On a family of Kepler problems with linear dissipation

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Abstract

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We consider the dissipative Kepler problem for a family of dissipations that is linear in the velocity. Under mild assumptions on the drag coefficient, we show that its forward dynamics is qualitatively similar to the one obtained in [15] and [16] for a constant drag coefficient. In particular, we extend to this more general framework the existence of a continuous vector-valued first integral I obtained as the limit along the trajectories of the Runge-Lenz vector. We also establish the existence of asymptotically circular orbits, so improving the result about the range of I contained in [16].

Keywords: Kepler equation, drag linear in the velocity, first integral

1 Introduction

In our previous papers [15] and [16] we studied the global dynamics of a Kepler problem with linear drag

$$\ddot{x} + \epsilon \dot{x} = -\frac{x}{|x|^3}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \quad \epsilon > 0. \tag{1}$$

The main conclusion was the existence of a vector-valued first integral $I = (I_1, I_2)$, $I_i = I_i(x, \dot{x})$. This integral was obtained in a rather indirect way and we do not know if it has an explicit formula. In contrast it has a very intuitive dynamical description. The vector $I(x, \dot{x})$ can be interpreted as the eccentricity vector of an ellipse $\mathcal{E}$ such that the solution $x(t)$ tends to the origin along a spiral modelled after $\mathcal{E}$ (see Figure 1). Also, we proved that

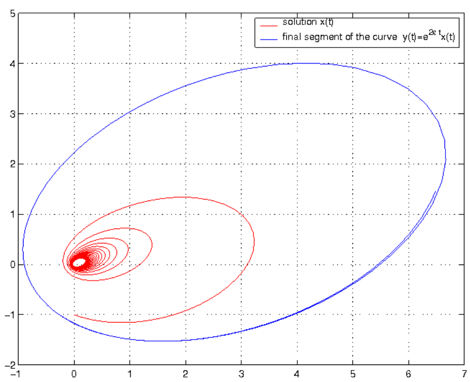

Figure 1: In red, an orbit $x(t)$ of (1) with $\epsilon = 0.01$ plotted for $t \in [0, 35]$. In blue, the approximate shape of $\mathcal{E}$, obtained by plotting the final segment of the curve $y(t) = e^{2\epsilon t}x(t) \to \mathcal{E}$ (see [16] for more details).

the existence of I implies that such spiral is described with angular velocity which increases exponentially with time.

The aim of this work is to extend this type of results to the family of dissipative Kepler problems

$$\ddot{x} + \mathcal{D}(|x|)\dot{x} = -\frac{x}{|x|^3}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{2}$$

where $\mathcal{D} : [0, +\infty[\to \mathbb{R}^+$ is a locally Lipschitz continuous function which satisfies

$$\mathcal{D}(r) \geq A_1 \quad \text{for any } r \geq 0 \tag{3}$$

for a suitable positive real number A_1.

It is a curious fact that the spiralling faster and faster towards the Sun of a celestial body was already described by Euler in a letter written in 1749 and published in Philosophical Transactions [8]. There Euler postulated the existence of small resistance forces around the planets and he described the consequent gradual approach of the Earth to the Sun as follows "...The effect of this Resistance will gradually bring the Planets nearer and nearer to the Sun; and as their Orbits thereby become less, their periodical Times will also be diminished."

More than one century later Poincaré went back to the study of the effect of a resistive medium on the motion of a planet in his Leçons sur les hypothéses cosmogoniques [20]. In that course he discussed several hypotheses on the formation of the solar system. In Chapter VI, devoted to an hypothesis due to T.J.J. See, Poincaré considered the class of dissipative Kepler problems

$$\ddot{x} + R\frac{\dot{x}}{|\dot{x}|} = -\frac{x}{|x|^3}, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{4}$$

where $R = h|x|^{-\beta}|\dot{x}|^{\alpha}$ and α and β are positive constants. After some computations with astronomical coordinates Poincaré found out that the semi major axis of an orbit of elliptic type is, *essentiellement*, decreasing with time and observed that this fact implies an increase of the orbital velocity of the planet.

Moreover, from his computations he concluded that if the exponents α and β are sufficiently large then the value of the orbital eccentricity decreases after each complete revolution. Poincaré also presented a qualitative argument[1] to justify the decrease of the eccentricity of an orbit in presence of a general resistive force.

Both these arguments suggest that dissipation has a circularizing effect on orbits, that is, that their eccentricity will eventually approach zero. In

[1]"en gros et sans calcul"

this connection, we note that our results for the linear drag ($\alpha = 1$, $\beta = 0$) imply that for an open set of initial conditions the eccentricity of the corresponding orbit will converge to a positive constant, and so we cannot expect a circularization effect for many orbits of (1). This fact has been observed previously in [12] (for more information on the notion of circularization see [9] and [1]).

When $\beta = 0$ the family (4) was already considered by Jacobi in his book on mechanics [13] but he only discussed some formal aspects. Another member of the family (4) that has been considered in the recent literature is the so called Poynting-Plummer-Danby drag (see [6], [3], [7] and the references therein), corresponding to $\alpha = 1$ and $\beta = 2$. In this case it is possible to obtain in closed form the equation of the orbits. We point out that for this family of resistive forces the qualitative behaviour of the solutions differs sharply from the one we obtained for (2). In fact, many non rectilinear solutions, corresponding to an open set of initial conditions, collide in finite time and with finite velocity with the singularity, winding around the origin just a finite number of times before collision. This is nicely described in the unpublished master thesis of Mauricio Misquero[2].

The Runge-Lenz vector, denoted by R, is a well-known first integral of the conservative Kepler problem. If its norm is less than one, then R corresponds to a family of elliptic orbits whose eccentricity is $|R|$. In the presence of friction this vector is no longer a constant of motion but it is still useful and it has been employed in the literature on dissipative problems (see [14], [10], [17]). We will show that for linear dissipations the Runge-Lenz vector has a limit $I = \lim_{t\to+\infty} R(t)$ that becomes a first integral such that $|I| \leq 1$. This approach to construct integrals is inspired by the ideas on asymptotic integrals developed by Moser in [18] for the study of the Störmer problem (see also [19]).

We notice that in our setting the circularization of an orbit is equivalent to $I = 0$. Orbits satisfying this condition were called asymptotically circular in our previous paper [16], but at that time we were unable to decide whether they existed or not. In this work we show that they actually exist, although they are not typical. In this aspect the results of this paper improve those in [16] even for the case $\mathcal{D} = \epsilon$. This approach to construct integrals is inspired by the ideas on asymptotic integrals developed by Moser in [18] for the study of the Störmer problem (see also [19]).

This paper can be seen as a contribution to the construction of a qualitative theory of the Kepler problem with dissipation but many interesting

[2] Castro, M. M., El problema de Kepler disipativo, Master thesis, Universidad de Granada, (2016)

problems in this topic are still to be addressed. For example, to determine the region of parameters (α, β) producing circularization on an open set of initial conditions seems a challenging question. Also, the study of more realistic drags involved in satellite dynamics appears to be relevant (see [2]).

The rest of the paper is organized as follows.

In the second section we study the forward dynamics of (2), showing that the singularity is a global attractor. Our proof makes use of an extension to singular systems of the LaSalle invariance principle, which may have an independent interest. In Section 3 we extend to (2) the results given in [15] about the asymptotic values of the energy of solutions. This is done by adapting the approach based on the Levi-Civita transformation for the dissipative setting already considered in [15] for the linear drag. We recall that the Levi-Civita regularization in a dissipative setting was introduced by [4] for the numerical study of the global dynamics of a restricted three body problem with drag. In the fourth section we construct the asymptotic first integral for (2) and we show that it is continuous and invariant under planar rotations. In the fifth section we prove that its range is the unit disk. This is achieved by establishing the existence of asymptotically circular orbits of (2). It is interesting to note that for this aim we employ the Brouwer degree to show that there is a continuation from the circular solutions of the conservative case. Finally, in the Appendix we sketch the proofs of some results about rectilinear motions.

2 Dynamics in forward time: attraction towards the singularity

In this section we study the behaviour of the solutions of (2) when $t \to \omega$, where ω is the right endpoint of their maximal interval of definition. We show that the singularity $x = 0$ is a global attractor of (2). First we prove that non rectilinear solutions are defined up to $\omega = +\infty$ and are bounded. Then we state and apply a version LaSalle's invariance principle which is well suited for singular equations. An analogous result is given in [1]. Finally, we show that all the rectilinear motions collide in finite time with $x = 0$.

We point out that in [15] the property that the origin is a global attractor was obtained by applying the results in [5], where such result is proved for a family of resistive forces of the form $F(x, \dot{x}) = -\frac{k(|\dot{x}|)}{|\dot{x}|}\dot{x}$ which includes the linear drag. Here, rather than trying to adapt to our setting such results we have preferred to provide a direct proof of the global attractiveness of the singularity.

The dissipative Kepler problem described by equation (2) is equivalent to the system

$$\begin{cases} \dot{x} = v \\ \dot{v} + \mathcal{D}(|x|)v = -\frac{x}{|x|^3} \end{cases} \tag{5}$$

in the phase space $\Omega = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$.

In what follows, for any fixed $(x_0, v_0) \in \Omega$, we will sometimes employ the notation $x(t; x_0, v_0)$ for the solution of (2) such that $x(0) = x_0$, $v(0) = v_0$. If we consider the functions of the real variables (x, v) given respectively by

$$E(x, v) = \frac{1}{2}|v|^2 - \frac{1}{|x|}, \quad (\textit{energy}) \tag{6}$$

and

$$C(x, v) = x \wedge v, \quad (\textit{angular momentum}) \tag{7}$$

then along the solutions of (2) it is

$$\dot{E}(t) := \frac{dE}{dt}(x(t), \dot{x}(t)) = -\mathcal{D}(|x(t)|)|\dot{x}(t)|^2. \tag{8}$$

and

$$\dot{C}(t) := \frac{dC}{dt}(x(t), \dot{x}(t)) = -\mathcal{D}(|x(t)|)C(t),$$

from which it follows

$$C(t) = C(0)\mathrm{e}^{-\int_0^t \mathcal{D}(|x(\tau)|)\,d\tau}, \quad C(0) := x(0) \wedge \dot{x}(0). \tag{9}$$

We rewrite now equation (2) using polar coordinates. If we consider the change of variables $x = r\mathrm{e}^{i\theta}$, the new coordinates satisfy the following differential system:

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 + \mathcal{D}(r)\dot{r} = -\frac{1}{r^2} \\ \frac{d}{dt}(r^2\dot{\theta}) = -\mathcal{D}(r)r^2\dot{\theta}. \end{cases} \tag{10}$$

Recalling that $|x \wedge \dot{x}| = \pm r^2\dot{\theta}$, by (9) we get that the radial component of the solutions of (2) satisfies the integro-differential equation

$$\ddot{r} - \alpha^2 \, \frac{\mathrm{e}^{-2\int_0^t \mathcal{D}(r(s))ds}}{r^3} + \mathcal{D}(r)\dot{r} = -\frac{1}{r^2} \tag{11}$$

where $\alpha = |C(0)|$.

We are now in a position to state our result about the non rectilinear motions of (2).

Proposition 2.1 *Let $x(t) = r(t)e^{i\theta(t)}$ be a maximal solution of* (2) *with $\alpha \neq 0$, and let $[0,\omega[$ be its domain in forward time. Then $\omega = +\infty$, and $r(t) = |x(t)|$ is bounded on $[0,+\infty[$.*

Proof. We prove first that $r(t)$ is bounded on $[0,\omega[$ and then we show that $\omega = +\infty$.

To get the boundedness of the solutions we argue as follows. Either $r(t) \leq \alpha^2$ when t is sufficiently close to ω and we have nothing to prove, or there exists a sequence $t_n \to \omega$ such that $r(t_n) > \alpha^2$. If this is the case, there are two possible occurrences:

i) $r(t) > \alpha^2$ in $[\tau,\omega[$ for some $\tau \in [0,\omega[$;

ii) there exists a sequence of intervals $I_n = [a_n, b_n] \subseteq [0,\omega[$ such that $r(t) > \alpha^2$ if and only if $t \in]a_n, b_n[$.

If i) holds, by (11) it follows that if $t \in [\tau,\omega[$

$$\frac{d}{dt}(e^{\int_\tau^t \mathcal{D}(r(s))ds}\dot{r}) = e^{\int_\tau^t \mathcal{D}(r(s))ds}(\ddot{r} + \mathcal{D}(r)\dot{r}) \leq 0 \tag{12}$$

and by integrating this inequality we obtain

$$\dot{r}(t) \leq e^{-\int_\tau^t \mathcal{D}(r(s))ds}\dot{r}(\tau) \leq e^{-A_1(t-\tau)}|\dot{r}(\tau)|, \quad t \in [\tau,\omega[,$$

which implies

$$r(t) \leq r(\tau) + \frac{|\dot{r}(\tau)|}{A_1}(1 - e^{-A_1(t-\tau)}) < r(\tau) + \frac{|\dot{r}(\tau)|}{A_1}, \quad t \in [\tau,\omega[.$$

The proof of boundedness of $r(t)$ on $[0,\omega[$ in case i) is concluded.

In case ii) we note that, since for any n we have $r(a_n) = |x(a_n)| = \alpha^2$, then from $E(a_1) \geq E(a_n)$, it follows that $|\dot{r}(a_n)| \leq |\dot{x}(a_n)| \leq |\dot{x}(a_1)|$ for any n. Then, taking into account that on I_n (12) holds, in a similar manner as above we get $\dot{r}(t) \leq e^{-A_1(t-a_n)}|\dot{r}(a_n)|$, $t \in I_n$, and then

$$r(t) \leq \alpha^2 + \frac{|\dot{x}(a_1)|}{A_1}, \quad t \in I_n.$$

Since the constant that bounds the solution is the same for all the intervals I_n and since $r(t) \leq \alpha^2$ on the set $[0,+\omega[\setminus \cup_n I_n$, the proof of the boundedness of $r(t)$ in case ii) is finished.

We conclude that $r(t)$ is bounded on $[0,\omega[$.

To prove that $\omega = +\infty$ assume by contradiction that $\omega < +\infty$. The standard theory for initial value problems implies that one of the following cases hold:

(i) there exists a sequence $t_n \uparrow \omega$ such that $x(t_n) \to 0$;

(ii) $|x(t)| \geq \delta$ if $t \in [0, \omega[$ for some $\delta > 0$ and $\lim_{t\uparrow\omega} |\dot{x}(t)| = +\infty$.

If condition (i) were valid then $E(t_n) \to +\infty$ as $n \to +\infty$ and this is incompatible with (8). In fact,

$$E(t_n) \geq \frac{\alpha^2}{r^2(t_n)} e^{-2\int_0^{t_n} \mathcal{D}(r(s))ds} - \frac{1}{r(t_n)} \geq \frac{1}{r^2(t_n)}[\alpha^2 e^{-2Mt_n} - r(t_n)],$$

where $M := \sup_{t\in[0,\omega[} \mathcal{D}(r(t))$ is finite since $\mathcal{D}(r)$ is continuous on $[0, +\infty[$ and $r(t)$ is bounded in $[0, \omega[$. Assume now that (ii) holds. From $|x(t)| \geq \delta$ for any $t \in [0, \omega[$ we get

$$\frac{1}{2}|\dot{x}(t)|^2 - \frac{1}{\delta} \leq E(t) \leq E(0).$$

Since this inequality gives a bound for $|\dot{x}(t)|$ on $[0, \omega[$ we get a contradiction with the limit in (ii).

■

To prove that all the non rectilinear solutions of equation (2) tend to the singularity as $t \to +\infty$ we need the following general auxiliary result, which is an extension to singular systems of the LaSalle invariance principle.

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^d$ be an open set and assume that the existence and uniqueness of solution holds for the system $\dot{x} = f(x)$ with $f : \Omega \to \mathbb{R}^d$ continuous. Let $\phi_t(x)$ denote the value at time t of the solution of $\dot{x} = f(x)$ which starts from x at $t = 0$ and let $I_x \subset \mathbb{R}$ be its maximal interval of definition. Assume there exists a continuous function $V : \Omega \to \mathbb{R}$ such that*

$$V(\phi_t(x)) < V(x), \quad t \in I_x,\ t > 0,\ x \in \Omega. \tag{13}$$

If $x_* \in \Omega$ is such that $[0, +\infty[\subset I_{x_*}$, then

$$L_\omega(x_*) \cap \Omega = \emptyset,$$

where $L_\omega(x_*)$ denotes the $\omega-$limit set of x_*.

Note that in the above statement the limit set is defined as

$$L_\omega(x_*) = \cap_{t\geq 0}\overline{\{\phi_\tau(x_*)\ :\ \tau \geq t\}},$$

where the closure is taken in $\mathbb{R}^d$.

Remark. The following variant of the Proposition will be useful later. We can assume that the condition (13) only holds for points x lying on a closed subset F of Ω. If the set F is invariant under the flow then the conclusion on the limit set will be valid for the orbits lying on F.

Proof. By contradiction, assume that there exists $t_n \to +\infty$ such that $\phi_{t_n}(x_*) \to \xi \in \Omega$. By the continuous dependence of the solutions of $\dot{x} = f(x)$ on the initial value, given $\sigma > 0$ such that $\sigma \in I_\xi$ we know that, for large n, $\sigma \in I_{\phi_{t_n}(x_*)}$ and $\phi_{\sigma+t_n}(x_*) \to \phi_\sigma(\xi)$. For each n there exists $\mu(n) > n$ such that $t_{\mu(n)} > t_n + \sigma$. Then,

$$V(\phi_{t_{\mu(n)}}(x_*)) < V(\phi_{t_n+\sigma}(x_*)).$$

Letting $n \to +\infty$ we get

$$V(\xi) \leq V(\phi_\sigma(\xi)),$$

and this is a contradiction

■

As a corollary we get:

Proposition 2.3 *Let $x(t) = r(t)e^{i\theta(t)}$ be a non rectilinear solution of* (2)*. Then*

$$\lim_{t\to+\infty} x(t) = 0.$$

Proof. Assume by contradiction that there exists a sequence $t_n \to +\infty$ such that $|x(t_n)| \geq \delta > 0$ for a suitable δ. From the energy inequality

$$E(t_n) = \frac{1}{2}|\dot{x}(t_n)|^2 - \frac{1}{|x(t_n)|} \leq E(0)$$

we deduce that $|\dot{x}(t_n)|^2 \leq E(0) + \frac{1}{\delta}$. As by Proposition 2.1 $x(t_n)$ is bounded, it must be $L_\omega(x(0), \dot{x}(0)) \cap \Omega \neq \emptyset$. Since $\ddot{x}(t) \neq 0$ when $\dot{x}(t) = 0$, we deduce that the zeros of $\dot{x}(t)$ are isolated. Then the formula $\dot{E} = -\mathcal{D}(|x|)|\dot{x}|^2$ implies that the energy function E is strictly decreasing on the solutions of (2) and (13) holds. Now we can apply the previous proposition with $V = E$ and get a contradiction.

□

As to the solutions with zero angular momentum, the so called rectilinear motions, they satisfy the equation

$$\ddot{r} + \mathcal{D}(r)\dot{r} = -\frac{1}{r^2}, \tag{14}$$

obtained from (11) by setting $\alpha = 0$.

For this class of solutions we state the following result. Its proof is analogous to the one of Proposition 3.1 in [15], and we only sketch it in the Appendix.

Proposition 2.4 *All solutions of* (14) *are collision solutions, that is ω is finite and*

$$\lim_{t\to\omega^-} r(t) = 0, \quad \lim_{t\to\omega^-} \dot{r}(t) = -\infty.$$

3 The Levi-Civita transformation and the asymptotic behaviour of the energy

In this section we study the behaviour of the energy of the solutions of (2) as they approach the singularity.

The starting point is to adapt to equation (2) the Levi-Civita regularization that was introduced in a dissipative setting in [15] to deal with the linear drag. We recall that, after the natural identification of $x = (x_1, x_2)$ with the complex number $x_1 + ix_2$, the Levi-Civita regularization is defined by the change of variables

$$x = w^2, \ ds = \frac{dt}{|x|}. \tag{15}$$

Using this regularization, equation (2) is transformed into the system of ODEs in the new time s

$$w' = v, \ \ v' = \frac{Ew}{2} - \mathcal{D}(|w|^2)|w|^2 v, \ \ E' = -2\mathcal{D}(|w|^2)(E|w|^2 + 1). \tag{16}$$

This system has to be considered on the invariant manifold

$$\mathcal{M} = \{(w, v, E) \in \mathbb{C}^2 \times \mathbb{R} : \ E|w|^2 + 1 - 2|v|^2 = 0\}, \tag{17}$$

which contains all the physically meaningful solutions. A solution of (2) starting from $(x_0, v_0) \in \Omega$ is transformed in a solution of (16) starting from $(w_0, \hat{v}_0, E_0) \in \mathcal{M}$, where w_0 is a square root of x_0, $\hat{v}_0 = \frac{|x_0|v_0}{2w_0}$ and $E_0 = \frac{1}{2}|v_0|^2 - \frac{1}{|x_0|}$. Vice-versa, a solution of (16) starting on $\mathcal{M}$ and such that $w(0) \neq 0$ corresponds to the solution $x(t) := w^2(S(t))$ of (2), where $S(t)$ is the inverse function of $T(s) := \int_0^s |w(\sigma)|^2 \, d\sigma$.

Notice that if the points (x_0, v_0) belong to a compact subset K of Ω, then the triplets $(w_0, \hat{v}_0, E_0)$ lie on a compact subset $\mathcal{K}$ of $\mathcal{M}$.

Lemma 3.1 *Let (w_0, v_0, E_0) be a point of $\mathcal{M}$ and let $(w(s), v(s), E(s))$ denote the solution of* (16) *passing through this point at $s = 0$. Then this solution is well defined on $[0, +\infty[$ and*

$$\lim_{s\to+\infty} E(s) = -\infty.$$

Proof. Let $[0, \sigma[$ be the maximal interval to the right of the solution. By a contradiction argument we assume that $\sigma < +\infty$. The third equation of (16) and the invariance of $\mathcal{M}$ imply that

$$E'(s) = -4\mathcal{D}(|w(s)|^2)|v(s)|^2 \leq 0.$$

In particular $E(s) \leq E(0)$ for each $s \in [0, \sigma[$. Again, the invariance of $\mathcal{M}$ leads to the differential inequality

$$\frac{d}{ds}|w(s)| \leq |w'(s)| = \sqrt{\frac{1 + E(s)|w(s)|^2}{2}} \leq \frac{1 + |E(0)|^{\frac{1}{2}}|w(s)|}{\sqrt{2}}.$$

It follows that $|w(s)|$ remains bounded in $[0, \sigma[$. This fact implies that the function $\mathcal{D}(|w(s)|^2)$ is bounded on $[0, \sigma[$ and by the last equation of (16) we conclude that the same is true for $|E(s)|$. The definition of $\mathcal{M}$ implies now that $|v(s)|$ is bounded on $[0, \sigma[$. It follows that the solution $(w(s), v(s), E(s))$ cannot blow up at $s = \sigma$ and this gives a contradiction with $\sigma < +\infty$. We conclude that the solution is well defined on $[0, +\infty[$.

Since on this interval we have $E'(s) \leq 0$, then $E_\infty = \lim_{s \to +\infty} E(s)$ exists and belongs to $[-\infty, E(0)]$. We prove now that $E_\infty = -\infty$. Let us assume by contradiction that $E_\infty \in \mathbb{R}$ and distinguish two cases:

(i) $E_\infty \geq 0$;

(ii) $E_\infty < 0$.

If (i) holds, we know that $E(s) \geq E_\infty \geq 0$ if $s \geq 0$. After integrating the third equation of (16), we have

$$E(s) = E(0) - 2\int_0^s \mathcal{D}(|w(\xi)|^2)(E(\xi)|w(\xi)|^2 + 1)\, d\xi \leq E(0) - 2A_1 s \to -\infty$$

as $s \to -\infty$, and we get a contradiction.

Assume now that (ii) holds. We note that system (16), defined on $\Omega = \mathbb{C}^2 \times \mathbb{R}$, is in the conditions of the remark after Proposition 2.2 with $F = \mathcal{M}$ and $V = E$. Once we are on $\mathcal{M}$ we know from the discussions of the case (i) that it is not restrictive to assume that $E(s) < 0$ if $s \geq 0$, and we claim that the zeros of $v(s)$ on $[0, +\infty[$ are isolated. Indeed, $v(s) = 0$ implies $|w(s)|^2 = \frac{1}{|E(s)|} > 0$ and then $v'(s) = \frac{E(s)w(s)}{2} \neq 0$. Thus

$$E(s) - E(0) = -4\int_0^s \mathcal{D}(|w(\xi)|^2)|v(\xi)|^2\, d\xi < 0$$

when $s > 0$ and then condition (13) holds on $\mathcal{M}$ with $V = E$. As a consequence, the ω-limit set of our solution is empty. From the identity

$$E(s)|w(s)|^2 + 1 = 2|v(s)|^2 \geq 0$$

we deduce that

$$\limsup_{s\to+\infty} |w(s)|^2 \leq \frac{1}{|E_\infty|}.$$

Also,

$$\limsup_{s\to+\infty} |v(s)|^2 \leq \frac{1}{2}.$$

Then the forward orbit $\{(w(s), v(s), E(s)) : s \geq 0\}$ is bounded and the ω-limit set is a non empty compact set of $\mathcal{M}$. This is the searched contradiction.

■

As an immediate consequence of this lemma we have the following:

Proposition 3.2 *If $x(t)$ is a solution of* (2) *with non zero angular momentum, then*

$$\lim_{t\to+\infty} E(t) = -\infty.$$

Proof. By choosing a branch of the square root, a non rectilinear solution $x(t)$ of (2) is transformed by (15) in a solution of (16) on $\mathcal{M}$ such that $w(s) = \sqrt{x(T(s))}$, where $T(s)$ is the inverse function of $s = S(t) = \int_0^t \frac{1}{|x(\tau)|}\, d\tau$. By Proposition 2.3 we conclude that $s \to +\infty$ when $t \to +\infty$, and the claim follows from Lemma 3.1.

■

As to the energy of the rectilinear solutions $x = r(t)$ of (2) we have the following result. Its proof is analogous to the one of the corresponding results given in [15] for the linear drag (see Proposition 3.1 and Proposition 4.2 therein) and therefore it is just outlined in the Appendix. Here we stress that the Levi-Civita regularization is used to get the second part of the statement.

Proposition 3.3 *Collisions occur with finite energy. Energy at collision may have any arbitrarily prescribed real value.*

4 Existence and properties of the Runge-Lenz-type first integral

As proved in the previous sections, a solution of (2) (and hence of (5)) such that $x(0) = x_0$ and $\dot{x}(0) = v_0$ is defined for $t \in [0, \omega[$ where $\omega = \omega(x_0, v_0)$

is finite in the case of a rectilinear motion, whereas $\omega = +\infty$ for a non rectilinear motion.

We recall that, if we consider the energy $E(x, v)$, the angular momentum $C(x, v)$ and the vector

$$R(x, v) = v \wedge (x \wedge v) - \frac{x}{|x|}, \quad (Runge - Lenz\ vector) \tag{18}$$

then the two following functional relationships hold among them as functions of the real variables (x, v) (see also [11], 3-9):

$$|x| + < R, x >= |C|^2, \ \text{ for any } x \in \mathbb{R}^2 \setminus \{0\}\,, \tag{19}$$

where $< v, w >$ denotes the inner product between the vectors v and w, and

$$|R|^2 - 1 = 2|C|^2 E. \tag{20}$$

In the conservative case, E, C and R are first integrals of the Kepler problem. In particular, if $0 < |R| < 1$, the vector R is the eccentricity vector corresponding to the Keplerian ellipse defined by (19), the unit vector $\dfrac{R}{|R|}$ is the direction of its major axis and $e = |R|$ is its eccentricity.

To end our preparatory work, we state the following lemma, needed to prove the continuity of I on Ω.

Lemma 4.1 *Let K be a compact subset of Ω. Then, there exist numbers $m_K > 0$ and $\mu_K > 0$ such that*

$$|x(t; x_0, v_0)| \le m_K \tag{21}$$

and

$$|\dot{x}(t; x_0, v_0)||x(t; x_0, v_0)|^{\frac{1}{2}} \le \mu_K \tag{22}$$

for any $(x_0, v_0) \in K$ and $t \in [0, \omega[$.

Proof. To prove the first estimate, we proceed as in the last part of the proof of Lemma 2.2 in [16], to which the reader should refer for the details. As pointed out in the previous section, the Levi-Civita regularization transforms the solutions of (2) starting in $(x_0, v_0) \in K$ into solutions of system (16) starting in a compact set $\mathcal{K} \subset \mathcal{M}$. By Lemma 3.1 these solutions are defined on $[0, +\infty[$ and are such that their energy $E(s)$ becomes, eventually, negative, say less than $-\frac{1}{2}$. [3] If we consider the w component of a solution of (16), the

[3] The discussion in [16] on the inequality (62) appearing in the proof of Lemma 2.2 was incomplete. This inequality is valid and follows from Lemma 3.1.

invariance of $\mathcal{M}$ gives the bound $|w(s)|^2 < 2$ for sufficiently large s. Then, by a standard compactness argument, solutions of (16) starting in $\mathcal{K}$ are such that the previous bound on $|w(s)|$ holds for s greater than a suitable s^* uniformly in $\mathcal{K}$. For such solutions the existence of a uniform bound for $|w(s)|$ on $[0, +\infty[$ easily follows. Going back to the original variables one gets (21) for $|x| = |w|^2$ when $(x_0, v_0) \in K$.

To prove the second estimate we observe that since the energy is decreasing, $E(t) \leq E(0)$ for any $t \in [0, \omega[$, and we get the following bound on the velocity:

$$|\dot{x}(t)| \leq \sqrt{2\left(E(0) + \frac{1}{|x(t)|}\right)}, \quad t \in [0, \omega[. \tag{23}$$

Multiplying (23) by $|x(t)|^{\frac{1}{2}}$ we obtain (22) with $\mu_K := \sqrt{2(E_K m_K + 1)}$ and $E_K := \max_K |E(x_0, v_0)|$.

■

We are now in a position to state the main result of this section. This result provides a continuous vector first integral $I = (I_1, I_2)$ which is invariant under the group of planar rotations and whose components are two functionally independent scalar first integrals (see Remark 1 in [16]).

As in the case of the linear drag, I can be interpreted as an asymptotic eccentricity vector and its norm as an asymptotic eccentricity. In particular, solutions with $|I| < 1$ tend to the origin along a spiral determined asymptotically by I.

Theorem 4.2 *There exists a continuous vector field*

$$I : \Omega \to \mathbb{R}^2, \;\; I = I(x, v)$$

satisfying

(i) $I(\sigma x, \sigma v) = \sigma I(x, v)$, *for each* $(x, v) \in \Omega$ *and each rotation* $\sigma \in SO(2)$.

(ii) The range of I *is the closed unit disk, that is*

$$I(\Omega) = \mathbb{D}, \tag{24}$$

where $\mathbb{D} = \{y \in \mathbb{R}^2 : |y| \leq 1\}$.

(iii) Each solution $(x(t), v(t))$ *of* (5)*, defined on a maximal right interval of the form* $[0, \omega[$*, satisfies*

$$I(x(t), v(t)) = \lim_{\tau \to \omega} R(x(\tau), v(\tau)). \tag{25}$$

Proof. Below we will prove the continuity of I and properties (i) and (iii). The proof of (ii) is postponed to the next section, to properly highlight the fact that it relies on the existence of asymptotically circular orbits.

Throughout the proof, K will be a fixed compact set contained in Ω, and (x_0, v_0) will be a point of K. Let $(x(t), v(t))$ be the solution of system (5) such that $(x(0), v(0)) = (x_0, v_0)$. We denote by $R(t) = R(x(t), v(t))$ and denote by $\dot{R}$ its derivative with respect to time.

Recall that we have $\dot{C} = -\mathcal{D}(|x(t)|)C$ and that

$$\frac{d}{dt}\left(\frac{x}{|x|}\right) = C \wedge \left(\frac{x}{|x|^3}\right) \tag{26}$$

for any smooth function $x = x(t)$. By differentiating the equality defining $R(t)$ and then integrating the result from 0 to t we get

$$R(t) = R(0) - 2\int_0^t \mathcal{D}(|x(\tau)|)\dot{x}(\tau) \wedge C(\tau)\, d\tau. \tag{27}$$

If $x_0 \wedge v_0 = 0$, then for the corresponding rectilinear motion we have $R(t) = R(0) = R(x_0, v_0) = -\frac{x_0}{|x_0|}$ for any $t \in [0, \omega[$ so that, trivially, we define $I(x_0, v_0) := \lim_{t\to\omega} R(t) = R(x_0, v_0)$. Let us consider the case $x_0 \wedge v_0 \neq 0$. We claim that the estimate below holds:

$$|(\mathcal{D}(|x(t)|) - \mathcal{D}(0))\dot{x}(t) \wedge C(t)| \leq Me^{-A_1 t} \text{ if } t \geq 0 \tag{28}$$

where the constant M is uniform with respect to K. To prove (28) let m_K and μ_K be the numbers provided by Lemma 4.1. Since $\mathcal{D}$ is locally Lipschitz continuous on $[0, +\infty[$, we can find a Lipschitz constant L_K on the compact interval $[0, m_K]$. In particular

$$|\mathcal{D}(r) - \mathcal{D}(0)| \leq L_K r \ \text{ if } \ 0 \leq r \leq m_K.$$

Thus, for any $t \geq 0$ we have

$$|(\mathcal{D}(|x(t)|) - \mathcal{D}(0))\dot{x}(t) \wedge C(t)| \leq L_K|x(t)||\dot{x}(t)||C(t)| \leq L_K m_K^{\frac{1}{2}} \mu_K |x_0||v_0|e^{-A_1 t}, \tag{29}$$

where we have used (3) and

$$|C(t)| \leq |x_0||v_0|e^{-A_1 t}, \ \ t \geq 0. \tag{30}$$

Once (28) has been proved, we rewrite the Runge-Lenz vector in the form

$$R(t) = R(0) + 2\mathcal{D}(0)x_0 \wedge C(0) - 2\mathcal{D}(0)x(t) \wedge C(t) - 2I_1(t) + 2\mathcal{D}(0)I_2(t), \tag{31}$$

with

$$I_1(t) = \int_0^t (\mathcal{D}(|x(\tau)|) - \mathcal{D}(0))\dot{x}(\tau) \wedge C(\tau)\, d\tau$$

and

$$I_2(t) = \int_0^t x(\tau) \wedge \dot{C}(\tau)\, d\tau.$$

Formula (31) is obtained by adding and subtracting $\mathcal{D}(0)$ in the scalar factor of the integral in (27) and then applying an integration by parts. From (30) we deduce that if $t \geq 0$

$$|x(t) \wedge C(t)| \leq m_K |x_0||v_0| e^{-A_1 t} \tag{32}$$

$$|x(t) \wedge \dot{C}(t)| \leq m_K D_K |x_0||v_0| e^{-A_1 t} \tag{33}$$

where $D_K = \max_{[0,m_K]} \mathcal{D}(r)$. Together with (28) these inequalities imply that $I = \lim_{t\to+\infty} R(t)$ exists.

At this point it is convenient to make explicit the functional dependence of I on the initial condition $(x_0, v_0) \in \Omega$ and write it as

$$\begin{aligned} I(x_0, v_0) \quad &= \quad R(x_0, v_0) + 2\mathcal{D}(0) x_0 \wedge C(x_0, v_0) - 2I_1(+\infty; x_0, v_0) + \\ &\quad\; +2\mathcal{D}(0) I_2(+\infty; x_0, v_0), \end{aligned} \tag{34}$$

where we set $I_1(+\infty; x_0, v_0) = I_2(+\infty; x_0, v_0) = 0$ if $C(x_0, v_0) = x_0 \wedge v_0 = 0$.

To prove the continuity of this function at each point we consider first the case $(x_0, v_0) \in \Omega$ with $x_0 \wedge v_0 \neq 0$. We can select a small closed ball centered at (x_0, v_0) such that the angular momentum does not vanish on it. This will be our set K. Then, estimates (28) and (33), together with the results on continuous dependence of solutions with respect to initial conditions, allow to get the continuity of $I_1(+\infty; \cdot, \cdot)$ and $I_2(+\infty; \cdot, \cdot)$ by applying standard results on functions defined by parametric Lebesgue integrals. In the case $x_0 \wedge v_0 = 0$ it must be noticed that if (x_{0n}, v_{0n}) is a sequence converging to (x_0, v_0) with $x_{0n} \wedge v_{0n} \neq 0$, then the corresponding solution satisfies

$$C_n(t) := x_n(t) \wedge \dot{x}_n(t) = e^{-\int_0^t \mathcal{D}(|x_n(\tau)|)\, d\tau} x_{0n} \wedge v_{0n} \to 0$$

as $n \to +\infty$ for each $t \geq 0$. Similarly, $\lim_{n\to+\infty} \dot{C}_n(t) = 0$ for each $t \geq 0$. From the estimates

$$|(\mathcal{D}(|x(t)|) - \mathcal{D}(0))\dot{x}(t) \wedge C(t)| \leq L_K m_K^{1/2} \mu_K |C(t)|, \quad |x(t) \wedge \dot{C}(t)| \leq m_K |\dot{C}(t)|$$

we deduce that $I_i(+\infty; x_{0n}, v_{0n}) \to 0$. Note that the estimates (29) and (33) imply that the convergence is dominated.

Then $I(x_{0n}, v_{0n}) \to I(x_0, v_0) = -\frac{x_0}{|x_0|}$ as $n \to \infty$. Since the same property trivially holds for sequences (x_{0n}, v_{0n}) converging to (x_0, v_0) and such that $x_{0n} \wedge v_{0n} = 0$, the continuity of I on Ω is proved.

Properties (i) and (iii) follow immediately from the definition of I.

■

5 Existence of asymptotically circular orbits

In this section we complete the proof of Theorem 4.2 by showing that the range of I is the closed unit disk. This property will be a consequence of the continuity of I, of its invariance under rotations, and of the existence of asymptotically circular orbits of (2), that is orbits for which $I = 0$. The decreasing towards zero of the eccentricity is an interesting effect associated to the motion of celestial bodies in a resistive medium, and the main focus of the section is on the proof that such effect, although not typical, occurs for the drag in (2).

We start by considering the set

$$\mathcal{C}_+ = \{(\xi, \eta) \in \Omega : \eta = |\xi|^{-\frac{3}{2}} J\xi\},$$

with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then $R(\xi, \eta) = 0, E(\xi, \eta) = -\frac{1}{2|\xi|} < 0$ and $C(\xi, \eta) \neq 0$. We fix a point $(\xi, \eta) \in \mathcal{C}_+$ and define the function $F : \Omega \to \mathbb{R}^4$

$$F(x, v) := \begin{pmatrix} R(x, v) \\ x - \xi \end{pmatrix}.$$

Lemma 5.1 *The point (ξ, η) is a nondegenerate zero of F. Actually*

$$\det F'(\xi, \eta) = 2|\xi| > 0.$$

Proof. Clearly $F(\xi, \eta) = 0$. We have

$$F'(\xi, \eta) = \begin{pmatrix} \partial_x R(\xi, \eta) & \partial_v R(\xi, \eta) \\ Id & 0 \end{pmatrix}$$

where Id denote the identity matrix of order two, so that $\det F'(\xi, \eta) = \det[\partial_v R(\xi, \eta)]$. Since

$$R(x, v) = \begin{pmatrix} x_1 v_2^2 - x_2 v_1 v_2 \\ x_2 v_1^2 - x_1 v_1 v_2 \end{pmatrix} - \frac{x}{|x|},$$

it follows that

$$\det[\partial_v R(\xi,\eta)] = |\xi|^{-3} \begin{vmatrix} -\xi_2\xi_1 & 2\xi_1^2+\xi_2^2 \\ -2\xi_2^2-\xi_1^2 & \xi_1\xi_2 \end{vmatrix} = 2|\xi|$$

and our proof is concluded.

■

Let us fix a small open ball $B \subset \mathbb{R}^4$ centred at (ξ,η) satisfying the following properties:

- (ξ,η) is the only zero of F in $\bar{B}$;
- there exists $\delta > 0$ such that $E(x_0,v_0) \le -\delta < 0$ if $(x_0,v_0) \in \bar{B}$;
- $C(x_0,v_0) \neq 0$ if $(x_0,v_0) \in \bar{B}$.

In particular the Brouwer degree of F in $\bar{B}$ is well defined and

$$\deg(F,B,0) = 1. \tag{35}$$

For each $\epsilon > 0$ the change of variables $x(t) = \epsilon^{\frac{2}{3}} y(\frac{t}{\epsilon})$ transforms equation (2) into

$$\ddot{y} + \epsilon\mathcal{D}(\epsilon^{\frac{2}{3}}|y|)\dot{y} = -\frac{y}{|y|^3}. \tag{36}$$

The Runge-Lenz vector has the invariance property

$$R(x,v) = R(\epsilon^{\frac{2}{3}}x, \epsilon^{-\frac{1}{3}}v)$$

and so

$$R(x(t),\dot{x}(t)) = R(y(t/\epsilon),\dot{y}(t/\epsilon)) \ \text{ for any } \ t \in [0,+\infty[.$$

Letting $t \to +\infty$ we obtain the identity $I_1(x_0,v_0) = I_\epsilon(\epsilon^{-\frac{2}{3}}x_0, \epsilon^{\frac{1}{3}}v_0)$ or, equivalently,

$$I_\epsilon(x_0,v_0) = I_1(\epsilon^{\frac{2}{3}}x_0, \epsilon^{-\frac{1}{3}}v_0) \tag{37}$$

where $I_\epsilon(x_0,v_0) := \lim_{t\to+\infty} R(y(t;x_0,v_0,\epsilon),\dot{y}(t;x_0,v_0,\epsilon))$ and $y(t;x_0,v_0,\epsilon)$ is the solution of the Cauchy problem for (36). The identity (37) shows that it is sufficient to prove the existence of an asymptotically circular motion for (36) for some $\epsilon > 0$.

Lemma 5.2 *The function $\tilde{I} : [0,1] \times \bar{B} \to \mathbb{R}^2$, given by $\tilde{I}(\epsilon,x_0,v_0) := I_\epsilon(x_0,v_0)$ is continuous.*

Proof. The continuity of $\tilde{I}$ on $]0,1]\times\bar{B}$ is a consequence of (37) and of the continuity of I_1 established in Theorem 4.2. The continuity at $\epsilon=0$ is a consequence of the expansion

$$I_\epsilon(x_0,v_0)=R(x_0,v_0)+O(\epsilon^{\frac{2}{3}}),\quad \text{uniformly in } (x_0,v_0)\in\bar{B}. \tag{38}$$

To prove (38) we simplify the notation by setting $y=y(t;x_0,v_0,\epsilon)$, $\dot{y}=\dot{y}(t;x_0,v_0,\epsilon)$, $C=y\wedge\dot{y}$ and observe that

$$|y|\le\frac{1}{\delta},\ \ |y|^{\frac{1}{2}}|\dot{y}|\le\sqrt{2}. \tag{39}$$

These estimates are a consequence of the inequality $\frac{1}{2}|\dot{y}|^2-\frac{1}{|y|}\le-\delta$. Also,

$$|C|\le|x_0||v_0|e^{-\epsilon A_1 t}\quad\text{and}\quad|\dot{C}|\le\epsilon M_\delta|x_0||v_0|e^{-\epsilon A_1 t}, \tag{40}$$

where $M_\delta:=\max_{r\in[0,\frac{1}{\delta}]}\mathcal{D}(r)$. From the proof of Theorem 4.2 we see that I_ϵ can be expressed in the form

$$\begin{aligned} I_\epsilon(x_0,v_0) &= R(x_0,v_0)+2\epsilon\mathcal{D}(0)x_0\wedge C(x_0,v_0)-2I_{1,\epsilon}(+\infty;x_0,v_0)+\\ &\quad +2\epsilon\mathcal{D}(0)I_{2,\epsilon}(+\infty;x_0,v_0). \end{aligned} \tag{41}$$

If we denote by L_δ the Lipschitz constant of $\mathcal{D}$ on $[0,\frac{1}{\delta}]$, by using (39) and the first inequality of (40) we get

$$\begin{aligned} |I_{1,\epsilon}| &= \epsilon\left|\int_0^{+\infty}(\mathcal{D}(\epsilon^{\frac{2}{3}}|y|)-\mathcal{D}(0))\dot{y}\wedge C\,dt\right|\le\epsilon L_\delta\epsilon^{\frac{2}{3}}\int_0^{+\infty}|y||\dot{y}||C|\,dt\le\\ &\le \epsilon^{\frac{2}{3}}L_\delta\sqrt{\frac{2}{\delta}}|x_0||v_0|\epsilon\int_0^{+\infty}e^{-\epsilon A_1 t}\,dt=\epsilon^{\frac{2}{3}}\frac{L_\delta}{A_1}\sqrt{\frac{2}{\delta}}|x_0||v_0|. \end{aligned} \tag{42}$$

Now, from the first inequality of (39) and the second inequality of (40) we get

$$|I_{2,\epsilon}|=\left|\int_0^{+\infty}y\wedge\dot{C}\,dt\right|\le\frac{1}{\delta}M_\delta|x_0||v_0|\epsilon\int_0^{+\infty}e^{-A_1\epsilon t}\,dt=\frac{M_\delta}{\delta A_1}|x_0||v_0|,$$

that together with (42) gives (38).

∎

We are now in a position to prove that for the drag in (2) we can have a circularizing effect on the orbits of (2).

Proposition 5.3 *There exists $(x_0,v_0)\in\Omega$ such that the corresponding solution of (2) is asymptotically circular, that is (x_0,v_0) satisfies $I(x_0,v_0)=0$.*

Proof. Consider the family of functions $F_\epsilon : [0,1] \times \Omega \to \mathbb{R}^4$, where

$$F_\epsilon(x,v) := \begin{pmatrix} I_\epsilon(x,v) \\ x-\xi \end{pmatrix}.$$

By Lemma 5.2 the family F_ϵ is continuous in $[0,1] \times \bar{B}$ and, moreover, by (38) we have that $F_0 = F$. Then, since $\deg(F, B, 0) = 1$, the homotopy invariance of the degree guarantees that for sufficiently small ϵ there exists a zero, necessarily of the form $(\xi, v(\epsilon))$, of F_ϵ in B. Hence, $I_\epsilon(\xi, v(\epsilon)) = 0$ and by (37) we conclude that the point $(x_0, v_0) := (\epsilon^{\frac{2}{3}}\xi, \epsilon^{-\frac{1}{3}}v(\epsilon)) \in \Omega$ is the initial condition of an asymptotically circular orbit of (2).

■

Finally, we prove our claim about the range of I.

5.1 Proof of (ii) of Theorem 4.2.

If $x_0 \wedge v_0 \neq 0$ then $C(t) \neq 0$ for any $t \in [0, +\infty[$ and, by Proposition 3.2, $E(t) \to -\infty$ when $t \to +\infty$. As a consequence, by (20) we get $|R(t)|^2 - 1 < 0$ if t is large enough, and $|I| \leq 1$ follows taking the limit in t. In the case $x_0 \wedge v_0 = 0$, we have $I(x_0, v_0) = -\dfrac{x_0}{|x_0|}$ so that $|I(x_0, v_0)| = 1$. Since by Proposition 5.3 the first integral I takes the value 0, by its continuity and by its invariance under planar rotations we get $I(\Omega) = \mathbb{D}$. ■

The geometrical and dynamical consequences of the existence of I are analogous to the ones described in [16] for the linear drag. Namely, if $x(t) = x(t; x_0, v_0) = r(t)e^{i\theta(t)}$ is a non rectilinear motion of (2), then the trajectory

$$y(t) = e^{2\int_0^t \mathcal{D}(|x(s)|)\,ds} x(t)$$

tends asymptotically to the curve

$$|y| + < y, I(x_0, v_0) >= |C(x_0, v_0)|^2.$$

When $|I(x_0, v_0)| < 1$ this is an ellipse whose eccentricity vector is $I(x_0, v_0)$, and in such a case the following holds: $|x(t)| = r(t)$ tends to zero exponentially with time, whereas the modulus of the angular velocity $|\dot{\theta}(t)|$ increases exponentially with time. The proofs of these facts follow taking into account that $A_1 \leq \mathcal{D}(|x(t; x_0, v_0)|) \leq M = \max_{t \geq 0} \mathcal{D}(|x(t; x_0, v_0)|)$ and following the steps in [16] to obtain the exponential estimates on the growth of $|x|$ and $|\dot{\theta}|$.

6 Appendix

6.1 Proof of Proposition 2.4

We start by regularizing the first order system equivalent to equation (14) by the time rescaling $\tau = \tau(t) = \int_0^t \frac{d\sigma}{r^2(\sigma)}$. We obtain the system

$$\begin{cases} r' = r^2 u \\ u' = -\mathcal{D}(r) r^2 u - 1 \end{cases} \tag{43}$$

where the derivatives are taken with respect to the time τ. Now we proceed in a manner that is analogous to the one employed in the proof of Proposition 3.2 of [15]. We start by noticing that $r = 0$ is an orbit of (43) and that this system does not have any equilibria. Also, the set $Q = \{(r, u) : r > 0,\ u < 0\}$ is a positively invariant set for (43) on which the r component of the solutions of (43) is decreasing. A key ingredient of the proof is the existence of the first integral of (43) given by

$$H := u + \Delta(r) + \tau, \tag{44}$$

where $\Delta(r) := \int_0^r \mathcal{D}(\sigma)\, d\sigma$ satisfies the estimate $\Delta(r) \geq A_1 r$. Using the first integral and the estimate, one proves that all solutions with $r(0) > 0$ eventually enter the set Q. In fact, by a contradiction argument, one sees that the negation of this property implies the existence of a bounded orbit having as its ω-limit an equilibrium of (43) in the first quadrant. Then, in an analogous manner, it is easily shown that all solutions are defined for $\tau \in [0, +\infty[$. Since if $(r(0), u(0)) = (r_0, u_0) \in Q$, then $r(\tau) \in [0, r_0]$ for any $\tau \in [0, +\infty[$, and since from (44) we have

$$u(\tau) + \Delta(r(\tau)) = u_0 + \Delta(r_0) - \tau,$$

we conclude that

$$\frac{u(\tau)}{\tau} \to -1 \quad \text{as} \quad \tau \to +\infty. \tag{45}$$

As a consequence, we get that $u(\tau) \to -\infty$ as $\tau \to +\infty$ and, integrating the first equation of (43), we get also that

$$\tau^2 r(\tau) = \frac{\tau^2}{\frac{1}{r_0} + \int_0^\tau |u(\sigma)|\, d\sigma} \to 2 \quad \text{as} \quad \tau \to +\infty. \tag{46}$$

We conclude that $r(\tau) \to 0$ as $\tau \to +\infty$. To end our proof we have to show that the maximal interval $[0, \omega[$ of a solution $r(t)$ of (14) is bounded. If

$t = T(\tau)$ is the inverse function of $\tau = \tau(t)$, then $r(\tau) := r(T(\tau))$ is the first component of a solution of (43), and we have

$$\omega = \int_0^{+\infty} T'(\tau)\, d\tau = \int_0^{+\infty} r^2(\tau)\, d\tau \in \mathbb{R},$$

since by (46) $r^2(\tau)$ behaves like $\frac{4}{\tau^4}$ for large τ. ■

6.2 Proof of Proposition 3.3

This proof follows the steps of the one given for the linear drag in Proposition 3.2 of [15]. Let $r(t)$ be a maximal solution of (14) defined on $[0, \omega[$, $\omega \in \mathbb{R}$. Its energy, expressed in the time τ, is given by $E(\tau) := E(r(\tau), u(\tau))$, where $(r(\tau), u(\tau)) = (r(T(\tau)), \dot{r}(T(\tau)))$ is a solution of (43) is defined on $[0, +\infty[$. Then,

$$E'(\tau) = -\mathcal{D}(r(\tau))u^2(\tau)r^2(\tau), \quad \tau \in [0, +\infty[.$$

By (45) and (46) we get that fixed any positive η

$$|E'(\tau)| \leq \max_{[0,M]} \mathcal{D}(r)\, \frac{4+\eta}{\tau^2}$$

for sufficiently large τ, where $M = \max_{[0,+\infty[} r(\tau) \in \mathbb{R}$ exists since by Proposition 2.4 all solutions of (14) tend to zero. We conclude that $E' \in L^1[0, +\infty[$ and hence $E(\omega) = E(0) + \int_0^{+\infty} E'(\tau)\, d\tau \in \mathbb{R}$.

The proof of the fact that the energy may take any arbitrarily prescribed value $E_1 \in \mathbb{R}$ at collision is completely analogous to the proof of Proposition 4.2 in [15], and we give it for the reader's sake. Let E_1 be a prescribed value of the energy. Let $(w(s), v(s), E(s)) \in \mathcal{M}$ be the solution of (16) such that $(w(0), v(0), E(0)) = (0, -1/\sqrt{2}, E_1) \in \mathcal{M}$. Let $s = S(t)$ be the local inverse of $T(s) = t_1 - \int_s^0 w^2(\sigma)\, d\sigma$ in a suitable left neighbourhood of $s = 0$, where t_1 is arbitrarily fixed in $\mathbb{R}$. Then, the function $r(t) := w(S(t))^2$, defined in a left neighbourhood of t_1, will solve

$$\ddot{r} = -\mathcal{D}(r)\dot{r} + \frac{1}{r^2}\mathcal{J} - \frac{1}{r^2},$$

where $\mathcal{J}(E, w, v) := E|w|^2 - 2|v|^2 + 1$. Since $\mathcal{M}$ is invariant, we get that $\mathcal{J} = 0$ along the solutions of (16) and we conclude that $r(t)$ is a solution (14) which collides with the singularity at time t_1 having energy E_1 at collision. ■

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