On the use of Morse index and rotation numbers for multiplicity results of resonant BVPs

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Abstract

Motivated by a recent series of papers by K. Li and co-workers ([15, 16, 17, 18, 19]), we consider the problem of the existence and multiplicity of solutions for the Neumann or periodic BVPs associated to a class of scalar equations of the form $x'' + f(t,x) = 0$. The class considered is such that the behaviour of its solutions near zero and near infinity may be compared with the behaviour of the solutions of two suitable linear

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systems, one considered near zero and the other near infinity. We show how a rotation numbers approach, together with the Poincaré-Birkhoff theorem and a recent variant of it, allow to obtain multiplicity results in terms of the gap between the Morse indexes of the referred linear systems at zero and at infinity. These systems may also be resonant. When the gaps are sufficiently large, our multiplicity results improve the ones obtained by variational methods in the quoted papers. Also, our approach allows a description of the solutions obtained in terms of their nodal properties.

Keywords: Morse index, rotation number, Neumann problem, periodic solutions, Poincaré-Birkhoff Theorem

1 Introduction

In a recent series of papers [15, 16, 17, 18, 19], K. Li and co-workers have studied the asymptotically linear Duffing equation with Neumann or periodic boundary conditions by using a variational approach combined with Morse index and Lusternik-Schnirelmann theories. A different approach to tackle this problem is to use more geometrical arguments like shooting methods, rotation numbers and variants of the Poincaré-Birkhoff theorem (see [11, 21, 10] and the references therein). In this paper we use this last approach and obtain results which complement or extend some of the results previously obtained.

We are interested on the existence and multiplicity of solutions of the equation

$$x'' + f(t,x) = 0 \tag{1}$$

satisfying Neumann or periodic conditions. We assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitz continuous with respect to the second variable in order that uniqueness for the associated Cauchy problems is guaranteed. Weaker conditions could be assumed as well, see [12]. Also, we assume that zero is a solution of (1), that is $f(t,0) \equiv 0$. Hence, due to the uniqueness of solutions $x(t)$ of (1), a non-trivial solution curve $(x(t), x'(t))$ in the phase plane never passes through the origin, so that its rotation number around the origin is well defined.

The problems that we consider will have some kind of asymptotically linear property at the origin and at infinity. More precisely, let us consider for $i = 0, \infty$ the following assumptions

(H_i^l) there exists a function $a_i \in L^1([0,T])$ such that $a_i(t) \leq \liminf_{x \to i} \dfrac{f(t,x)}{x}$ uniformly a.e. in $t \in [0,T]$

(H_i^r) there exists a function $b_i \in L^1([0,T])$ such that $\displaystyle\limsup_{x\to i} \frac{f(t,x)}{x} \leq b_i(t)$ uniformly a.e. in $t \in [0,T]$.

In all our results but the ones of the last section, we will assume that either (H_0^l) or (H_0^r) holds and that *both* (H_∞^l) and (H_∞^r) hold. In fact, although for each i only one of the two assumptions (H_i^l), (H_i^r) will be used at a time to estimate the rotations of the solutions of (1) at zero and at infinity, assumptions (H_∞^l) and (H_∞^r) together guarantee that all solutions of Cauchy problems associated to (1) are defined globally on $[0,T]$.

Existence and multiplicity of solutions of boundary value problems associated to equation (1) under asymptotically linear conditions have been studied by many authors. The multiplicity results are obtained assuming a gap between the behaviour at zero and infinity, we refer to [12, 24] for recent results in this line and for more references about the problem. In what concerns the periodic problem, in [24] the author analyses a relation obtained in [13] between rotation numbers and eigenvalues of a Hill's equation. Multiplicity results are obtained using this relation and the Poincaré-Birkhoff theorem. With respect to the Dirichlet problem, in [12] the authors give a relation between rotation numbers and eigenvalues of the linear problems. In this way multiplicity results are obtained using a topological approach. Their results are obtained using a shooting approach without uniqueness.

In Section 2 of our paper, using the same approach as in [12], a detailed relation between Morse index and rotation numbers, even in the case of resonance, is given and multiplicity results for the Neumann problem are obtained. For the periodic problem, in Section 3 we generalize the auxiliary result given in [24] obtaining a relation between Morse index and rotation numbers which also contemplates the case of non-zero nullities. As in [24] our auxiliary result follows from previous results in [13]. Then we can apply the Poincaré-Birkhoff and the modified Poincaré-Birkhoff theorems ([12, 21, 10]) obtaining multiplicity of solutions. When the modified Poincaré-Birkhoff theorem is applied we assume (H_0^l) and (H_0^r) and $a_0 = b_0$. Hence we can guarantee the existence of a linearization of (1) at zero. In this way we can extend the polar coordinates Poincaré map to all the closed half plane $r \geq 0$ and apply the theorem.

In Sections 3 and 4 we compare the results previously mentioned obtained by the use of variational methods with ours. When the gaps between the Morse-indexes at zero and at infinity are large, we obtain a larger number of solutions. In the case when the indexes at zero and at infinity are the same, with our approach we are not able to obtain solutions. The solutions obtained in these cases by K. Li and his co-authors depend on a Landesman-

Lazer type condition, relations between this condition and rotation numbers deserve an independent investigation.

2 Neumann conditions

Our aim is to study the problem with Neumann conditions

$$\begin{cases} x'' + f(t,x) = 0 \\ x'(0) = 0 = x'(T) \end{cases} . \tag{2}$$

Let us recall the following classical result (see [9], [20]) .

Proposition 2.1 *Given $q \in L^1([a,b])$ there exists a sequence of eigenvalues*

$$\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \ldots < \lambda_j \to +\infty, \ \text{as } j \to +\infty$$

of the problem

$$\begin{cases} x'' + (\lambda + q(t))x = 0 \\ x'(0) = 0 = x'(T) \end{cases} . \tag{3}$$

For each j, there exists a space of dimension one of nontrivial solutions (eigenvectors) of the problem.

Definition 1 *Given $q \in L^1([a,b])$, the Morse index $i(q)$ is defined as the number of negative eigenvalues of the problem* (3)*, whereas the nullity $\nu(q)$ is the dimension of the space of solutions of the problem* (3) *associated to $\lambda = 0$.*

With this definition, the following results are straightforward.

Lemma 2.2 *Let $q \in L^1(a,b)$. Then, $i(q) = j$ if and only if $\lambda_j(q) < 0 \leq \lambda_{j+1}(q)$. Moreover, if $\nu(q) = 0$ then $\lambda_{j+1}(q) > 0$, while $\lambda_{j+1}(q) = 0$ if $\nu(q) = 1$.*

Theorem 2.3 *Let us assume (H_0^r), (H_∞^l) and (H_∞^r). If $i(a_\infty) > i(b_0)+\nu(b_0)$, there are at least $i(a_\infty) - i(b_0) - \nu(b_0)$ pairs of solutions of problem* (2)*, that can be ordered according to the exact number of zeroes h on $[0,T]$ with $h = i(b_0) + \nu(b_0) + 1, \ldots, i(a_\infty)$.*

Proof. This result is a direct consequence of the results contained in [12]. Assume for instance that $i(a_\infty) > i(b_0)$, $\nu(b_0) = 0$. By Lemma 2.2, calling for convenience $i(a_\infty) = n, i(b_0) = m - 1$, we get

$$\lambda_n(a_\infty) < 0 \leq \lambda_{n+1}(a_\infty), \qquad \lambda_{m-1}(b_0) < 0 < \lambda_m(b_0).$$

Hence,

$$\lambda_n(a_\infty) < 0 < \lambda_m(b_0),$$

and $n \geq m$ which is just [12, Condition (3.25)]. Then, [12, Theorem 3.13] (in the version for the eigenvalue problem (3) with Neumann conditions, see the detailed explanation at the end of Section 3) provides the existence of $n - m + 1$ (that is, $i(a_\infty) - i(b_0)$) pairs of solutions of problem (2).

Consider now that $i(a_\infty) > i(b_0) + 1$, $\nu(b_0) = 1$. Calling again $i(a_\infty) = n, i(b_0) = m - 1$, we get

$$\lambda_n(a_\infty) < 0 < \lambda_{m+1}(b_0).$$

By using again Theorem 3.13 from [12], we get $n-m$ (that is $i(a_\infty)-i(b_0)-1$) pairs of solutions of problem (2). ■

Theorem 2.4 *Let us assume* (H_0^l), (H_∞^l) *and* (H_∞^r). *If* $i(a_0) > i(b_\infty) + \nu(b_\infty)$, *there are at least* $i(a_0) - i(b_\infty) - \nu(b_\infty)$ *pairs of solutions of problem* (2), *that can be ordered according to the exact number of zeroes* h *on* $[0, T]$ *with* $h = i(b_\infty) + \nu(b_\infty) + 1, \ldots, i(a_0)$.

Proof. The proof is analogous. ■

Theorem 2.4 generalizes [17, Theorem 1.1]. It turns out that the Landesman-Lazer condition as well as condition (A_1) assumed in [17, Theorem 1.1] can be dropped. On the other hand, [17, Theorem 1.2] is complemented. In fact condition (H_1') therein means that the Morse index of a_0 (b_∞ in our case) is 1. Our result also generalizes most of the situations considered by the results in [15], although two particular cases are not covered, where the Landesman-Lazer condition seems to play a key role. Such cases are

$$i(a_\infty) = \nu(b_\infty) = 0 < i(b_\infty) = \nu(a_\infty) = 1 \text{ and } i(a_0) = \nu(a_0) = i(b_0) = \nu(b_0) = 0,$$

and

$$i(b_\infty) = 0 < \nu(b_\infty) = 1 \text{ and } \nu(a_0) = \nu(b_0) = 0 < i(a_0) = i(b_0) = 1.$$

Landesman-Lazer conditions are classical and recently a relation with the rotation number has been reported in [6], this opens a possibility that should be explored elsewhere.

3 Periodic problem

In this section we address the study of the periodic problem

$$\begin{cases} x'' + f(t,x) = 0 \\ x(0) = x(T) \\ x'(0) = x'(T), \end{cases} \tag{4}$$

where we assume that $f(t,0) = 0$.
In the case of periodic conditions we have the following result (see [9], [14]) :

Proposition 3.1 *Given $q \in L^1([a,b])$ there exists a sequence of eigenvalues*

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \ldots < \lambda_{2j-1} \leq \lambda_{2j} \to +\infty, \ \text{as } j \to +\infty$$

of the problem

$$\begin{cases} x'' + (\lambda + q(t))x = 0 \\ x(0) = x(T) \\ x'(0) = x'(T) \end{cases} .$$

For each j, there exists a space of dimension one of nontrivial solutions (eigenvectors) of the problem.

In this case the definitions of Morse index and nullity of $q(\cdot)$ are analogous to the ones given in Definition 1, replacing the Neumann boundary conditions with the periodic ones and counting each eigenvalue with its multiplicity.

Let $q \in L^1([0,T])$. Following [13] we consider the system

$$\begin{cases} x' = -y \\ y' = q(t)x \end{cases} \tag{5}$$

and the change of variables $x = r\cos\theta$, $y = r\sin\theta$. For each $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we consider the angular coordinate $\theta(t;(x_0,y_0))$, $t \in [0,T]$ of the solution of (5) satisfying $x(0) = x_0$ and $y(t_0) = y_0$. The corresponding rotation number, defined by

$$\mathrm{rot}_{(x_0,y_0)}(q) = \frac{\theta(T;(x_0,y_0)) - \theta(0;(x_0,y_0))}{2\pi}$$

counts the total angular variation around the origin in the time interval $[0,T]$.

From [13] we have the following result, which links the rotation number of the solutions of the Hill's equation $x'' + q(t)x = 0$ to the index and nullity of $q(\cdot)$.

Theorem 3.2 *Let $q \in L^1(0,T)$ and consider the problem*

$$\begin{cases} x'' + q(t)x = 0 \\ x(0) = x(T) \\ x'(0) = x'(T). \end{cases} \tag{6}$$

Assume that $i(q) = j$ and $\nu(q) \leq 1$. Then

i) if j is even different from zero then for each (x_0, y_0) we have

$$\frac{j}{2} - 1 < rot_{(x_0,y_0)}(q) < \frac{j}{2} + 1.$$

Moreover if $\nu(q) = 0$ then there exist (x_i, y_i), $i = 1, 2$, such that

$$rot_{(x_1,y_1)}(q) < \frac{j}{2} < rot_{(x_2,y_2)}(q)$$

while in the case that $\nu(q) = 1$ there exist (x_i, y_i), $i = 1, 2$, such that

$$rot_{(x_1,y_1)}(q) > \frac{j}{2} = rot_{(x_2,y_2)}(q)$$

ii) if j is odd then for each (x_0, y_0) we have

$$\frac{j-1}{2} < rot_{(x_0,y_0)}(q) < \frac{j+1}{2} + \nu(q).$$

iii) if $j = 0$ then for each (x_0, y_0) we have

$$rot_{(x_0,y_0)}(q) < 1$$

and in the case $\nu(q) = 0$ there is (x_0, y_0) such that

$$rot_{(x_0,y_0)}(q) < 0.$$

Proof. The result is a consequence of [13, Proposition 2.3.] taking into account that the rotation of the solutions of $x'' + (\lambda + q(t))x = 0$ is increasing in λ. ∎

The relation established above between $i(q)$, $\nu(q)$ and the rotation numbers of solutions of Hill's equation allow to use the Poincaré-Birkhoff Theorem (PB for short) and its variant (MPB, for Modified PB Theorem) to obtain multiplicity results of periodic solutions of (4) in terms of the gap between the Morse indexes and the nullities of two suitable linear systems which allow to estimate the rotations of the solutions of the first order system associated to (4) near, respectively, $(0,0)$ and ∞.

In fact, in our framework, the PB theorem (see [21] for a precise statement) gives the existence of two T- periodic solutions with rotation number j for any integer j in the open interval $]\max \mathrm{rot}(b_0), \min \mathrm{rot}(a_\infty)[$ or for any integer j in the open interval $]\max \mathrm{rot}(b_\infty), \min \mathrm{rot}(a_0)[$. Here $\max \mathrm{rot}(b_0)$ denotes the maximum value of the rotation of a solution of equation $x'' + b_0(t)x = 0$, and analogous definitions hold for the other quantities. Moreover, the MPB theorem (see [21] for the exact statement) gives *one* T-periodic solution with rotation number j if the equation $x''+b_0(t)x = 0 \quad (x''+a_0(t)x = 0)$ has *one* solution which rotates less than j (more than j) whereas *all* solutions of the equation $x'' + a_\infty(t)x = 0 \quad (x'' + b_\infty x = 0)$ rotate more than j (less than j). It is important to recall that, unlike for the PB theorem, in the MPB setting the origin and infinity do not play a symmetric role, in the sense that it is not possible, in general, to have a periodic solution interchanging the conditions about the rotations of solutions near zero and near infinity (see [10]). Also in order to apply this theorem we must have invariance of one of the boundaries, in this case we will ask that the lifting of the Poincaré map to the polar coordinate system is defined in $r = 0$ which is invariant. This will be guaranteed by assuming (H_0^l) and (H_0^r) and $a_0 = b_0$. As a consequence, the MPB allows to increase the number of solutions only in some cases when the Morse index at the origin is even.

Theorem 3.3 *Assume (H_∞^r) and (H_∞^l). Then:*

i) If (H_0^r) holds, $\nu(b_0) \leq 1$ and $\nu(a_\infty) \leq 1$ then:

- *if $i(b_0)$ is odd and*
  - *$i(a_\infty)$ is even and $i(a_\infty) - i(b_0) > 1 + 2\nu(b_0)$ there are at least $i(a_\infty) - i(b_0) - 1 - 2\nu(b_0)$ nontrivial solutions of* (4);
  - *$i(a_\infty)$ is odd and $i(a_\infty) - i(b_0) > 2\nu(b_0)$ there are at least $i(a_\infty) - i(b_0) - 2\nu(b_0)$ nontrivial solutions of* (4).
- *if $i(b_0)$ is even and*
  - *$i(a_\infty)$ is even and $i(a_\infty) - i(b_0) > 2$ there are at least $i(a_\infty) - i(b_0) - 2$ nontrivial solutions of* (4);
  - *$i(a_\infty)$ is odd and $i(a_\infty) - i(b_0) > 1$ there are at least $i(a_\infty) - i(b_0) - 1$ nontrivial solutions of* (4).
  - *if moreover (H_0^l) holds and $a_0 = b_0$ then in the last two cases one more solution exists if $\nu(b_0) = 0$.*

ii) If (H_0^l) *is satisfied,* $\nu(a_0) \leq 1$ *and* $\nu(b_\infty) \leq 1$ *then:*

- *if* $i(b_\infty)$ *is even and*
  - $i(a_0)$ *is even and* $i(a_0) - i(b_\infty) > 2$ *there are at least* $i(a_0) - i(b_\infty) - 2$ *nontrivial solutions of* (4). *If moreover* (H_0^r) *holds and* $a_0 = b_0$ *then one more solution exists.*
  - $i(a_0)$ *is odd and* $i(a_0) - i(b_\infty) > 1$ *there are at least* $i(a_0) - i(b_\infty) - 1$ *nontrivial solutions of* (4).
- *if* $i(b_\infty)$ *is odd then*
  - *if* $i(a_0)$ *is even and* $i(a_0) - i(b_\infty) > 1 + 2\nu(b_\infty)$ *there are at least* $i(a_0) - i(b_\infty) - 1 - 2\nu(b_\infty)$ *nontrivial solutions of* (4). *If moreover* (H_0^r) *holds and* $a_0 = b_0$ *then one more solution exists.*
  - *if* $i(a_0)$ *is odd and* $i(a_0) - i(b_\infty) > 2\nu(b_\infty)$ *there are at least* $i(a_0) - i(b_\infty) - 2\nu(b_\infty)$ *nontrivial solutions of* (4).

Proof. The result is an easy consequence of Theorem 3.2, of [12, Lemmas 3.4, 3.6, 3.7], and of the PB and MPB theorems [21, 10]. Although there are many occurrences to consider, to avoid the repeated use of the very similar arguments (which can also be found in an analogous proof of [21, Theorem 2]) we just outline the proof of a case which shows how to use the results quoted above.

More precisely, we prove $i)$ with $i(b_0)$ *even* (possibly 0) and $i(a_\infty)$ *odd.*

By [12, Lemma 3.6, 3.7] the so called *elastic property* for the solutions of the first order planar system

$$x' = -y, \; y' = f(t, x) \tag{7}$$

associated to $x'' + f(t, x) = 0$ holds. Namely, for any $0 < r < R$ there exists $r_1 < r$ and $R_1 > R$ such that for any solution of (7) with initial value $(x(0), y(0)) = (x_0, y_0)$ satisfying $|(x_0, y_0)| < r_1$ it is $|(x(t), y(t))| \leq r, \; t \in [0, T]$ and for any solution of (7) for which $|(x_0, y_0)| > R_1$ it is $|(x(t), y(t))| \geq R, \; t \in [0, T]$. As a consequence, denoting by $\mathrm{rot}_{(x_0,y_0)}(f)$ the rotation of the solution of (7) starting from (x_0, y_0) when $t = 0$, by [12, Lemma 3.4] we have that, fixed any ϵ, there exist $0 < r < R$ such that

$$\mathrm{rot}_{(x_0,y_0)}(f) \leq \mathrm{rot}_{(x_0,y_0)}(b_0) + \epsilon, \; \forall |(x_0, y_0)| \leq r,$$

and

$$\mathrm{rot}_{(x_0,y_0)}(f) \geq \mathrm{rot}_{(x_0,y_0)}(a_\infty) - \epsilon, \; \forall |(x_0, y_0)| \geq R.$$

By (H_0^r) and Theorem 3.2 we get that for all solutions of (7) starting sufficiently close to $(0,0)$ it is

$$\mathrm{rot}_{(x_0,y_0)}(f) < \frac{i(b_0)}{2} + 1 \tag{8}$$

Moreover, the same theorem guarantees that if $\nu(b_0) = 0$ there exists (x_1, y_1) such that the corresponding solution of (7) satisfies

$$\mathrm{rot}_{(x_1,y_1)}(f) < \frac{i(b_0)}{2}. \tag{9}$$

On the other hand, (H_∞^l) and Theorem 3.2 imply that all solutions of (7) with sufficiently large initial condition satisfy

$$\mathrm{rot}_{(x_0,y_0)}(f) > \frac{i(a_\infty) - 1}{2}. \tag{10}$$

Then, the PB theorem guarantees that for any integer k in the interval $[\frac{i(b_0)}{2} + 1, \frac{i(a_\infty)-1}{2}]$ there exist two T-periodic solutions of (4) with rotation number k. This gives $i(a_\infty) - i(b_0) - 1$ solutions. Notice that in order to apply the PB theorem we do not need to have the lifting of the Poincaré map associated to the polar coordinates defined in $\mathbb{R} \times [0, +\infty[$ and hence we do not need to have the linearization in zero. Indeed it is not needed to guarantee the invariance of one of the boundaries of the annulus in which we apply the PB unlike in the MPB theorem.

Now let us assume (H_0^l) holds and $a_0 = b_0$. In this case we can consider the lifting of the Poincaré map associated to the polar coordinates defined in $\mathbb{R} \times [0, +\infty[$. Moreover notice that if $\nu(b_0) = 0$ then (9) holds and since $\frac{i(b_0)}{2} \le \frac{i(a_\infty)-1}{2}$ we can apply the MPB theorem, to get a T-periodic solution which rotates $\frac{i(b_0)}{2}$ and is therefore different from the ones detected with the PB theorem. ∎

Remark 1 *The proof of Theorem 3.3 provides precise information about the rotation number of the solutions, which can be used to fix the exact number of zeroes. This happens as well in the next results, but further details are omitted for the sake of brevity.*

Remark 2 *If compared with [18], when the gaps of the linearizations at zero and at infinity are large, even in the case of nonzero nullities, Theorem 3.3 generalizes theorems 1 and 2 in that paper. However, some special resonant* *cases when* $i(b_0) = i(a_\infty)$ *or* $i(a_0) = i(b_\infty)$ *are not covered by our results. Hence, [18, Theorem 1] applies to* $i(b_0) = i(a_\infty) = 0$ *and [18, Theorem 2] to* $i(a_0) = i(b_\infty), \nu(a_0) = 2$*, at the cost of assuming a kind of Landesman-Lazer condition. As in the case of Neumann boundary conditions, a more detailed study of these resonant cases should be done in the future.*

Remark 3 *In the previous theorem we assumed the nullities less or equal to one. In the case of nullities two we can also obtain multiplicity results provided that the gaps between the indexes are larger. Note that due to the structure of the spectrum in the case of periodic boundary conditions, nullity two can appear only associated to odd indexes. Moreover in the case of* $i(q) = 2j + 1$ *for some nonnegative* j*, if* $\nu(q) = 2$ *we have that all the solutions of the linear problem*

$$\begin{cases} x'' + q(t)x = 0 \\ x(0) = x(T) \\ x'(0) = x'(T). \end{cases} \tag{11}$$

rotate $j + 1$ *times around the origin [13, Proposition 2.3.] .*

Let us for example assume, in the setting of the previous theorem, that $i(b_0) = 2j + 1$ *odd and* $\nu(b_0) = 2$ *and that* $i(a_\infty) = 2l$ *even and* $\nu(a_\infty) < 2$*. Then we have that near the origin the solutions of the problem we are considering rotate less than* $j + 2$ *and large solutions rotate at least* $l - 1$*. Let us assume that* $i(a_\infty) - i(b_0) > 3$ *In this way we can obtain, by applying the PB theorem, at least* $2(l - j - 2)$ *periodic solutions, that is, at least* $i(a_\infty) - i(b_0) - 3$ *solutions. The other cases can be treated similarly. We opted, for conciseness not to analyse all the variants.*

Next we obtain simple variants of Theorem 3.3 which generalize or complement the results in [16, 18, 19]. In our results less conditions are required, and in particular we do not need to assume Landesman- Lazer conditions. Moreover when the difference of the indexes at zero and at infinity is large, a large number of nontrivial solutions is obtained while in the result in the mentioned papers the existence of at most two nontrivial solutions is guaranteed.

These results are not a direct consequence of the previous one when there is not a gap between the indexes at zero and at infinity. Nevertheless they can be obtained in a similar way.

Proposition 3.4 *Assume* $(H_0^r), (H_0^l)$ *and* $(H_\infty^r), (H_\infty^l)$ *and that* $a_0 = b_0$*. Then if* $i(b_0) = 0$*,* $\nu(b_0) = 0$ *and there exists* $r > 0$ *such that* $\dfrac{f(t,x)}{x} \geq 0$ *for* $|x| \geq r$*, there exists at least* 1 *nontrivial solution of* (4)*.*

Proof. Note that under the condition assumed at ∞, we have that either there is an equilibrium with large norm, and hence a nontrivial periodic solution, or, for initial conditions (x_0, y_0) with large norm the rotation of the corresponding solutions will be positive. In fact, by the elastic property (see for example [12]), we can guarantee that there exists $R > 0$ such that if $\|(x_0, y_0)\| > R$ then for each $t \in [0, T]$ either $x(t)f(t, x(t)) \geq 0$ or $y^2(t)+x(t)f(t, x(t)) \geq 0$. Now the result follows from the Modified Poincaré-Birkhoff theorem as by Theorem 3.2 there is an initial condition near zero such that the corresponding solution rotates in a negative sense. ■

This result, together with Theorem 3.3, generalizes [16, Theorem 1.1].

In what concerns the main result in [19], if we interchange the roles of zero and ∞ we have the following complementary result.

Proposition 3.5 *Assume* $(H_0^r), (H_0^l)$ *and* $(H_\infty^l), (H_\infty^r)$ *and that* $a_0 = b_0$. *Suppose that* $b_0(\cdot) \leq 0$ *with* $\int_0^T b_0(t)dt < 0$. *Then if* $i(a_\infty) \geq 1$ *there exist at least* $i(a_\infty)$ *nontrivial solutions of* (4) *if* $i(a_\infty)$ *is odd and* $i(a_\infty) - 1$ *nontrivial solutions of* (4) *in the case it is even.*

Proof. First of all note that as $\limsup_{x\to 0} \dfrac{f(t,x)}{x} \leq b_0(t) \leq 0$ and $\int_0^T b_0(t)dt < 0$, for each (x, y) near the origin $\mathrm{rot}_{(x,y)(f)} < 1$. Also there exists an initial condition (x_0, y_0) and a positive ε such that for each $0 < r < \varepsilon$, $\mathrm{rot}_{r(x_0,y_0)(f)} < 0$. These facts can be proved following the same reasoning as in [5, Proof of Claim 1 in Theorems 3.1. and 3.2]. Then we conclude the existence of the nontrivial periodic solutions as in the previous results using the Poincaré-Birkhoff and the modified Poincaré-Birkhoff theorems. ■

We will come back to this result ([19, Theorem 1.1]) in more detail in the next section.

4 The periodic problem with a sign condition at infinity.

In this section we focus on the main result of [19], which is recalled here for convenience.

Theorem 4.1 *[19, Theorem 1.1] Assume that* f *satisfies the following conditions*

(H_1) $\int_0^T f(t,+\infty)dt < 0 < \int_0^T f(t,-\infty)dt.$

(H_2) *There exists a constant* $r > 0$ *such that* $\frac{f(t,x)}{x} \leq 0$ *for* $|x| \geq r$,

(H_3) $f(t,0) = 0, i(f'(t,0)) \geq 1$ *and* $\nu(f'(t,0)) = 0$.

Then (4) *has at least two nontrivial solutions.*

This result can be applied directly to the model example

$$x'' + a_0(t)x = g(t)x^3, \tag{12}$$

under suitable conditions on a_0 and g. Eq. (12) is of special interest for a variety of physical models related with Bose-Einstein condensates and optical fiber transmission (see [1, 2, 3, 22, 23] and the references therein). It is easy to verify that Theorem 4.1 generalizes the main result in [22].

Clearly, condition (H_2) is just a sign condition at infinity, and it is equivalent to (H^r_∞) with $b_\infty(t) \equiv 0$. Note that the results of Section 3 are not directly applicable because to get them we need the additional condition (H^l_∞) . However, by using a truncation argument, we are able to prove the following result.

Theorem 4.2 *Assume that* f *satisfies* (H^l_0) *and* (H_2). *Suppose* $i(a_0) > 1$ *and* $\nu(a_0) \leq 1$. *Then* (4) *has at least* $i(a_0) - 1$ *nontrivial solutions if* $i(a_0)$ *is odd at least* $i(a_0) - 2$ *nontrivial solutions if* $i(a_0)$ *is even. If moreover* (H^r_0) *holds and* $a_0 = b_0$ *then also in the case* $i(a_0)$ *even the existence of* $i(a_0) - 1$ *nontrivial solutions is guaranteed.*

Before going to the proof, some comments are convenient. Under the conditions of Theorem 4.1, our result does not cover the case $i(a_0) = 1$ and for $i(a_0) = 2$ we only get one solution. On the other hand, our result covers resonant cases and shows a novel link between multiplicity and Morse index at the origin. This result is very much related with the recent papers [4, 7, 8] and in fact the general strategy of proof is similar, but one can see that our results are independent.

Proof. We perform the proof in several steps:

Step 1: First, we prove that it is not restrictive to assume that there exists $R > r$ such that

$$x\int_0^T f(t,x)dt < 0 \qquad \text{for all } |x| \geq R. \tag{13}$$

In fact, if (13) does not hold there exists a sequence $\{x_n\}_n \to +\infty$ (or $-\infty$) such that $\int_0^T f(t,x_n)dt = 0$. Then (H_2) implies that $f(t,x_n) = 0$ for a.e. t, and hence x_n is a sequence of periodic (constant) solutions, and our result is proved.

Step 2: Every solution of problem (4) verifies $|x(t)| \leq R$ for all t. By contradiction, let us assume that $\max_t x(t) > R$ (the argument with the minimum is analogous). Let $x(t_1) = \max_t x(t)$. The first remark is that $x(t)$ cannot be constant because of condition (13) (integrating the equation over a period would give a contradition). Then, there must be $t_0 < t_1$ such that $x'(t_0) > 0$ and $x(t) > R$ for all $t_0 < t < t_1$. Integrating the equation on $[t_0, t_1]$ we get

$$x'(t_0) = \int_{t_0}^{t_1} f(t,x(t))dt < 0,$$

which is a contradiction.

Step 3: Consider the truncated equation $x'' + \tilde{f}(t,x) = 0$ with

$$\tilde{f}(t,x) = \begin{cases} f(t,x) & \text{if } |x| < R \\ f(t,R) & \text{if } |x| \geq R \end{cases}.$$

Then, $\tilde{f}(t,x)$ is bounded and fully verifies the hypothesis of Theorem 3.3. Note that $b_\infty(t) \equiv 0$, so $i(b_\infty) = 0, \nu(b_\infty) = 1$. Theorem 3.3 provides the existence of nontrivial periodic solutions of the truncated equation, but they are in fact solutions of the original problem (4) by Step 2. The proof is finished.

■

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