subseteq \mathbb{R}^2$ is simply connected, and $\mathfrak{D}$ is given by* (3.1). *Then every connected component of $\mathfrak{D}$ is simply connected.*

Proof. It will rely on a well known fact: if $\Omega \subseteq \mathbb{R}^2$ is open and connected, then it is simply connected if and only if $R_i(J) \subseteq \Omega$ for every Jordan curve $J \subseteq \Omega$. Here $R_i(J)$ denotes the bounded connected component of $\mathbb{R}^2 \setminus J$.

We are going to apply this characterisation with $\Omega = \widehat{\mathfrak{D}}$, where $\widehat{\mathfrak{D}}$ is any connected component of $\mathfrak{D}$. Given a Jordan curve $J \subseteq \widehat{\mathfrak{D}}$. We consider the set defined by

$$C := \{(\mathbf{x}(t;\mathbf{p}), t) \in \mathcal{D} \times [0,T] : \mathbf{p} \in J,\ t \in [0,T]\},$$

and we make the following **topological claim**:

C is a topological cylinder (that is, homeomorphic to $S^1 \times [0,T]$), and $(\mathbb{R}^2 \times [0,T]) \setminus C$ has two connected components $\mathfrak{B}$ and $\mathfrak{U}$. Moreover, $\mathfrak{B}$ is bounded, $R_i(J) \times \{0\} \subseteq \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathcal{D} \times [0,T]$.

By now this claim will be taken as evident, and we will try to prove that every point $\mathbf{p} \in R_i(J)$ also belongs to $\widehat{\mathcal{D}}$.

Given $\mathbf{p} \in R_i(J)$, since $\mathcal{D}$ is simply connected, and $J \subseteq \mathcal{D}$, we know that $\mathbf{p} \in \mathcal{D}$. Then we can consider the solution $\mathbf{x}(t;\mathbf{p})$ defined on a certain interval $[0, \omega(\mathbf{p})[$. The corresponding graph

$$G := \{(\mathbf{x}(t;\mathbf{p}), t) : t \in [0,T] \cap [0, \omega(\mathbf{p})[\},$$

is a connected subset of $\mathbb{R}^2 \times [0,T]$. The uniqueness of the initial value problem implies that $G \cap C = \emptyset$. Then G is contained in one of the components of $(\mathbb{R}^2 \times [0,T]) \setminus C$. The point $(\mathbf{p}, 0)$ belongs to $G \cap \mathfrak{B}$, therefore, $G \subseteq \mathfrak{B}$. We can conclude from here that $\omega(\mathbf{p}) > T$. In fact, the solution $\mathbf{x}(t;\mathbf{p})$ cannot blow up in $[0,T]$ because $\mathfrak{B}$ is bounded and it cannot touch the boundary $\partial\mathcal{D}$ because

$$\overline{\mathfrak{B}} \subseteq \mathfrak{B} \cup C \subseteq \mathcal{D} \times [0,T].$$

Summing up, we have proved that $\omega(\mathbf{p}) > T$ if $\mathbf{p} \in R_i(J)$. This implies that $R_i(J) \subseteq \mathfrak{D}$. The set $R_i(J) \cup J$ is connected and contained in $\mathfrak{D}$, then it must remains inside some component of $\mathfrak{D}$. But J is contained in $\widehat{\mathfrak{D}}$ by assumption, and so also $R_i(J)$ must be contained in $\widehat{\mathfrak{D}}$. $\square$

Proof of topological claim. Since C is compact, we can find two bounded and open subsets U and V of $\mathbb{R}^2 \times [0,T]$ such that

$$C \subseteq U \subseteq \overline{U} \subseteq V \subseteq \mathcal{D} \times [0,T].$$

Note that U and V are open with respect to the relative topology of $\mathbb{R}^2 \times [0,T]$. Then we can find a C^∞ function $\alpha : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ such that

$$\alpha(\mathbf{x}, t) = 1 \text{ if } (\mathbf{x}, t) \in U, \quad \text{and} \quad \alpha(\mathbf{x}, t) = 0 \text{ if } (\mathbf{x}, t) \notin V.$$

With the help of this function, we construct the modified vector field

$$\widetilde{\mathbf{X}} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \widetilde{\mathbf{X}}(t, \mathbf{x}) = \begin{cases} \alpha(\mathbf{x}, t)\mathbf{X}(t, \mathbf{x}), & \text{if } (\mathbf{x}, t) \in V, \\ 0, & \text{otherwise.} \end{cases}$$

This vector field is smooth and the solution $\widetilde{\mathrm{x}}(t;\mathbf{p})$ of the initial value problem

$$\mathbf{x}' = \widetilde{\mathbf{X}}(t,\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{p},$$

is well defined on $[0,T]$. In addition, the map

$$\Phi : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2 \times [0,T], \quad (\mathbf{p},t) \longmapsto (\widetilde{\mathbf{x}}(t;\mathbf{p}),t)$$

is a homeomorphism. Since $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ coincide in a neighborhood of C, $\mathbf{x}(t;\mathbf{p}) = \widetilde{\mathbf{x}}(t;\mathbf{p})$ for each $t \in [0,T]$ and $\mathbf{p} \in J$. Then

$$\Phi(\widetilde{C}) = C \quad \text{with} \quad \widetilde{C} = J \times [0,T].$$

The set $(\mathbb{R}^2 \times [0,T]) \setminus \widetilde{C}$ has two connected components

$$\widetilde{\mathfrak{B}} = R_i(J) \times [0,T], \quad \widetilde{\mathfrak{U}} = R_e(J) \times [0,T],$$

where $R_e(J)$ is the unbounded component of $\mathbb{R}^2 \setminus J$. In consequence, $\mathfrak{B} = \Phi(\widetilde{\mathfrak{B}})$, $\mathfrak{U} = \Phi(\widetilde{\mathfrak{U}})$ are the components of $(\mathbb{R}^2 \times [0,T]) \setminus C$. Since $\widetilde{\mathfrak{B}} \cup \widetilde{C}$ is compact, the same can be said about $\mathfrak{B} \cup C$. Moreover, if we consider the Jordan curve

$$J_t = \{\mathbf{x}(t;\mathbf{p}) : \mathbf{p} \in J\},$$

we observe that $J_t \subseteq \mathcal{D}$ for each $t \in [0,T]$. Then, since $\mathcal{D}$ is simply connected, $R_i(J_t) \subseteq \mathcal{D}$. The proof of claim is completed from the decomposition

$$\mathfrak{B} = \bigcup_{t \in [0,T]} \left(R_i(J_t) \times \{t\}\right).$$

$\square$

Proof of Theorem 3.1. In view of Proposition 3.1, each member the family $\{\mathfrak{D}_i\}_{i \in I}$ of connected components of $\mathfrak{D}$ is simply connected.

Define now the Poincaré map associated to (1.1) by

$$\mathbf{P}_T : \mathfrak{D} \subseteq \mathcal{D} \to \mathbb{R}^2, \quad \mathbf{p} \mapsto \mathbf{x}(t;\mathbf{p}).$$

The set $\mathrm{Fix}(\mathbf{P}_T)$ is in a one-to-one correspondence with the set of T-periodic solutions, and we will prove that $\mathrm{Fix}(\mathbf{P}_T)$ is finite by an application of Theorem 2.1 to the map $\mathbf{P}_T$. The general theorems of the initial value problem imply that $\mathbf{P}_T$ is one-to-one and analytic (see [5]). Moreover, (A2) is a consequence of Liouville formula

$$\det \mathbf{P}_T'(\mathbf{p}) = \exp\left\{\int_0^T \mathrm{div}_{\mathbf{x}} \mathbf{X}(t,\mathbf{x}(t;\mathbf{p}) dt\right\}.$$

We must prove that $\mathrm{Fix}(\mathbf{P}_T)$ is compact. From the assumption $(\alpha 1)$, we know that this set is contained in the ball centered at the origin of radius C. Once we know that $\mathrm{Fix}(\mathbf{P}_T)$ is bounded, we prove that it is also closed in $\mathbb{R}^2$.

By a contradiction argument, assume that $\{\mathbf{p}_n\}$ is a sequence in $\mathrm{Fix}(\mathbf{P}_T)$ with $\mathbf{p}_n \to \mathbf{p}$ and $\mathbf{p} \notin \mathrm{Fix}(\mathbf{P}_T)$. A first observation is that $\mathbf{p}$ lies in $\mathcal{D}$. In principle, we know that it belongs to the closure of $\mathcal{D}$ but the condition $(\alpha 1)$ implies that

$$\mathrm{dist}(\mathbf{p}_n, \partial\mathcal{D}) \geq \frac{1}{C}$$

and, letting $n \to +\infty$, we deduce that $\mathbf{p} \notin \partial\mathcal{D}$. Also, $\mathbf{p} \in \partial\mathfrak{D}$, for otherwise $\mathbf{p}$ should be in $\mathfrak{D}$ and a passage to the limit in the identity $\mathbf{P}_T(\mathbf{p}_n) = \mathbf{p}_n$ would imply that also $\mathbf{p}$ is a fixed point. From there we deduce that $\omega(\mathbf{p}) \leq T$ and so there exists a sequence $t_n \to \omega(\mathbf{p})$ such that one of the following alternatives holds

(i) $\|\mathbf{x}(t_n;\mathbf{p})\| \to \infty$;

(ii) There exists $\xi \in \partial\mathcal{D}$ such that $\mathbf{x}(t_n;\mathbf{p}) \to \xi$.

Then we can find an instant $\tau \in [0,\omega(\mathbf{p})[$ such that either $\|\mathbf{x}(\tau;\mathbf{p})\| > C$ or $\operatorname{dist}(\mathbf{x}(\tau;\mathbf{p}),\partial\mathcal{D}) < \frac{1}{C}$. By continuous dependence, we know that $\mathbf{x}(\cdot;\mathbf{p}_n)$ converges uniformly to $\mathbf{x}(\cdot;\mathbf{p})$ on the interval $[0,\tau]$. Therefore, for large n, either $\|\mathbf{x}(\tau;\mathbf{p}_n)\| > C$ or $\operatorname{dist}(\mathbf{x}(\tau;\mathbf{p}_n),\partial\mathcal{D}) < \frac{1}{C}$. Since $\mathbf{x}(\cdot;\mathbf{p}_n)$ is a T-periodic solution, this is not compatible with $(\alpha 1)$.

The compact set $\operatorname{Fix}(\mathbf{P}_T)$ is contained in

$$\mathfrak{D} = \bigcup_{i\in I} \mathfrak{D}_i,$$

and there exists a finite set $J \subseteq I$ such that

$$\operatorname{Fix}(\mathbf{P}_T) \subseteq \bigcup_{i\in J} \mathfrak{D}_i.$$

We can now apply Theorem 2.1 on the simply connected domain $\mathfrak{D}_i$ with $i \in J$. The set of fixed points is $\operatorname{Fix}(\mathbf{P}_T) \cap \mathfrak{D}_i$, a compact set. $\square$

4 Application to Hamiltonian systems

To finish the paper, we present a result on Hamiltonian systems. Assume that $\mathcal{D}$ is an open and connected subset of $\mathbb{R}^2$ and $H : \mathbb{R}\times\mathcal{D} \to \mathbb{R}$ is real analytic and T-periodic in t. The system

$$\dot{\mathbf{x}} = \mathbf{J}\nabla_x H(t,\mathbf{x}), \qquad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.1}$$

does not satisfy $(\alpha 2)$ because the divergence vanishes. We replace this assumption by

$(\alpha 3)$ $\frac{\partial^2 H}{\partial x_2^2}(t,\mathbf{x}) > \mathbf{0}$, $D^2_{\mathbf{x}} H(t,\mathbf{x}) < \frac{2\pi}{T}\mathbf{I}$, for $(t,\mathbf{x}) \in \mathbb{R}\times\mathcal{D}$.

The Hessian matrix of $H(t,\cdot)$ has been denoted by $D^2_{\mathbf{x}} H(t,\mathbf{x})$ and we are using the ordering in the space of symmetric matrices. More precisely, given $\mathbf{A},\mathbf{B} \in \mathbb{R}^{2\times 2}$ symmetric, $\mathbf{A} < \mathbf{B}$ means that

$$\langle \mathbf{A}\mathbf{x},\mathbf{x}\rangle < \langle \mathbf{B}\mathbf{x},\mathbf{x}\rangle \qquad \forall\, x \in \mathbb{R}^2 \setminus \{0\}.$$

Theorem 4.1. *In the previous setting assume that $(\alpha 1)$ and $(\alpha 3)$ hold. Then the set of T-periodic solutions of* (4.1) *is finite.*

To prepare the proof, we recall a property of linear Hamiltonian systems of the type

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{S}(t)\mathbf{y}, \tag{4.2}$$

where $\mathbf{S} : \mathbb{R} \to \mathbb{R}^{2\times 2}$, $\mathbf{S}(t) = (s_{ij}(t))_{1\le i,j\le 2}$, is continuous, T-periodic and symmetric.

Lemma 4.1. *Assume that, for each $t \in \mathbb{R}$,*

$$s_{22}(t) > 0, \qquad \mathbf{S}(t) < \frac{2\pi}{T}\mathbf{I}.$$

Then every nontrivial T-periodic solution $\mathbf{y}(t)$ of (4.2) *satisfies $y_1(t) \neq 0$ for each $t \in \mathbb{R}$.*

Proof. Given a nontrivial T-periodic solution $\mathbf{y}(t)$, we write it in polar coordinates

$$\mathbf{y}(t) = r(t)\begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is C^1 function satisfying

$$\theta(t+T) = \theta(t) + 2\pi N,$$

for some integer N. The condition $s_{22}(t) > 0$ implies that $y_1'(t_0) > 0$ if $y_1(t_0) = 0$ and $y_2(t_0) > 0$. In a similar way, $y_1'(t_0) < 0$ if $y_1(t_0) = 0$ and $y_2(t_0) < 0$. Then N cannot be positive. Either $\mathbf{y}(t)$ remains in $\{y_1 \neq 0\}$ and $N = 0$, or $\mathbf{y}(t)$ crosses $\{y_1 = 0\}$ and $N < 0$. We are going to prove that the second alternative is impossible. From (4.2),

$$\theta'(t) = \frac{1}{r^2(t)} \langle \mathbf{y}(t), \mathbf{J}\mathbf{y}'(t) \rangle = -\frac{1}{r^2(t)} \langle \mathbf{y}(t), \mathbf{S}(t)\mathbf{y}(t) \rangle > -\frac{2\pi}{T},$$

implying $\theta(T) - \theta(0) > -2\pi$ and $N \geq 0$. □

Proof of Theorem 4.1. As in the proof of Theorem 3.1, we consider the Poincaré map $\mathbf{P}_T$, now associated to (4.1). The condition (α1) allows to prove that $\mathrm{Fix}(\mathbf{P}_T)$ is a compact subset of $\mathfrak{D}$. We apply Theorem 2.2 at each connected components of $\mathfrak{D}$, say $\widehat{\mathfrak{D}}$. Note that this time we do not need a simply connected domain.

To prove that (H1) holds, we must show that the projection

$$\mathrm{Fix}(\mathbf{P}_T) \cap \widehat{\mathfrak{D}} \to \mathbb{R}, \quad \mathbf{p} = (p_1, p_2) \mapsto p_1$$

is one-to-one. Since $\mathrm{Fix}(\mathbf{P}_T) \cap \widehat{\mathfrak{D}}$ is compact, we conclude that it is homeomorphic to its projection in $\mathbb{R}$.

Given $\mathbf{p}, \mathbf{q} \in \mathrm{Fix}(\mathbf{P}_T) \cap \widehat{\mathfrak{D}}, \mathbf{p} \neq \mathbf{q}$, we define

$$\mathbf{x}_0(t) = \mathbf{x}_0(t; \mathbf{p}), \qquad \mathbf{x}_1(t) = \mathbf{x}_0(t; \mathbf{q}).$$

Then $\mathbf{y}(t) = \mathbf{x}_1(t) - \mathbf{x}_0(t)$ is a non-trivial T-periodic solution of (4.2) with

$$\mathbf{S}(t) = \int_0^1 D_x^2 \mathbf{H}(t, \mathbf{x}_\lambda(t)) d\lambda,$$

and $\mathbf{x}_\lambda = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_0$. Since we are in the conditions of Lemma 4.1, $y_1(t) \neq 0$ for each $t \in \mathbb{R}$. In particular, for $t = 0$, $q_1 \neq p_1$.

To prove (H2), we observe that if $\mathbf{p} \in \mathrm{Fix}(\mathbf{P}_T)$, then

$$\mathbf{P}_T'(\mathbf{p}) = \boldsymbol{\Upsilon}(T),$$

where $\boldsymbol{\Upsilon}(T)$ is the matrix solution of (4.2) with $\boldsymbol{\Upsilon}(0) = \mathbf{I}$ and $\mathbf{S}(t) = D_{\mathbf{x}}^2 H(t, \mathbf{x}(t; \mathbf{p}))$. Again Lemma 4.1 can be applied to prove that $\boldsymbol{\Upsilon}(T)$ cannot be the identity. Otherwise, $\boldsymbol{\Upsilon}(T) = \mathbf{I}$ would imply that all the solutions of (4.2) are T-periodic and this is not compatible with the conclusion of the Lemma. □

Example 4.1. The NS property holds for (1.2) when $c = 0, \beta < 0, \alpha < \frac{1}{n^2}$ and $T = 2\pi n$.

We take $\mathcal{D} = \mathbb{R}^2$ and $\mathbf{x} = \left(\varepsilon u, \frac{1}{\varepsilon} u'\right)$, where $\varepsilon > 0$ will be chosen later. Then (1.2) is equivalent to (4.1) with

$$H(t, x_1, x_2) = \frac{1}{2}\varepsilon^2 x_2^2 + V\left(\frac{x_1}{\varepsilon}\right) - \frac{1}{\varepsilon} x_1 B \sin t, \quad V(u) = \frac{\alpha u^2}{2} - \frac{\beta u^4}{4}.$$

To check the condition (α1), we can employ the same argument as in Example 3.1. To check the condition (α3), we compute

$$D_x^2 H(t, \mathbf{x}) = \begin{pmatrix} \frac{1}{\varepsilon^2} V''(\frac{x_1}{\varepsilon}) & 0 \\ 0 & \varepsilon^2 \end{pmatrix}$$

and observe that $V''(u) \leq \alpha$.

The case $\alpha = \frac{1}{n^2}$ can be treated using an improved version of Lemma 4.1 under the condition

$$\mathbf{S}(t) \leq \frac{2\pi}{T} \mathbf{I}, \qquad \text{for } t \in \mathbb{R},$$

with strict inequality somewhere.

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