Almost surely recurrent motions in the Euclidean space

MARKUS KUNZE[1] & RAFAEL ORTEGA[2]

[1] Universität Köln, Institut für Mathematik, Weyertal 86-90,
D - 50931 Köln, Germany

[2] Departamento de Matemática Aplicada, Universidad de Granada,
E-18071 Granada, Spain

Key words: discrete dynamical systems, infinite measure space, recurrence

2010 *Mathematics Subject Classification.* 37A05, 37B20, 37J10, 37J45

Abstract

We will show that measure-preserving transformations of $\mathbb{R}^n$ are recurrent if they satisfy a certain growth condition depending on the dimension n. Moreover, it is also shown that this condition is sharp.

1 Introduction

To illustrate the topic of this paper, consider a fluid in $\mathbb{R}^3$ whose motion is governed by a C^1-velocity field $v = v(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ which is T-periodic in time and satisfies

$$\mathrm{div}_x v(t, x) = 0 \quad \text{for each} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3$$

and

$$v(t, x) = \mathcal{O}(|x|^{-\alpha}) \quad \text{as} \quad |x| \to \infty, \text{ uniformly in } t.$$

We are interested in the recurrence properties of the solutions to the system $\dot{x} = v(t, x)$. Indeed it is sufficient to ask about the recurrence properties of the measure-preserving Poincaré map

$$h : x(0) \mapsto x(T) = x(0) + \int_0^T v(s, x(s))\, ds.$$

In general the Poincaré recurrence theorem will not be applicable, since the underlying space has infinite measure. However, it will be a consequence of our main result, Theorem 2.1, that the map is recurrent for $\alpha > 2$. This theorem will be presented in a general framework which is valid for general maps h of $\mathbb{R}^n$.

For the proof we are going to use the following result due to Dolgopyat [3, Lemma 4.1], see also [4, Lemma 1.4]. It provides a useful extension of the finite-measure Poincaré recurrence theorem.

Lemma 1.1 *Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose that the map $T : X \to X$ is one-to-one and such that the following holds:*

(a) T is measurable, in the sense that $T(B), T^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{F}$,

(b) T is measure-preserving, in the sense that $\mu(T(B)) = \mu(B)$ for $B \in \mathcal{F}$, and

(c) there is a set $A \in \mathcal{F}$ such that $\mu(A) < \infty$ with the property that almost all points from X visit A in the future.

Then for every measurable set $B \subset X$ almost all points of B visit B infinitely many times in the future.

We say that a point $x \in X$ visits a set $S \subset X$ if some iterate $T^k(x)$ with $k \geq 1$ belongs to S.

A central issue in the proof of the main result will be to construct a suitable set A as in (c), and to this end the growth condition is needed. Another key insight is to realize that Lemma 1.1 has to be applied to $T = h|_U : U \to U$, where U denotes the set of initial conditions which lead to unbounded orbits. As a corollary we will show that, if h is continuous, the set of non-recurrent points is of first Baire category, and hence small also in a topological sense. We expect the main theorem to have many applications, and apart from the one sketched at the beginning of this introduction we will derive a new fixed point theorem for area preserving homeomorphisms on the plane and furthermore a result on the existence of periodic solutions to non-autonomous Hamiltonian systems for one degree of freedom.

2 Main result

For a map $h : \mathbb{R}^n \to \mathbb{R}^n$ we denote by $(z_k) = (h^k(z_0))$, $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, the forward orbit of $z_0 \in \mathbb{R}^n$ under the iteration of h. The map h will be said to be recurrent, if for every measurable set $B \subset \mathbb{R}^n$ such that $\lambda^n(B) > 0$ and almost every point $z_0 \in B$ there is $k = k(z_0) \in \mathbb{N}$ such that $z_k = h^k(z_0) \in B$; here λ^n is the Lebesgue measure on $\mathbb{R}^n$ and the term 'measure-preserving' will always be understood as w.r. to λ^n.

Our main result is the following theorem on recurrence; in Section 3 we will outline an example which implies that the assumptions are sharp in what concerns the exponent α in condition (2.1).

Theorem 2.1 *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one, measurable, and measure-preserving map such that there are constants $C > 0$ and $\alpha > n - 1$ so that*

$$|h(z)| \leq |z| + \frac{C}{1 + |z|^\alpha} \quad \text{for} \quad z \in \mathbb{R}^n, \tag{2.1}$$

where $|\cdot|$ denotes any fixed norm on $\mathbb{R}^n$. Then h is recurrent.

Proof: Step 1: Let $(\varepsilon_j)_{j \in \mathbb{N}}$ and $(b_j)_{j \in \mathbb{N}}$ be sequences of positive numbers satisfying

$$\varepsilon_j \leq 1 \leq b_j, \quad \sum_{j=1}^{\infty} \varepsilon_j b_j^{n-1} < \infty, \quad \text{and} \quad \lim_{j \to \infty} (\varepsilon_j b_j^\alpha) = \infty. \tag{2.2}$$

Appropriate choices in (2.2) are, for instance,

$$\varepsilon_j = \frac{1}{j^{2+\frac{3(n-1)}{\alpha-(n-1)}}} \quad \text{and} \quad b_j = j^{\frac{3}{\alpha-(n-1)}}.$$

Denote

$$A_* = \bigcup_{j\in\mathbb{N}} \{z \in \mathbb{R}^n : |z| \in [b_j - \varepsilon_j, b_j + \varepsilon_j]\}.$$

First we are going to show that A_* has finite measure and every unbounded orbit of h enters A_*. Let $\omega_n = \frac{2\pi}{\Gamma(\frac{n}{2})}$ denote the surface of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then

$$\lambda^n(A_*) \leq \omega_n \sum_{j=1}^{\infty} \int_{b_j-\varepsilon_j}^{b_j+\varepsilon_j} dr\, r^{n-1} = \frac{\omega_n}{n} \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) (1-(-1)^{n-k}) \sum_{j=1}^{\infty} b_j^k \varepsilon_j^{n-k} \leq \frac{2^n \omega_n}{n} \sum_{j=1}^{\infty} b_j^{n-1} \varepsilon_j,$$

which is finite by (2.2). Now let $(z_k) = (h^k(z_0))$ be an unbounded orbit of h and assume that it does not enter A_*. Then we have $\limsup_{k\to\infty} |z_k| = \infty$ and $||z_k| - b_j| > \varepsilon_j$ for all $k, j \in \mathbb{N}$. Since $\lim_{j\to\infty} b_j = \infty$ there is $j_0 \in \mathbb{N}$ such that $b_j \geq 3C$ and moreover $b_j > |z_1|$ for $j \geq j_0$. Fix $j \geq j_0$. Owing to $\limsup_{k\to\infty} |z_k| = \infty$ and $|z_1| < b_j$ we can select a first index $K \geq 2$ so that $|z_K| > b_j$. In particular, this implies that $|z_{K-1}| \leq b_j$. Hence by (2.1):

$$\varepsilon_j < ||z_K| - b_j| = |z_K| - b_j \leq |z_K| - |z_{K-1}| = |h(z_{K-1})| - |z_{K-1}| \leq \frac{C}{1 + |z_{K-1}|^\alpha}. \tag{2.3}$$

If we had $|z_{K-1}| \leq b_j/2$, then

$$b_j < |z_K| \leq |z_{K-1}| + \frac{C}{1+|z_{K-1}|^\alpha} \leq \frac{b_j}{2} + \frac{C}{1+|z_{K-1}|^\alpha} \tag{2.4}$$

implies that $1 \leq 1+|z_{K-1}|^\alpha \leq 2C/b_j$, which is impossible by the choice of j_0. Thus $|z_{K-1}| > b_j/2$ and therefore (2.3) leads to

$$\varepsilon_j \leq \frac{C}{1+|z_{K-1}|^\alpha} \leq \frac{2^\alpha C}{b_j^\alpha}, \quad j \geq j_0,$$

but this contradicts $\lim_{j\to\infty}(\varepsilon_j b_j^\alpha) = \infty$. <u>Step 2:</u> Let

$$U = \{z_0 \in \mathbb{R}^n : \limsup_{k\to\infty} |z_k| = \infty\}$$

denote the set of unbounded orbits. Since

$$U = \bigcap_{m=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{k\geq l} h^{-k}(\{z \in \mathbb{R}^n : |z| \geq m\}),$$

we see that U is measurable. Now we are going to apply Lemma 1.1 to $X = U$, $T = h|_U : U \to U$, and $A = U \cap A_*$. From Step 1 it follows that $\lambda^n(A) \leq \lambda^n(A_*) < \infty$ and every $z_0 \in U$ visits

A in the future. Hence Lemma 1.1 implies that for every measurable $C \subset U$ almost every point $z_0 \in C$ returns to C infinitely often under the iteration of h. Step 3: Denote by

$$B = \mathbb{R}^n \setminus U = \{z_0 \in \mathbb{R}^n : \limsup_{k\to\infty} |z_k| < \infty\}$$

the set of bounded orbits and define $B_M = \{z_0 \in B : |z_k| \le M \text{ for all } k \in \mathbb{N}_0\}$ for $M \in \mathbb{N}$. Then $B = \bigcup_{M=1}^{\infty} B_M$, h maps B_M into itself, and B_M has finite measure, since it is a measurable set contained in the closed ball of radius M. Therefore the standard Poincaré recurrence theorem applies, which allows us to deduce that for every measurable $C \subset B_M$ almost every point $z_0 \in C$ returns to C infinitely often under the iteration of h. Step 4: Completion of the proof. Let $C \subset \mathbb{R}^n$ be measurable and such that $\lambda^n(C) > 0$. Since

$$C = (C \cap U) \cup \bigcup_{M=1}^{\infty} (C \cap B_M),$$

we apply the previous two steps to find sets $V \subset C \cap U$ and $Z_M \subset C \cap B_M$ for every $M \in \mathbb{N}$ of zero measure, and with the additional property that every point $z_0 \in (C \cap U) \setminus V$ returns to $C \cap U$ infinitely often, and every point $z_0 \in (C \cap B_M) \setminus Z_M$ returns to $C \cap B_M$ infinitely often. Then $Z = V \cup \bigcup_{M=1}^{\infty} Z_M$ has zero measure and every point $z_0 \in C \setminus Z$ returns to C, in fact infinitely often. $\square$

Recall that the ω-limit set $\omega(z_0) \subset \mathbb{R}^n$ of a point $z_0 \in \mathbb{R}^n$ is given by the accumulation points of $(z_k)_{k\in\mathbb{N}_0}$. A point $z_0 \in \mathbb{R}^n$ is said to be recurrent, if $z_0 \in \omega(z_0)$. Let $G = \{z \in \mathbb{R}^n : z \in \omega(z)\}$ denote the set of recurrent points.

The set of non-recurrent points is not only small in measure, but also topologically.

Corollary 2.2 *Let the assumptions of Theorem 2.1 be satisfied.*

(a) Then almost all $z_0 \in \mathbb{R}^n$ are recurrent.

(b) If, in addition, h is supposed to be continuous, then the set of non-recurrent points is of first Baire category.

Proof: (a) For every $N \in \mathbb{N}$ we cover $\mathbb{R}^n$ by a countable family $(B_j^{(N)})_{j\in\mathbb{N}}$ of balls of radius $1/N$. Applying Theorem 2.1, we find sets $Z_j^{(N)} \subset B_j^{(N)}$ of measure zero such that every $z_0 \in B_j^{(N)} \setminus Z_j^{(N)}$ returns to $B_j^{(N)}$ infinitely often. Let $Z = \bigcup_{j=1}^{\infty} \bigcup_{N=1}^{\infty} Z_j^{(N)}$. Then $\lambda^n(Z) = 0$ and $\mathbb{R}^n \setminus G \subset Z$. To establish the last relation take $z_0 \in \mathbb{R}^n \setminus Z$. Since $\bigcup_{j=1}^{\infty} B_j^{(N)} = \mathbb{R}^n$ for every $N \in \mathbb{N}$, there is $j_N \in \mathbb{N}$ such that $z_0 \in B_{j_N}^{(N)}$. But $z_0 \notin Z_{j_N}^{(N)}$, and this means that z_0 returns to $B_{j_N}^{(N)}$ infinitely often. Therefore we can fix $k_N \ge N$ such that $h^{k_N}(z_0) \in B_{j_N}^{(N)}$, which implies that $|h^{k_N}(z_0) - z_0| < 1/N$ and $k_N \to \infty$. Thus $z_0 \in \omega(z_0)$ and hence $z_0 \in G$, which proves that $\mathbb{R}^n \setminus G \subset Z$. This in turn yields $\lambda^n(\mathbb{R}^n \setminus G) = 0$, as desired. (b) For $j \in \mathbb{N}$ define $G_j = \{z \in \mathbb{R}^n : |h^k(z) - z| < 1/j \text{ for some } k \in \mathbb{N}\}$. Then $G = \bigcap_{j=1}^{\infty} G_j = \{z \in \mathbb{R}^n : z \in \omega(z)\}$ is the set of recurrent points, for which one has $\lambda^n(\mathbb{R}^n \setminus G) = 0$ by (a). Hence $\lambda^n(\mathbb{R}^n \setminus G_j) = 0$ for every $j \in \mathbb{N}$ and this implies that $\operatorname{int}(\mathbb{R}^n \setminus G_j) = \emptyset$ for the interior of this set. Since h is continuous, $G_j \subset \mathbb{R}^n$ is open, thus $\mathbb{R}^n \setminus G_j$ is nowhere dense and $\mathbb{R}^n \setminus G = \bigcup_{j=1}^{\infty}(\mathbb{R}^n \setminus G_j)$. $\square$

Remark 2.3 (a) Theorem 2.1 is useful in many respects, for example it yields a new fixed point theorem on the plane: Suppose that $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism preserving area and orientation and such that there are constants $C > 0$ and $\alpha > 1$ so that $|h(z)| \le |z| + \frac{C}{1+|z|^\alpha}$ for $z \in \mathbb{R}^2$. Then h has a fixed point.

To establish this claim, note that in particular h is continuous and one-to-one. Owing to a result of Brouwer (see [1] and [5, Cor. 12]), it suffices to find a point $z_0 \in \mathbb{R}^2$ such that $\liminf_{k\to\infty} |h^k(z_0)| < \infty$, which is equivalent to $\omega(z_0) \neq \emptyset$. But Corollary 2.2(a) implies that in fact $\omega(z) \neq \emptyset$ for almost all $z \in \mathbb{R}^2$.

This result should be contrasted with earlier work, [2, Cor. 2.2], which says the following: Let (M, ω) be a symplectic manifold where M is diffeomorphic to $\mathbb{R}^{2n}$. Then there is a neighborhood in the C^1-fine topology of $\mathrm{Diff}(M, \omega)$ which contains id_M, such that every mapping in this neighborhood has a fixed point. Thus our fixed point theorem provides a quantitative generalization of [2, Cor. 2.2] in the planar case. Observe that the condition $|h(z) - z| \le \frac{C}{1+|z|^\alpha}$ describes a certain neighborhood of id_M, but in the C^0-fine topology, and also note that (2.1) is satisfied if this condition on h holds.

(b) We outline a further application of our results. Consider a non-autonomous Hamiltonian system $\dot{z} = J\nabla_z H(t, z)$ of one degree of freedom, i.e. $z \in \mathbb{R}^2$, such that H is C^2, T-periodic in time and such that $\nabla_z H(t, z) = \mathcal{O}(|z|^{-\alpha})$ as $|z| \to \infty$ for some $\alpha > 1$, uniformly in t. Then the system has a T-periodic solution. To prove this, it is sufficient to apply the fixed point theorem from (a) to the Poincaré map of the system.

3 A counterexample

We will show that the assertion of Theorem 2.1 may fail to hold, if only

$$|h(z)| \le |z| + \frac{C}{1 + |z|^{n-1}} \quad \text{for} \quad z \in \mathbb{R}^n \tag{3.1}$$

is assumed, i.e., the condition $\alpha > n - 1$ is sharp. For $r \ge 1$ consider the function $\varphi(r) = (1 + \frac{1}{r^n})^{1/n}$ and define

$$h : \mathbb{R}^n \to \mathbb{R}^n, \quad h(z) = \left\{ \begin{array}{ccc} z & : & |z| \le 1 \\ \varphi(|z|)z & : & |z| > 1 \end{array} \right. .$$

Then h is one-to-one. To establish this fact, take $z, \tilde{z} \in \mathbb{R}^n$ such that $h(z) = h(\tilde{z})$. If $|z|, |\tilde{z}| \le 1$, then $z = \tilde{z}$. If $|z| \le 1$ and $|\tilde{z}| > 1$, then $\varphi(r) > 1$ yields the contradiction $1 \ge |z| = |h(z)| = |h(\tilde{z})| = \varphi(|\tilde{z}|)|\tilde{z}| > |\tilde{z}| > 1$. If $|z|, |\tilde{z}| > 1$, then $\varphi(|z|)z = \varphi(|\tilde{z}|)\tilde{z}$ implies that $\tilde{z} = \lambda z$ for some $\lambda > 0$. From the definition it follows that $(1 + \frac{1}{|z|^n})|z|^n = \varphi(|z|)^n |z|^n = (\lambda^n + \frac{1}{|z|^n})|z|^n$, and hence $\lambda = 1$ and $z = \tilde{z}$. Therefore h is one-to-one. Next we claim that h is measure-preserving. Since $h(z) = z$ for $|z| \le 1$ and h maps $h : \{|z| > 1\} \to \{|z| > 1\}$, it suffices to prove that $h_2(z) = \varphi(|z|)z$ for $|z| > 1$ is measure-preserving. The Jacobi matrix of h_2 is calculated to be

$$Dh_2(z) = \varphi(r)I_n + \frac{\varphi'(r)}{|z|}\, z \otimes z, \quad r = |z|,$$

where I_n is the $n \times n$–unit matrix and $(z \otimes z)_{ij} = z_i z_j$. It is proved by induction that in general $\det(aI_n + b\, z \otimes z) = a^{n-1}(a + b|z|^2)$. As a consequence, and by the definition of φ,

$$\det Dh_2(z) = \varphi(r)^{n-1}(\varphi(r) + r\varphi'(r)) = 1, \quad |z| > 1,$$

which shows that h_2 is measure-preserving. Concerning condition (3.1), if $|z| \le 1$, then $h(z) = z$. If $|z| > 1$, then $|h(z) - z| = |\varphi(|z|) - 1|\,|z|$. From the ODE $\varphi(r)^{n-1}(\varphi(r) + r\varphi'(r)) = 1$ one has

$$0 \le -\varphi'(r) = \frac{\varphi(r)^n - 1}{r\varphi(r)^{n-1}} = \frac{1}{r^{n+1}\varphi(r)^{n-1}} \le \frac{1}{r^{n+1}},$$

and accordingly $|\varphi(r) - 1| = -\int_r^\infty \varphi'(s)\, ds \le \int_r^\infty \frac{ds}{s^{n+1}} \le \frac{1}{r^n}$. This leads to

$$|h(z) - z| = |\varphi(|z|) - 1|\,|z| \le \frac{1}{|z|^{n-1}} \le \frac{2}{1 + |z|^{n-1}}, \quad |z| > 1,$$

and completes the proof that (3.1) is verified. Next we are going to show that the set G of recurrent points is $G = \{|z| \le 1\}$. Since $h(z) = z$ for $|z| \le 1$, this is equivalent to proving that all $|z_0| > 1$ are non-recurrent. So let $z_0 \in \mathbb{R}^n$ satisfy $|z_0| > 1$ and denote $z_k = h^k(z_0)$. From $\varphi(r) > 1$ it follows that $|z_k| > 1$ for $k \in \mathbb{N}_0$. Then $z_{k+1} = h(z_k) = \varphi(|z_k|)z_k$ shows that $|z_{k+1}|^n = (1 + \frac{1}{|z_k|^n})|z_k|^n = |z_k|^n + 1$, and in particular $\lim_{k\to\infty} |z_k| = \infty$. This proves that $G = \{|z| \le 1\}$, and clearly h is a non-recurrent map.

4 Additional remarks

In this section we will briefly address some generalizations and questions that have kindly been brought to our attention by the anonymous referee.

(a) For Theorem 2.1 to hold it in fact suffices to assume the weaker condition

$$|h(z)| \le |z| + \frac{C}{1 + |z|^{n-1}\phi(|z|)} \quad \text{for} \quad z \in \mathbb{R}^n,$$

where, for instance, $\phi : [0, \infty[\to [0, \infty[$ is a continuous and increasing function such that $\lim_{r\to\infty} \phi(r) = \infty$; the authors thank Prof. Dolgopyat who also suggested to extend condition (2.1). To see this, choose $b_j \to \infty$ such that $\phi(b_j/2) = j^3$ for j sufficiently large. Define $\varepsilon_j = \frac{1}{j^2 b_j^{n-1}}$. Then once again $\sum_{j=1}^\infty b_j^{n-1}\varepsilon_j < \infty$. A straightforward inspection of the proof to Theorem 2.1 now shows that (2.3) and (2.4) will be replaced by

$$\varepsilon_j < \frac{C}{1 + |z_{K-1}|^{n-1}\phi(|z_{K-1}|)} \tag{4.1}$$

and

$$b_j < \frac{b_j}{2} + \frac{C}{1 + |z_{K-1}|^{n-1}\phi(|z_{K-1}|)}, \tag{4.2}$$

respectively. The case where $|z_{K-1}| \le b_j/2$ is impossible as before by (4.2) and $b_j \to \infty$, and if $|z_{K-1}| > b_j/2$, then (4.1) implies that

$$\varepsilon_j < \frac{C}{|z_{K-1}|^{n-1}\phi(|z_{K-1}|)} \le \frac{2^{n-1}C}{b_j^{n-1}\phi(b_j/2)},$$

but $\varepsilon_j b_j^{n-1}\phi(b_j/2) = j \to \infty$ gives a contradiction as before.

(b) The example in Section 3 can be modified so that h becomes a one-to-one measure-preserving map satisfying (3.1) and having no recurrent points. Define

$$h : \mathbb{R}^n \to \mathbb{R}^n, \quad h(z) = \begin{cases} z_* & : \quad z = 0 \\ \varphi(|z|)z & : \quad z \neq 0 \end{cases},$$

where z_* is chosen arbitrarily such that $0 < |z_*| \le 1$. Then $|z_k| \to \infty$ along all orbits.

(c) It is an interesting open problem whether one can construct a one-to-one measure-preserving map h satisfying (3.1) such that h is continuous (or even a homeomorphism), but non-recurrent.

References

[1] Brown M.: Homeomorphisms of two-dimensional manifolds, *Houston J. Math.* **11**, 455-469 (1985)

[2] Colvin M. & Morrison K.: A symplectic fixed point theorem on open manifolds, *Proc. Amer. Math. Soc.* **84**, 601-604 (1982)

[3] Dolgopyat D.: *Lectures on Bouncing Balls*, lecture notes for a course in Murcia, 2013; available at `http://www2.math.umd.edu/~dolgop/BBNotes.pdf`

[4] Kunze M. & Ortega R.: Escaping orbits are rare in the quasi-periodic Fermi-Ulam ping-pong, preprint 2015; available at `http://www.ugr.es/~ecuadif/fuentenueva.htm`

[5] Ortega R.: *Topology of the Plane and Periodic Differential Equations. Chapter 3. Free Embeddings of the Plane*, book project; available at `http://www.ugr.es/~ecuadif/files/libro3.pdf`