The spin-spin model and the capture into the double synchronous resonance

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Abstract

The aim of this article is to propose a model, that is a planar version of the Full Two-Body Problem, and discuss the existence and stability of a relevant periodic solution. Consider two homogeneous ellipsoids orbiting around each other in fixed coplanar Keplerian orbits. Moreover, their respective spin axes are assumed to be perpendicular to the orbital plane, that is also a common equatorial plane. The spin-spin model deals with the coupled rotational dynamics of both ellipsoids. For a non-zero orbital eccentricity, it has the structure of a non-autonomous system of coupled pendula. This model is a natural extension of the classical spin-orbit problem for two extended bodies. In addition, we consider dissipative tidal torques, that can trigger the capture of the system into spin-orbit and spin-spin resonances. In this paper we give some theoretical results for both the conservative model and the dissipative one. The conservative model has a Hamiltonian structure. We use properties of Hamiltonian systems to give some sufficient conditions in the space of parameters of the model, that guarantee existence, uniqueness and linear stability of an odd periodic solution. This solution represents a double synchronous resonance in the conservative regime. Such solution can be continued to the dissipative regime, where it becomes asymptotically stable. We see asymptotic stability as a dynamical mechanism for the capture into the double synchronous resonance. Finally we apply our results to several cases including the Pluto-Charon binary system and the Trojan binary asteroid 617 Patroclus, target of the LUCY mission.

Keywords: Celestial mechanics, Hamiltonian systems, Dissipative systems, Rotational dynamics, Coupled oscillators, Two-Body problem.

1 Introduction

1.1 Motivation

The model we propose here is a natural extension of the well known spin-orbit problem of celestial mechanics. The spin-orbit model is an elementary, but not trivial, model to study the rotational dynamics of a satellite about its center of mass when it orbits around a planet. Here the planet acts as a point mass and the satellite is an extended body whose spin axis is perpendicular to the orbital plane. This model has the structure of a nearly integrable and periodically forced pendulum. It has attracted much attention not only for its accurate

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physical implications but also for its mathematical richness. Some pioneer papers are [4] for the conservative case and [20] for the dissipative case. This model contributes to explain the synchronization of the rotational motion of the Moon and its orbital motion around the Earth. In other words, the Moon is in a 1:1 spin-orbit resonance. This phenomenon is indeed very common in the solar system for natural satellites that are close enough to their respective planets, [30]. Besides, Mercury, as an orbiting body around the Sun, is locked in a 3:2 spin-orbit resonance. According to [13], in its chaotic evolution, Mercury could have reached large orbital eccentricities that made possible the capture into this higher order resonance. It is accepted that the phenomenon of capture into resonances is driven by dissipative torques, caused by internal frictions within the satellite, [22]. The concept of stability of a resonance in the conservative regime is linked to the concept of capture in the dissipative case and both can be related. In one hand, [8] studies the KAM stability in the conservative case, whereas [9] proves the existence of quasiperiodic attractors for the dissipative problem, that bifurcate from the KAM tori of the conservative case. On the other hand, [29] proves the existence of an asymptotically stable solution in 1:1 resonance that is a continuation of a linearly stable odd periodic solution of the conservative case. The onset of chaos is another interesting feature of this problem. The oblateness of the satellite produces chaotic regions in the phase space that surround the libration regions of resonances. Chaotic zones can be very large due to overlapping of different resonances, [10]. A large eccentricity emphasizes this behavior, as in the case of Hyperion, [36], [35].

The Full Two-Body Problem (F2BP) deals with the dynamics of two extended bodies interacting gravitationally. It has been extensively investigated, especially in the last two decades, due to an increasing interest on binary systems. Due to its complexity, most of the studies are numerical explorations of particular cases, see [18] or [12]. There are some works with a more analytical approach dealing with relative equilibria and stability, [31] and [23]. The spin-spin model is motivated mainly by [5], [32], [14] and [3]. In one hand, [5] is focused on the evolution of the orbit and the spin axes of the bodies in the secular F2BP (averaging over fast angles). This paper points out that the mutual influence in the spin dynamics is contained in the terms of order $1/r^5$ of the expansion of the potential energy of the system, where r is the distance between the bodies. On the other hand, [32] studies the relative equilibria and stability in the planar case, i.e., the spin axes of the bodies are perpendicular to the orbital plane, that is also a common equatorial plane. [14] studies the observability of non-planar stable oscillations around the double synchronous equilibrium in binary asteroids. In [32] and [14], only terms up to $1/r^3$ of the potential energy are considered, so the resulting system is equivalent to two uncoupled spin-orbit problems. The planar spin-spin coupling was first studied in [3], making an analogous study as the classical paper [20] on the spin-orbit coupling. Particularly, [3] studies the spin of the body 1, identified with two point masses slightly separated from each other (dumbbell model), that moves in a circular orbit around the body 2, an ellipsoid with uniform rotation. They focus on the case when the orbital motion is slow and the angular velocity of the body 1 becomes commensurable with the angular velocity of the body 2 (spin-spin resonance).

The model we propose in this paper deals with the complete coupled dynamics of the F2BP in the planar and ellipsoidal case. As usual in the spin-orbit problem, we also assume that the orbital motion takes place in Keplerian ellipses. This reduces the high dimensional phase space of the F2BP to a problem of two degrees of freedom (spins) plus time-dependence (orbit). For a small non-zero orbital eccentricity, it has the structure of a nearly integrable system of coupled pendula that is periodically forced. This setting is suited to study the phenomena

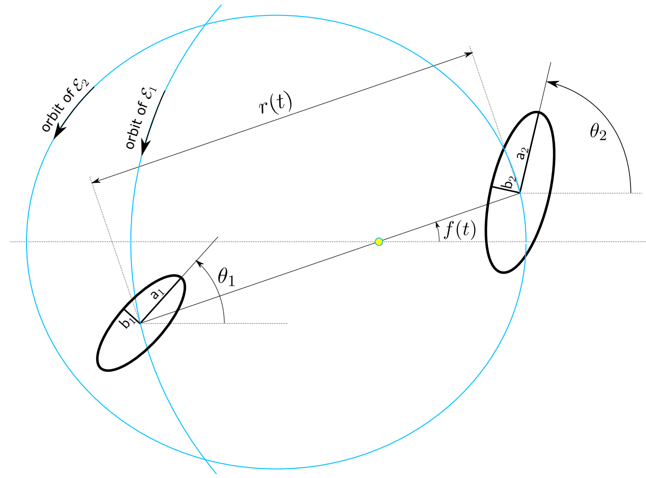

Figure 1: The planar spin-spin problem.

related to spin-orbit and spin-spin resonances. Furthermore, the intrinsic dissipative nature of the capture into resonances supports the relevance of this model. The reason is that the most used family of dissipative torques, see [22], is of order $1/r^6$, whereas the spin-spin coupling appears at order $1/r^5$. In addition to the questions related to the spin-orbit problem, this model of coupled oscillators opens new questions that were not possible to consider before. We will discuss this in Section 6.

1.2 Setting of the model.

Consider two homogeneous ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ with respective masses M_j, $j = 1, 2$, principal moments of inertia $\mathcal{A}_j < \mathcal{B}_j < \mathcal{C}_j$ and corresponding principal semi-axes $\mathsf{a}_j > \mathsf{b}_j > \mathsf{c}_j$. Assume that the orbital motion of the ellipsoids is the same as for two point masses, say, the centers of the ellipsoids describe coplanar Keplerian orbits of eccentricity $e \in [0, 1)$ with a common focus at the center of mass of the system. Moreover, assume that the spin axis of each body is the principal axis associated to c_j and is perpendicular to the orbital plane.

Let us identify the orbital plane with the complex plane $\mathbb{C}$. Consider the center of mass of the system fixed at the origin and let the center of each ellipsoid be $\mathbf{r}_j$, then, $M_1\mathbf{r}_1 + M_2\mathbf{r}_2 = 0$. If we define the relative position vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and choose the units of mass such that $M_1 + M_2 = 1$, then, $\mathbf{r}_1 = -M_2\mathbf{r}$ and $\mathbf{r}_2 = M_1\mathbf{r}$. The orbital motion is defined by $\mathbf{r}$, which can be written as $\mathbf{r} = r\exp(if) \in \mathbb{C}$, where $r > 0$ and f are real functions of the time. Note that $\mathbf{r}$ describes an ellipse of eccentricity $e \in [0, 1)$ and semimajor axis a with focus at the origin, so, the polar coordinates r and f vary periodically with time, and are known by the Kepler problem. Let us take convenient units of time so that the period is 2π. In the usual terminology, f is called *true anomaly* and the time t is the *mean anomaly*. There is a third useful angle u, the *eccentric anomaly*, which is defined by the famous Kepler's equation

$$t = u - e\sin u, \tag{1}$$

and let us determine the Keplerian ellipse simply by

$$r = a(1 - e\cos u). \tag{2}$$

Also, using the graphical definition of the eccentric anomaly and some geometrical relations of ellipses, we can write the position of $\mathbf{r}$ in terms of the eccentric anomaly as

$$r\exp(if) = a(\cos u - e + i\sqrt{1-e^2}\sin u). \tag{3}$$

Note that for $t = 0$ we assumed that $f = u = 0$, and consequently, $f = u = \pi$ when $t = \pi$. The expressions eqs. (2) and (3) relate the true and eccentric anomalies. Moreover, Equations (1) to (3) define $u = u(t,e)$, $\frac{r}{a} = \frac{r}{a}(t,e)$ and $f = f(t,e)$ as analytic functions in both entries.

Recall Kepler's third law for the Two-Body Problem

$$G(M_1 + M_2)\left(\frac{T}{2\pi}\right)^2 = a^3, \tag{4}$$

where G is the Gravitational constant and T is the orbital period. In consequence, $G = a^3$ in our units. For our model to be completely non-dimensional and adequate to the scale of the system, we take convenient units of length such that $\mathcal{C}_1 + \mathcal{C}_2 = 1$. In these units the semi-major axes a_j of the ellipsoids are of order 1, whereas a should be much larger. See Appendix A for specific conversion of units.

Let θ_j be the polar angle of the principal direction associated to a_j with respect to the orbit's major axis. See Figure 1. The spin dynamics of the ellipsoids is modelled by the following coupled system of ordinary differential equations

$$\mathcal{C}_j\ddot{\theta}_j = \mathcal{T}_j^{\mathrm{C}}(t,\theta_1,\theta_2) + \mathcal{T}_j^{\mathrm{D}}(t,\dot{\theta}_j), \qquad j = 1,2, \tag{5}$$

where $\mathcal{T}_j^{\mathrm{C}}$ and $\mathcal{T}_j^{\mathrm{D}}$ are respectively the *conservative* and *dissipative* torques acting on $\mathcal{E}_j$.

The *conservative* torque is derived from the potential gravitational energy, see Section 2, and it takes the form

$$\begin{aligned}\mathcal{T}_j^{\mathrm{C}}(t,\theta_1,\theta_2) = &-\left(\frac{a}{r(t)}\right)^3 \frac{\Lambda_j}{2}\sin(2\theta_j - 2f(t)) \\ &- \left(\frac{a}{r(t)}\right)^5 \sum_{(m_1,m_2)\in\Xi} \frac{m_j\,\Lambda_{m_2}^{m_1}}{2}\sin(2m_1(\theta_1 - f(t)) + 2m_2(\theta_2 - f(t))),\end{aligned} \tag{6}$$

where

$$\Xi = \{(m_1,m_2) \in \mathbb{Z}^2 : \ |m_1| + |m_2| \le 2\}.$$

In (6) we have ignored terms of order $(a/r(t))^n$ with $n \ge 7$. The parameters Λ_j and $\Lambda_{m_2}^{m_1}$ are positive small quantities depending on the physical parameters of the bodies and on a. These parameters satisfy $\Lambda_{m_2}^{m_1} = \Lambda_{-m_2}^{-m_1} < \Lambda_j < 3\mathcal{C}_j$. Note that if all the constants $\Lambda_{m_2}^{m_1}$ in (6) vanish, the system (5) is formed by two uncoupled spin-orbit problems in θ_1 and θ_2. The coupling of the system is contained in the terms (m_1,m_2) of type $(\pm1,\pm1)$ and $(\pm1,\mp1)$, whereas the rest of them are high order spin-orbit terms.

On the other hand, the *dissipative* torque $\mathcal{T}_j^D$ has different forms depending on the model. We will use a linear MacDonald torque [22]

$$\mathcal{T}_j^D(t,\dot{\theta}_j) = -C_{M,j}\left(\frac{a}{r(t)}\right)^6 \sin(2\Delta t_j(\dot{\theta}_j - \dot{f}(t))) \approx -\delta_j\mathcal{C}_j\left(\frac{a}{r(t)}\right)^6(\dot{\theta}_j - \dot{f}(t)), \tag{7}$$

where $C_{M,j}$ are constants depending on the parameters of the bodies. Here we assumed that $|\Delta t_j(\dot{\theta}_j - \dot{f}(t))| \ll 1$ because the parameters Δt_j and δ_j are very small positive numbers.

This type of torque has been extensively used, taking as reference [20] or [30], for example. According to [16], to obtain (7), the dissipation is modelled by assuming that there is a time delay between the deforming disturbance and the actual deformation of each body. That delay is a small fixed amount Δt_j (time lag), leading to an angular lag of $(\dot{f}(t,e)-\dot{\theta}_j)\Delta t_j$ (geometric lag). It is worth mentioning that there is no physical reason for both lags (or both δ_j) to match.

Note that if $\mathcal{T}_j^{\mathrm{D}} = 0$, the system (5) has a Hamiltonian structure. The corresponding Hamiltonian has two degrees of freedom and time dependence and it is given by

$$H(\theta_1,\theta_2,p_{\theta_1},p_{\theta_2},t) = \frac{p_{\theta_1}^2}{2\mathcal{C}_1} + \frac{p_{\theta_2}^2}{2\mathcal{C}_2} + \mathcal{V}(t,\theta_1,\theta_2), \tag{8}$$

where

$$\begin{aligned}\mathcal{V}(t,\theta_1,\theta_2) = &-\frac{1}{4}\left(\frac{a}{r(t)}\right)^3 \sum_{j=1}^{2} \Lambda_j \cos(2\theta_j - 2f(t)) \\ &-\frac{1}{4}\left(\frac{a}{r(t)}\right)^5 \sum_{(m_1,m_2)\in\Xi} \Lambda_{m_2}^{m_1} \cos(2m_1(\theta_1 - f(t)) + 2m_2(\theta_2 - f(t))).\end{aligned}$$

Due to the explicit time dependence of the Hamiltonian, the energy of the system is not constant even though $\mathcal{T}_j^{\mathrm{D}} \equiv 0$. However, if $\mathcal{T}_j^{\mathrm{D}} \equiv 0$, the system (5) will be called *conservative*, because no dissipative forces are involved in the physical derivation of the model. On the other hand, if $\mathcal{T}_j^{\mathrm{D}}$ is not identically zero for all time, then we will call it *dissipative*. The italic font will remark this point. In Section 2 we will see also a purely conservative version of the model involving $(r, f, \theta_1, \theta_2)$ as unknown functions of time.

There are solutions of (5) that are especially relevant. Since the spin-orbit problem is a particular case of (5), a solution satisfying $\theta_1(t + 2\pi n_o) = \theta_1(t) + 2\pi n_s$, with $n_s, n_o \in \mathbb{Z}$, is called $n_s : n_o$ spin-orbit resonance of the ellipsoid $\mathcal{E}_1$. The same is true for $\mathcal{E}_2$. Spin-spin resonances arise when the spin rates of the two ellipsoids become commensurable. In [3] these resonances were studied independently from the orbital rate. There are some solutions in which the ellipsoids are simultaneously in a spin-orbit and a spin-spin resonance. The simplest of these resonances is the double synchronous resonance of equation (5), that is, solutions satisfying $\theta_j(t + 2\pi) = \theta_j(t) + 2\pi$, for both $j = 1, 2$. In other words, the spin of both ellipsoids synchronize with the orbital motion at the same time.

1.3 Setting of our approach and results

We are going to deal with the capture into the double synchronous resonance. In the same way as in [29], in this paper we will approach this phenomenon from an analytical point of view. We will look for conditions resulting in the existence of a double synchronous solution of the *conservative* model that can be continued to an asymptotically stable solution of the *dissipative* model. In this context, the asymptotic stability of the solution represents the phenomenon of capture into the resonance: solutions in the vicinity of the asymptotically stable solution get closer and closer to it as $t \to +\infty$.

Let us take the change of variable $\Theta_j = 2(\theta_j - f)$, such that the system (5) turns into

$$\mathcal{C}_j\ddot{\Theta}_j + \delta_j\mathcal{C}_j\left(\frac{a}{r(t)}\right)^6\dot{\Theta}_j + \left(\frac{a}{r(t)}\right)^3\Lambda_j\sin\Theta_j + \left(\frac{a}{r(t)}\right)^5\sum_{(m_1,m_2)\in\Xi} m_j\,\Lambda_{m_2}^{m_1}\sin(m_1\Theta_1+m_2\Theta_2) = -2\mathcal{C}_j\ddot{f}(t). \quad (9)$$

The system (9) models a couple of damped and forced pendula of variable length. Since $f(t+2\pi) = f(t)+2\pi$, then, double synchronous resonances correspond to solutions of (9) satisfying $\Theta_j(t+2\pi) = \Theta_j(t)$.

In Section 2 we will make the derivation of the *conservative* model from the Lagrangian of the physical system and obtain the expression of Λ_j and $\Lambda_{m_2}^{m_1}$ in terms of physical parameters. In Section 3, we will deal with the *conservative* system, say, (9) with $\delta_j = 0$,

$$\mathcal{C}_j\ddot{\Theta}_j + \left(\frac{a}{r(t)}\right)^3\Lambda_j\sin\Theta_j + \left(\frac{a}{r(t)}\right)^5\sum_{(m_1,m_2)\in\Xi} m_j\,\Lambda_{m_2}^{m_1}\sin(m_1\Theta_1+m_2\Theta_2) = -2\mathcal{C}_j\ddot{f}(t), \quad (10)$$

and discuss the existence, uniqueness and linear stability of an odd 2π-periodic solution. This solution is a continuation of the trivial solution $\Theta(t)\equiv 0$ for $e=0$. This will lead us to a region of linear stability in the space of parameters of the system. In this section we will use some properties of symmetric matrices and linear Hamiltonian systems with periodic coefficients. We are interested in the linear stability of the periodic solution of (10) because it will allow us to find, by continuation, an asymptotically stable periodic solution for the *dissipative* case (9) for $\delta_j > 0$. This will be proved in Section 4, provided that $|\Lambda_{m_2}^{m_1}|$ and $|\delta_j|$ are small enough. In Section 5 we will explain how to apply our results to real cases and use the Pluto-Charon system and the binary asteroid 617 Patroclus as two representative examples. We will also compare our estimates with some numerical experiments and with the spin-orbit problem. Finally in Section 6 we will make a discussion about the model and our results.

2 Derivation of the conservative spin-spin model

In this section we will compute the equations of motion of the ellipsoids with respect to the inertial frame with origin at the barycenter of the system. Section 2.1 is devoted to find the equations of motion of the full system of four variables $(r, f, \theta_1, \theta_2)$, in terms of the gravitational potential energy $V = V(r, f, \theta_1, \theta_2)$. In Section 2.2 we fix the Keplerian orbit and obtain the final model in terms of physical parameters of the system.

2.1 The planar Lagrangian model

Let the Lagrangian of the system be $L = T - V$, where T is the kinetic energy and V the potential energy of the system. Recall that the positions of the bodies are $\mathbf{r}_1 = -M_2\mathbf{r}$ and $\mathbf{r}_2 = M_1\mathbf{r}$, where the relative position vector is defined by $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = r\exp(if)$. Besides, for each body, the angle θ_j defines the orientation of the axis associated to a_j. We are going to use r, f, θ_1 and θ_2, depicted in Figure 1, as the Lagrangian variables of our system. The total orbital kinetic energy is given by

$$T_{orb} = \frac{1}{2}(M_1\dot{\mathbf{r}}_1{}^2 + M_2\dot{\mathbf{r}}_2{}^2) = \frac{\mu}{2}\dot{\mathbf{r}}^2 = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{f}^2),$$

where $\mu = M_1 M_2$ is the the reduced mass of the system (recall $M_1 + M_2 = 1$). While the rotational kinetic energy is $T_{rot} = \frac{1}{2}\mathcal{C}_1\dot{\theta}_1^2 + \frac{1}{2}\mathcal{C}_2\dot{\theta}_2^2$. See Appendix B for the derivation of the full expression of the potential energy of the system $V = V(r, f, \theta_1, \theta_2)$, equation (56). The Euler-Lagrange equations corresponding to the Lagrangian $L = T_{orb}(r, \dot{r}, \dot{f}) + T_{rot}(\dot{\theta}_1, \dot{\theta}_2) - V(r, f, \theta_1, \theta_2)$ are

$$\mathcal{C}_1\ddot{\theta}_1 = -\partial_{\theta_1}V, \quad \mathcal{C}_2\ddot{\theta}_2 = -\partial_{\theta_2}V, \tag{11}$$

$$\mu\ddot{r} = \mu r\dot{f}^2 - \partial_r V, \quad \ddot{f} = -\frac{1}{\mu r^2}\partial_f V - 2\frac{\dot{r}\dot{f}}{r}. \tag{12}$$

In Appendix B.2 we give the expansion of the potential energy. In the case of ellipsoids, it has the form $V = \sum_{n=0}^{\infty} V_{2n}$, where V_{2n} is proportional to $1/r^{2n+1}$. The first terms of the expansion are

$$V_0 = -\frac{GM_1M_2}{r}.$$

$$V_2 = -\frac{GM_2}{4r^3}\left(q_1 + 3d_1\cos(2(\theta_1 - f))\right) - \frac{GM_1}{4r^3}\left(q_2 + 3d_2\cos(2(\theta_2 - f))\right),$$

$$\begin{aligned} V_4 = -\frac{3G}{4^3r^5}\{ \quad & 12q_1q_2 + \tfrac{15}{7}[\tfrac{M_2}{M_1}d_1^2 + 2\tfrac{M_2}{M_1}q_1^2 + \tfrac{M_1}{M_2}d_2^2 + 2\tfrac{M_1}{M_2}q_2^2] \\ & +d_1M_2\left\{[20\tfrac{q_2}{M_2} + \tfrac{100}{7}\tfrac{q_1}{M_1}]\cos(2(\theta_1 - f)) + 25\tfrac{d_1}{M_1}\cos(4(\theta_1 - f))\right\} \\ & +d_2M_1\left\{[20\tfrac{q_1}{M_1} + \tfrac{100}{7}\tfrac{q_2}{M_2}]\cos(2(\theta_2 - f)) + 25\tfrac{d_2}{M_2}\cos(4(\theta_2 - f))\right\} \\ & +6d_1d_2\cos(2(\theta_1 - \theta_2)) + 70d_1d_2\cos(2(\theta_1 + \theta_2) - 4f)\}, \end{aligned} \tag{13}$$

where we defined the parameters

$$d_j = \mathcal{B}_j - \mathcal{A}_j, \qquad q_j = 2\mathcal{C}_j - \mathcal{B}_j - \mathcal{A}_j. \tag{14}$$

Note that d_j is proportional to $C_{22}^{j)}$, whereas q_j is proportional to $C_{20}^{j)}$, where $C_{nm}^{j)}$ are the usual coefficients in the expansion of the gravitational potential of the ellipsoid $\mathcal{E}_j$. The quantity $d_j/\mathcal{C}_j$ measures the oblateness of the section of the ellipsoid in the plane of motion, whereas, $q_j/\mathcal{C}_j$ measures the flattening with respect to the plane. If $\mathcal{A}_j \leq \mathcal{B}_j \leq \mathcal{C}_j$, then, $q_j \geq d_j \geq 0$. Note that the term V_0 contains the dynamics of two point masses, V_2 the uncoupled spin-orbit dynamics and V_4 the spin-spin coupled dynamics between θ_1 and θ_2. The coupling terms appear in the last line of (13).

2.2 The Keplerian assumption and the spin-spin model

The complete dynamics of the system is given by Equations (11) and (12), with V in (56). In this paper we impose that the orbital motion is Keplerian, i.e., we keep only V_0 in the orbital part (12). Besides, in the spin part (11), we truncate V ignoring terms of order $1/r^7$ and higher, then $V \approx V_0 + V_2 + V_4$. The resulting system is

$$\mathcal{C}_1\ddot{\theta}_1 = -\partial_{\theta_1}(V_0 + V_2 + V_4), \quad \mathcal{C}_2\ddot{\theta}_2 = -\partial_{\theta_2}(V_0 + V_2 + V_4), \tag{15}$$

$$\mu\ddot{r} = \mu r\dot{f}^2 - \partial_r V_0, \quad \ddot{f} = -\frac{1}{\mu r^2}\partial_f V_0 - 2\frac{\dot{r}\dot{f}}{r}. \tag{16}$$

Note that, since $\partial_{\theta_j}V_0 = 0$, the system (16) is now decoupled from (15). Its solution is $r = r(t)$, $f = f(t)$ given by Equations (1) to (3) and depends on the eccentricity of the orbit e and its semi-major axis a.

Let us now write V_2 and V_4 in a more convenient way. The quantity $M_j a^2$ is a sort of orbital moment of inertia of the body $\mathcal{E}_j$. Then, we can define

$$\hat{d}_j = \frac{d_j}{M_j a^2}, \quad \hat{q}_j = \frac{q_j}{M_j a^2}, \tag{17}$$

so that $\hat{d}_j$ measures the equatorial oblateness of $\mathcal{E}_j$ with respect to the size of the orbit and $\hat{q}_j$ measures the flattening of $\mathcal{E}_j$ with respect to the size of the orbit.

Taking into account that in our units $G = a^3$, the terms V_2 and V_4 can be written in a compact way as

$$V_2 = -\frac{1}{4}\left(\frac{a}{r(t)}\right)^3 (\Lambda_0 + \Lambda_1 \cos(2\theta_1 - 2f(t)) + \Lambda_2 \cos(2\theta_2 - 2f(t))) \tag{18}$$

and

$$V_4 = -\frac{1}{4}\left(\frac{a}{r(t)}\right)^5 \sum_{(m_1,m_2)\in\Xi} \Lambda^{m_1}_{m_2} \cos(2m_1(\theta_1 - f(t)) + 2m_2(\theta_2 - f(t))) \tag{19}$$

where

$$\Xi = \{(m_1, m_2) \in \mathbb{Z}^2 : \ |m_1| + |m_2| \le 2\},$$

and the following Λ parameters are defined by

$$\Lambda_1 = 3d_1 M_2, \qquad \Lambda_2 = 3d_2 M_1, \tag{20}$$

$$\Lambda^1_0 = \Lambda^{-1}_0 = \frac{5}{56}(7\hat{q}_2 + 5\hat{q}_1)\Lambda_1, \qquad \Lambda^0_1 = \Lambda^0_{-1} = \frac{5}{56}(7\hat{q}_1 + 5\hat{q}_2)\Lambda_2, \tag{21}$$

$$\Lambda^2_0 = \Lambda^{-2}_0 = \frac{25}{32}\hat{d}_1\Lambda_1, \qquad \Lambda^0_2 = \Lambda^0_{-2} = \frac{25}{32}\hat{d}_2\Lambda_2, \tag{22}$$

$$\Lambda^1_1 = \Lambda^{-1}_{-1} = \frac{35}{16}\hat{d}_1\Lambda_2 = \frac{35}{16}\hat{d}_2\Lambda_1, \qquad \Lambda^{-1}_1 = \Lambda^1_{-1} = \frac{3}{16}\hat{d}_1\Lambda_2 = \frac{3}{16}\hat{d}_2\Lambda_1 \tag{23}$$

$$\Lambda_0 = q_1 M_2 + q_2 M_1, \qquad \Lambda^0_0 = \frac{9}{4}\hat{q}_1 q_2 M_1 + \frac{15}{112}(\Lambda_1\hat{d}_1 + 6\hat{q}_1 q_1 M_2 + \Lambda_2\hat{d}_2 + 6\hat{q}_2 q_2 M_1).$$

With the last definitions we can write equations (15) as $\mathcal{C}_j\ddot{\theta} = \mathcal{T}^C_j$, where $\mathcal{T}^C_j = -\partial_{\theta_j}(V_2 + V_4)$ are the *conservative* torques of the spin-spin model shown in (6). This can be checked with the expressions (18) and (19). Note that $m_j\Lambda^{m_1}_{m_2}$ is in all cases proportional to the corresponding Λ_j. Then, the equations of the *conservative* spin-spin model (15) can be written in terms of the physical parameters in the following symmetric way for $j = 1, 2$,

$$\begin{aligned} 0 = \ddot{\theta}_j + \frac{\lambda_j}{2}\Bigg\{ &\left(\frac{a}{r(t)}\right)^3 \sin(2\theta_j - 2f(t)) + \\ &+ \left(\frac{a}{r(t)}\right)^5 \Bigg[\frac{5}{4}\left(\hat{q}_{3-j} + \frac{5}{7}\hat{q}_j\right)\sin(2\theta_j - 2f(t)) + \frac{25\hat{d}_j}{8}\sin(4\theta_j - 4f(t)) \\ &+ \frac{3\hat{d}_{3-j}}{8}\sin(2\theta_j - 2\theta_{3-j}) + \frac{35\hat{d}_{3-j}}{8}\sin(2\theta_{3-j} + 2\theta_j - 4f(t))\Bigg]\Bigg\}, \end{aligned} \tag{24}$$

where

$$\lambda_j = \frac{\Lambda_j}{\mathcal{C}_j} = 3\frac{d_j}{\mathcal{C}_j}\frac{\mu}{M_j}.$$

It is worth mentioning that the terms with $\hat{q}_j$ and $\hat{d}_j$ in (24) were missing in the model used in [3] due to the dumbbell simplification for one of the bodies in the derivation of the equations. Not all the parameters appearing in (24) are free because the following identities hold

$$\mathcal{C}_1 + \mathcal{C}_2 = 1, \quad \Lambda_1 \hat{d}_2 = \Lambda_2 \hat{d}_1, \quad \Lambda_1 \hat{q}_2 = \Lambda_2 \hat{q}_1. \tag{25}$$

In consequence, our model depends on six independent parameters with physical meaning $(e; \mathcal{C}_1, \lambda_1, \lambda_2, \hat{d}_1, \hat{q}_1)$. Moreover, in (24) we see that spin of the ellipsoid $\mathcal{E}_2$ is affected by the spin-spin coupling with a strength essentially given by $\hat{d}_1$, and vice versa.

3 Linear stability of the double synchronous resonance in the conservative model

In this section we deal with the *conservative* system with the notation in (10), that is more convenient for our purpose. The main result is Theorem 2. It determines a region of linear stability of the double synchronous resonance in the space of parameters of the system.

3.1 Existence of the odd 2π-periodic solution

The system (10) can be written as

$$\mathcal{C}\ddot{\Theta} + F(t, \Theta) = 0, \tag{26}$$

where

$$\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & 0 \\ 0 & \mathcal{C}_2 \end{pmatrix}, \quad \mathcal{C}_j > 0,$$

and $F(t, \Theta)$ is the bounded function given by

$$F(t, \Theta) = \left(\frac{a}{r(t)}\right)^3 \begin{pmatrix} \Lambda_1 \sin \Theta_1 \\ \Lambda_2 \sin \Theta_2 \end{pmatrix} + \\ + \left(\frac{a}{r(t)}\right)^5 \sum_{(m_1, m_2) \in \Xi} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \Lambda_{m_2}^{m_1} \sin(m_1 \Theta_1 + m_2 \Theta_2) + 2\ddot{f}(t) \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}. \tag{27}$$

Note that equation (26) is invariant under the change $(t, \Theta) \to (-t, -\Theta)$, since $f(-t) = -f(t)$ and $r(-t) = r(t)$. Then, if $\Theta(t)$ is a solution of (26), so it is $-\Theta(-t)$. On the other hand, for $e = 0$, we have $f(t) = t$ and $r(t) = a$, meaning that the system (26) is that of two coupled free pendula. For this case, the trivial solution $\Theta(t) \equiv 0$ is a stable equilibrium. Then, for $e \neq 0$, it is natural to look for the 2π-periodic continuation of $\Theta(t) \equiv 0$ in the family of the odd solutions of (26), say, solutions satisfying $\Theta(-t) = -\Theta(t)$. This is equivalent to solve the Dirichlet problem

$$\begin{cases} \mathcal{C}\ddot{\Theta} + F(t, \Theta) = 0, \\ \Theta(0) = \Theta(\pi) = 0. \end{cases} \tag{28}$$

It is well known from nonlinear analysis that the system (28) has at least one solution because $F(t, \Theta)$ is bounded. We can give a simple proof for this. Let $\Theta(t) = \vartheta(t, v)$ be the solution of (26) satisfying initial conditions $\Theta(0) = 0$, $\dot{\Theta}(0) = v \in \mathbb{R}^2$. Solutions of the problem (28) are in correspondence with the solutions of the equation $\vartheta(\pi, v) = 0$. From (26), we know that ϑ satisfies the following integral equation

$$\vartheta(t, v) = vt - \int_0^t (t - s)\mathcal{C}^{-1}F(s, \vartheta(s, v))\,\mathrm{d}s. \tag{29}$$

Let $||\cdot||$ be a norm in $\mathbb{R}^2$, for instance, the maximum norm or the Euclidean one. We will employ the same notation for the corresponding induced matrix norm in $\mathbb{R}^{2\times 2}$. Since there exists a positive number $M \geq ||\mathcal{C}^{-1}F(t, \Theta)||$, then

$$||\vartheta(t, v) - vt|| \leq M\frac{t^2}{2},$$

for each $t \in \mathbb{R}$. If we take $t = \pi$, then, $||\Phi(v)|| \leq M\pi/2$, with $\Phi(v) = v - \vartheta(\pi, v)/\pi$ and $v \in \mathbb{R}^2$. Hence, we can apply Brouwer's fixed-point theorem to guarantee that $\Phi(v)$ has a fixed point for some v_0 satisfying $||v_0|| \leq M\pi/2$. For such point we have that $\vartheta(\pi, v_0) = 0$, and the corresponding $\vartheta(t, v_0)$ satisfyies (28).

3.2 Uniqueness of the solution

We know now that the Dirichlet problem (28) has a solution, however, it is not necessarily unique. For instance, if $\Lambda > 1$, there is not a unique solution for the free pendulum equation $\ddot{x} + \Lambda \sin x = 0$, $x \in \mathbb{R}$, with Dirichlet conditions $x(0) = x(\pi) = 0$. See [27]. We would like to determine sufficient conditions on the space of parameters of the system such that there is uniqueness for the problem (28).

We can prove uniqueness by a contradiction argument. Define the following matrix

$$\mathcal{C}^{1/2} = \begin{pmatrix} \sqrt{\mathcal{C}_1} & 0 \\ 0 & \sqrt{\mathcal{C}_2} \end{pmatrix}$$

and its inverse $\mathcal{C}^{-1/2} = (\mathcal{C}^{1/2})^{-1}$. Let $\Theta^{(0)}(t)$ and $\Theta^{(1)}(t)$ be two non-identical solutions of (28). Then, we can check that $y(t) = \mathcal{C}^{1/2}(\Theta^{(1)}(t) - \Theta^{(0)}(t))$ is a solution of the Dirichlet problem

$$\begin{cases} \ddot{y} + A(t)y = 0, \\ y(0) = y(\pi) = 0, \end{cases} \tag{30}$$

with $A(t)$ a symmetric[1] matrix given by

$$\mathcal{C}^{1/2}A(t)\mathcal{C}^{1/2} = \int_0^1 \partial_\Theta F(t, \Theta^{(\lambda)}(t))\,\mathrm{d}\lambda, \tag{31}$$

where $\Theta^{(\lambda)}(t) = \lambda\Theta^{(1)}(t) + (1 - \lambda)\Theta^{(0)}(t)$ and

$$F(t, \Theta) = \begin{pmatrix} F_1(t, \Theta) \\ F_2(t, \Theta) \end{pmatrix}, \quad \partial_\Theta F(t, \Theta) = \begin{pmatrix} \frac{\partial F_1}{\partial \Theta_1} & \frac{\partial F_1}{\partial \Theta_2} \\ \frac{\partial F_2}{\partial \Theta_1} & \frac{\partial F_2}{\partial \Theta_2} \end{pmatrix}.$$

[1] In this paper we use properties of linear systems with symmetric coefficient matrices. $\mathcal{C}^{-1}\partial_\Theta F(t, \Theta)$ is not symmetric, but we obtain the desired structure using $\mathcal{C}^{1/2}$. See [37].

The statement of uniqueness is given in Theorem 1. We can prove it by guaranteeing that (30) has only the trivial solution. In the proof we are going to apply the following lemma to (30) for a generic matrix $A(t) \in \mathbb{R}^{d\times d}$. But first we need some definitions. Let $\langle\cdot,\cdot\rangle$ be the Euclidean inner product in $\mathbb{R}^d$ and $||\cdot||$ its corresponding norm. Let $\mathbb{1}$ be the unit matrix in $R^{d\times d}$.

Definition 1 *Let $A_1, A_2 \in \mathbb{R}^{d\times d}$ be two symmetric matrices. We say that $A_1 \le A_2$ if, for the corresponding quadratic forms, $\langle A_1\, y, y\rangle \le \langle A_2\, y, y\rangle$ for all $y \in \mathbb{R}^d$.*

Lemma 1 *Assume that, for some $\gamma < 1$, the matrix $A(t) \in \mathbb{R}^{d\times d}$ is such that $A(t) \le \gamma\mathbb{1}$ for each $t \in [0,\pi]$. Then, the only solution of $\ddot{y} + A(t)y = 0$, $y \in \mathbb{R}^d$, with Dirichlet conditions $y(0) = y(\pi) = 0$ is the trivial one.*

Proof. Proceed by contradiction. Let $y(t)$ be a non-trivial solution of $\ddot{y} + A(t)y = 0$, $y(0) = y(\pi) = 0$, then,

$$\int_0^\pi \langle \ddot{y}(t), y(t)\rangle + \int_0^\pi \langle A(t)y(t), y(t)\rangle = 0,$$

integrating by parts it follows that

$$\int_0^\pi ||\dot{y}(t)||^2 = \int_0^\pi \langle A(t)y(t), y(t)\rangle.$$

Let $y_n(t)$ be the components of the vector $y(t)$. From the Sobolev inequality $\int_0^\pi |y_n(t)|^2 \le \int_0^\pi |\dot{y}_n(t)|^2$, see [38] or [29], we get that

$$\int_0^\pi ||y(t)||^2 \le \int_0^\pi \langle A(t)y(t), y(t)\rangle.$$

This contradicts the hypothesis $A(t) \le \gamma\mathbb{1}$ for some $\gamma < 1$. Then, $y(t)$ must be the trivial solution. ■

Let us define the matrix $\tilde{A}(t,\Theta) = \mathcal{C}^{-1/2}\partial_\Theta F(t,\Theta)\mathcal{C}^{-1/2}$,

$$\tilde{A}(t,\Theta) = \left(\frac{a}{r(t)}\right)^3 \begin{pmatrix} \frac{\Lambda_1}{\mathcal{C}_1}\cos\Theta_1 & 0 \\ 0 & \frac{\Lambda_2}{\mathcal{C}_2}\cos\Theta_2 \end{pmatrix} + \left(\frac{a}{r(t)}\right)^5 \sum_{(m_1,m_2)\in\Xi} \begin{pmatrix} \frac{m_1^2}{\mathcal{C}_1} & \frac{m_1 m_2}{\sqrt{\mathcal{C}_1\mathcal{C}_2}} \\ \frac{m_1 m_2}{\sqrt{\mathcal{C}_1\mathcal{C}_2}} & \frac{m_2^2}{\mathcal{C}_2} \end{pmatrix} \Lambda_{m_2}^{m_1}\cos(m_1\Theta_1 + m_2\Theta_2). \quad (32)$$

We will use the maximum norm

$$||y|| = \max\{|y_1|, |y_2|\}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

and its induced norm in matrices

$$||A|| = \max\{|A_{11}| + |A_{12}|, |A_{21}| + |A_{22}|\}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Theorem 1 *Assume that $e \in [0,1)$ and the parameters of the problem satisfy*

$$1 > \frac{1}{(1-e)^3} \max\left\{\frac{\Lambda_1}{\mathcal{C}_1}(1+\alpha_1), \frac{\Lambda_2}{\mathcal{C}_2}(1+\alpha_2)\right\}, \tag{33}$$

where

$$\alpha_j \frac{\Lambda_j}{\mathcal{C}_j} = \frac{1}{(1-e)^2} \sum_{(m_1,m_2)\in\Xi} \left(\frac{m_j^2}{\mathcal{C}_j} + \frac{|m_1 m_2|}{\sqrt{\mathcal{C}_1\mathcal{C}_2}}\right) \Lambda_{m_2}^{m_1}. \tag{34}$$

Then, there exists a unique solution of the Dirichlet problem (28), denoted by $\Theta^*(t)$.

Proof. Using the fact that $a/r \le 1/(1-e)$ by (2), equations (33) and (34) imply that $1 > ||\tilde{A}(t,\Theta)||$ for all $(t,\Theta) \in \mathbb{R}^3$, where we use the maximum norm. Furthermore, if $\rho(A)$ is the spectral radius of A, the well known inequality $||\tilde{A}(t,\Theta)|| \ge \rho(\tilde{A}(t,\Theta))$ guarantees that $\gamma\mathbb{1} \ge \tilde{A}(t,\Theta)$ for some $\gamma < 1$. Then, $\gamma\mathbb{1} \ge A(t)$ for $A(t)$ defined in (31). Now a direct application of Lemma 1 finishes the proof. ■

Remark 1 *Note that, as in the spin-orbit problem, there are two special cases for which Θ^* can be computed explicitly for some combination of parameters satisfying (33). If $\Lambda_j = 0$ and $\Lambda_{m_2}^{m_1} = 0$ for $m_1 m_2 \ne 0$, for each $e \in (0,1)$ the solution is the synchronous resonance of the uncoupled system*

$$\Theta^*(t) = 2(t - f(t,e)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{35}$$

On the other hand, if $e = 0$ the solution is $\Theta^*(t) = 0$.

3.3 Linear stability of the solution

Now we are interested in the stability properties of the solution $\Theta^*(t)$, which should be seen as 2π-periodic and odd from now on. In the following we will find a region of parameters guaranteeing stability of the (scaled) linearized system of (10) at the periodic solution Θ^*, say,

$$\ddot{y} + A(t)y = 0, \tag{36}$$

where we take the symmetric matrix $A(t)$ now defined by

$$A(t) = \tilde{A}(t,\Theta^*(t)) = \mathcal{C}^{-1/2}\partial_\Theta F(t,\Theta^*(t))\mathcal{C}^{-1/2},$$

and $\tilde{A}(t,\Theta)$ was defined in (32).

Recall from Section 1.2 that the *conservative* spin-spin model has a time-dependent Hamiltonian structure given by (8). The variational equations associated to periodic solutions, like (36), are linear Hamiltonian systems with periodic coefficients. We will abbreviate them by LPH systems[2]. These systems have some special properties that we will use in the following. For the general theory see [37] or [17]. For example, assume that φ is a Floquet multiplier of an LPH system. Then, its inverse φ^{-1}, its complex conjugate $\bar{\varphi}$ and $\bar{\varphi}^{-1}$ are also multipliers and have the same multiplicity as φ. This is stated in Corollary 6 of Chapter 1.1 of [17]. Let us point out two interesting consequences. First, a necessary condition for stability of an LPH system is that all its Floquet multipliers must have modulus 1. Second, an LPH system can never be asymptotically stable. In order to do continuation of periodic solutions to the *dissipative* regime we will need the concept of strong stability for LPH systems.

[2]The linear system $\mathcal{C}\ddot{y} + \partial_\Theta F(t,\Theta^*(t))y = 0$ is an LPH system in the general sense. However, for simplicity, we particularize the general theory to (36).

Definition 2 *Let $A_0(t) \in \mathbb{R}^{d\times d}$ be a fixed symmetric and T-periodic matrix. Assume that the there exists a number $\varepsilon > 0$ such that the equation $\ddot{y} + A_*(t)y = 0$ is stable for all $A_*(t) \in \mathbb{R}^{d\times d}$ symmetric and T-periodic satisfying $\int_0^T ||A_*(t) - A_0(t)|| < \varepsilon$. Then, $\ddot{y} + A_0(t)y = 0$ is strongly stable.*

In other words, if an LPH system is strongly stable, then, any sufficiently small perturbation of it is stable. The perturbation should keep the Hamiltonian structure. Let us illustrate this with an example of the so-called Mathieu equation. Consider the 2π-periodic equation

$$\ddot{x} + \frac{1}{4}(1 + \epsilon \cos t)x = 0, \quad x \in \mathbb{R}.$$

For $\epsilon = 0$ it is stable, but not strongly stable, because we can always find a small number $\epsilon \neq 0$ such that the corresponding equation is not stable. This is called parametric resonance, see [1].

Strong stability can be characterized with the Floquet multipliers of the system. For example, take an LPH system whose multipliers belong to the unit circle. If the multiplicity of all the multipliers is one, then the system is strongly stable. However, the converse is not true. M. Krein developed a theory to determine if a system is strong stable with further algebraic properties of the multipliers. For our purpose of making continuation of periodic solutions the following property is relevant.

Proposition 1 *Assume that $\ddot{y} + A(t)y = 0$, with $A(t) \in \mathbb{R}^{d\times d}$ symmetric and T-periodic, is strongly stable. Then, neither 1 nor -1 are Floquet multipliers of the system.*

We will not prove this property because it is a particular result of the general theory. Nonetheless, it can be inferred by the paragraph previous to Theorem 10 in Chapter 1.2 of [17], that is the main result of Krein's theory.

Some sufficient conditions for strong stability of (36) are given by the following Lyapunov-like stability criterion, from Test 4, in [37], Chapter III, Section 7.

Stability test 1 *The equation $\ddot{y} + A(t)y = 0$, with $A(t) \in \mathbb{R}^{d\times d}$ symmetric and 2π-periodic, is strongly stable provided that, for all $x \in \mathbb{R}^d \backslash \{0\}$,*

$$\int_0^{2\pi} \langle A(t)x, x\rangle \, \mathrm{d}t > 0 \quad \textit{and} \quad \int_0^{2\pi} \mathrm{Tr}(A(t)) \, \mathrm{d}t < \frac{2}{\pi}. \tag{37}$$

This stability test is the main tool for the proof of the next theorem.

Theorem 2 *Assume that the parameters of the model satisfy the following conditions.*

$$\frac{1}{\pi^2} > \frac{1}{(1-e)^3}\left(\frac{\Lambda_1}{\mathcal{C}_1} + \frac{\Lambda_2}{\mathcal{C}_2}\right) + \frac{1}{(1-e)^5} \sum_{(m_1,m_2)\in\Xi} \left(\frac{m_1^2}{\mathcal{C}_1} + \frac{m_2^2}{\mathcal{C}_2}\right) \Lambda_{m_2}^{m_1}, \tag{38}$$

$$\frac{1}{4\pi} > M := \frac{1}{(1-e)^3} \max\left\{\frac{\Lambda_1}{\mathcal{C}_1}, \frac{\Lambda_2}{\mathcal{C}_2}\right\} + \frac{1}{(1-e)^5} \sum_{(m_1,m_2)\in\Xi} \max\left\{\frac{|m_1|}{\mathcal{C}_1}, \frac{|m_2|}{\mathcal{C}_2}\right\} \Lambda_{m_2}^{m_1} + \frac{4e\sqrt{1-e^2}}{(1-e)^4}, \tag{39}$$

$$\cos(2\pi^2 M) \min\left\{\frac{\Lambda_1}{\mathcal{C}_1}, \frac{\Lambda_2}{\mathcal{C}_2}\right\} > \max\left\{\alpha_1\frac{\Lambda_1}{\mathcal{C}_1}, \alpha_2\frac{\Lambda_2}{\mathcal{C}_2}\right\}, \tag{40}$$

with α_j defined in (34). Then the solution $\Theta^*(t)$ is strongly linearly stable.

Note that the second condition of (37) is guaranteed by (38). The first condition of (37) is a bit more complicated, but its proof is immediate by the following two lemmas.

Lemma 2 *The components of the solution $\Theta^*(t)$ satisfy the following bounds $|\Theta_j^*(t)| \leq 2\pi^2 M$, $|\dot{\Theta}_j^*(t)| \leq 2\pi M$ provided that $M \geq ||\mathcal{C}^{-1}F(t, \Theta^*(t))||$.*

Proof. Integrating the identity $\ddot{\Theta}^*(t) + \mathcal{C}^{-1}F(t, \Theta^*(t)) = 0$ and taking the first component,

$$\dot{\Theta}_1^*(t) = \dot{\Theta}_1^*(t_0) - \int_{t_0}^{t} u_1\, \mathcal{C}^{-1}F(s, \Theta^*(s))\, \mathrm{d}s,$$

where u_1 is the row vector $(1, 0)$. Then, for $t \in [t_0, t_0 + 2\pi]$,

$$|\dot{\Theta}_1^*(t)| \leq |\dot{\Theta}_1^*(t_0)| + \int_{t_0}^{t_0+2\pi} ||\mathcal{C}^{-1}F(s, \Theta^*(s))||\, \mathrm{d}s \leq |\dot{\Theta}_1^*(t_0)| + 2\pi M,$$

where $||\cdot||$ indicates a matrix norm induced by a norm in $\mathbb{R}^2$. Since $\Theta_1^*(t)$ is 2π-periodic, we can choose t_0 such that $\dot{\Theta}_1^*(t_0) = 0$. The same is applicable to Θ_2 for a possibly different t_0, consequently, $|\dot{\Theta}_j^*(t)| \leq 2\pi M$ for all t. Furthermore, since $\Theta_1^*(0) = 0$,

$$\Theta_1^*(t) = \int_0^t \dot{\Theta}_1^*(s)\, \mathrm{d}s,$$

and, due to the odd symmetry of $\Theta_1^*(t)$, it is enough to consider $t \in [0, \pi]$. Then, $|\Theta_1^*(t)| \leq 2\pi^2 M$. The same is true for $\Theta_2^*(t)$. ■

Lemma 3 *The conditions (39) and (40) are sufficient so that $A(t) = \tilde{A}(t, \Theta^*(t)) \geq \gamma \mathbb{1}$ for some $\gamma > 0$.*

Proof. The proof this lemma is based on the following fact. Considering the partial ordering of symmetric matrices given by Definition 1, the conditions (39) and (40) imply that the term proportional to $1/r^3$ in (32) dominates the other term, that is proportional to $1/r^5$. Let us prove it. We can compute the derivatives of $f(t)$ using Equations (1) to (3) and get

$$\ddot{f}(t) = -\frac{2e\sqrt{1-e^2}\sin(u(t))}{(1 - e\cos(u(t)))^4},$$

where u is the eccentric anomaly. Using the maximum norm we see from (39) and (27) that $1/(4\pi) > M \geq ||\mathcal{C}^{-1}F(t, \Theta^*(t))||$. Furthermore, from Lemma 2 we know that $|\Theta_j^*(t)| \leq 2\pi^2 M$, then, we can see graphically that

$$\cos\Theta_1^*(t) \geq \cos(2\pi^2 M) > 0,$$

therefore,

$$\begin{pmatrix} \frac{\Lambda_1}{\mathcal{C}_1}\cos\Theta_1^*(t) & 0 \\ 0 & \frac{\Lambda_2}{\mathcal{C}_2}\cos\Theta_2^*(t) \end{pmatrix} \geq \cos(2\pi^2 M) \min\left\{\frac{\Lambda_1}{\mathcal{C}_1}, \frac{\Lambda_2}{\mathcal{C}_2}\right\} \mathbb{1}. \qquad (41)$$

On the other hand, let us define

$$B = -\left(\frac{a}{r(t)}\right)^2 \sum_{(m_1,m_2)\in\Xi} \begin{pmatrix} \frac{m_1^2}{\mathcal{C}_1} & \frac{m_1 m_2}{\sqrt{\mathcal{C}_1\mathcal{C}_2}} \\ \frac{m_1 m_2}{\sqrt{\mathcal{C}_1\mathcal{C}_2}} & \frac{m_2^2}{\mathcal{C}_2} \end{pmatrix} \Lambda_{m_2}^{m_1} \cos(m_1\Theta_1^*(t) + m_2\Theta_2^*(t)).$$

As we did in the Proof of Theorem 1, we can take the maximum norm and obtain that

$$\max\left\{\alpha_1\frac{\Lambda_1}{\mathcal{C}_1},\alpha_2\frac{\Lambda_2}{\mathcal{C}_2}\right\}\geq ||B||\geq\rho(B)$$

where $\rho(B)$ is the spectral radius of B, then,

$$\max\left\{\alpha_1\frac{\Lambda_1}{\mathcal{C}_1},\alpha_2\frac{\Lambda_2}{\mathcal{C}_2}\right\}\mathbb{1}\geq B.$$

From this inequality, (41) and the definition (32) of $\tilde{A}(t,\Theta)$, we prove that $\tilde{A}(t,\Theta^*(t))\geq\gamma\mathbb{1}$ with

$$\gamma=\cos(2\pi^2 M)\min\left\{\frac{\Lambda_1}{\mathcal{C}_1},\frac{\Lambda_2}{\mathcal{C}_2}\right\}-\max\left\{\alpha_1\frac{\Lambda_1}{\mathcal{C}_1},\alpha_2\frac{\Lambda_2}{\mathcal{C}_2}\right\}>0.$$

■

Now we see that Lemma 3 implies the first condition of (37) because $\langle A(t)x,x\rangle\geq\gamma||x||^2>0$.

4 The synchronous resonance in the dissipative regime

Recall from (9) that the *dissipative* spin-spin model takes the form of the system

$$\ddot{\Theta}+\mathrm{diag}(\delta)D(t)\dot{\Theta}+\mathcal{C}^{-1}F(t,\Theta)=0,\qquad\delta=\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix},\ \ \delta_j\geq 0,\tag{42}$$

with $D(t)=(a/r(t))^6$. We know from Theorem 2 that, for $\delta=0$, there exists an odd 2π-periodic solution $\Theta^*(t)$, that is strongly linearly stable in the set of the parameters space satisfying the conditions given in Equations (38) to (40).

The main result of this section is Theorem 3. There we will see that the *conservative* periodic solution $\Theta^*(t)$ can be continued in the presence of friction to an asymptotically stable periodic solution $\Psi^*(t,\delta)$. However, the odd symmetry of the solution is lost because (42) is not invariant under the change $(t,\Theta)\to(-t,-\Theta)$ as in the *conservative* case. The proof of Theorem 3 is mainly based on Theorem 2 in [29] and on classical results on continuation of periodic solutions summarized in the next proposition.

Proposition 2 *Let $\mathcal{F}$ be a real analytic function $\mathcal{F}=\mathcal{F}(t,x,\zeta)$, such that $\mathcal{F}(t+T,x,\zeta)=\mathcal{F}(t,x,\zeta)$, with $t\in\mathbb{R}$, $x\in\mathbb{R}^n$, $\zeta\in\mathbb{R}^d$. Assume that the equation $\dot{x}=\mathcal{F}(t,x,0)$ has a T-periodic solution $x=p(t)$.*

1. *Suppose that 1 is not a Floquet multiplier of the corresponding variational equation at $x=p(t)$,*
$$\dot{y}=\partial_x\mathcal{F}(t,p(t),0)y.$$
Then, for $\zeta\neq 0$, with small enough norm $||\zeta||$, the equation $\dot{x}=\mathcal{F}(t,x,\zeta)$ has a T-periodic solution $x=p_c(t,\zeta)$ such that $p_c(t,0)=p(t)$. Moreover, $p_c(t,\zeta)$ is an analytic function and it is unique of each ζ.

2. *If additionally, $p(t)$ is asymptotically stable, then this is also true for $p_c(t,\zeta)$.*

For the detailed proof of this proposition, see Theorems 1.1 and 1.2 in Chapter 14, [11]. Now we can state the main theorem.

Theorem 3 *Assume that the parameters of the system satisfy the conditions in Theorem 2. If $|\delta_j|$ are small enough, then there exists a function $\Psi^*(t,\delta)$, analytic in both entries, satisfying*

i) $\Psi^*(t,0) = \Theta^*(t)$ *for each* $t \in \mathbb{R}$.

ii) $\Psi^*(t,\delta)$ *is a 2π-periodic solution of (42). Moreover, if $|\Lambda_{m_2}^{m_1}|$ are small enough, then, $\Psi^*(t,\delta)$ is asymptotically stable.*

Proof. Recall that the *conservative* periodic solution $\Theta^*(t)$ is strongly linearly stable. Proposition 1 guarantees that 1 is not a Floquet multiplier of the variational equation at $\Theta^*(t)$. Then, we can apply the first item of Proposition 2 to make the analytic continuation of the periodic solution from the *conservative* ($\delta_j = 0$) to the *dissipative* regime ($\delta_j > 0$). We conclude that there exists a unique analytic 2π-periodic solution $\Psi^*(t,\delta)$ of (42) such that $\Psi^*(t,0) = \Theta^*(t)$ for small enough δ_j.

Let us explain more in detail the proof that the continuation is asymptotically stable. If $\Lambda_{m_2}^{m_1} = 0$ for all $(m_1, m_2) \in \Xi$, then (42) takes the form of two uncoupled *dissipative* spin-orbit equations

$$\ddot{\Theta}_j + \delta_j \left(\frac{a}{r(t)}\right)^6 \dot{\Theta}_j + \frac{\Lambda_j}{\mathcal{C}_j}\left(\frac{a}{r(t)}\right)^3 \sin\Theta_j = 0. \tag{43}$$

Besides, conditions in Equations (38) to (40) guarantee that, for $\Lambda_{m_2}^{m_1} = 0$, the *conservative* solution $\Theta^*(t)$ is strongly linearly stable. We can see the solution $\Theta^*(t)$ split in two components $\Theta_j^*(t)$, each of them is a solution of the *conservative* spin-orbit problem (43) with $\delta_j = 0$. Now we can apply Theorem 2 in [29] that guarantees that each equation in (43) has an asymptotically stable 2π-periodic solution $\Theta_{j,\delta_j}^*(t)$ provided that $\delta_j \in (0, \bar{\delta}_j]$. Here $\bar{\delta}_j$ are small numbers quantified in [29]. Moreover, $\Theta_{j,\delta_j}^*(t)$ is the unique continuation of $\Theta_j^*(t) = \Theta_{j,0}^*(t)$.

Let us consider (43) as a system of two equations. This system has an asymptotically stable 2π-periodic solution $\Psi^*(t,\delta) = (\Theta_{1,\delta_1}^*(t), \Theta_{2,\delta_2}^*(t))^{\mathsf{T}}$ such that $\Psi^*(t,0) = \Theta^*(t)$. If $|\Lambda_{m_2}^{m_1}|$ are small, we can see (42) as a perturbation of the system (43) and apply the second item of Proposition 2. In this way we guarantee that $\Psi^*(t,\delta)$ has a 2π-periodic continuation for $\Lambda_{m_2}^{m_1} \neq 0$ that is asymptotically stable if $|\Lambda_{m_2}^{m_1}|$ are small enough. ∎

Note that for asymptotic stability we require not only that $|\delta_j|$ should be small, but also $|\Lambda_{m_2}^{m_1}|$. We would like to erase this condition on the coupling parameters $\Lambda_{m_2}^{m_1}$. However, from a theoretical point of view, this is certainly difficult to address in general since we deal with systems of differential equations. Let us explain this point. The variational equation of (42) near $\Psi^*(t,\delta)$ is

$$\ddot{\eta} + \operatorname{diag}(\delta) D(t)\dot{\eta} + \mathcal{C}^{-1}\partial_\Theta F(t, \Psi^*(t,\delta))\eta = 0, \quad \delta = \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}. \tag{44}$$

For $e \neq 0$, (44) is a linear 2π-periodic system of two equations of second order. In [29], asymptotic stability was proved for the spin-orbit problem taking advantage of the following fact. Any second order periodic equation $\ddot{x} + a_1(t)\dot{x} + a_0(t)x = 0$, $x \in \mathbb{R}$, $a_n(t) = a_n(t+T)$, can be converted into a Hill's equation $\ddot{\chi} + \alpha(t)\chi = 0$, $\alpha(t) = a_0(t) - \frac{1}{4}a_1(t)^2 - \frac{1}{2}\dot{a}_1(t)$, by the change of variables $\chi(t) = x(t)\exp(\frac{1}{2}\int_0^t a_1(s)\,\mathrm{d}s)$. See [24]. We can see the *dissipative* problem ($a_1(t) \neq 0$) as a perturbation of the *conservative* one ($a_1(t) = 0$). Assume that $\ddot{x} + a_0(t)x = 0$ is strongly stable, then $\ddot{\chi} + \alpha(t)\chi = 0$ is stable. Since it is a Hill's equation (also a LPH system), the modulus of the Floquet multipliers of $\ddot{\chi} + \alpha(t)\chi = 0$ is 1. Now we undo the change of variables and conclude that the modulus of the Floquet multipliers of $\ddot{x} + a_1(t)\dot{x} + a_0(t)x = 0$ is smaller than 1, therefore, it is asymptotically stable. However, it is not clear how to perform an analogous procedure in (44). The main obstacle is the non-commutativity of matrices due to the asymmetric nature of the *dissipative* problem ($\delta_1 \neq \delta_2$). Actually, if we follow the same steps, we end up with a system of equations that is no longer periodic for $\delta_1 \neq \delta_2$. The numbers δ_j depend on several parameters of the bodies and we do not see any good physical reason to impose both dissipative parameters to be equal. In fact, if $\delta_1 \neq \delta_2$, in principle the *dissipative* spin-spin model cannot be considered conformally symplectic as the spin-orbit problem. See [7]. From this discussion, we conclude that this it is necessary a deeper theoretical study, but it is beyond the scope of this paper.

On the other hand, let us see that for $e = 0$, the solution of (44) is asymptotically stable. The solution given by Theorem 3 is $\Psi^*(t, \delta) \equiv 0$. Taking $y = \mathcal{C}^{1/2}\eta$, the corresponding variational equation is

$$\ddot{y} + \operatorname{diag}(\delta)\dot{y} + Ay = 0, \tag{45}$$

where A is the symmetric constant matrix given by

$$A = \begin{pmatrix} \xi_1 & \sigma \\ \sigma & \xi_2 \end{pmatrix} = \begin{pmatrix} \frac{\Lambda_1}{\mathcal{C}_1} & 0 \\ 0 & \frac{\Lambda_2}{\mathcal{C}_2} \end{pmatrix} + \sum_{(m_1,m_2)\in\Xi} \begin{pmatrix} \frac{m_1^2}{\mathcal{C}_1} & \frac{m_1 m_2}{\sqrt{\mathcal{C}_1\mathcal{C}_2}} \\ \frac{m_1 m_2}{\sqrt{\mathcal{C}_1\mathcal{C}_2}} & \frac{m_2^2}{\mathcal{C}_2} \end{pmatrix} \Lambda^{m_1}_{m_2}.$$

Note that, by conditions (39) and (40), A is a positive definite matrix. See Lemma 3. The characteristic polynomial of equation (45) is

$$p(\omega) = \omega^4 + (\delta_1 + \delta_2)\omega^3 + (\xi_1 + \xi_2 + \delta_1\delta_2)\omega^2 + (\xi_1\delta_2 + \xi_2\delta_2)\omega + \det A.$$

Equation (45) is asymptotically stable if and only if all the roots of $p(\omega)$ have negative real parts. This can be checked with the Routh-Hurwitz criterion, see [19]. According to it, all the roots of the polynomial have negative real parts if and only if the associated Hurwitz determinants of the polynomial are strictly positive, say,

$$D_1 = \delta_1 + \delta_2, \quad D_2 = \delta_1^2\delta_2 + \delta_2^2\delta_1 + \xi_1\delta_1 + \xi_2\delta_2,$$

$$D_3 = D_1^2\sigma^2 + \delta_1\delta_2(D_1(\xi_1\delta_2 + \xi_2\delta_2) + (\xi_1 - \xi_2)^2), \quad D_4 = D_3 \det A.$$

Since A is positive definite, we get asymptotic stability for all δ_1 and δ_2 such that both are non-negative and at least one is different from zero.

5 Applications

Recall from the end of Section 2 that our model depends on six independent physical parameters $(e; \mathcal{C}_1, \lambda_1, \lambda_2, \hat{d}_1, \hat{q}_1)$, where e is the orbital eccentricity, $\mathcal{C}_j$ the moment of inertia of $\mathcal{E}_j$ with respect to the c_j-axis, $\lambda_j = \Lambda_j/\mathcal{C}_j$ is the oblateness of $\mathcal{E}_j$ in the plane of motion, and $\hat{d}_j$ and $\hat{q}_j$ are, respectively, the oblateness and the flatness of $\mathcal{E}_j$ with respect to the size of the orbit.

We have two type of estimates. The first type in (33) guarantees uniqueness of the synchronous resonance in the *conservative* regime. The second one in Equations (38) to (40) guarantees linear stability of the same solution. Our estimates depend on certain values α_j in

System	M_j	a_j	$\mathcal{C}_j$	λ_j	$\hat{d}_j$	$\hat{q}_j$	a	e
Pluto	0.89	1.65	0.97	$3.3 \cdot 10^{-5}$	$1.5 \cdot 10^{-7}$	$1.2 \cdot 10^{-6}$	27.2	$2.0 \cdot 10^{-4}$
Charon	0.11	0.84	0.03	$2.4 \cdot 10^{-3}$	$3.5 \cdot 10^{-7}$	$8.2 \cdot 10^{-7}$		
Patroclus	0.56	1.7	0.60	0.11	$2.6 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$	18.2 ± 0.5	0.02 ± 0.02
Menoetius	0.44	1.6	0.40	0.14	$2.2 \cdot 10^{-4}$	$9.9 \cdot 10^{-4}$		

Table 1: Real physical parameters for two binary systems. For Pluto and Charon, we take the largest values of λ_j, $\hat{d}_j$ and $\hat{q}_j$ obtained from data in [21]. The parameters of Patroclus and Menoetius are obtained from data in [14] and the orbital parameters from [25].

(34). To write them in terms of the physical parameters, we use the definitions in Equations (20) to (23), then

$$\sum_{(m_1,m_2)\in\Xi} \frac{m_j^2}{\mathcal{C}_j}\Lambda_{m_2}^{m_1} = \lambda_j \left(\frac{25}{4}\hat{d}_j + \frac{25}{28}\hat{q}_j + \frac{19}{4}\hat{d}_{3-j} + \frac{5}{4}\hat{q}_{3-j}\right), \tag{46}$$

$$\sum_{(m_1,m_2)\in\Xi} \frac{|m_1 m_2|}{\sqrt{\mathcal{C}_1\mathcal{C}_2}}\Lambda_{m_2}^{m_1} = \frac{19}{4}\sqrt{\frac{\mathcal{C}_1}{\mathcal{C}_2}}\lambda_1\hat{d}_2 = \frac{19}{4}\sqrt{\frac{\mathcal{C}_2}{\mathcal{C}_1}}\lambda_2\hat{d}_1, \tag{47}$$

$$\sum_{(m_1,m_2)\in\Xi} \frac{|m_j|}{\mathcal{C}_j}\Lambda_{m_2}^{m_1} = \lambda_j \left(\frac{25}{8}\hat{d}_j + \frac{25}{28}\hat{q}_j + \frac{19}{4}\hat{d}_{3-j} + \frac{5}{4}\hat{q}_{3-j}\right). \tag{48}$$

Now we are ready to apply our estimates to specific cases.

5.1 Real systems

In one hand, the Pluto-Charon binary is the largest known system that is in double synchronous resonance. The physical parameters of the system relevant for the spin-spin model are shown in Table 1. Pluto is almost twice the size of Charon, contains the 89% of the mass and the 97% of the body moment of inertia ($\mathcal{C}_j$) of the system. Besides, the size of the orbit is quite large ($a = 27.2$) compared to the sizes of the bodies. This results in very small values of $\hat{d}_j$ of order 10^{-7}, which means this is a certainly weak spin-spin coupling. The orbit has a very small eccentricity $e = 0.0002$. Recall that the double synchronous resonance of the circular case ($e = 0$) is the trivial solution $\Theta(t) \equiv 0$, both for the *conservative* case (10) and the *dissipative* case (9). The asymptotic stability of the solution for any value of the dissipative parameters is easily guaranteed, as it was shown at the end of Section 4 using equation (45). For the real eccentricity, the solution $\Theta^*(t)$ of (10) oscillates very close to zero and our estimates guarantee the uniqueness and linear stability of solution. Furthermore, Theorem 3 shows the existence of an asymptotically stable solution $\Psi^*(t,\delta)$ of the *dissipative* model provided that δ_j, $\hat{d}_j$ and $\hat{q}_j$ are small enough. Unfortunately, this last result is not quantified in this paper for the real parameters.

On the other hand, the Trojan binary asteroid 617 Patroclus is a system whose components are of similar size, mass and moment of inertia. See the physical parameters of its components, Patroclus and Menoetius, in Table 1. Each body has a diameter of around one hundred kilometres, almost ten times smaller than Charon. Patroclus and Menoetius have a more oblate ellipsoidal shape than Pluto and Charon and the size of the orbit in this case ($a = 18.2 \pm 0.5$) is smaller. In consequence, the corresponding dynamical parameters λ_j, $\hat{d}_j$ and $\hat{q}_j$ are several orders of magnitude larger. The orbital eccentricity is not measured with enough precision, $e = 0.02 \pm 0.02$. With our estimates, we are able to guarantee the uniqueness of the solution $\Theta^*(t)$ of (10) for eccentricities up to $e = 0.04$. However, we fail to guarantee linear stability even for $e = 0$. The main reason is that the stability test given by the conditions (37) is not fine enough for such large values of λ_j. In the following subsection we will explain what is the range of parameters that is covered by our study.

5.2 Stability diagrams in the space of parameters

Note that all the terms appearing in Equations (46) to (48) are positive. Since $\hat{q} \geq \hat{d}$, and, in order to reduce the parameters in the upper bounds for the expressions in Equations (46) to (48), we can take $\hat{d}_j = \hat{q}_j$. In this way, we reduce the independent parameters to five $(e; \lambda_1, \lambda_2, \mathcal{C}_1, \hat{q}_1)$. Note now that, to take $\hat{q}_1 = 0$ is equivalent to break the coupling of the system, resulting in two independent spin-orbit problems.

We will consider two special cases with three free parameters. In one hand, the case of identical bodies, that we compare with the asteroid 617 Patroclus. Here the parameters are e, $\lambda_j = \lambda$ and $\hat{q}_j = \hat{q}$. On the other hand, the case when $\mathcal{E}_1$ is twice the size of $\mathcal{E}_2$, that we compare with the Pluto-Charon system. Here we consider the same density and the free parameters are e, λ_2 and $\hat{q}_1$, whereas the dependent parameters are $\lambda_1 = 2^{-3}\lambda_2$ and $\hat{q}_2 = 2^{-5}\hat{q}_1$.

Figure 2 shows regions in the space of parameters for which there is uniqueness and linear stability of the double synchronous resonance according to our theoretical estimates. We see that we cover the Patroclus-Menoetius system (top panels) only for the uniqueness of the solution but not for the linear stability. In contrast, the Pluto-Charon system (bottom panels) is covered for linear stability as well. We can compare the diagrams of $\hat{q} = 0$ and $\hat{q}_1 = 0$ with the theoretical estimates obtained in [29], shown in Figure 3. We see that, although the uniqueness region is similar, the stability region (in yellow) is considerably larger in Figure 3 than those in Figure 2. This shows that the mathematical techniques used in [29] are much finer than in this paper. In [29] we used generalized Lyapunov criteria using L^p-norms, with $p \in [1, \infty]$, see [38], and upper and lower solutions to bound the amplitude of the solution. Instead, in this paper we use the stability test given by (37), that is of type L^∞, and a rougher bound for the amplitude of the solution in Lemma 2. Since the model is quite new, here we initiate the analysis with a simpler approach. Besides, the mathematical tools are not as well developed for systems of equations as for standard second order scalar equations.

We see in Figure 2 that an increase in the value of $\hat{q}$ results in a global reduction of the regions that we estimated theoretically, both for stability and uniqueness regions. This behavior can be compared with the numerical plots in Figure 4. We focus only on the case of equal bodies. Here we see how the instability region changes when we increase $\hat{q}$. There are some interesting phenomena.

1. For $\hat{q} = 0$ there is only one bifurcation point for the unstable solution in the λ-axis at $(e, \lambda) = (0, 0.25)$. However, for $\hat{q} > 0$, it becomes two bifurcation points at $(0, \lambda_{(1)})$ and $(0, \lambda_{(2)})$, with $0 < \lambda_{(1)} < \lambda_{(2)} < 0.25$. This opens a small window of stability at the points (e, λ) with e close to 0 and $\lambda \in (\lambda_{(1)}, \lambda_{(2)})$.

2. For $\hat{q} = 0$, apart from the instability region bifurcating from the λ-axis, there is another one bifurcating from the e-axis at $(e, \lambda) \approx (0.682, 0)$. The existence of such bifurcation

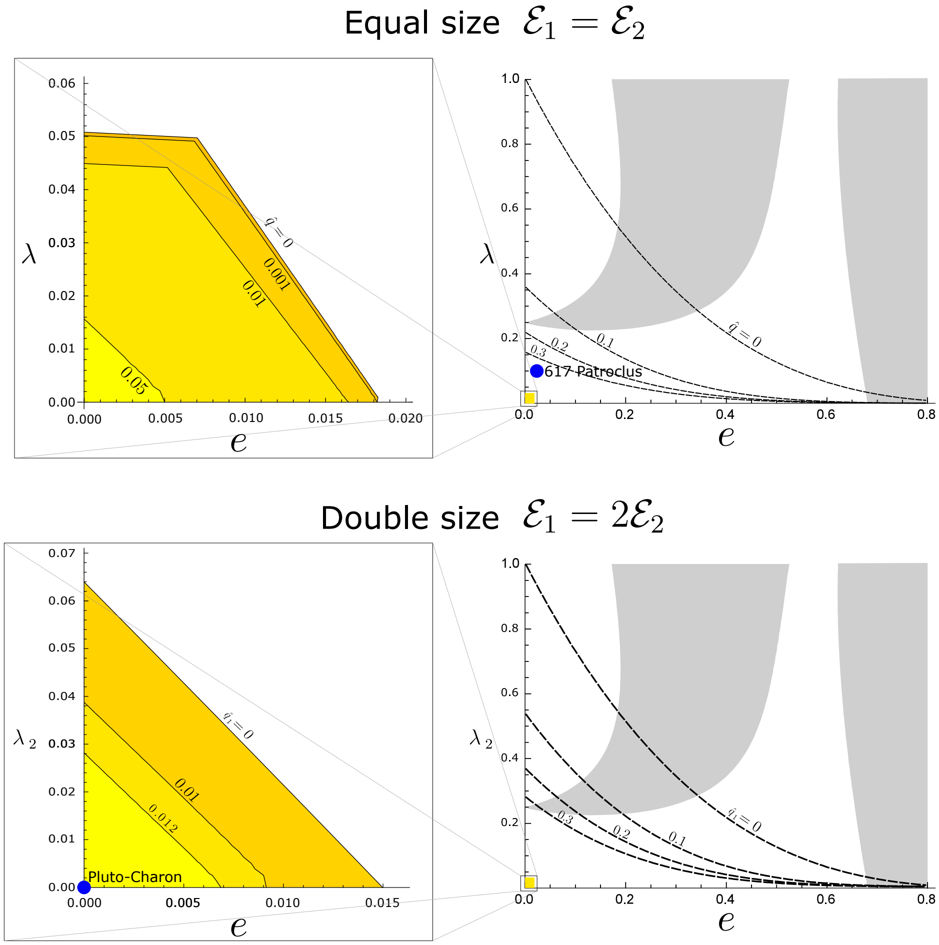

Figure 2: Stability diagrams in the (e, λ)-plane of the synchronous resonance of the spin-spin model. Top: both bodies are equal. Bottom: one body is double the size of the other. The double synchronous resonance is unique under the dashed lines (right) and linearly stable under the black lines (left) for the indicated value of $\hat{q}$. In the left we see zoomed views of the stable regions. The more yellow is the region indicates that stability is guaranteed for larger values of $\hat{q}$. The gray regions in the right are unstable for the uncoupled system (spin-orbit), i.e., with $\hat{q} = 0$.

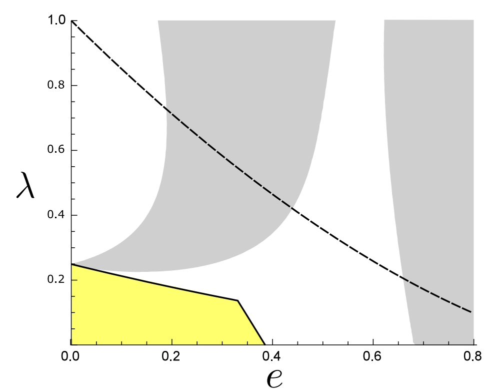

Figure 3: Stability diagram in the (e, λ)-plane of the spin-orbit in [29], Figure 3.

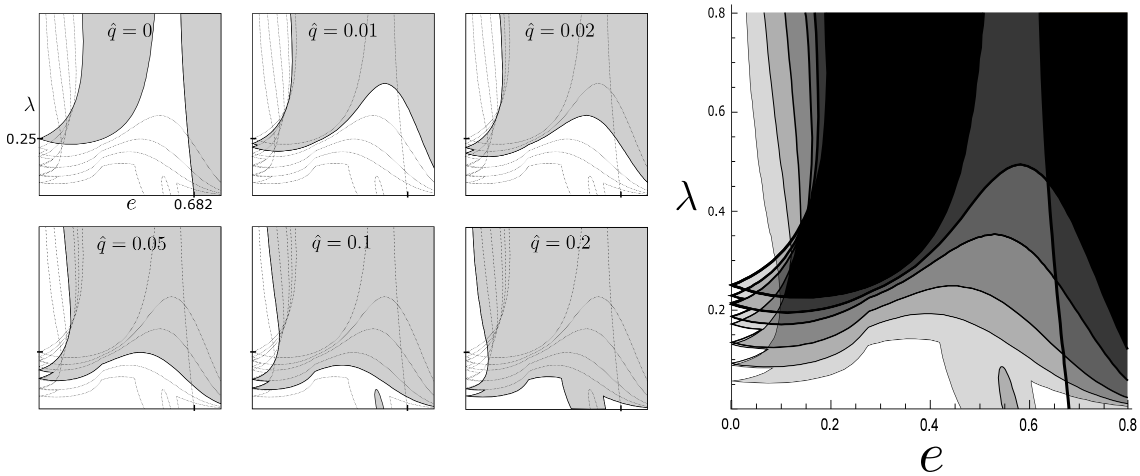

Figure 4: Stability diagrams in the (e, λ)-plane in the case of equal bodies. The six plots in the left show the unstable region in gray for different values of $\hat{q}$. The image in the right shows the six diagrams superimposed. Darker tones of gray indicate more overlapping between unstable regions.

was studied in [29]. However, for $\hat{q} > 0$, it looks that the last bifurcation point moves to the right, at the same time that the two instability regions merge into a single one. This shows that turning on the coupling has a stabilizing effect of the synchronous resonance for large e and small λ. This holds up to a critical $\hat{q} \in (0.05, 0.1)$ for which another unstable region bifurcates from the e-axis. This region merges with the large one at some $\hat{q} \in (0.1, 0.2)$. This leaves an island of stability for large e and small λ.

3. In the right panel of Figure 4 we see that there are some regions (the darkest ones), that remain unstable, not very affected by changes in $\hat{q}$. Instead, the lighter regions show more susceptibility to change their stability when $\hat{q}$ changes.

In Figure 4 we have taken large values of $\hat{q}$, compared to the real values in Table 1. From its definition in (17) and (14) we see that $\hat{q} \leq 1/a^2$ for equal bodies ($M = 0.5, \mathcal{C} = 0.5$). In order to be consistent with the Keplerian orbit approximation, a should be quite larger than 1, that gives the scale of the objects. For example, a of order 10 would give an upper estimate of $\hat{q}$ of order 10^{-2}. In consequence, for more realistic parameters, we should not consider the appearance of the additional instability region bifurcating from the e-axis from large $\hat{q}$.

6 Discussion

In this paper we have proposed a simplified mathematical model for the rotational dynamics in the Full Two-Body Problem. This model is a straightforward continuation of the spin-orbit problem. In consequence, we hope it will be of interest for physical applications as well as for theoretical studies. We have approached the problem from a theoretical point of view, but always keeping what we think is the essence of the physical problem: the dissipative effects are fundamental to explain the universe we observe today. In this sense, the spin-spin model not only broaden the scope of the spin-orbit problem in a higher dimensional phase space, but also contributes to fill the gap between the conservative and the dissipative effects considered in the spin-orbit problem. More precisely, if the *dissipative* torque (of order $1/r^6$) is important in the evolution of a satellite, then, we should consider also the spin-spin interaction (of order $1/r^5$). Of course this two effects are more important when the bodies are closer to each other. In fact, in the spin-spin model the strength of the terms of order $1/r^5$ is given by parameters that compare the shape of the bodies with the size of the orbit, say, $\hat{d}_j$ and $\hat{q}_j$. In contrast, the spin-orbit problem only regards the equatorial oblateness of the satellite $d_j/\mathcal{C}_j$. It is reasonable to think that the different types of interactions, say, point-point, spin-orbit and spin-spin, must have their own specific relevance in different ranges of parameters. This shows that the non-Keplerian behavior of the full Lagrangian model (11), (12), should be investigated more deeply. Here the full expansion of the potential energy, given in (56), may also play a role. Moreover, as [14] shows, non-planar oscillations around solutions of the planar problem can be studied and are of practical interest.

In the present research, we have made a brief theoretical study that allowed us to point out the importance of the double synchronous resonance and compare it with the synchronous resonance of the spin-orbit problem. Particularly, in a similar way than [29], we determine sufficient conditions for the existence of an asymptotically stable periodic solution (capture into resonance). Besides, note that our estimates do not pretend to be optimal at all. Instead, we illustrate a way to extend to the spin-spin model the tools used for the spin-orbit model, as well as to compare them. Furthermore, in this sense we have included some numerical diagrams of linear stability in Figure 4 that show us how the spin-spin interaction alters the schemes of the spin-orbit model.

We have applied our study to two real systems in double synchronous resonance. In one hand, Pluto and Charon are representative of a large binary with one body much larger than the other one, see [15]. On the other hand, the binary asteroid 617 Patroclus is an archetype of a small system of similar components, see [14], [25]. Here we propose a way how to make an effective comparison between different systems. Note that the convenient choice of units and parameters helps to clarify the comparison. As we expected, the best candidates to apply the spin-spin model are binary asteroids. They are very abundant in the solar system, e.g., about 15% of the near-Earth asteroids are thought to be binaries. For a detailed discussion on the applications of the general spin-spin model and its full Lagrangian version, we refer to [3] and the bibliography therein. With our study on the double synchronous resonance we hope to contribute to the study of the spin-spin resonances made in [3]. Whereas they focus on the synchronization of both spins for slow circular orbital motion ($\dot{f} \ll \dot{\theta}_j$), we consider the full synchronization including the orbit with arbitrary eccentricity. According to [14], most of the equal mass binaries are expected to be in the double synchronous state. In [30], Section 4.14, they provide a formula for a critical mass ratio of the components for this state to be possible. We want to remark also that, apart from the application to binary asteroids and large natural satellites, the spin-spin interaction can be relevant for artificial satellites whose rotation state along an orbit is important. For instance, communication satellites in equatorial orbits or even spacecraft exploring small bodies.

Finally, we think that the theoretical interest of the model is large, even beyond the phenomena already observed in the spin-orbit problem. For example, in the spin-orbit problem we can apply the notion of KAM stability because KAM tori confine regions in the phase space. However this does not happen in the spin-spin model due to the increase in the phase space dimension (two degrees of freedom and time dependence). In fact, it is expected that Arnold diffusion takes place in this case. In general, the weak coupling and the Hamiltonian character of the system makes it suitable to apply perturbative techniques. Particular questions may be investigated, such as chaos by overlapping of resonances, stochastic phenomena, normally

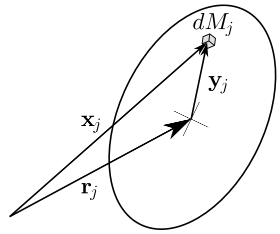

Figure 5: Position vectors of the element of mass of an ellipsoid.

hyperbolic manifolds, scattering maps, among other phenomena, see [10].

A Units

If t, M and l stand for time, mass and length respectively, the relation between our system of units and any other one is the following

$$t_{\text{ours}} = \frac{2\pi}{T} t, \quad M_{\text{ours}} = \frac{M}{M_1 + M_2}, \quad l_{\text{ours}} = l\sqrt{\frac{M_1 + M_2}{\mathcal{C}_1 + \mathcal{C}_2}}.$$

It is worth mentioning that, if I is any magnitude with units of moment of inertia, then the conversion is given simply by

$$I_{\text{ours}} = \frac{I}{\mathcal{C}_1 + \mathcal{C}_2}.$$

The value of the gravitational constant G in any system of units must respect Kepler's third law (4).

B Derivation of the potential of the spin-spin problem

B.1 Potential of the Full Two-Body Problem

The expansion of the potential energy in the Full Two-Body Problem has been obtained in several papers, see [34] for example. In this subsection, and in order to introduce some notation, we present a short derivation of the spherical harmonics expansion, following the approach of [23] and [6]. See also a similar approach in [26] and [12]. We start from the formula

$$V = -G \int\int \frac{\mathrm{d}M_1(\mathbf{x}_1)\, \mathrm{d}M_2(\mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|},$$

where each $\mathbf{x}_j \in \mathbb{R}^3$ is the position vector (with respect to the barycenter of the system) of the mass element $\mathrm{d}M_j(\mathbf{x}_j)$ corresponding to the ellipsoid $\mathcal{E}_j$. Making the change of variables $\mathbf{y}_j = \mathbf{x}_j - \mathbf{r}_j$, illustrated in Figure 5, and defining $\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$, we obtain

$$V = -G \int\int \frac{dM_1(\mathbf{y}_1)\, dM_2(\mathbf{y}_2)}{|\mathbf{r} - \mathbf{y}|}.$$

Recall that $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The usual expansion in spherical harmonics gives us

$$V = -G \sum_{(l,m)\in\Upsilon} Q_{l,m} \frac{Y_{l,m}(\hat{\mathbf{r}})}{|\mathbf{r}|^{l+1}}, \tag{49}$$

where

$$\Upsilon = \{(l,m) \in \mathbb{Z}^2 :\ 0 \le |m| \le l\}$$

and the multipolar moments of the system $Q_{l,m}$ are defined by

$$Q_{l,m} = \int\int |\mathbf{y}|^l \bar{Y}_{l,m}(\hat{\mathbf{y}})\,\mathrm{d}M_1(\mathbf{y}_1)\ \mathrm{d}M_2(\mathbf{y}_2), \tag{50}$$

where the upper bar indicates complex conjugation. We use the Schmidt semi-normalization[3] of the spherical harmonics in the same way as in [6]. Assume that, in the inertial frame, $\mathbf{r}$ has spherical coordinates (r, ϑ, ϕ), then, the spherical harmonics are defined by

$$Y_{l,m}(\vartheta,\phi) = (-1)^m \sqrt{\frac{(l-m)!}{(l+m)!}} P_{l,m}(\cos\vartheta)\exp(im\phi),$$

where the associated Legendre polynomials are given by

$$P_{l,m}(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l, \quad x \in [-1,1].$$

Note that, since $\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$, we cannot factorize the integral in (50) into factors that involve quantities associated to each body separately. However we can express this integral as a sum of factorized terms. For this we can define the auxiliary normalized solid harmonics

$$\mathcal{Y}_{l,m}(\mathbf{x}) = \frac{|\mathbf{x}|^l Y_{l,m}(\hat{\mathbf{x}})}{\sqrt{(l-m)!(l+m)!}}, \quad \mathbf{x} \in \mathbb{R}^3,$$

and apply the translation formula, given in equation (313) in [33],

$$\mathcal{Y}_{l,m}(\mathbf{y}_1 - \mathbf{y}_2) = \sum_{\lambda_1,\mu_1}\sum_{\lambda_2,\mu_2} \mathcal{Y}_{\lambda_1,\mu_1}(\mathbf{y}_1)\mathcal{Y}_{\lambda_2,\mu_2}(-\mathbf{y}_2),$$

where λ_j and μ_j are integers running all the values such that

$$0 \le \lambda_j \le l, \quad \lambda_1 + \lambda_2 = l; \quad -\lambda_j \le \mu_j \le \lambda_j, \quad \mu_1 + \mu_2 = m.$$

Then, using the parity relation $Y_{l,m}(-\hat{\mathbf{x}}) = (-1)^l Y_{l,m}(\hat{\mathbf{x}})$, the expression (50) becomes

$$\frac{Q_{l,m}}{\sqrt{(l-m)!(l+m)!}} = \sum_{\lambda_1,\mu_1}\sum_{\lambda_2,\mu_2}(-1)^{\lambda_2}\frac{M_1 R_1^{\lambda_1} Z^{1)}_{\lambda_1,\mu_1}}{\sqrt{(\lambda_1-\mu_1)!(\lambda_1+\mu_1)!}}\frac{M_2 R_2^{\lambda_2} Z^{2)}_{\lambda_2,\mu_2}}{\sqrt{(\lambda_2-\mu_2)!(\lambda_2+\mu_2)!}}, \tag{51}$$

where, the complex Stokes coefficients[4] of each ellipsoid are given by

$$Z^{j)}_{\lambda,\mu} = \frac{1}{M_j R_j^\lambda}\int |\mathbf{y}_j|^\lambda \bar{Y}_{\lambda,\mu}(\hat{\mathbf{y}}_j)\,\mathrm{d}M_j(\mathbf{y}_j), \tag{52}$$

[3] With this choice, the Legendre polynomials can be written in terms of the spherical harmonics as

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}) = \sum_{m=-l}^{l} Y_{l,m}(\hat{\mathbf{r}})\bar{Y}_{l,m}(\hat{\mathbf{y}}).$$

[4] The quantities $Z^{j)}_{l,m}$ provide the expansion of the potential created for the body $\mathcal{E}_j$. They are related to the usual parameters $C^{j)}_{l,m}$ and $S^{j)}_{l,m}$ by

$$C^{j)}_{l,m} + iS^{j)}_{l,m} = (-1)^m \frac{2}{1+\delta_{m,0}}\sqrt{\frac{(l-m)!}{(l+m)!}}\bar{Z}^{j)}_{l,m}, \quad m \ge 0,$$

where $\delta_{m,n}$ is the Kronecker delta.

and R_j is the mean radius of $\mathcal{E}_j$.

Finally, since in the potential energy the summation range is $0 \le l \le \infty$, $-l \le m \le l$, which are all the possible terms, then, from (49) and (51) we can write

$$V = -\frac{GM_1M_2}{|\mathbf{r}|} \sum_{\substack{(\lambda_1,\mu_1)\in\Upsilon \\ (\lambda_2,\mu_2)\in\Upsilon}} (-1)^{\lambda_2} \gamma^{\lambda_1,\mu_1}_{\lambda_2,\mu_2} \left(\frac{R_1}{|\mathbf{r}|}\right)^{\lambda_1} \left(\frac{R_2}{|\mathbf{r}|}\right)^{\lambda_2} Z^{1)}_{\lambda_1,\mu_1} Z^{2)}_{\lambda_2,\mu_2} Y_{\lambda_1+\lambda_2,\mu_1+\mu_2}(\hat{\mathbf{r}}), \qquad (53)$$

where we defined the constants

$$\gamma^{\lambda_1,\mu_1}_{\lambda_2,\mu_2} = \sqrt{\frac{(\lambda_1+\lambda_2-\mu_1-\mu_2)!(\lambda_1+\lambda_2+\mu_1+\mu_2)!}{(\lambda_1-\mu_1)!(\lambda_1+\mu_1)!(\lambda_2-\mu_2)!(\lambda_2+\mu_2)!}}.$$

B.2 Potential of the ellipsoidal spin-spin model

Note that the terms in the expansion (53), and in particular $Z^{j)}_{\lambda,\mu}$, have to be computed with respect to the inertial frame. Let us call $\mathcal{E}_j$-frame to the fixed body frame of each ellipsoid, formed by its center and its principal directions associated respectively to a_j, b_j and c_j. Let $\mathcal{Z}^{j)}_{\lambda,\mu}$ be the Stokes coefficients computed with respect to the $\mathcal{E}_j$-frame. The $\mathcal{E}_j$-frame is rotated, with respect to the inertial frame, with the rotation labelled by the Euler z-y-z angles $(\alpha,\beta,\gamma) = (\theta_j,0,0)$.

Let $\mathbf{x} \in \mathbb{R}^3$ be a vector with spherical coordinates $(|\mathbf{x}|, \vartheta_j, \phi_j)$ with respect to the $\mathcal{E}_j$-frame and $(|\mathbf{x}|, \vartheta, \phi)$ with respect to the reference frame formed by the center of the body $\mathcal{E}_j$ and the directions parallel to those of the inertial frame. The relation between spherical harmonics $Y_{l,m}(\hat{\mathbf{x}})$ computed with respect to both systems of reference is the following

$$Y_{l,m}(\vartheta_j,\phi_j) = \sum_{m'=-l}^{l} Y_{l,m'}(\vartheta,\phi) \bar{D}^l_{m,m'}(\alpha,\beta,\gamma)$$

where $D^l_{m,m'}(\alpha,\beta,\gamma)$ is the (m,m')-element of the Wigner D-matrix associated to the rotation given by the Euler z-y-z angles (α,β,γ), see [33]. Then, from (52),

$$Z^{j)}_{\lambda,\mu} = \sum_{\mu'=-\lambda}^{\lambda} D^{\lambda}_{\mu,\mu'}(\alpha,\beta,\gamma)\ \mathcal{Z}^{j)}_{\lambda,\mu'}.$$

From the definition of the Wigner D-matrices, see for instance equation (186) in [33], in our planar case they are diagonal $D^{\lambda}_{\mu,\mu'}(\theta_j,0,0) = \delta_{\mu,\mu'} \exp(-i\mu'\theta_j)$, where $\delta_{\mu,\mu'}$ is the Kronecker delta. Then,

$$Z^{j)}_{\lambda,\mu} = \mathcal{Z}^{j)}_{\lambda,\mu} \exp(-i\mu\theta_j).$$

Now we can express (53) in terms of $\mathcal{Z}^{j)}_{\lambda,\mu}$. In [2] an expansion of the potential created by a homogeneous ellipsoid was computed. Incidentally, a complicated general expression for $\mathcal{Z}^{j)}_{\lambda,\mu}$ was computed there as well. In the next Proposition we summarize some remarkable properties of those quantities.

Proposition 3 *Let $\mathcal{Z}_{\lambda,\mu}$ be Stokes coefficients of an homogeneous ellipsoid computed in its own fixed body frame. They have the following properties*

1. $\mathcal{Z}_{\lambda,\mu} \in \mathbb{R}$.

2. $\mathcal{Z}_{\lambda,\mu} \equiv 0$ *if either* λ *or* μ *are odd numbers.*

3. $\mathcal{Z}_{\lambda,-2n} = \mathcal{Z}_{\lambda,2n}$*, with* n *integer.*

We will not reproduce the whole proof here but it can be found in [28]. We just want to remark that it is based on the symmetry properties of the spherical harmonics and the geometrical symmetries of the ellipsoids.

Remark 2 *Regarding these properties, a convenient expression to compute numerically* $\mathcal{Z}_{2k,2n}$*, with* $k \geq 0$ *and* n *integers, is*

$$\mathcal{Z}_{2k,2n} = \frac{3}{4\pi R^{2k}}\sqrt{\frac{(2k-2n)!}{(2k+2n)!}}\int_{\mathsf{B}} \mathrm{Re}((\mathsf{a}Z - i\mathsf{b}Y)^{2n})\frac{[(\mathsf{a}X)^2+(\mathsf{b}Y)^2+(cZ)^2]^k}{[(\mathsf{a}X)^2+(\mathsf{b}Y)^2]^n}\times$$
$$\times P_{2k,2n}\left(\frac{\mathsf{c}Z}{\sqrt{(\mathsf{a}X)^2+(\mathsf{b}Y)^2+(\mathsf{c}Z)^2}}\right)\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z,$$

where R *is the mean radius of the ellipsoid,* a*,* b *and* c *are its principal semi-axes,* Re *indicates the real part and* B *is the unit ball, defined by* $X^2+Y^2+Z^2 \leq 1$*. Moreover,* $\mathcal{Z}_{2k,2n}$ *can be written only in terms of* M *and the principal moments of inertia because*

$$\mathsf{a} = \sqrt{\frac{5(-\mathcal{A}+\mathcal{B}+\mathcal{C})}{2M}},\quad \mathsf{b} = \sqrt{\frac{5(\mathcal{A}-\mathcal{B}+\mathcal{C})}{2M}},\quad \mathsf{c} = \sqrt{\frac{5(\mathcal{A}+\mathcal{B}-\mathcal{C})}{2M}}.$$

Recalling the definitions of q *and* d *in (14), the first non-vanishing Stokes coefficients are given by*

$$\mathcal{Z}_{0,0} = 1, \qquad \mathcal{Z}_{2,0} = -\frac{1}{2}\frac{q}{MR^2}, \qquad \mathcal{Z}_{2,2} = \sqrt{\frac{3}{8}}\,\frac{d}{MR^2}, \tag{54}$$

$$\mathcal{Z}_{4,0} = \frac{15}{56}\frac{d^2+2q^2}{M^2R^4}, \qquad \mathcal{Z}_{4,2} = -\frac{15}{28}\sqrt{\frac{5}{3}}\,\frac{d\,q}{M^2R^4}, \qquad \mathcal{Z}_{4,4} = \frac{15}{8}\sqrt{\frac{5}{14}}\,\frac{d^2}{M^2R^4}, \tag{55}$$

and it seems that, in general, $\mathcal{Z}_{2k,2n}$ *has the form of a homogeneous polynomial of degree* k *with respect to* $q/(MR^2)$ *and* $d/(MR^2)$*.*

In order to simplify expression (53), recall that $\mathbf{r}$ is the vector pointing from the center of $\mathcal{E}_1$ to the center of $\mathcal{E}_2$. Then, the spherical coordinates of $\mathbf{r}$ with respect to the inertial frame are $(r, \vartheta = \pi/2, \phi = f)$. The non-vanishing terms of (53) are such that $\lambda_j = 2l_j$ and $\mu_j = 2m_j$. Let us call from now on $l = l_1 + l_2$ and $m = m_1 + m_2$. We can apply the formula

$$Y_{2l,2m}(\pi/2, f) = \sqrt{\frac{(2l-2m)!}{(2l+2m)!}}P_{2l,2m}(0)e^{2imf},$$

and the following property of the associated Legendre polynomials

$$P_{2l,2m}(0) = \frac{(-1)^{l-m}}{4^l}\frac{(2l+2m)!}{(l-m)!(l+m)!},$$

see for instance equation (68) in [33]. Then, we can write the potential keeping only the real part of V, so that the final expression potential is

$$V = -\frac{GM_1M_2}{r} \sum_{\substack{(l_1,m_1)\in\Upsilon \\ (l_2,m_2)\in\Upsilon}} \Gamma^{l_1,m_1}_{l_2,m_2} \left(\frac{R_1}{r}\right)^{2l_1} \left(\frac{R_2}{r}\right)^{2l_2} \mathcal{Z}^{1)}_{2l_1,2m_1} \mathcal{Z}^{2)}_{2l_2,2m_2} \cos(2m_1(\theta_1 - f) + 2m_2(\theta_2 - f)), \tag{56}$$

where

$$\Gamma^{l_1,m_1}_{l_2,m_2} = \frac{(-1)^{l-m}}{4^l\sqrt{(2l_1-2m_1)!(2l_1+2m_1)!(2l_2-2m_2)!(2l_2+2m_2)!}} \frac{(2l-2m)!(2l+2m)!}{(l-m)!(l+m)!}. \tag{57}$$

The first terms of the expansion (56) can be computed using (54) and (55). The terms corresponding to $l = l_1 + l_2$, for $l = 0, 1$ and 2, are shown in (13).

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