Existence of periodic and solitary waves for a Nonlinear Schrödinger Equation with nonlocal integral term of convolution type

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Abstract

We prove the existence of periodic solutions and solitons in the nonlinear Schrödinger equation with a nonlocal integral term of convolution type. By separating phase and ampllitude, the problem is reduced to an integro-differential formulation that can be written as a fixed point problem for a suitable operator on a Banach space. Then a fixed point theorem due to Krasnoselskii can be applied.

Key words: Nonlinear Schrödinger Equation, non-local term, Krasnoselskii fixed point theorem, periodic Solution, Solitary wave, Green's function.

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1 Introduction and main results

Our aim in this paper is to study the existence of periodic and solitary waves of the Schrödinger equation with nonlocal term

$$iu_t + u_{xx} + V(x)u + u(x)\int_{-\infty}^{\infty} K(x,s)|u(s)|^2 ds = 0 \tag{1}$$

where the kernel $K(x,s)$ is assumed to be of the form

$$K(x,s) = \gamma(x)W(x-s),$$

being W a function (or distribution) with non-negative values such that

$$\|W\|_1 = \int_{-\infty}^{\infty} W(s)ds < +\infty. \tag{2}$$

In Bose-Einstein condensates, the non-local integral term in (1) is a rather general form of the nonlinearity due to the two-body interactions [1]. A typical approximation is that of the local interactions $K(x,s) = \gamma(x)\delta(x-s)$, where $\gamma(x)$ plays the role of the x-dependent scattering length (such dependence on the spatial coordinate can be induced through the technique called Feshbach resonance, see [2] for a review). If one limits the consideration by non-long-range interactions, then the natural generalization of form for the kernel $K(x,s)$ could be a Gaussian function $K(x,s) = \gamma(x)\exp(-(x-s)^2)$, super-Gaussian $K(x,s) = \gamma(x)\exp(-(x-s)^4)$, or the step-like function $\gamma(x)\theta(a - |x-s|)$ simply indicating that two atoms interact as two hard spheres of the radius a. Then the integral in (3) is reduced to

$$\gamma(x)u(x)\int_{x-a}^{x+a} K(x,s)|u(s)|^2 ds. \tag{3}$$

See also [3] and references therein. On the other hand, the linear term $V(x)u$ is relevant in Bose-Einstein condensates as a model of a possible external magnetic trap.

By setting $u(x,t) = e^{i\delta t}u(x)$, the above partial differential equation can be directly reduced to de second order integro-differential equation

$$-u''(x) + a(x)u(x) = \gamma(x)u(x)\int_{-\infty}^{+\infty} W(x-s)|u(s)|^2 ds \tag{4}$$

where $a(x) = \delta - V(x)$. We look for an analytical proof of the existence of two types of solutions:

(i) Periodic waves:

$$u(x) = u(x+T), \quad \text{for all } x$$

(ii) Solitary waves:

$$u(-\infty) = 0 = u(+\infty).$$

Recently, huge amount of work has been done in the literature analyzing the existence and properties of solutions of nonlinear Schrödinger equation with different kinds of nonlinearities (we cite [4]-[14] only to mention some of them). Many authors use substitution based techniques to find explicit solutions, by forcing the solution to have a given specific form. In other cases, iterative methods are used without checking the convergence and the existence of the solution before applying it. The consideration of an integral term of convolution type like in our case makes the application of such methods much more difficult or even impossible in most of the cases.

In this paper we will deal with the problem from a different point of view by using the classical Fixed Point Theory for operators on Banach spaces, and more concretely the Krasnoselskii's fixed point Theorem for compression-expansion of conical sections. This tool has been successfully applied to related local problems (see [15,16] for the periodic problem and [17–19] for solitary waves).

At this point, it is worth to precise the notion of solution for equation (4). In the rest of the paper, the coefficients $V(x), \gamma(x)$ belong to the space of bounded functions $L^\infty(\mathbb{R})$ and solutions of (4) are understood in the Caratheodory sense, that is, belonging to the Sobolev space $W^{2,\infty}$. To avoid trivialities, $\gamma(x)$ is always assumed to be not identically zero. Below we state our main results.

Theorem 1 *Assume that $V(x), \gamma(x)$ are T-periodic functions. If γ takes non-negative values, $\delta > \|V\|_\infty$ and W verifies condition* (2)*, then eq.* (4) *has at least one positive T-periodic solution $u \in W^{2,\infty}(0, T)$.*

Theorem 2 *If $\gamma(x)$ is a non-negative function with non-empty compact support, $\delta > \|V\|_\infty$ and W verifies condition* (2)*, then eq.* (4) *has at least one non-negative solution (not identically zero) such that $u(-\infty) = 0 = u(+\infty)$. Besides, it has finite energy in the sense that $u \in W^{2,\infty}(\mathbb{R})$.*

This paper is organized in the following way. In Section 2, we review the main ingredients of Krasnoselskii fixed point Theorem. Section 3 and 4 develop the proofs of the main results. The paper is finished with Section 5, where we present the conclusions as well as further remarks.

2 Terminology and statement of Krasnoselskii fixed point Theorem

Krasnoselskii theorem is a famous result in Fixed Point Theory introduced by Krasnoselskii in 1960 [20,21] that has been extensively used in the study of boundary value problems. In an abstract formulation, a boundary value problem (BVP) is written as $L(u) = F(u)$, where L is an invertible linear operator. Then, to solve the BVP is equivalent to find a fix point of the operator $H = L^{-1} \circ F$ on a given Banach space X. The following notion is crucial.

Definition 1 *A non-empty subset K of X is a* cone *if it satisfies the following properties: (i) K is closed (ii) If $x \in K$ and $-x \in K$, then $x = 0$ (iii) If $x,\ y \in K$, then $x + y \in K$ (iv)If $x \in K$, then $\lambda x \in K$ for all real number $\lambda > 0$.*

A map $H : K \to K$ is completely continuous (or compact) if it is continuous and the image of a bounded set is relatively compact. Thereafter, we state a version of the well known Krasnoselskii fixed point Theorem.

Theorem 3 *Let X be a Banach space, and $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ and let $H : K \bigcap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that one of the following conditions holds:*

1. *$\|Hu\| \leq \|u\|$, if $u \in K \bigcap \partial\Omega_1$, and $\|Hu\| \geq \|u\|$, if $u \in K \bigcap \partial\Omega_2$.*
2. *$\|Hu\| \geq \|u\|$, if $u \in K \bigcap \partial\Omega_1$, and $\|Hu\| \leq \|u\|$, if $u \in K \bigcap \partial\Omega_2$.*

Then, H has at least one fixed point in $K \bigcap (\overline{\Omega}_2 \backslash \Omega_1)$.

This is the basis for the extensive work in analyzing the existence of the solutions of nonlinear boundary value problems with separated or periodic boundary conditions in the literature. In the next sections we will apply this theorem to prove the existence of the solution for the nonlinear Schrödinger equation with nonlocal integral term.

3 Existence of Periodic Solutions

Let us denote by X_T the Banach space of bounded and periodic solutions with minimal period T endowed with the uniform norm $\|u\|_\infty$. Following the methodology described in Section 2, the first task is to invert the linear part. Consider the equation

$$-u''(x) + a(x)u(x) = w(x) \tag{5}$$

subjected to periodic boundary conditions. Given $w \in X_T$, eq. (5) admits a unique T-periodic solution by Fredholm's alternative, and it can be expressed as

$$u(x) = \int_0^T G(x,y)w(y)dy \tag{6}$$

where $G(x,y)$ is the associated Green's function. For a detailed exposition of the theory of Green's functions, we recommend the monograph [22].

Recall that $a(x) = \delta - V(x)$. When $V(x) \equiv 0$, the Green's function has an explicit expression (see for instance [15]). In the more general case under consideration, such explicit expression is not available anymore, but the condition $\delta > \|V\|_\infty$ implies that $G(x,y) > 0$ for all $(x,y) \in [0,T] \times [0,T]$ (see [16]). Let us define

$$m = \min_{x,y} G(x,y), \qquad M = \max_{x,y} G(x,y).$$

Now, we can define the operator $H : X_T \to W^{2,\infty}(0,T) \subset X_T$ by

$$Hu(x) = \int_0^T G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy. \tag{7}$$

A function $u \in W^{2,\infty}(0,T)$ is a T-periodic solution of eq. (4) if and only if it is a fixed point of H. The compactness of H is a direct consequence of Ascoli-Arzela Theorem.

Thereafter, we define a cone of the form,

$$K = \{u \in X_T \,:\, \min_x u \geq \frac{m}{M}\|u\|_\infty\}.$$

As a first step of the proof of Theorem 1, in the next Lemma we prove that H leaves invariant the cone K.

Lemma 1 $H(K) \subset K$.

Proof: Take $u \in X_T$ and fix x_0 such that $u(x_0) = \min_{x\in[0,T]} Hu(x)$. Then,

$$\begin{aligned}
Hu(x_0) &= \int_0^T G(x_0,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \\
&\geq m\int_0^T \frac{\max_x G(x,y)}{M}\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \\
&= \frac{m}{M}\int_0^T \max_x G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \\
&= \tfrac{m}{M}\left\|Hu\right\|_\infty,
\end{aligned}$$

therefore the cone is invariant by H. $\square$

Now we are ready to prove the main result for periodic solutions.

Proof of Theorem 1. Define $\Omega_1 = \{u \in X_T : \|u\|_\infty \leq r\}$. Given $u \in K \bigcap \partial\Omega_1$, it is evident that $\|u\|_\infty = r$. Then,

$$\begin{aligned} Hu(x) = \int_0^T G(x,y) \left[\gamma(y)u(y) \int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \leq \\ \leq M \|\gamma\|_\infty r^3 \int_0^T \int_{-\infty}^{+\infty} W(y-s) ds dy = M \|\gamma\|_\infty T \|W\|_1 r^3 < r, \end{aligned}$$

if r is small enough. Therefore $\|Hu\|_\infty < \|u\|_\infty$ for any $u \in K \bigcap \partial\Omega_1$.

On the other hand, define $\Omega_2 = \{u \in X_T : \|u\|_\infty \leq R\}$. Assume that $u \in K \bigcap \partial\Omega_2$, then by the own definition of the cone $\min_x u \geq \frac{m}{M} R$. Hence,

$$\begin{aligned} Hu(x) = \int_0^T G(x,y) \left[\gamma(y)u(y) \int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \geq \\ \geq \left(\frac{m}{M} R\right)^3 \|W\|_1 \int_0^T G(x,y)\gamma(y) dy \geq \left(\frac{m}{M} R\right)^3 \|W\|_1 m \int_0^T \gamma(y) dy. \end{aligned}$$

Note that γ is not identically zero, so $\int_0^T \gamma(y)dy > 0$ and the latter inequality holds for any x. In consequence, taking R big enough we get

$$\|Hu\|_\infty > R = \|u\|_\infty .$$

Therefore, the assumptions of Theorem 1 are fulfilled, in consequence H has a fixed point in $K \bigcap (\bar{\Omega}_2 \backslash \Omega_1)$, which is equivalent to a positive T-periodic solution of eq. (4). $\square$

4 Existence of solitary waves

Along this section, we assume that $\gamma \in L^\infty(\mathbb{R})$ is a non-negative function with non-trivial compact support D. The objective is to prove the existence of solutions of (4) with boundary conditions $u(-\infty) = 0 = u(+\infty)$. The fact that the BVP is defined on the whole real line will lead to some technical difficulties, specially concerning the compactness of the operator.

Let us denote by $C_b(\mathbb{R})$ the Banach space of the bounded and continuous functions in $\mathbb{R}$ with the uniform norm. The following result is well-known.

Lemma 2 *Assume that there exists a_* such thath $a(x) \geq a_* > 0$ for a.e. x. If $w \in L^\infty(\mathbb{R})$, then the linear equation*

$$-u''(x) + a(x)u(x) = w(x)$$

admits a unique bounded solution and it can be expressed as

$$u(x) = \int_{-\infty}^{+\infty} G(x,y)w(y)dy.$$

Besides, if $w \in L^1(\mathbb{R})$*, then* $u \in W^{2,\infty}(\mathbb{R})$.

When $V(x) \equiv 0$ then $a(x) \equiv \delta$ and the Green's function has the simple expression

$$G(x,y) = \frac{1}{2\sqrt{\delta}} e^{-\sqrt{\delta}|x-y|}.$$

However, as remarked in the periodic case, the Green's function for the general case of a variable $a(x)$ does not have such an explicit formula and requires a more careful study of its properties. Following [17,18], the Green's function $G(x,y)$ can be written as

$$G(x,y) = \begin{cases} u_1(x)u_2(y), & \text{if } x \leq s \\ u_1(y)u_2(x), & \text{if } s \leq x \end{cases} \tag{8}$$

where u_1, u_2 are solutions such that $u_1(-\infty) = 0, u_2(+\infty) = 0$ and $u_1'(0)u_2(0) - u_1(0)u_2'(0) = 1$. Such u_1, u_2 are monotone and intersect in a unique point x_0. Let us define

$$p(x) = \begin{cases} \dfrac{1}{u_2(x)}, & \text{if } x \leq x_0, \\ \\ \dfrac{1}{u_1(x)}, & \text{if } x > x_0. \end{cases} \tag{9}$$

The main properties of $G(x,y)$ are recorded in the following result.

Proposition 1 ([18]) *The following properties for the Green's function defined by* (8) *hold*

($P1$) $G(x,y) > 0$ *for every* $(x,s) \in \mathbb{R}^2$.
($P2$) $G(x,y) \leq G(y,y)$ *for every* $(x,y) \in \mathbb{R}^2$.
($P3$) *Given a non-empty compact subset* $P \subset \mathbb{R}$*, we define*

$$m_1(P) = \min\{u_1(\inf P), u_2(\sup P)\}. \tag{10}$$

Then,

$$G(x,y) \geq m_1(P)p(y)G(y,y) \text{ for all } (x,y) \in P \times \mathbb{R}^2.$$

($P4$) $G(y,y)p(y) \geq G(x,y)p(y)$ *for every* $(x,y) \in \mathbb{R}^2$.

By Lemma 2, to find a solitary wave of eq. (4) is equivalent to find a fixed point of the operator $H : C_b(\mathbb{R}) \to W^{2,\infty}(\mathbb{R}) \subset C_b(\mathbb{R})$ defined by

$$Hu(x) = \int_{-\infty}^{+\infty} G(x,y) \left[\gamma(y)u(y) \int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy. \qquad (11)$$

The compactness of H is a consequence of the following lemma.

Lemma 3 ([18]) *Let $\Omega \subset C_b(\mathbb{R})$. Let us assume that the functions $u \in \Omega$ are equicontinuous in each compact interval of $\mathbb{R}$ and that for all $u \in \Omega$ we have*

$$|u(x)| \le \xi(x), \qquad \forall x \in \mathbb{R} \qquad (12)$$

where $\xi \in C_b(\mathbb{R})$ satisfies

$$\lim_{|x|\to+\infty} \xi(x) = 0. \qquad (13)$$

Then, Ω is relatively compact.

Proposition 2 *The operator H defined by* (11) *is compact.*

Proof. The continuity of H is trivial so that we focus on the compactness property. We will use Lemma 3. Let $(u_n)_n \subset C_b(\mathbb{R})$ be a bounded sequence, say by M. Define the sequence $v_n(x) = Hu_n(x)$. We just need to prove that, up to a subsequence, v_n converges uniformly in $\mathbb{R}$. Therefore, we compute

$$\begin{aligned} |v_n(x)| &= |\textstyle\int_{-\infty}^{+\infty} G(x,y) \left[\gamma(y)u(y) \int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy| \le \\ &\le M^3 \|W\|_1 \textstyle\int_{-\infty}^{+\infty} G(x,y)\gamma(y)dy. \end{aligned}$$

Now,since $\gamma(x)$ has compact support, Lemma 2 implies that the function $\int_{\mathbb{R}} G(x,y)\gamma(y)dy$ goes to zero at $\pm\infty$. Moveover, the equicontinuity of the v_n sequence is clear as it follows directly from the continuity of the Green function. Hence, we are able to apply the previous result. $\square$

The next step is to define a suitable cone of the form

$$K = \{u \in C_b(\mathbb{R}) : \; u(x) \ge 0 \;\text{ for all } x, \quad \min_D u(x) \ge m_1 p_0 \|u\|_\infty\}.$$

Recall that D is the (compact) support of γ. In this definition, $p_0 = \inf_D p(x)$, $p(x)$ being defined by (9), and the constant $m_1 \equiv m_1(D)$ is defined by (10). Note that the compactness of D implies that $p_0 > 0$. Also, it is easy to see, by definition, that $m_1 p_0 \le 1$, hence $u \equiv 1 \in K$ and this cone is non-empty.

Lemma 4 $H(K) \subset K$.

Proof. Assume that the $Hu(x_0) = \min_{x\in D} Hu(x)$. Using the properties (P3) and (P2) of the Green's function, we obtain

$$
\begin{aligned}
Au(x_0) &= \textstyle\int_D G(x_0,s)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \geq \\
&\geq m_1 \textstyle\int_D G(s,s)p(s)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \geq \\
&\geq m_1 \textstyle\int_D G(x,s)p_0\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \\
&= m_1 p_0 Hu(x)
\end{aligned}
$$

for every $x \in \mathbb{R}$. This implies the invariance of the cone K. $\square$

Proof of Theorem 2. Define $\Omega_1 = \{u \in C_b(\mathbb{R}) : \|u\|_\infty \leq r\}$. Given $u \in K \bigcap \partial\Omega_1$,

$$
\begin{aligned}
\|Hu\|_\infty &= \max_x \int_D G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \leq \\
&\leq r^3 \max_x \int_D G(x,s)\gamma(s)\int_{-\infty}^{+\infty} W(y-s) ds dy = \\
&= \|W\|_1 r^3 \max_x \textstyle\int_D G(x,y)\gamma(y) dy.
\end{aligned}
$$

Note that by Lemma 2, $\int_D G(x,y)\gamma(y)dy$ is the unique solution belonging to $W^{2,\infty}(\mathbb{R})$ of the linear problem $-u'' + a(x)u = \gamma(x)$. Of course, such a solution is bounded and the maximum in the previous inequality makes sense. In conclusion, if r is small enough,

$$
\|Hu\|_\infty \leq \|W\|_1 r^3 \max_x \int_D G(x,y)\gamma(y)dy < r = \|u\|_\infty
$$

for every $u \in K \bigcap \partial\Omega_1$.

On the other hand, define $\Omega_2 = \{u \in C_b(\mathbb{R}) : \|u\|_\infty \leq R\}$. Assume that $u \in K \bigcap \partial\Omega_2$, then by definition of the cone $\min_{x\in D} u \geq m_1 p_0 R$. Hence,

$$
\begin{aligned}
\|Hu\|_\infty &= \max_x \int_D G(x,y)\left[\gamma(y)u(y)\int_{-\infty}^{+\infty} W(y-s)u(s)^2 ds\right] dy \geq \\
&\geq (m_1 p_0 R)^3 \int_D G(x,y)\left[\gamma(y)\int_{-\infty}^{+\infty} W(y-s) ds\right] dy \\
&\geq (m_1 p_0 R)^3 \|W\|_1 \max_x \textstyle\int_D G(x,y)\gamma(y) dy.
\end{aligned}
$$

Note that γ is not identically zero, so $\max_x \int_D G(x,y)\gamma(y)dy > 0$. In consequence, taking R big enough we get

$$
\|Hu\|_\infty > R = \|u\|_\infty .
$$

Therefore, the assumptions of Theorem 1 are fulfilled, and H has a fixed point in $K \bigcap (\bar{\Omega}_2 \backslash \Omega_1)$, which is equivalent to a non-trivial non-negative solution u of eq. (4) belonging to $W^{2,\infty}(\mathbb{R})$. $\square$

5 Conclusions and further remarks

In this paper we have considered solutions of a nonlinear Schrödinger equation with a non-local term of convolution type with the ansantz $u(x,t) = e^{i\delta t}u(x)$. Hence, the time dependence is periodic. For the spatial dependence, we have studied the periodic problem and the convergence to zero at $\pm\infty$ (solitary waves) separately. In both cases, the introduction of the ansantz into the Schrödinger equation leads to an integro-differential equation that can be analyzed via a fixed point formulation and a suitable fixed point theorem for operators defined on conical sections of a Banach space. It is interesting to remark that δ is not a parameter of the equation, hence the condition $\delta > \|V\|_\infty$ is not a real restriction for the equation and provides a whole uniparametric family of solutions.

An interesting open problem is to know wheter the non-negative solutions obtained in Theorem 2 are in fact positive. This is true in the local case $K(x,s) = \gamma(x)\delta(x-s)$. In this case, eq. (4) is a simple ODE and since u is non-negative, if $u(x_0) = 0$ then by force $u'(x_0) = 0$, then $u \equiv 0$ by uniqueness of the initial value problem. This simple argument can not be applied in the general case. Finally, a possible line of future research is the extension of the presented results to coupled systems, in the line of the study developed in [19].

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