A dissipative Kepler problem with a family of singular drags

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Abstract

In this work we consider the Kepler problem with a family of singular dissipations of the form $-\frac{k}{|x|^\beta}\dot{x},\ \ k>0, \beta>0$. We present some results about the qualitative dynamics as β increases from zero (linear drag) to infinity. In particular, we detect some threshold values of β, for which qualitative changes in the global dynamics occur. In the case $\beta = 2$, we refine some results obtained by Diacu and prove that, unlike for the case of the linear drag, the asymptotic Runge-Lenz vector is discontinuous.

Keywords: Kepler problem, drag, singularity, asymptotic Runge-Lenz vector

1 Introduction

The aim of this paper is to investigate the changes that occur in the forward dynamics of a Kepler problem with the family of singular dissipative forces $F_\beta(x,\dot{x}) := -\frac{k}{|x|^\beta}\dot{x}$, as the positive parameter β increases. We assume that k is a fixed positive number and $(x,\dot{x}) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$. The corresponding equation is

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$$\ddot{x} + \frac{k}{|x|^\beta}\dot{x} = -\frac{x}{|x|^3}. \tag{1}$$

In general, dissipative forces in celestial mechanics are crucial for the understanding of the time-asymptotic motion of celestial bodies, and they are also important during planetary system formation (see [9] for a comprehensive survey on the topic). These forces may be due to the motion of a body in a fluid, as it is the case for a satellite moving in the Earth's atmosphere ([2]), or for a very small particle of dust moving in a gas ([9]). It is usual to take these forces proportional to the velocity, as, for example, in equation (1). In particular, the case $\beta = 0$ corresponds to the so-called Stokes drag or linear drag, which is valid for homogeneous viscous fluids and small Reynolds numbers.

Dissipation may also be of electromagnetic nature, due for instance to re-radiation of photons by small bodies ([3], [9]). In this class one can find the dissipation which corresponds to equation (1) with $\beta = 2$, called Poynting-Plummer-Danby (PPD) drag ([1], [7]) or, sometimes, Poynting-Robertson drag ([9]).

The linear drag and the PPD drag generate two very different global dynamics, whose main features we recall below. Some of our results will provide insights on the role of the parameter β in the transition between them.

We note that (1) is a member of the larger class of dissipative Kepler problems considered by Poincaré in [20], namely

$$\ddot{x} + P_{\alpha,\beta}(x,\dot{x})\frac{\dot{x}}{|\dot{x}|} = -\frac{x}{|x|^3}, \ \ x \in \mathbb{R}^2 \setminus \{0\}, \tag{2}$$

where $P_{\alpha,\beta}(x,\dot{x}) = h|x|^{-\beta}|\dot{x}|^\alpha$, h is a positive fixed real number, and the parameters α and β are positive. Essentially, Poincaré found that orbits with negative energy spiral towards the singularity with increasing velocity[1]. Moreover, for α and β sufficiently large, after each revolution their eccentricity decreases, leading to the circularization of orbits. Poincaré presented also a qualitative argument supporting that this effect occurs for a general dissipative force. However, it tuns out that such circularization of orbits does not take place for the linear drag ($\alpha = 1$, $\beta = 0$) and the PPD drag ($\alpha = 1, \beta = 2$).

In the first case, the results presented in [13] and [14] show that, although some orbits circularize as they spiral down toward the singularity, for an open set of initial conditions, the value of the eccentricity of the corresponding orbit converges to a positive constant, being all values in $]0,1[$ attainable. Geometrically, these trajectories are spirals made of asymptotically self similar ellipses which shrink to the singularity. Moreover, it turns out that these spirals are traveled with an angular velocity which increases exponentially with time. These results are obtained from the existence and continuity on the phase space of a first integral I, defined as the limit along the solutions of the Runge-Lenz vector. We recall that this vector, also called eccentricity vector, is a first integral of the conservative Kepler problem, and that it defines the type of conic section corresponding to the orbit (its modulus is the eccentricity of the orbit) as well as its orientation (it is parallel to the axis containing the focus). Then, we can think of I as an asymptotic eccentricity vector.

In [14] it is proved that the range of I is the closed unit disk in the plane. This property expresses that all the non rectilinear orbits are of elliptic type, meaning that, eventually, their energy becomes negative. This last fact is stated in [12], where it is also established that the

[1] This fact had already been mentioned by Euler when discussing the motion of a planet in a resistive medium, see [8].

singularity is a global attractor, reached in infinite time by non rectilinear motions and in finite time by rectilinear ones. In each case, it is proved that the velocity tends to infinity.

The fact that the linear drag does not circularize orbits has been observed previously in [10].

In the case of the PPD drag (see [1], [3], [6], [7]) all the orbits which tend to the singularity spiral only a finite number of times around it and achieve an asymptotic direction. This last property implies that their eccentricity tends to one. Moreover, all collisions occur in finite time and with finite velocity. Also, for this drag there exist solutions which escape to infinity. Essentially, these results are obtained in [7] by means of a qualitative study of (1). The fact that collision orbits are asymptotically rectilinear was previously observed in [1], where a more a general class of drags is treated using a suitable transformation (called generalized Robertson transformation) to find explicit analytic solutions. The first step of such transformation is the Binet change of variables, exploited in [15], [17] to transform (1) with $\beta = 2$ into a forced harmonic oscillator, so obtaining a closed form for the orbit equation.

In Section 4 we show that a careful analysis of such orbit equation allows to find another interesting difference between the linear drag and the PPD drag. More precisely, we prove in Theorem 8 that for the PPD drag the first integral I, although well defined on the whole phase space, presents jump discontinuities on the set of parabolic solutions (solutions which escape to infinity with energy tending to zero). The discontinuity arises as we cross this set going from collision solutions to hyperbolic ones (solutions which escape to infinity with positive energy). In Theorem 8, we show also that the range of I is the exterior of the open unit disk in the plane.

In Section 3 we present an analysis of the dynamics as β increases. In particular, we detect some thresholds for different global behaviors, for non rectilinear motions as well as for rectilinear ones.

We show that the global attractiveness of the singularity and the unboundedness of the angular velocity of solutions, which hold for $\beta = 0$, can be continued, respectively, for $\beta \in]0, 1]$ and for $\beta \in]0, 1[$ (see, respectively, Theorem 1 and Theorem 5). These results suggest that, when $\beta \in]0, 1[$, all solutions collide with the singularity winding faster and faster infinite times around it, as they do in the case of the the linear drag. In Theorem 1 we also prove that escape solutions exist for any $\beta > 1$, and in Theorem 3 we show that the variation of their polar angle must be less than 2π.

The case $\beta \in [\frac{3}{2}, +\infty[$ is addressed in Theorem 6, where we give a fairly complete description of the qualitative dynamics of collision orbits. Namely, we show that when $\beta > \frac{3}{2}$ such orbits are asymptotically rectilinear, a behavior that, as already mentioned above, was observed for $\beta = 2$ in [1] and [7]. Moreover, we prove that the approach to zero occurs in finite time if and only if $\beta \in]\frac{3}{2}, 3[$, and that $\beta = 2$ is the threshold for the value of the terminal velocity. When β crosses this value, the terminal velocity passes from $-\infty$ to 0. We extended these results to $\beta = \frac{3}{2}$, but only imposing that $k > 2\sqrt{2}$. This restriction is related to the fact that $\beta = \frac{3}{2}$ is the only case in which it is not possible to rescale the variables so as to eliminate k from equation (1). We do not know if this constraint on k is just a technical condition which arises due to our technique of proof, or if it reflects some deeper aspects of the dynamics.

Unfortunately, we could not provide any result about the rotational properties, collision time or terminal velocity of non rectilinear solutions when $\beta \in [1, \frac{3}{2}[$. However, in Section 5 we present some conjectures about the dynamics at collision for $\beta \in]0, \frac{3}{2}[$.

Still in Section 3, we present two results valid for any $\beta > 0$ (actually, they hold for more general dissipations, see Remark 1). In the first, Theorem 2, we prove that escape and collision solutions are the only kind of solutions that can occur for (1). In the second, Theorem 4, we show that non rectilinear collision orbits of (1) are such that their angular momentum tends to zero and their energy tends to $-\infty$.

Concerning rectilinear collisions, in Theorem 7 we present a complete description, as β increases, of their collision time (discussing whether it is finite or not), and of the asymptotic behavior of their velocity and energy. We postpone its proof to the Appendix. Theorem 7 implies that the asymptotic expansions of solutions around the collision time given for $\beta = 0$ in [12] still hold when $\beta \in]0, \frac{1}{2}[$. However, we show that, unlike for the case $\beta = 0$, the presence of the singularity in the dissipation does not allow to regularize collisions by means of a Levi-Civita type transformation (see Remark 3).

Some preliminary facts and notations are established in Section 2.

2 Some preliminaries

In this section we present some preliminary considerations and establish some notation for the dissipative Kepler problem (1). We rewrite equation (1) in an equivalent form as the first order system

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\dfrac{k}{|x|^\beta} v - \dfrac{x}{|x|^3}, \end{cases} \tag{3}$$

defined in the phase space $\Omega = (\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}^3$. Given an initial condition $(x_0, v_0) \in \Omega$, the unique maximal solution of (3) such that $x(0) = x_0$ and $v(0) = v_0$ will be denoted by $(x(t), v(t))$ or by $\phi^t(x_0, v_0)$, where ϕ^t is the flow of the system (3). The corresponding interval of definition will be indicated by $]\alpha, \omega[$.

The energy E and the angular momentum $\mathcal{M}$, defined respectively by

$$E(x, v) = \frac{1}{2}|v|^2 - \frac{1}{|x|}, \qquad \mathcal{M}(x, v) = x \wedge v,$$

are no longer conserved quantities for (3), since their derivatives along any solution of (3) satisfy

$$\dot{E}(t) = -k\frac{|v(t)|^2}{|x(t)|^\beta}, \qquad \dot{\mathcal{M}}(t) = -\frac{k}{|x(t)|^\beta}\mathcal{M}(t). \tag{4}$$

It follows that the energy is strictly decreasing along the solutions of (3), and that the angular momentum of any solution satisfies

$$\mathcal{M}(t) = \mathcal{M}_0 e^{-\int_0^t \frac{k}{|x(s)|^\beta} ds}, \quad \mathcal{M}_0 := x_0 \wedge v_0. \tag{5}$$

Then, either $\mathcal{M}(t) \equiv 0$ on $]\alpha, \omega[$, and the corresponding orbit is rectilinear, or $\mathcal{M}(t) \neq 0$ on $]\alpha, \omega[$, and the corresponding orbit is planar since $\frac{\mathcal{M}(t)}{|\mathcal{M}(t)|}$ is a conserved vector. In this second case, $|\mathcal{M}(t)|$ is strictly decreasing.[2]

Due to the invariance of our problem with respect to the group of isometries of $\mathbb{R}^3$, we can study the dynamics of (3) in the phase space $\Omega = \{(x, v) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}\}$, where we have

[2] Jacobi, in his book on mechanics [11], had already a considered the dissipative Kepler problem corresponding to (2) with drag $P_{1,\beta}$, finding that the motions are planar and have decreasing energy and decreasing scalar angular momentum. We note that these properties actually hold for any dissipation opposite to the velocity.

set $x = x_1 + ix_2$, $v = v_1 + iv_2$. Moreover, denoting by $M = x_1v_2 - x_2v_1$ the scalar angular momentum of a solution, in order to study the rotational properties of non rectilinear motions, it will be sufficient to restrict ourselves to the set $\Omega^+ = \{(x, v) \in \Omega : M > 0\}$. The manifold $\Omega^0 = \{(x, v) \in \Omega : M = 0\}$ corresponds to rectilinear motions.

We rewrite now system (3) using polar coordinates in $\mathbb{C} \setminus \{0\}$. Considering the change of variables $x = re^{i\theta}$ we see that the new coordinates satisfy the system

$$\begin{cases} \dot{r} = u \\ \dot{u} = r\varphi^2 - k\dfrac{u}{r^\beta} - \dfrac{1}{r^2} \\ \dot{\varphi} = -\dfrac{k + 2ur^{\beta-1}}{r^\beta}\varphi, \end{cases} \tag{6}$$

where $\varphi = \dot{\theta} \geq 0$ and $u \in \mathbb{R}$. Of course, motions in Ω^+ will correspond to $\varphi > 0$, whereas the equality $\varphi = 0$ singles out the rectilinear motions in Ω^0. When dealing with rectilinear motions in Subsection 3.4, we will identify Ω^0 with the set $\{(r, u) : r > 0, u \in \mathbb{R}\}$.

In what follows, we will consider just the forward dynamics of (1). Accordingly, all the solutions will be considered on their right maximal interval $[0, \omega[$.

Throughout the paper, the subscript ω attached to a time dependent function will denote the limit of that function as $t \to \omega^-$. For simplicity, we will generally omit the dependence of such limit on initial conditions (x_0, v_0). For example, since the energy and the scalar angular momentum are decreasing along the solutions of (1) we will write

$$E_\omega = \lim_{t \to \omega^-} E(\phi^t(x_0, v_0)) \in [-\infty, E(0)] \quad \text{and} \quad M_\omega = \lim_{t \to \omega^-} M(\phi^t(x_0, v_0)) \in [0, M(0)].$$

Throughout the paper we will make use of the following definition.

Definition 1 *A solution of* (1) *defined on the right maximal interval* $[0, \omega[$ *is called a collision solution if*

$$\lim_{t \to \omega} x(t) = 0. \tag{7}$$

We will say that the collision occurs in finite time if ω is finite, and that it occurs in infinite time if $\omega = +\infty$.

3 Forward dynamics

3.1 A threshold for the existence of escape orbits and non existence of oscillatory ones

In this subsection we first address the problem of the existence of escape orbits for (1). From [12] we know that escapes do not exist for $\beta = 0$, since in this case the singularity is a global attractor. However, the influence of the singularity at infinity becomes weaker and weaker as β increases, and escapes are expected to exist when β crosses some threshold value. The results in [7] show that escape rectilinear orbits exist for $\beta = 2$, implying that such threshold is less than or equal to 2. In Theorem 1 below we show that the threshold is $\beta = 1$. We also show, in Theorem 2, that no oscillatory solutions exist for (1). We first give the following auxiliary lemma. We point out that this result holds for ω finite or infinite.

Lemma 1 *For any $\beta > 0$, let $x(t)$ be a solution of* (1) *defined on the right maximal interval* $[0, \omega[$.

i) If $\liminf_{t\to\omega^-} |x(t)| < +\infty$, *then*

$$\liminf_{t\to\omega^-} |x(t)| = 0. \tag{8}$$

ii) If $x(t)$ *is bounded on* $[0,\omega[$, *then it is a collision solution.*

Proof. To prove $i)$, we argue by contradiction. Assume that there exists a positive real number δ_* such that

$$\liminf_{t\to\omega^-} |x(t)| = 2\delta_*. \tag{9}$$

Then, there exists a sequence $\{t_n\} \subset [0,\omega[$ that satisfies $t_n \to \omega$, $|x(t_n)| \to 2\delta_*$, and

$$|x(t_n)| \geq \delta_*, \quad \text{for any } n. \tag{10}$$

Since the energy is decreasing along solutions, from (10) we have

$$\frac{|v(t_n)|^2}{2} \leq E(0) + \frac{1}{|x(t_n)|} \leq E(0) + \frac{1}{\delta_*}, \tag{11}$$

for any n. It follows that there exists $(x_*, v_*) \in \Omega$ which is a limit point of $(x(t_n), v(t_n))$. By the general theory of ODEs and the maximality of $(x(t), v(t))$, we conclude that $\omega = +\infty$. Then, we can apply the extension of La Salle's principle to singular systems given in [14], Proposition 2.2, with $V = E$, and conclude that

$$L_\omega(x,v) \cap \Omega = \emptyset, \tag{12}$$

where $L_\omega(x,v)$ is the ω-limit set of the solution $t \mapsto (x(t), v(t))$. Since $(x_*, v_*) \in L_\omega(x,v) \cap \Omega$ we get a contradiction, and our proof of $i)$ is concluded.

To get $ii)$ we start by noticing that, if $\omega = +\infty$, the conclusion follows immediately as in [14]. If $\omega < +\infty$, we show that

$$\limsup_{t\to\omega^-} |x(t)| = 0, \tag{13}$$

arguing by contradiction. Assume that (13) does not hold, and let $2\delta^* > 0$ be the value of the upper limit. Then, there exists a sequence $\{t_n\} \subset [0,\omega[$ converging to ω as $n \to \infty$, such that $|x(t_n)| \to 2\delta^*$ and $|x(t_n)| \geq \delta^*$. Now, the same argument already used in $i)$ leads to the contradiction that $\omega = +\infty$. Our proof is concluded. ■

Theorem 1 *If* $0 < \beta \leq 1$ *all the orbits of* (1) *tend to the singularity, whereas if* $\beta > 1$ *there are also escape orbits. Escapes occur in infinite time with a finite velocity, which can have an arbitrarily large modulus, and with non-negative finite energy.*

Proof. We start the proof of the first claim by showing that for $0 < \beta \leq 1$ all the solutions are bounded in the future.

Let $w = |\dot{x}|$ and consider the following family of functions

$$\Lambda_\beta(r,w) := \begin{cases} \frac{k}{1-\beta} r^{1-\beta} + w & \text{if} \quad 0 < \beta < 1, \\ k \ln r + w & \text{if} \quad \beta = 1. \end{cases}$$

Let $t \mapsto x(t)$ be a solution of (1) defined on the right maximal interval $[0,\omega[$. Define $r_0 := r(0)$ and $w_0 := w(0)$. We will show that

$$\Lambda_\beta(r(t), 0) \leq \Lambda_\beta(r_0, w_0), \quad t \in [0,\omega[,$$

which implies immediately that $r(t)$ is bounded for all $t \in [0, \omega[$. We argue by contradiction. Assume that there exists $t_1 \in [0, \omega[$ such that $r_1 := r(t_1)$ satisfies

$$\Lambda_\beta(r_1, 0) > \Lambda_\beta(r_0, w_0) \geq \Lambda_\beta(r_0, 0).$$

Then, $r_1 > r_0$, and, if we let $w_1 := w(t_1)$ we have

$$\Lambda_\beta(r_1, w_1) > \Lambda_\beta(r_0, w_0). \tag{14}$$

Let $\Lambda_\beta(t) := \Lambda_\beta(r(t), w(t))$. If $t \in [0, t_1]$ is such that $\dot{r}(t) > 0$, then

$$\dot{\Lambda}_\beta(t) = k\frac{\dot{r}}{r^\beta} - k\frac{w}{r^\beta} - \frac{\dot{r}}{r^2 w} < 0, \quad 0 < \beta \leq 1.$$

The inequality holds since $w \geq \dot{r}$. Moreover, let t' and t'' be two values such that $0 < t' < t'' \leq t_1$ and $r(t') = r(t'')$, then, from $E(t'') < E(t')$, it follows that $w(t'') < w(t')$, and consequently, $\Lambda_\beta(t'') \leq \Lambda_\beta(t')$. By Lemma 6 in [5], applied with $a = 0$, $b = t_1$, $y(t) = r(t)$ and $z(t) = -\Lambda_\beta(t)$, we get that

$$-\Lambda_\beta(t_1) = -\Lambda_\beta(r_1, w_1) \geq -\Lambda_\beta(0) = -\Lambda_\beta(r_0, w_0),$$

contradicting (14). We conclude that all solutions of (1) are bounded on $[0, \omega[$. The first part of the statement follows now from *ii*) of Lemma 1.

In order to prove the second claim, we rewrite system (6) introducing the scalar angular momentum $M = r^2\dot{\theta}$ as a variable, getting the following system:

$$\begin{cases} \dot{r} &= u \\ \dot{u} &= \dfrac{M^2}{r^3} - k\dfrac{u}{r^\beta} - \dfrac{1}{r^2} \\ \dot{M} &= -k\dfrac{M}{r^\beta}. \end{cases} \tag{15}$$

Consider the set

$$B := \{(r, u, M) : r > 1, u \in \mathbb{R}, M \geq 0\},$$

and the following family of functions $f_c : B \to \mathbb{R}$ depending on the positive parameter c:

$$f_c(r, u, M) := c + \frac{1}{\ln r} - u.$$

We claim that, fixed $c > 0$, there exists $r_* > 1$ such that the set

$$B_c := \{(r, u, M) \in B \ : \ r \geq r_*, \ f_c(r, u, M) \leq 0\}$$

is positively invariant with respect to the flow of system (15). Denote by $\mathcal{V}(r, u, M)$ the vector field associated to (15) and by $\phi^t(r, u, M)$ the corresponding flow. A computation shows that on the surface $f_c(r, u, M) = 0$ we have

$$(\mathcal{V} \cdot \nabla f_c)|_{f_c=0} = -\frac{c}{r\ln^2 r} - \frac{1}{r\ln^3 r} - \frac{M^2}{r^3} + \frac{kc}{r^\beta} + \frac{k}{r^\beta \ln r} + \frac{1}{r^2},$$

where the dot denotes the euclidean inner product. As $\beta > 1$, it follows that there exists a sufficiently large $r_* > 1$ such that, for any initial condition $(r, u, M) \in B_c$ satisfying $f_c(r, u, M) = 0$, the following inequality holds

$$\frac{d}{dt} f_c(\phi^t(r, u, M))_{t=0} = (\mathcal{V} \cdot \nabla f_c)(r, u, M) < 0.$$

Since for $(r_*, u, M) \in B_c$ we have that $u = \dot{r} > 0$, and since the set $\{M = 0\} \cap B$ is positively invariant, we conclude that B_c is positively invariant for ϕ^t. Taking into account that $M(t)$ is decreasing, we have

$$\dot{u} = \frac{M^2}{r^3} - k\frac{u}{r^\beta} - \frac{1}{r^2} \leq \frac{M^2(0)}{r^3} - \frac{kc}{r^\beta} - \frac{k}{r^\beta \ln r} - \frac{1}{r^2},$$

and, by choosing if necessary a larger r_*, we may assume that $\dot{u} < 0$ in B_c. Consider now a solution of (15) with initial condition in B_c. Then, $u(t)$ is decreasing on $[0, \omega[$ and

$$c \leq \lim_{t \to \omega} u(t) = u_\omega \leq u(0).$$

If ω were finite, from $\dot{r} = u$ we would get that $r_* \leq r(t) \leq r(0) + u(0)\omega$ on $[0, \omega[$. Since on this interval it holds also $0 \leq M(t) \leq M(0)$, we would have a maximal solution of (15) contained in a compact set of the phase space. We conclude that $\omega = +\infty$, and then

$$r_* + ct \leq r(t) \to +\infty$$

as $t \to +\infty$. The existence of escape solutions is proved.

By the identity

$$|v(t)|^2 = u^2(t) + r^2(t)\dot{\theta}^2(t) = u^2(t) + \frac{M^2(t)}{r^2(t)},$$

and the boundedness of $M(t)$, we see that the value of the asymptotic velocity for escapes is exactly u_ω. For initial conditions in B_c, we get that $u_\omega \geq c$, where c may be fixed arbitrarily large. Furthermore, the limit energy verifies $E_\omega = u_\omega^2/2$. Our proof is concluded. ■

The arguments used in Lemma 1 to prove the attractiveness of the singularity can be adapted to show that collisions and escapes are the only possible types of solutions for (1). This is done in the next result.

We recall that a solution $t \mapsto x(t)$ of (1), defined on the right maximal interval $[0, \omega[$, is called oscillatory if it satisfies

$$\limsup_{t \to \omega^-} |x(t)| = +\infty, \quad \liminf_{t \to \omega^-} |x(t)| = 0. \tag{16}$$

Actually, in the definition of an oscillatory solution, it is usually required only that the lower limit of $|x(t)|$ is finite, but Lemma 1 rules out any value different from zero.

Theorem 2 *Equation* (1) *does not admit oscillatory solutions.*

Proof. If $\beta \in]0, 1]$ the statement is trivially true, since all solutions are collision solutions. The case $\beta > 1$ can be proved arguing by contradiction. If $x(t)$ is an oscillatory solution, there exist a $d > 0$ and a sequence $\{t_n\} \subset [0, \omega[$ converging to ω such that $|x(t_n)| = d$ for any n. Then, we get a contradiction in the same way as in the proof of the item *ii*) in Lemma 1. We omit the details. ■

Remark 1 According to Lemma 1, a bounded solution is attracted to the singularity. Actually, a closer look at the proof shows that this behavior, as well as the statement of Theorem 2, hold for any (sufficiently regular) drag such that the energy along the motions is strictly decreasing. In our framework, a general example is provided by a force of the form $-D(x,\dot{x})\dot{x}$, where the function D is strictly positive and sufficiently smooth on Ω. In fact, in this case $\dot{E}(t) = -D(x(t),\dot{x}(t))|\dot{x}(t)|^2 < 0,\ t \in [0,\omega[$. We conclude that, for such class of dissipations, the only attractor is the singularity.

It may be interesting to observe that a dissipative Kepler problem with a different kind of attractor is obtained in [4], where a simplified model of tidal dissipation is discussed. The general class of dissipations considered there is radial, as illustrated by the example of dissipative force $-\epsilon(x \cdot \dot{x})x$. For this dissipative Kepler problem, each orbit is attracted to a circular orbit depending on the initial conditions.

3.2 Non-rectilinear motions

In this subsection we focus on some qualitative properties of the non rectilinear motions of (1). We show that, for any $\beta > 1$, an escape solution cannot make a full turn around the singularity. As to collision solutions, we prove several facts. Firstly, for any $\beta > 0$, their angular momentum tends to zero and their energy goes to $-\infty$. Secondly, if $\beta \in]0,1[$, their angular velocity is unbounded. Thirdly, collision solutions are asymptotically rectilinear for $\beta > \frac{3}{2}$. In this case we discuss the time to collision and the terminal velocity as β increases. We are able to extend our results to the case $\beta = \frac{3}{2}$, but just for k sufficiently large. This is due to the fact that $\frac{3}{2}$ is the only value of the parameter β for which k cannot be eliminated from equation (1) by a rescaling of the solutions.

3.2.1 Variation of the polar angle of escape orbits

In order to prove our result about escapes, we rewrite (1) using the well known Binet transformation, which we recall here. Since for any solution in Ω^+ we have $M(t) = r^2(t)\dot{\theta}(t) > 0$, for any $t \in [0,\omega[$, the function $t \mapsto \theta(t)$ is an increasing diffeomorphism between $J_t = [0,\omega[$ and the interval $J_\theta = [\theta_0 := \theta(0), \theta_\omega = \theta(\omega^-)[$. The inverse function $\theta \mapsto t(\theta)$ is then used to re-parameterize the solutions of system (6), in which r is replaced by the new variable $\rho = \frac{1}{r}$. Then, the maximal solutions $t \mapsto (x(t), v(t)),\ t \in J_t,$ of (3) in the set Ω^+ are transformed into the maximal solutions $\theta \mapsto y(\theta) = (\rho(\theta), \zeta(\theta), \theta, M(\theta)),\ \theta \in J_\theta,$ of the differential system

$$y' = g(y) := \left(\zeta, \frac{1}{M^2} - \rho, 1, -\frac{k}{\rho^{2-\beta}}\right), \tag{17}$$

where the prime denotes derivation with respect to θ and

$$y = (\rho, \zeta, \theta, M) \in]0,+\infty[\times\mathbb{R}\times\mathbb{R}\times]0,+\infty[.$$

The previous transformation may be defined by the time rescaling $t = t(\theta)$ and by the change of variables

$$(x,v) = U(y) := \left(\frac{1}{\rho}e_r(\theta), -M\zeta e_r(\theta) + \rho M e_\theta(\theta)\right), \quad e_r(\theta) = e^{i\theta},\ e_\theta(\theta) = ie_r(\theta), \tag{18}$$

where we identify θ modulo 2π.

The first two equations of system (17) are equivalent to the second order scalar equation of a forced linear oscillator, namely $\rho'' + \rho = \frac{1}{M^2}$. Given an initial condition $(x_0, v_0) \in \Omega^+$, we set $y_0 := (\rho_0, \zeta_0, \theta_0, M_0) = U^{-1}(x_0, v_0)$, in which $\theta_0 = \arg x_0$ is defined modulo 2π. In what follows we will consider $\theta_0 \in [0, 2\pi[$. By the variation of constants formula we see that the corresponding solution satisfies the coupled system

$$\begin{aligned} \rho(\theta) &= \rho_0 \cos(\theta - \theta_0) + \zeta_0 \sin(\theta - \theta_0) + \int_{\theta_0}^{\theta} \frac{\sin(\theta-\eta)}{M^2(\eta)}\, d\eta \\ \zeta(\theta) &= \rho'(\theta) \\ M(\theta) &= M_0 - \int_{\theta_0}^{\theta} \frac{k}{\rho^{2-\beta}(\eta)}\, d\eta \end{aligned} \tag{19}$$

on $[\theta_0, \theta_\omega[$. We can rewrite the first equation as

$$\rho(\theta) = -\kappa \sin(\theta - \alpha_0 - \theta_0) + \Phi(\theta), \tag{20}$$

where $\alpha_0 = \arccos\left(-\frac{\zeta_0}{\kappa}\right) \in]0, \pi[$ is the angle between x_0 and v_0, and where we have defined

$$\Phi(\theta) := \int_{\theta_0}^{\theta} \frac{\sin(\theta - \eta)}{M^2(\eta)}\, d\eta, \qquad \kappa = \sqrt{\rho_0^2 + \zeta_0^2}. \tag{21}$$

Note that a solution of $y' = g(y)$ defined on $[0, \theta_\omega[$ corresponds to an escape solution of (1) if and only if

$$\rho(\theta) > 0 \ \text{ on } \ [0, \theta_\omega[\qquad \text{and} \qquad \lim_{\theta \to \theta_\omega^-} \rho(\theta) = 0.$$

We are now in a position to prove our result about escapes. To simplify our notations, we assume that $\theta_0 = 0$.

Theorem 3 *Escapes can only occur during the first turn around the origin. Moreover, the limit angle θ_ω satisfies*

$$\theta_\omega < \alpha_0 + \pi.$$

If a solution does more than one turn, then it corresponds to a collision orbit.

The theorem will be an immediate consequence of the following lemma, whose proof is based on the fact that $M(\theta)$ is a strictly decreasing function.

Lemma 2 *The function Φ defined in (21) satisfies the following properties:*

i) $\Phi(\theta) > 0$ *for all* $\theta \in]0, \theta_\omega[$.

ii) *If* $\theta_\omega > 2\pi$, *then* $\Phi(\theta) > \Phi(\theta - 2\pi)$ *for all* $\theta \in [2\pi, \theta_\omega[$.

Proof. From the sign of $\sin(\theta - \eta)$ we see that

$$I_1 := \int_{\theta-\pi}^{\theta} \frac{\sin(\theta - \eta)}{M^2(\eta)} d\eta > 0, \quad I_2 := \int_{\theta-2\pi}^{\theta-\pi} \frac{\sin(\theta - \eta)}{M^2(\eta)} d\eta < 0.$$

By the second inequality, it follows that

$$|I_2| = \int_{\theta-2\pi}^{\theta-\pi} \frac{|\sin(\theta - \eta)|}{M^2(\eta)} d\eta = \int_{\theta-\pi}^{\theta} \frac{|\sin(\theta - \bar{\eta} + \pi)|}{M^2(\bar{\eta} - \pi)} d\bar{\eta} = \int_{\theta-\pi}^{\theta} \frac{\sin(\theta - \bar{\eta})}{M^2(\bar{\eta} - \pi)} d\bar{\eta}.$$

Since $\frac{1}{M(\eta)}$ is strictly increasing, we have $I_1 > |I_2|$, and then

$$\int_{\theta-2\pi}^{\theta} \frac{\sin(\theta-\eta)}{M^2(\eta)} d\eta = I_2 + I_1 > 0, \quad \text{for any } \theta \in [0, \theta_\omega[.$$

This inequality implies both assertions of the lemma. ■

Proof. (*of Theorem 3*)

If a solution of (1) makes one complete turn around the origin, it follows that

$$\rho(\theta) > 0, \text{ for any } \theta \in [0, 2\pi] \subset [0, \theta_\omega[.$$

Property *ii*) of Lemma 2 implies

$$\liminf_{\theta \to \theta_\omega} \rho(\theta) \geq \rho(\theta_\omega - 2\pi) > 0,$$

and $|x(\theta)| = \frac{1}{\rho(\theta)}$ is bounded on $[0, \theta_\omega[$. From Remark 1, such solution is a collision solution. To end our proof, we observe that *i*) of Lemma 2 and (20) imply that, for an escape solution, $\theta_\omega \in]\alpha_0, \alpha_0 + \pi[$. ■

3.2.2 Asymptotic behavior of energy and angular momentum of collision orbits

Next result collects some general facts about collision solutions of (1).

Theorem 4 *Collisions always occur with zero angular momentum and energy equals to minus infinity.*

Proof. From the expression of the energy

$$E(t) = \frac{u^2(t)}{2} + \frac{r^2(t)\dot{\theta}^2(t)}{2} - \frac{1}{r(t)} = \frac{u^2(t)}{2} + \frac{M^2(t)}{2r^2(t)} - \frac{1}{r(t)},$$

we see that in the case of a collision, since $r(t) \to 0^+$ as $t \to \omega^-$, it must be $M_\omega = 0$. Otherwise, we would get $E_\omega = +\infty$, which is not possible since $E(t)$ is a decreasing function.

To prove the second part of the statement, we start by observing that, by (5),

$$M(t) = M(0)e^{-k\,\tau(t)},$$

where

$$\tau(t) = \int_0^t \frac{ds}{r^\beta(s)}. \tag{22}$$

Since $M_\omega = 0$, we get that, for collision solutions,

$$\tau_\omega = \int_0^\omega \frac{ds}{r^\beta(s)} = +\infty. \tag{23}$$

Now we can conclude our proof arguing by contradiction. Assume that $E_\omega \in \mathbb{R}$. Since for a collision solution there exists a $t_0 \in [0, \omega[$ such that

$$E(t) > E_\omega \geq \frac{1}{2} - \frac{1}{r(t)}$$

for all $t \in [t_0, \omega[$, then, on this interval,

$$|v(t)|^2 = 2E(t) + \frac{2}{r(t)} > 1.$$

As a consequence, we get

$$\dot{E}(t) = -k\frac{|v(t)|^2}{r^\beta(t)} < -\frac{k}{r^\beta(t)}.$$

Integrating this inequality from t_0 to t, we get $E(t) < E(t_0) - k\,\tau(t)$. But now, from (23) we get $E_\omega = -\infty$, contradicting the assumption. Then, $E_\omega = -\infty$ and our proof is complete. ■

Remark 2 Note that Theorem 2 and Theorem 4 imply that Ω^+ can be partitioned in the following three sets: the set of initial conditions of collisions orbits

$$\Omega_C^+ := \{(x,v) \in \Omega^+ \,:\, E_\omega = -\infty\},$$

the set of initial conditions of hyperbolic escapes, and the set of initial conditions of parabolic escapes, defined respectively by

$$\Omega_H^+ := \{(x,v) \in \Omega^+ \,:\, E_\omega > 0\} \quad \text{and} \quad \Omega_P^+ := \{(x,v) \in \Omega^+ \,:\, E_\omega = 0\}.$$

The last two sets are empty when $\beta \in [0,1]$.

Of course, an analogous partition holds for Ω, but we will not need this fact in what follows.

3.2.3 A rotational property of collision orbits for $\beta \in]0,1[$

Next result shows that the rotational property of solutions obtained for the linear drag in Proposition 2.5 in [12] continue to hold when $\beta \in]0,1[$.

Theorem 5 *Let $\beta \in]0,1[$. Given a non rectilinear solution $t \mapsto x(t) = r(t)e^{i\theta(t)}$, there exists a sequence $t_n \to \omega^-$ such that*

$$\dot{\theta}(t_n) \to +\infty.$$

Proof. We start by regularizing system (3) using the time rescaling $d\mu = \frac{dt}{r^2}$. We obtain the C^1 system

$$\begin{cases} \dfrac{dr}{d\mu} = r^2 u \\ \dfrac{du}{d\mu} = -kur^{2-\beta} + r^3\varphi^2 - 1 \\ \dfrac{d\varphi}{d\mu} = -(kr^{2-\beta} + 2ur)\varphi \end{cases} \tag{24}$$

on the set $r \geq 0, u \in \mathbb{R}$, $\varphi > 0$.

Then, multiplying the first equation by $kr^{-\beta}$, adding it to the second equation and integrating the result, we obtain that any solution $\mu \mapsto (r(\mu), u(\mu), \varphi(\mu))$ satisfies the following equality:

$$u(\mu) + k\frac{r^{1-\beta}(\mu)}{1-\beta} = \int_0^\mu r^3(\sigma)\varphi^2(\sigma)d\sigma + C_0 - \mu, \tag{25}$$

where $C_0 := u(0) + k\frac{r^{1-\beta}(0)}{1-\beta}$, on its right maximal interval $I_\mu = [0, \mu_\omega[$.

Now we argue by contradiction. Assume that $\varphi(\mu)$ is bounded on I_μ. Then, it must be $\mu_\omega = +\infty$. Otherwise, if we assume that $\mu_\omega < +\infty$, we are led to a contradiction as follows.

From (25) we infer that $u(\mu)$ is bounded on I_μ. As a consequence, from the third equation of the system we see that $\varphi(\mu)$ is bounded away from zero on I_μ. But then the solution $\mu \mapsto (r(\mu), u(\mu), \varphi(\mu))$ is contained in a compact set of the phase space for all $\mu \in I_\mu$, contradicting its maximality. We conclude that $\mu_\omega = +\infty$. Now the argument proceeds as in [12]. Since $r(\mu) \to 0$ as $\mu \to +\infty$, there exists a sequence $\mu_n \to +\infty$ such that $u(\mu_n) \to 0$. By the boundedness of φ on I_μ, there exists $\bar{\mu}$ such that $r^3(\sigma)\varphi^2(\sigma) < \frac{1}{2}$ for any $\sigma \geq \bar{\mu}$. Then, from (25) we get that for any $\mu_n > \bar{\mu}$ it holds the inequality

$$u(\mu_n) + k\frac{r^{1-\beta}(\mu_n)}{1-\beta} \leq \int_0^{\bar{\mu}} r^3(\sigma)\varphi^2(\sigma)d\sigma + \frac{\mu_n - \bar{\mu}}{2} + C_0 - \mu_n.$$

Taking the limit for $n \to +\infty$, we get the contradiction $0 \leq -\infty$. We conclude that $\varphi(\mu)$ is unbounded on $[0, +\infty[$, and the same property holds for $\dot{\theta}(t)$ on $[0, \omega[$. ■

3.2.4 Asymptotic dynamics of collision solutions for $\beta \geq \frac{3}{2}$

To get our next results we start with a suitable rescaling of time in system (6), given by $d\tau = \frac{dt}{|x|^\beta}$. We obtain the following equivalent system in the new time τ

$$\begin{cases} \dfrac{dr}{d\tau} = r^\beta u, \\ \dfrac{du}{d\tau} = -ku + r^{\beta+1}\varphi^2 - r^{\beta-2}, \\ \dfrac{d\varphi}{d\tau} = -(k + 2ur^{\beta-1})\varphi. \end{cases} \tag{26}$$

The associated vector field is non singular for $\beta \geq 2$. In this case, it can be extended continuously to the collision manifold $r = 0$, on which it possesses a unique equilibrium $r = 0, u = 0, \varphi = 0$. For $\beta \geq 3$ the vector field is C^1 on the set $r \geq 0$, $u \in \mathbb{R}, \varphi \geq 0$.

By (23), collision solutions in Ω^+ are defined on the right maximal interval $[0, \tau_\omega = +\infty[$, and on such interval, $\varphi(\tau) = \dot{\theta}(t(\tau)) > 0$. Since $\tau \mapsto t(\tau)$ is an increasing diffeomorphism, the polar angle $\theta(t(\tau))$ is an increasing function of τ and, moreover, by (22) we get

$$\frac{d\theta}{d\tau} = \dot{\theta}\frac{dt}{d\tau} = \dot{\theta}(t(\tau))r^\beta(t(\tau)) > 0. \tag{27}$$

Theorem 6 *The following properties hold for collision orbits:*

i) If $\beta > \frac{3}{2}$, or if $\beta = \frac{3}{2}$ and $k > 2\sqrt{2}$, there exists a limit polar angle at collision, achieved with zero angular velocity.

ii) If $\frac{3}{2} < \beta < 3$, or if $\beta = \frac{3}{2}$ and $k > 2\sqrt{2}$, collisions occur in finite time, whereas if $\beta \geq 3$ they occur in infinite time.

iii) If $\frac{3}{2} < \beta < 2$, or if $\beta = \frac{3}{2}$ and $k > 2\sqrt{2}$, the limit velocity at collision is infinite, if $\beta = 2$, the limit velocity is finite and with modulus $\frac{1}{k}$, and if $\beta > 2$ the limit velocity is zero.

Proof.

i) Since $E_\omega = -\infty$ and r is bounded for collision orbits, we can take initial conditions such that $E(0) < 0$ and $0 < r(t) < 1$ on $[0, \omega[$. Then,

$$\frac{u^2(t)}{2} - \frac{1}{r(t)} < E(t) = \frac{|v(t)|^2}{2} - \frac{1}{r(t)} < 0,$$

or equivalently, $u^2 r < 2$. It follows that, for any $\beta \geq \frac{3}{2}$, we have

$$u^2 r^{2\beta-2} < 2r^{2\beta-3} \leq 2,$$

which implies

$$ur^{\beta-1} > -\sqrt{2}r^{\frac{2\beta-3}{2}} \geq -\sqrt{2}, \quad \text{for any} \quad \beta \geq \frac{3}{2}. \tag{28}$$

Consider now the evolution of the angular velocity given by the last equation of (26). By (28) we get the inequality

$$\frac{d\varphi}{d\tau} = -(k + 2ur^{\beta-1})\varphi < -(k - 2\sqrt{2})\varphi, \tag{29}$$

and hence

$$0 < \varphi(t(\tau)) < \varphi(0)e^{-(k-2\sqrt{2})\tau}. \tag{30}$$

Taking into account that $\varphi = \dot{\theta}$ and (27), we can integrate (30) obtaining

$$\theta(t(\tau)) < \theta(0) + \varphi(0) \int_0^\tau r^\beta(t(\bar{\tau}))e^{-(k-2\sqrt{2})\bar{\tau}} d\bar{\tau}.$$

For $k > 2\sqrt{2}$ the integrand function in the right hand side of the inequality is integrable on $[0, +\infty[$, since r is bounded on this interval. Then, $\tau \mapsto \theta(t(\tau))$ is an increasing function, bounded from above on $[0, +\infty[$. We conclude that, if $\beta \geq \frac{3}{2}$ and $k > 2\sqrt{2}$, there exists $\lim_{\tau\to+\infty} \theta(t(\tau)) = \lim_{t\to\omega^-} \theta(t) = \theta_\omega$, and is finite. Moreover, (30) implies that $\lim_{\tau\to+\infty} \dot{\theta}(t(\tau)) = \lim_{t\to\omega^-} \dot{\theta}(t) = 0^+$.

To finish the proof of *i*), we show that, if $\beta > \frac{3}{2}$, the restriction on the values of k may be removed. In fact, one can check that, when $\beta \neq \frac{3}{2}$, fixed arbitrarily two different values of k, the solutions of the two corresponding equations (1) can be transformed ones into the others by a suitable scaling of the form $\tilde{x}(t) = p\, x(qt)$, $p, q > 0$. In particular, for $\beta > \frac{3}{2}$, let us consider $k = k_1 \leq 2\sqrt{2}$. Then, the scaling $\tilde{x}(t) = p\, x(p^{-\frac{3}{2}}t)$, with $p > 0$ and $p^{\beta-\frac{3}{2}} > 2\sqrt{2}/k_1$, transforms any solution of (1) with $k = k_1$ into a solution of the same equation with $k = k_2 = p^{\beta-\frac{3}{2}}k_1 > 2\sqrt{2}$. Since the scaling preserves the asymptotic behavior of the solutions as well as the orientation of time, the proof of *i*) is concluded.

ii) To study if ω is finite or not, it will be convenient to deal directly with system (6). As above, without loss of generality, we may assume that $E(t) < 0$, $0 < r(t) < 1$ on $[0, \omega[$. Moreover, since we are considering $\beta > \frac{3}{2}$ or $\beta = \frac{3}{2}$ and $k > 2\sqrt{2}$, by (30) we may assume also $0 < \varphi(t) = \varphi(\tau(t)) < 1$ on $[0, \omega[$.

From the second equation in (6) we see that, whenever $u \geq 0$, it holds

$$\dot{u} = -k\frac{u}{r^\beta} + r\left(\varphi^2 - \frac{1}{r^3}\right) < -k\frac{u}{r^\beta} + r\left(1 - \frac{1}{r^3}\right) < 0.$$

Then, there exists $t_1 \in]0, \omega[$ such that $u(t) < 0$ on $[t_1, \omega[$. As a consequence, $r(t)$ is a decreasing function on $[t_1, \omega[$, and we can take r as independent variable by considering the time rescaling $t = t(r), \ \ r \in]0, r_1 := r(t_1)]$. It follows that ω satisfies the equality

$$\omega - t_1 = \int_{r_1}^{0} \frac{dr}{u(t(r))}.$$

Since $r^3\varphi^2 \to 0$ as $t \to \omega^-$, we can assume that $r^3(t)\varphi^2(t) < 1/2$ on $[t_1, \omega[$. Then, from (6) we get the inequality

$$r^2\frac{du}{dr} = \frac{r^3\varphi^2}{u} - \frac{k}{r^{\beta-2}} - \frac{1}{u} > -\frac{k}{r^{\beta-2}} - \frac{1}{2u}. \tag{31}$$

Integrating (31) on any interval of the form $[r_*, r_1]$, with $0 < r_* < r_1$, we obtain

$$r_1^2 u(t_1) - r_*^2 u(t(r_*)) - 2\int_{r_*}^{r_1} r\, u(t(r))dr > -k\int_{r_*}^{r_1} \frac{dr}{r^{\beta-2}} - \frac{1}{2}\int_{r_*}^{r_1} \frac{dr}{u(t(r))}. \tag{32}$$

By the inequality $u^2 r < 2$, it follows that, for all $\gamma > 1/2$, $\ \ ur^\gamma \to 0$ as $r \to 0^+$. Then, the left hand side of (32) has a finite limit, say l^*, when $r_* \to 0^+$. Passing to the limit in (32), we arrive to the following inequality:

$$l^* > -k\int_0^{r_1} \frac{dr}{r^{\beta-2}} + \frac{1}{2}(\omega - t_1).$$

Now, if $\beta < 3$, the integral is convergent, and then it must be $\omega < +\infty$.

In the case $\beta \geq 3$, we can proceed analogously, by integrating on the interval $[r_*, r_1]$ the inequality

$$r^2\frac{du}{dr} = \frac{r^3\varphi^2}{u} - \frac{k}{r^{\beta-2}} - \frac{1}{u} < -\frac{k}{r^{\beta-2}} - \frac{1}{u},$$

and then taking the limit as $r_* \to 0^+$. We obtain the inequality

$$l_* < -k\int_0^{r_1} \frac{dr}{r^{\beta-2}} + (\omega - t_1),$$

where l_* denotes the finite limit of the left hand side of the integrated inequality. Since when $\beta \geq 3$ the integral is divergent, it must be $\omega = +\infty$.

The proof of *ii*) is concluded.

iii) We know that, if $\beta > \frac{3}{2}$, or if $\beta = \frac{3}{2}$ and $k > 2\sqrt{2}$, the limit velocity at collisions depends only on the radial component u, because the angular component $r\varphi$ goes to zero. Additionally, in the previous item it was proved that u gets eventually negative.

To prove our claims we will use the second equation of system (26). Note that $r^{\beta+1}\varphi^2 \to 0$ as $r \to 0^+$ for any $\beta > 0$, but the behavior of the term $r^{\beta-2}$ will depend on the sign of $\beta - 2$.

If $\beta < 2$, we have that $r^{\beta-2} \to +\infty$ as $r \to 0^+$. Then, for any $a > 0$, we can take initial conditions such that, for all $\tau > 0$,

$$\frac{du}{d\tau} = -ku + r^{\beta+1}\varphi^2 - r^{\beta-2} < -ku - a,$$

or, equivalently,

$$\frac{d}{d\tau}\left(ue^{k\tau}\right) < -ae^{k\tau}.$$

Integrating this inequality we get

$$u(t(\tau)) < -\frac{a}{k} + \left(u(0) + \frac{a}{k}\right)e^{-k\tau},$$

which implies

$$\limsup_{\tau\to+\infty} u(t(\tau)) \leq -\frac{a}{k}.$$

Since a can be chosen arbitrary large, we conclude that $u \to u_\omega = -\infty$.

On the other hand, if $\beta > 2$, we have that $r^{\beta-2} \to 0$ as $r \to 0^+$. Then, for any $a > 0$ we can take initial conditions such that, for all $\tau > 0$,

$$\frac{du}{d\tau} = -ku + r^{\beta+1}\varphi^2 - r^{\beta-2} > -ku - a.$$

Analogously to the previous case, we conclude that

$$-\frac{a}{k} \leq \liminf_{\tau\to+\infty} u(t(\tau)) \leq 0,$$

for any arbitrarily small a. Then, $u \to u_\omega = 0$.

Finally, for the threshold value $\beta = 2$, since $r^3\varphi^2 \to 0$, fixed any arbitrarily small positive ϵ, we can use the inequality $0 < r^3\varphi^2 < \epsilon$ in the second equation of (26). We obtain, in a similar manner than above, that

$$-\frac{1}{k} \leq \liminf_{\tau\to+\infty} u(t(\tau)) \leq \limsup_{\tau\to+\infty} u(t(\tau)) \leq -\frac{1-\epsilon}{k},$$

which leads to the conclusion that $u \to u_\omega = -1/k$.

■

3.3 Rectilinear motions

In order to give a more complete description of the forward dynamics of (1), in this subsection we address the rectilinear collisions of (1).

We recall that the rectilinear motions of (1) verify the second order equation

$$\ddot{r} + k\frac{\dot{r}}{r^\beta} + \frac{1}{r^2} = 0, \tag{33}$$

equivalent to the following first order system (which, of course, corresponds to the first two equations of system (6) with $\varphi = 0$),

$$\begin{cases} \dot{r} = u \\ \dot{u} = -k\frac{u}{r^\beta} - \frac{1}{r^2}, \end{cases} \tag{34}$$

on the phase space $\Omega^0 = \{(r,u):\ r > 0, u \in \mathbb{R}\}$.

We have the following result.

Theorem 7 *The terminal time ω, the terminal velocity u_ω and the terminal energy E_ω of collision solutions depend on β according to the following table:*

<table>
<tr><th>β</th><th>$]0,1/2[$</th><th>$[1/2,2[$</th><th>2</th><th>$]2,3[$</th><th>$[3,\infty[$</th></tr>
<tr><td>ω</td><td colspan="4">finite</td><td>$+\infty$</td></tr>
<tr><td>u_ω</td><td colspan="2">$-\infty$</td><td>$-\frac{1}{k}$</td><td colspan="2">0</td></tr>
<tr><td>E_ω</td><td>finite</td><td colspan="4">$-\infty$</td></tr>
</table>

The corresponding proof is given in the Appendix.

Remark 3 From the previous theorem, we see that, when $\beta \in]0, \frac{1}{2}[$, the collision time, as well as the corresponding energy, are finite. It is not difficult to prove that also the ejection time and ejection energy[3] are finite. These properties are sufficient to infer that the asymptotic expansions of solutions around collision and ejection times obtained in [12] for $\beta = 0$ are still valid for $\beta \in]0, \frac{1}{2}[$. Namely, at collision we have

$$r(t) = \left(\frac{9}{2}\right)^{1/3} (\omega - t)^{2/3} + O((\omega - t)^{4/3}), \quad t \to \omega^-, \tag{35}$$

$$\dot{r}(t) = -\frac{2}{3}\left(\frac{9}{2}\right)^{1/3} (\omega - t)^{-1/3} + O((\omega - t)^{1/3}), \quad t \to \omega^-. \tag{36}$$

The analogous expansion at ejection is obtained just by replacing $\omega - t$ with $t - \alpha$ and by reversing the sign of $\dot{r}$ in (36). We point out that these expansions hold also for the rectilinear motions of a periodically forced Kepler problem ([19]), as well as for a perturbed two body problem, with a perturbation of the form $P(t, x, \dot{x})$. which, unlike in our case, is bounded ([21]).

[3] An ejection solution is a solution such that $\lim_{t\to\alpha^+} x(t) = 0$, and α is the ejection time.

However, unlike in the case of the linear drag, when $\beta \in]0, \frac{1}{2}[$ the ejection-collision solutions cannot be embedded in a regular flow. Let us explain this point. The Levi-Civita transformation, developed in the conservative setting, was used effectively in [12] to regularize the dynamics of (1) for $\beta = 0$. But, for $\beta > 0$, the natural generalization of this transformation leads to a system that, although non singular, does not define a flow. To show this, consider the Levi-Civita-like transformation $x = x_1 + ix_2 = w^2$, $d\nu = \frac{dt}{|x|^{\beta+1}}$ (the classical Levi-Civita transformation corresponds to $\beta = 0$). This transformation maps solutions of (1) into solutions of the non singular system

$$\frac{dw}{d\nu} = |w|^{2\beta}v, \ \frac{dv}{d\nu} = \frac{E}{2}|w|^{2\beta}w - |w|^2 v, \ \frac{dE}{d\nu} = -(2E|w|^2 + 1) \tag{37}$$

contained in the invariant manifold $\mathcal{M} \subset \mathbb{C}^2 \times \mathbb{R}$ of equation $E|w|^2 + 1 - 2|v|^2 = 0$.

We notice that an ejection-collision solution, $t \mapsto r(t)$, defined on the maximal finite interval $]\alpha, \omega[$ is transformed into a solution $\nu \mapsto \Xi(\nu) = (w(\nu), v(\nu), E(\nu))$ of (37), defined on the finite, but not maximal, interval

$$I_S = \left] \nu_\alpha := -\int_\alpha^0 \frac{1}{r^{\beta+1}(\sigma)} d\sigma, \ \nu_\omega := \int_0^\omega \frac{1}{r^{\beta+1}(\sigma)} d\sigma \right[.$$

A maximal solution of (37) which extends Ξ outside I_S is obtained by considering the function $\Xi_\alpha(\nu) := \left(0, \frac{1}{\sqrt{2}}, E_\alpha - \nu + \nu_\alpha\right)$ on $]-\infty, \nu_\alpha[$, where E_α is the energy at ejection, and the function $\Xi_\omega(\nu) := \left(0, -\frac{1}{\sqrt{2}}, E_\omega - \nu + \nu_\beta\right)$ on $[\nu_\omega, +\infty[$. Since $\Xi_\alpha(\nu)$, $\nu \in \mathbb{R}$, is also a maximal solution of (37), it follows that uniqueness os solutions fails at the point $(0, \frac{1}{\sqrt{2}}, E_\alpha)$.

This behavior agrees with the fact that, when $\beta \in]0, \frac{1}{2}[$, the regularized vector field is not locally Lipschitz continuous at points of the form $(0, v, E)$ with $v \neq 0$.

In a conservative setting, the study of the existence of a regularized flow by means of a change of variables analogous to the Levi-Civita transformation has been considered in [16].

4 $\beta = 2$: properties of the asymptotic Runge-Lenz vector

In this section we consider the case of the Poynting-Plummer-Danby drag. We show that a careful analysis of equation (20) allows a complete description of the orbit structure for such a case, improving the results in [7]. Such description will be contained in the main result of this section, where we discuss the properties of the limit I of the Runge-Lenz vector

$$R(x, v) = v \wedge (x \wedge v) - \frac{x}{|x|} \tag{38}$$

along the solutions of (1) with $\beta = 2$. We recall that R is a first integral of the classical Kepler problem, with the following geometrical meaning: for non rectilinear orbits, which are conic sections, R is parallel to the symmetry axis which contains the focus, and its modulus is the eccentricity of the orbit. For this reason it is also referred to as eccentricity vector.

The vector I, which can be thought of as an asymptotic eccentricity vector, was considered in [13] to study the dynamics of (1) when $\beta = 0$ (see Remark 4 below). In our case, its properties will be obtained by taking the limit of R along the solutions of (17) and then going back to the (x, v) variables. This is equivalent to take the limit of R along the flow of (3). In fact, if γ^θ denotes the flow of (17), we have

$$\phi^t = U \circ \gamma^\theta \circ U^{-1}, \tag{39}$$

where U was defined in (18) and t and θ are explicitly related by $t = t(\theta, y) = \int_{\theta_0}^{\theta} \lambda(\gamma^\sigma(y))\, d\sigma$, with $\lambda(y) = \frac{1}{\rho^2 M}$. The pair (U, λ) establishes the so called equivalence in the extended sense of the vector fields (3) and (17), see [18].

Theorem 8 *There exists a vector field*

$$I : \Omega \to \mathbb{R}^2, \quad I = I(x, v),$$

satisfying

i) $I(\sigma x, \sigma v) = \sigma I(x, v)$*, for each* $(x, v) \in \Omega$ *and each rotation*

$$\sigma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

ii) I *is smooth on the sets* Ω_H^+ *and* Ω_C^+*, corresponding, respectively, to hyperbolic and collision orbits, and is discontinuous on the set* Ω_P^+*, corresponding to parabolic orbits.*

Moreover,

$$I(\Omega) = \mathbb{R}^2 \setminus \operatorname{int}(\mathbb{D}),$$

where $\mathbb{D}$ *is the closed unit disk in* $\mathbb{R}^2$*.*

iii) Each solution $t \mapsto (x(t), v(t))$ *of (3) with* $\beta = 2$*, defined on the right maximal interval* $[0, \omega[$*, satisfies*

$$I(x(t'), v(t')) = \lim_{t \to \omega^-} R(x(t), v(t))$$

for each $t' \in [0, \omega[$*.*

Remark 4 This theorem is analogous to Theorem 2.1 in [13], where the properties of I for $\beta = 0$ are considered. In particular, in [13] it is found that I is continuous on Ω. Moreover, taking also into account an improvement of Theorem 2.1 presented in [14], it is established that the range of I is the closed unit disk. When $\beta = 2$, item *ii*) shows that I has significantly different properties: I is not continuous on Ω, and its range is the exterior of the open unit disk. We will see that the discontinuity arises along any fixed parabolic orbit since such orbit is the limit of hyperbolic orbits and of collision ones. We prove this fact only for $\beta = 2$, because in this case the problem is integrable, and we could make use of explicit closed formulas in our computations. As to the range of I, it somewhat expresses that, unlike for the case $\beta = 0$, there are parabolic and hyperbolic orbits, and there are no elliptic motions winding infinite times around the singularity as they approach it.

Proof.

We start by proving that I, as defined in *iii*), exists on Ω^+ and that the properties stated in *ii*) hold. To carry out our analysis we will use the Binet variables. Consider the initial condition

$$(x_0, v_0) = U(\rho_0, \zeta_0, \theta_0, M_0) = \left(\frac{1}{\rho_0} e_r(\theta_0), -\zeta_0 M_0 e_r(\theta_0) + \rho_0 M_0 e_\theta(\theta_0) \right) \in \Omega^+,$$

and consider the corresponding solution of system (17), given by (19), (20) and (21), with $\beta = 2$. In this case

$$M(\theta) = M_0 - k(\theta - \theta_0) = k(\theta_M - \theta + \theta_0),$$

where $\theta_M = M_0/k$, and we have the following explicit formula:

$$\rho(\theta) = -\kappa \sin(\theta - \alpha_0 - \theta_0) + \int_{\theta_0}^{\theta} \frac{\sin(\theta - \eta)}{k^2(\theta_M - \eta + \theta_0)^2} d\eta. \tag{40}$$

Collision solutions correspond to $\theta_\omega = \theta_M$, and are such that

$$\rho(\theta) > 0,\ \theta \in [\theta_0, \theta_\omega[,\ \ \rho(\theta_\omega^-) = +\infty.$$

Escape solutions correspond to $\theta_\omega < \theta_{M_0}$, and satisfy

$$\rho(\theta) > 0,\ \theta \in [\theta_0, \theta_\omega[,\ \ \rho(\theta_\omega^-) = 0.$$

By Theorem 3, we also know that for escapes $\theta_\omega \in]\theta_0 + \alpha_0, \theta_0 + \alpha_0 + \pi[$. If we define the set

$$A = \{(\theta, \theta_M)\ :\ \theta_0 < \theta < \theta_0 + \theta_M,\ \theta_0 + \alpha_0 \leq \theta \leq \theta_0 + \alpha_0 + \pi\}$$

and the smooth family of functions $F_{\rho_0,\zeta_0,\theta_0} : A \to \mathbb{R}$,

$$F_{\rho_0,\zeta_0,\theta_0}(\theta, \theta_M) := \rho(\theta) = -\kappa \sin(\theta - \alpha_0 - \theta_0) + \int_{\theta_0}^{\theta} \frac{\sin(\theta - \eta)}{k^2(\theta_M - \eta + \theta_0)^2} d\eta, \tag{41}$$

then the limit angle θ_ω of an escape orbit will satisfy the implicit equation

$$F_{\rho_0,\zeta_0,\theta_0}(\theta_\omega, \theta_M) = 0.$$

We will show below that this equation defines implicitly θ_ω as a function of $y_0 = U^{-1}(x_0, v_0)$ which is continuous in $\Omega_H^+ \cup \Omega_P^+$ and smooth on Ω_H^+.

For simplicity, in what follows we omit the dependence on the parameters of F and of the functions implicitly defined. We also set $\theta_0 = 0$ for our computations, but will remove this restriction in our conclusions.

In the same way we proved that $\Phi(\theta) > 0$ in $i)$ of Lemma 2, we obtain the inequality

$$\partial_{\theta_M} F(\theta, \theta_M) = -2 \int_0^{\theta} \frac{\sin(\theta - \eta)}{k^2(\theta_M - \eta)^3} d\eta < 0,$$

so that for each fixed $\theta \in]0, \theta_M[\cap [\alpha_0, \alpha_0 + \pi]$, the map $\theta_M \mapsto F(\theta, \theta_M)$ is strictly decreasing on $]\theta, +\infty[$. For each $\theta \in]\alpha_0, \alpha_0 + \pi[$ we have

$$\lim_{\theta_M \to \infty} F(\theta, \theta_M) = -\kappa \sin(\theta - \alpha_0) < 0, \tag{42}$$

and since

$$\frac{\sin(\theta_M - \eta)}{(\theta_M - \eta)^2} = \frac{1}{\theta_M - \eta} + O\left((\theta_M - \eta)^2\right),$$

we get

$$\lim_{\theta \to \theta_M^-} F(\theta, \theta_M) = +\infty,$$

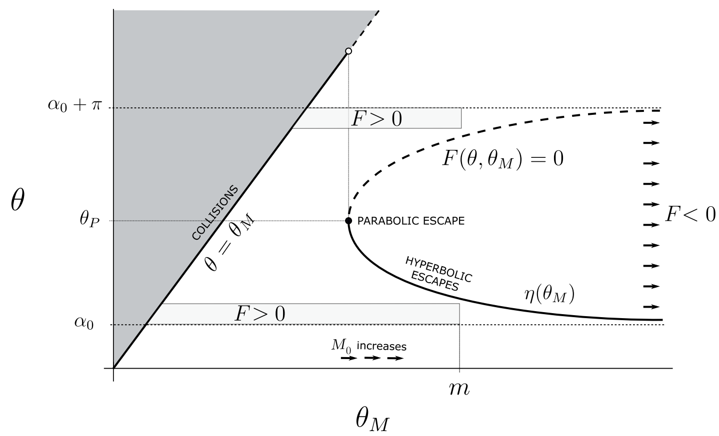

Figure 1: Discontinuity of θ_ω : graphical illustration of the proof.

for any fixed $\theta_M > \alpha_0$. By Bolzano's theorem and the implicit function theorem, there exists a smooth function $\theta_M = \psi(\theta)$ defined on $]\alpha_0, \alpha_0 + \pi[$ such that $F(\theta, \psi(\theta)) = 0$.

From (41) and $i)$ of Lemma 2, it follows that $F(\alpha_0, \theta_M) > 0$ and $F(\alpha_0 + \pi, \theta_M) > 0$. Then, fixed any $m > 0$, there exists $\delta > 0$ such that $F > 0$ on $([\alpha_0, \alpha_0 + \delta] \times [0, m]) \cap A$ and on $([\pi + \alpha_0 - \delta,] \times [0, m]) \cap A$, see Figure 1. From (42) we conclude that

$$\lim_{\theta \searrow \alpha_0} \psi(\theta) = \lim_{\theta \nearrow \alpha_0 + \pi} \psi(\theta) = +\infty,$$

and as a consequence $\psi(\theta)$ has at least a minimum. Actually, for all θ such that $\psi'(\theta) = 0$, we have

$$\psi''(\theta) = -\frac{\partial^2_\theta F(\theta, \psi(\theta))}{\partial_{\theta_M} F(\theta, \psi(\theta))} > 0,$$

since $\partial_{\theta_M} F(\theta, \psi(\theta)) < 0$ and $\partial^2_\theta F(\theta, \psi(\theta)) = \frac{1}{k^2(\psi(\theta)-\theta)^2} > 0$. Then, the minimum point, say $\theta = \theta_P$, is unique and it is the only critical point of ψ. The subscript indicates that the minimum point is associated to a parabolic escape, as we will see.

It follows that ψ admits a decreasing left inverse $\eta : [\psi(\theta_P), +\infty[\to]\alpha_0, \theta_P]$, which is continuous on $[\psi(\theta_P), +\infty[$ and smooth on $]\psi(\theta_P), +\infty[$.

Fixed $\theta_M \in [\psi(\theta_P), +\infty[$, the angle $\theta_\omega = \eta(\theta_M) \in]\alpha_0, \theta_P]$ is the limit angle of the escape solution $\theta \mapsto \rho(\theta) = F_{\rho_0, \zeta_0, 0}(\theta, \theta_M), \theta \in [0, \theta_\omega[$, corresponding to initial conditions $(\rho_0, \zeta_0, 0, M)$. We notice that if $\theta_M \in [\psi(\theta_P), +\infty[$, it is

$$\zeta(\theta_\omega) = \rho'(\theta_\omega) = \partial_\theta F(\theta_\omega, \psi(\theta_\omega)) \leq 0,$$

where the equality holds if and only if $\theta_\omega = \theta_P$. By definition of U, we have

$$v = -\zeta M e_r(\theta) + \rho M e_\theta(\theta).$$

We infer that, if $\theta_M > \psi(\theta_P)$, the corresponding orbit is hyperbolic, since the terminal velocity v_ω satisfies

$$v_\omega = -\zeta(\theta_\omega) M(\theta_\omega) e_r(\theta_\omega) \neq 0, \quad M(\theta_\omega) > 0,$$

whereas, if $\theta_M = \psi(\theta_P)$, the corresponding orbit is parabolic, since $v_\omega = 0$.

To complete our analysis, we note that, if $\alpha_0 < \theta_M < \psi(\theta_P)$, then the solution $\theta \mapsto \rho(\theta) = F_{\rho_0,\zeta_0,0}(\theta,\theta_M)$, $\theta \in [0,\theta_\omega[$ corresponds to a collision solution with limit angle $\theta_\omega = \theta_M$. Summarizing, the limit angle θ_ω, as shown by Figure 1, is defined by the function

$$\theta_\omega = \begin{cases} \theta_M & \text{if} \quad \theta_M < \psi(\theta_P), \\ \eta(\theta_M) & \text{if} \quad \theta_M \geq \psi(\theta_P). \end{cases}$$

We conclude that this map is a smooth function of $y_0 = (\rho_0, \zeta_0, \theta_0, M_0)$ if $\theta_\omega \neq \theta_P$ (that is, on hyperbolic and on collision orbits) and is discontinuous at $\theta_M = \psi(\theta_P)$ (that is, on parabolic orbits), since

$$\lim_{\theta_M \to \psi(\theta_P)^+} \theta_\omega = \theta_P < \psi(\theta_P) = \lim_{\theta_M \to \psi(\theta_P)^-} \theta_\omega. \tag{43}$$

Then, taking into account (39), if $(x_0, v_0) \in \Omega^+$ there exists

$$I(x_0,v_0) = \lim_{t\to\omega} R(x(t),v(t)) = \lim_{\theta\to\theta_\omega} R \circ U \circ \gamma^\theta \circ U^{-1}(x_0,v_0)$$

and is given by

$$I(x_0,v_0) = \begin{cases} -e_r(\theta_M) & \text{if } \theta_M < \psi(\theta_P), \\ -\zeta(\theta_\omega)M^2(\theta_\omega)e_\theta(\theta_\omega) - e_r(\theta_\omega) & \text{if } \theta_M \geq \psi(\theta_P), \end{cases}$$

where for simplicity of notations we have not indicated explicitly the composition with U^{-1} of the functions θ_M, θ_ω and θ_P. It follows that I is smooth on $\Omega_H^+ \cup \Omega_C^+$, since this property holds for θ_ω.

To prove that I is discontinuous on parabolic orbits, fix any $(x_0,v_0) = U(\rho_0,\zeta_0,\theta_0,M_0) \in \Omega_P^+$. Then, we can consider solutions of (17) with initial conditions of the form

$$(x_0,v_0) = (\rho_0,\zeta_0,\theta_0,M_*) \in \Omega_H^+ \cup \Omega_C^+$$

with M_* in a neighborhood of M_0. By (43)

$$\lim_{\theta_{M_*}\to\psi(\theta_P)^+} I \circ U = -e_r(\theta_P) \neq \lim_{\theta_{M_*}\to\psi(\theta_L)^-} I \circ U = -e_r(\psi(\theta_P)),$$

so that I is discontinuous on Ω_P^+.

Note that I, as defined by property *iii*) of the statement, exists on all Ω. In fact, on the rectilinear motions, it is $I(x_0,v_0) = -\frac{x_0}{|x_0|}$. On solutions with negative scalar angular momentum, one can easily adapt the argument used in Ω^+, getting also the corresponding regularity results.

To complete the proof of the theorem, we observe that *i*) holds since the $SO(2)$ invariance of R is inherited by I. Then, by *i*), the continuity of I on Ω_H^+ and the property

$$\lim_{\theta_\omega\to(\alpha_0)^+} |I \circ U| = +\infty,$$

it follows that

$$I(\Omega) = \mathbb{R}^2 \setminus \operatorname{int}(\mathbb{D}),$$

where $\mathbb{D}$ is the closed unit disk in $\mathbb{R}^2$. ■

5 Conclusions

In this paper we have have considered a family of dissipative Kepler problems with drags of the form $-\frac{k}{|x|^\beta}\dot{x}$, and studied the changes in the forward dynamics as β increases. This family includes two physically meaningful dissipations: the Stokes drag ($\beta = 0$) and the Poynting-Plummer-Danby drag ($\beta = 2$).

We were able to detect a threshold value for the existence of escape orbits, namely $\beta = 1$, and we gave a fairly complete description of non rectilinear collision orbits for $\beta \geq \frac{3}{2}$, showing in particular that they are asymptotically rectilinear. Moreover, the integrability of the equation for $\beta = 2$ allowed us to prove that the asymptotic Runge-Lenz vector, which is a non-trivial first integral, is not continuous on the phase space, unlike for the case $\beta = 0$. We think that the jump discontinuity in the parabolic orbits is a general property for those values of β for which there are hyperbolic escapes and asymptotically rectilinear collisions. The reason is that discontinuity follows essentially from the fact that θ_ω is defined by $\rho(\theta_\omega^-) = 0$ for hyperbolic escapes, whereas for collisions $\rho(\theta_\omega^-) = +\infty$.

The dynamic behavior of non rectilinear collisions for the values of β in the complementary interval $]0, \frac{3}{2}[$ remains an open question. Our only contribution in this direction is the unboundedness of angular velocity of these solutions when $\beta \in]0, 1[$. However, the following informal argument, which we were not able to make rigorous, leads to a conjecture: substituting the third equation of (19) into the first, we obtain a fixed point equation of the form $\rho = T(\rho)$. From the results in [13], we see that, when $\beta = 0$, any fixed point defined for every $\theta \geq 0$, satisfies $\rho(\theta) \approx \theta^{2/3}$ for large θ. Then, if we look for fixed points defined for every $\theta \geq 0$ when $\beta > 0$, we can try to find a space of functions satisfying $\rho(\theta) \approx \theta^\alpha$, $\alpha > 0$, for large θ, and which is invariant under T. We are led, heuristically, to $\alpha = \frac{2}{3-2\beta}$, which is correct for $\beta = 0$. Now, if $\beta \in]0, \frac{3}{2}[$, it is not difficult to see that the previous asymptotic growth for $\rho(\theta)$ implies that the collision time ω is finite and that $\dot{\theta}(t) \to +\infty$ as $t \to \omega^-$. This last property is consistent with Theorem 5. Due to the difficulty of numerical integration of singular systems, the simulations made to support our conjecture were inconclusive.

Finally, in our work we gave a complete description of the rectilinear collisions as β increases, including their asymptotic development for $\beta \in]0, \frac{1}{2}[$. As a consequence, we were able to show that the presence of the singularity in the dissipation is an obstruction to the regularization of collisions.

6 Appendix

Proof. (*of Theorem 6*) This proof borrows some ideas from the ones of Proposition 2.4 in [14] and of Proposition 3.1 in [12], mainly in Case I. The corresponding steps are presented below with less detail.

We recall that our phase space is the half-plane $\Omega^0 = \{(r, u) : \ r > 0, u \in \mathbb{R}\}$. The isocline of system (34) associated to $\dot{r} = 0$ is the half line defined by $u = 0$, whereas for $\dot{u} = 0$ the isocline is defined by

$$u = -\frac{r^{\beta-2}}{k}.$$

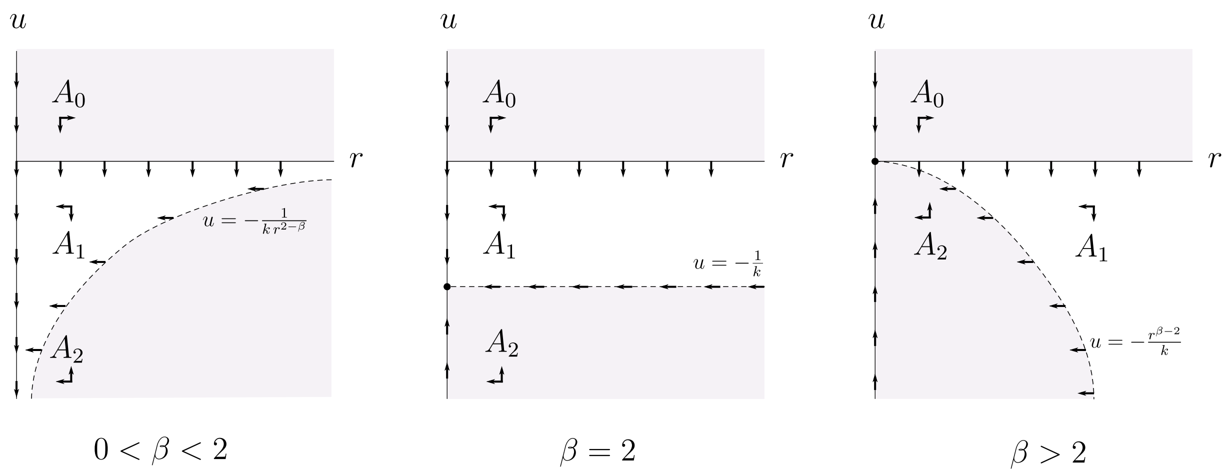

Figure 2: Dependence on β of the regions defined by the isoclines in the regularized systems.

These curves determine the following disjoint open regions in the phase space:

$$\begin{aligned} &A_0 = \{(r,u) \in \Omega^0 :\ u > 0\}, \quad A_1 = \left\{(r,u) \in \Omega^0 :\ 0 > k\,u > -r^{\beta-2}\right\}, \\ &A_2 = \left\{(r,u) \in \Omega^0 :\ k\,u < -r^{\beta-2}\right\}. \end{aligned} \tag{44}$$

The set A_0 is negatively invariant with respect to the flow of (34) for all $\beta > 0$, whereas for $\beta \in]0,2[$, A_1 is positively invariant and A_2 is negatively invariant. For $\beta > 2$, the set A_2 is positively invariant. We distinguish three cases.

Case 1: $\beta \in]0,2[$.

We prove first that ω is finite. If it were infinite, there should exist a sequence $t_n \to +\infty$ such that $u(t_n) \to 0$. However, this is not possible, because one can easily check that all collision solutions enter eventually into the positively invariant set A_1, where $u = \dot r$ is negative and decreasing.

In what follows, we consider the regularization of system (34) given by the time rescaling $d\mu = \frac{dt}{r^2}$, already considered in the proof of Theorem 5. Of course, we obtain a system made by the first two equations of (24) with $\varphi \equiv 0$, namely,

$$\begin{cases} \frac{dr}{d\mu} = r^2 u, \\ \frac{du}{d\mu} = -k\,r^{2-\beta}u - 1. \end{cases} \tag{45}$$

System (45) is defined in the extended phase space $\bar\Omega^0 = \Omega^0 \cup \{(r,u) :\ r = 0, u \in \mathbb{R}\}$ and the line $r = 0$ is an isocline orbit associated to $\frac{dr}{d\mu} = 0$ (see the left panel of Figure 2).
Let us prove now that, when $\beta \in]0,2[$, the velocity at collision satisfies $u_\omega = -\infty$. We argue by contradiction. Assume that $\mu \mapsto (r(\mu), u(\mu))$ is a collision solution such that $u_\omega \in]-\infty, 0[$. From the first equation of (45), we see that

$$\frac{1}{r(\mu)} = \frac{1}{r(0)} + \int_0^\mu |u(\sigma)| d\sigma. \tag{46}$$

Letting $\mu \to \mu_\omega$, we have that the left hand side tends to $+\infty$, and since $u(\tau)$ is bounded on

$[0, \mu_\omega[$, from (46) we get $\mu_\omega = +\infty$. But then, from the second equation of (45), it follows that

$$\lim_{\mu \to +\infty} \frac{du}{d\mu} = -1,$$

and this would imply that $u_\omega = -\infty$, contradicting the hypothesis that u_ω is finite. We conclude that $u_\omega = -\infty$.

Now let us see how the energy behaves when an orbit approaches the collision.

Consider first the case $\beta \in]0, 1[$. System (45) has the first integral

$$H(r, u, \mu) = u + \frac{k}{1-\beta} r^{1-\beta} + \mu, \tag{47}$$

obtained by setting $\varphi \equiv 0$ in (25).

Let (r_0, u_0) be an initial condition in A_1, and let $H_0 := H(r_0, u_0, 0)$, so that by (47) we have

$$u(\mu) + \frac{k}{1-\beta} r^{1-\beta}(\mu) + \mu = H_0. \tag{48}$$

Since $u(\mu) \to -\infty$ and $r(\mu) \to 0$ as $\mu \to \mu_\omega$, from (48) we infer that $\mu_\omega = +\infty$. Then, from the same equality it follows that $\frac{u(\mu)}{\mu} \to -1$ as $\mu \to +\infty$, and from (46) we get that $\mu^2 r(\mu) \to 2$ as $\mu \to +\infty$.

By (4), we see that $\frac{dE}{d\mu} = r^2 \frac{dE}{dt} = -k\, r^{2-\beta} u^2$. Then, using the two limits established above for u and r, we get that

$$\mu^{2(1-\beta)} \frac{dE}{d\mu}(\mu) \to -2^{2-\beta} k,$$

as $\mu \to \infty$. As a consequence,

$$E_\omega = E(0) + \int_0^\infty \frac{dE}{d\mu}(\sigma) d\sigma$$

is finite if $\beta \in]0, 1/2[$, whereas $E_\omega = -\infty$ if $\beta = [1/2, 1[$.

Let us consider now $\beta \in [1, 2[$.

In this case the approach through the first integral (given by $H = u + k \log r + \mu$, when $\beta = 1$, and by (47), when $\beta > 1$) does not allow to find out the asymptotic expansion of the solutions as they approach collision. However, we argue as follows. It is easy to see that in A_1 the trajectories of system (45) may be written in the form $u = \chi(r)$, $r \in]0, r_0]$. If we evaluate the slope of such orbits at the points of the form $u = -r^{\beta-1}$, we get

$$\left.\frac{du}{dr}\right|_{u=-r^{\beta-1}} = \frac{1}{r^\beta} \left(\frac{1}{r} - k \right),$$

which is positive for $0 < r < 1/k$. This implies that the region

$$D := \left\{ (r, u) : 0 < r < \frac{1}{k},\ -\frac{1}{k r^{2-\beta}} < u < -r^{\beta-1} \right\} \subset A_1$$

is positively invariant. On the other hand, note that

$$\frac{dE}{dr} = \frac{1}{u} \frac{dE}{dt} = -k \frac{u}{r^\beta} > 0.$$

Then, for every orbit $u = \chi(r)$ such that $(r_0, \chi(r_0)) \in D$, as a consequence of the invariance of D, we have

$$\frac{k}{r} < \left.\frac{dE}{dr}\right|_{u=\chi(r)} < \frac{1}{r^2},$$

and we conclude that

$$E_\omega = E(0) + \int_{r_0}^{0} \left.\frac{dE}{dr}\right|_{u=\chi(r)} dr = -\infty.$$

Case 2: $\beta = 2$.

This case is solved in [7]. The explicit solution is given by

$$u(\mu) = \left(u_0 + \frac{1}{k}\right) e^{-k\mu} - \frac{1}{k}, \qquad \frac{1}{r(\mu)} = \frac{1}{r(0)} - \int_0^\mu u(\sigma) d\sigma, \qquad \mu \in [0, \infty[.$$

From these expressions it is easy to see that $\omega < +\infty$, $u_\omega = -1/k$ and $E_\omega = -\infty$.

Case 3: $\beta > 2$.

In order to study the collisions for $\beta > 2$, it is convenient to consider the time rescaling $d\tau = \frac{dt}{r^\beta}$, introduced previously to get system (26). We obtain the following regular system, which corresponds to the first two equations of (26) with $\varphi \equiv 0$,

$$\begin{cases} \frac{dr}{d\tau} = r^\beta u \\ \frac{du}{d\tau} = -k\, u - r^{\beta-2}, \end{cases} \tag{49}$$

on the extended phase space $\bar{\Omega}^0$.

We note that the origin is an equilibrium of system (49) which attracts the points of the invariant line $r = 0$ (see the right panel of Figure 2).

One can see easily that all collision orbits will enter eventually in the positively invariant region A_2. Then, without loss of generality, we will consider only initial conditions in A_2. Actually, since in this region we have $\frac{du}{d\tau} > 0$ and $\frac{dr}{d\tau} < 0$, all solutions are bounded on $[0, \tau_\omega[$. This fact implies that all solutions starting in A_2 are collision ones, since otherwise an equilibrium of the system should exist in A_2. Moreover, any segment of orbit contained in A_2 may be expressed in the form $u = \chi(r)$, with r in a suitable interval of the form $]0, \tilde{r}[$, with $0 < r_0 < \tilde{r}$. Notice that, on this interval, $\chi(r)$ satisfies the scalar differential equation

$$\frac{du}{dr} := f(r, u) = -\frac{k}{r^\beta} - \frac{1}{r^2 u}. \tag{50}$$

When $\beta \geq 3$ the vector field associated to (49) is continuously differentiable on $\bar{\Omega}^0$ and, by the general theory of ODEs, we conclude that all solutions starting in A_2 tend to the equilibrium $(0, 0)$ in infinite τ time. We conclude that $u_\omega = 0$.

This cannot be guaranteed without further considerations for $\beta \in]2, 3[$. In fact, in this range of values the regularized vector field is not Lipschitz continuous in the points of the form $(0, u) \in \bar{\Omega}^0$. As a consequence, in such points uniqueness of solutions may fail, and all we can conclude by the general theory is that, for collision solutions, we have $-\infty < u_\omega \leq 0$. Let us prove that, actually, $u_\omega = 0$. Define the function $h_\lambda(r) := -\lambda r^{\beta-2}$, where $\lambda > 1/k$. Note that $A_2 = \cup_{\{\lambda>1/k\}}\{(r, h_\lambda(r)) : r > 0\}$. Evaluating the slope field du/dr at $u = h_\lambda(r)$, we get

$$\left(\frac{du}{dr}\bigg|_{h_\lambda}\right) \bigg/ \frac{dh_\lambda}{dr} = \frac{k\lambda - 1}{\lambda^2(\beta-2)r^{2\beta-3}}. \tag{51}$$

Hence, for any fixed $\lambda > 1/k$ there exists only one point $r = r_\lambda$ such that the graph of the function $h_\lambda(r)$ is tangent to an orbit, and is given by

$$r_\lambda := \left(\frac{k\lambda - 1}{\lambda^2(\beta-2)}\right)^{\frac{1}{2\beta-3}}. \tag{52}$$

Moreover, by (51) it follows that

$$\frac{dh_\lambda}{dr} \lesseqgtr f(r, h_\lambda(r)), \quad \text{if} \quad r \gtreqless r_\lambda. \tag{53}$$

As a consequence, by the comparison theorem for ODEs, the orbit $u = \chi(r)$ of (50) that is tangent to the curve $u = h_\lambda(r)$ in $r = r_\lambda$ satisfies

$$\chi(r) \geq h_\lambda(r),$$

for any $r \in]0, r_\lambda[$.

Note that (52) is a second degree equation in λ. Solving it, we obtain two local inverses of the function $\lambda \mapsto r_\lambda$, namely the functions

$$\lambda_\pm(r) := \frac{1}{r^{2\beta-3}} \frac{k \pm \sqrt{k^2 - 4(\beta-2)r^{2\beta-3}}}{2(\beta-2)},$$

defined for $0 < r \leq R$, where

$$R := \left(\frac{k^2}{4(\beta-2)}\right)^{\frac{1}{2\beta-3}}.$$

Now we use $\lambda_\pm(r)$ to construct the two following auxiliary functions:

$$h_\pm(r) = h_{\lambda_\pm(r)}(r) := -\frac{1}{r^{\beta-1}} \frac{k \pm \sqrt{k^2 - 4(\beta-2)r^{2\beta-3}}}{2(\beta-2)},$$

also defined for $0 < r \leq R$.

The functions h_+ and h_- satisfy the following properties. They are, respectively, strictly decreasing and strictly increasing on $]0, R]$, and such that $h_-(r) > h_+(r)$ on $]0, R[$, with $h_-(R) = h_+(R) := u_R$. Moreover, they have the following behavior as $r \to 0^+$:

$$h_-(r) = -\frac{r^{\beta-2}}{k} + o(r^{\beta-2}) \ \to 0 \ \text{ and } \ h_+ \to -\infty. \tag{54}$$

Finally, the range of h_+ is $]-\infty, u_R]$, and the one of h_- is $[u_R, 0[$. We are now ready to prove that on collision orbits $u_\omega = 0$.

We start by defining the positively invariant set

$$G := \{(r, u) : \ 0 < r \leq R, \ h_+(r) \leq u \leq h_-(r)\} \subset A_2.$$

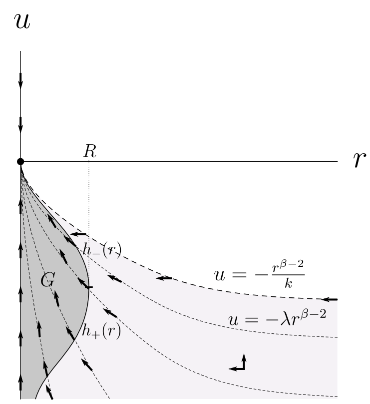

Figure 3: Illustration of the proof for $\beta \in]2, 3[$.

Given an initial condition $(r_0, u_0) \in G$, let us consider the corresponding orbit $u = \chi(r)$, $r \in]0, \tilde{r}[$. Since there exists a value $\lambda > 1/k$ such that $h_-(r) > \chi(r) > -\lambda r^{\beta-2}$ for all $0 < r < R$, we conclude that the orbit will go towards the equilibrium as $r \to 0^+$. If $(r_0, u_0) \in A_2 \backslash G$ the corresponding orbit will eventually enter in G. In fact, if there exists an orbit $u = \bar{\chi}(r)$ for which this is not the case, we can find $\bar{\lambda} = \lambda_-(\bar{r})$ such that $u = h_{\bar{\lambda}}(r)$ intersects $u = \bar{\chi}(r)$ in a point $\hat{r} > \bar{r}$ for which it holds the inequality $\frac{dh_{\bar{\lambda}}}{dr}(\hat{r}) > f(\hat{r}, h_{\bar{\lambda}}(\hat{r}))$. By (53), there exists a second point, $r_* > \hat{r} > \bar{r}$, such that the curve $u = h_{\bar{\lambda}}(r)$ is tangent to an orbit (the first being $\bar{r}$), which is absurd.

Then, $u_\omega = 0$ for all collision orbits also for $\beta \in]2, 3[$. Taking into account what was proved previously, we conclude that $u_\omega = 0$ for any $\beta > 2$. It follows immediately that the energy $E = u^2/2 - 1/r$ tends to $E_\omega = -\infty$.

Also, since

$$\omega = \int_{r_0}^{0} \frac{dr}{\chi(r)}, \tag{55}$$

by (54) and by the inequality $-\lambda r^{\beta-2} < \chi(r) < h_-(r)$ for any $r \in]0, R[$, we see that $1/|\chi(r)|$ is integrable on the interval $]0, r_0]$ if $\beta \in]2, 3[$, in which case ω is finite, whereas it is not integrable if $\beta \geq 3$, and then $\omega = +\infty$. Our proof is concluded. ■

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