Periodic solutions of singular second order differential equations: upper and lower functions

Robert Hakl[1], Pedro J. Torres[2] and Manuel Zamora[2]

(1) *Institute of Mathematics AS CR, Žižkova 22, 616 62 Brno, Czech Republic,*
e-mail: `hakl@ipm.cz`

(2) *Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva s/n, 18071 Granada, Spain,*
e-mails: `ptorres@ugr.es, mzc0708@ugr.es`

Abstract

In this paper we continue the study of the periodic problem for the second–order equation $u'' + f(u)u' + g(u) = h(t,u)$, where h is a Carathéodory function and f, g are continuous functions on $(0, +\infty)$ which may have singularities at zero. Both attractive and repulsive singularities are considered. The method relies on a novel technique of construction of lower and upper functions. As an application, we obtain new sufficient conditions for existence of periodic solutions to the Rayleigh–Plesset equation.

1 Introduction

In this paper, we are concerned with the periodic problem

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h(t, u(t)) \qquad \text{for a. e. } t \in [0, \omega], \tag{1.1}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega), \tag{1.2}$$

where $f, g \in C\big(\mathbb{R}^+; \mathbb{R}\big)$ may have singularities at zero, and $h \in \mathrm{Car}\,\big([0, \omega] \times \mathbb{R}_+; \mathbb{R}\big)$. In the related literature, g is said to present an attractive (resp. repulsive) singularity if $\lim\limits_{x \to 0^+} g(x) = +\infty$ (resp. $\lim\limits_{x \to 0^+} g(x) = -\infty$). By a positive solution to the problem (1.1), (1.2) we understand a function $u : [0, \omega] \to \mathbb{R}^+$ which is absolutely continuous together with its first derivative, satisfies (1.1) almost everywhere on $[0, \omega]$, and verifies (1.2).

In the paper [3], we obtain some sufficient conditions for the existence of solutions to the problem (1.1), (1.2) by using the Schaefer's fixed point theorem. This paper can be considered the second part of our previous work [3]. Our initial motivation was the

[1]Supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503; corresponding author.

[2]Supported by Ministerio de Educación y Ciencia, Spain, project MTM2008-02502, and by Junta de Andalucía, Spain, Project FQM2216.

Rayleigh-Plesset equation (see, e.g., [2])

$$\rho\left[R\ddot{R}+\frac{3}{2}\dot{R}^2\right]=[P_v-P_\infty(t)]+P_{g_0}\left(\frac{R_0}{R}\right)^{3k}-\frac{2S}{R}-\frac{4\mu\dot{R}}{R}.$$

In Physics of Fluids, this is a famous model for the oscillations of the radius $R(t)$ of a spherical bubble immersed in a fluid under the influence of a periodic acoustic field P_∞. A detailed explanation of the physical meaning of the involved parameters can be found in [3, Section 3] and the references therein. It is observed in [3] that the change of variable $R=u^{\frac{2}{5}}$ leads to

$$\ddot{u}=\frac{5\,[P_v-P_\infty(t)]}{2\rho}u^{\frac{1}{5}}+\left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)\frac{1}{u^{\frac{6k-1}{5}}}-\frac{5S}{u^{\frac{1}{5}}}-4\mu\frac{\dot{u}}{u^{\frac{4}{5}}},$$

which is a particular case of (1.1). Therefore, we will pay a special attention to the model equation

$$u''(t)+f(u(t))u'(t)+\frac{g_1}{u^\nu(t)}-\frac{g_2}{u^\gamma(t)}=h_0(t)u^\delta(t)\qquad\text{for a. e. } t\in[0,\omega],\tag{1.3}$$

where $g_1,g_2,\delta\in\mathbb{R}_+$, $\nu>0$, $\gamma\in\mathbb{R}$, $h_0\in L\big([0,\omega];\mathbb{R}\big)$ and $f\in C\big(\mathbb{R}^+;\mathbb{R}\big)$.

Our purpose is to develop a novel method of construction of lower and upper functions, giving rise to new abstract existence theorems for the general equation (1.1) which are easily applicable to the model equation (1.3) and make more complete the results of [3]. Incidentally, it turns out that our results are new even for the classical equation of Liénard type

$$u''(t)+f(u(t))u'(t)+g(u(t))=h_0(t)\qquad\text{for a. e. } t\in[0,\omega].\tag{1.4}$$

When $f\in C\big(\mathbb{R}_+;\mathbb{R}\big)$, this equation has been extensively studied in the last two decades (see the bibliography of [3]).

The paper is structured in 4 sections: after this Introduction, Section 2 is devoted to develop a new technique of construction of upper and lower functions. The main results are presented and proved in Section 3, with special attention the the model equation (1.3). Finally, in Section 4 the results of Section 3 are applied to the Rayleigh-Plesset equation.

For convenience, we finish this introduction with a list of notation which is used throughout the paper:

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+=(0,+\infty)$, $\mathbb{R}_+=[0,+\infty)$, $[x]_+=\max\{x,0\}$, $[x]_-=\max\{-x,0\}$.

$C\big([0,\omega];\mathbb{R}\big)$ is the Banach space of continuous functions $u:[0,\omega]\to\mathbb{R}$ with the norm

$$\|u\|_\infty=\max\{|u(t)|:t\in[0,\omega]\}.$$

$C(D_1;D_2)$, where $D_1,D_2\subseteq\mathbb{R}$, is the set of continuous functions $u:D_1\to D_2$.

$C^1\big([0,\omega];\mathbb{R}\big)$ is the Banach space of continuous functions $u:[0,\omega]\to\mathbb{R}$ with continuous derivative, with the norm $\|u\|_{C^1}=\|u\|_\infty+\|u'\|_\infty$.

$AC\big([0,\omega];\mathbb{R}\big)$ is a set of all absolutely continuous functions.

$AC^1\big([0,\omega];\mathbb{R}\big)$ is a set of all functions $u:[0,\omega]\to\mathbb{R}$ such that u and u' are absolutely continuous.

$L\big([0,\omega];\mathbb{R}\big)$ is the Banach space of Lebesgue integrable functions $p:[0,\omega]\to\mathbb{R}$ with the norm

$$\|p\|_1=\int_0^\omega |p(s)|ds.$$

$L\big([0,\omega];\mathbb{R}_+\big)=\big\{p\in L\big([0,\omega];\mathbb{R}\big):p(t)\geq 0 \text{ for a.e. } t\in[0,\omega]\big\}$.

For a given $p\in L\big([0,\omega];\mathbb{R}\big)$, its mean value is defined by

$$\overline{p}=\frac{1}{\omega}\int_0^\omega p(s)ds.$$

Finally, a function $f:[0,\omega]\times D_1\to D_2$ belongs to the set of Carathéodory functions $\text{Car}\big([0,\omega]\times D_1;D_2\big)$ if and only if $f(\cdot,x):[0,\omega]\to D_2$ is measurable for all $x\in D_1$, $f(t,\cdot):D_1\to D_2$ is continuous for a.e. $t\in[0,\omega]$, and for each compact set $K\subset D_1$, there exists $m_K\in L\big([0,\omega];\mathbb{R}_+\big)$ such that $|f(t,x)|\leq m_K(t)$ for a.e. $t\in[0,\omega]$ and all $x\in K$.

Throughout the paper, speaking about periodic function u we mean that both u and u' are periodic functions; i.e.,

$$u(0)=u(\omega),\qquad u'(0)=u'(\omega).$$

2 The method of upper and lower functions

The method of upper and lower functions is one of the most fruitful techniques in Nonlinear Analysis and the main idea can be traced back at least to Picard. The monograph [1] presents a nice and complete historical review of the subject. In our context, the definition of upper and lower functions is as follows.

Definition 2.1. A function $\alpha\in AC^1\big([0,\omega];\mathbb{R}\big)$ is called a *lower–function* to the problem (1.1), (1.2) if $\alpha(t)>0$ for every $t\in[0,\omega]$ and

$$\begin{gathered}\alpha''(t)+f(\alpha(t))\alpha'(t)+g(\alpha(t))\geq h(t,\alpha(t))\qquad\text{for a. e. } t\in[0,\omega],\\ \alpha(0)=\alpha(\omega),\qquad \alpha'(0)\geq\alpha'(\omega).\end{gathered}$$

Definition 2.2. A function $\beta\in AC^1\big([0,\omega];\mathbb{R}\big)$ is called an *upper–function* to the problem (1.1), (1.2) if $\beta(t)>0$ for every $t\in[0,\omega]$ and

$$\begin{gathered}\beta''(t)+f(\beta(t))\beta'(t)+g(\beta(t))\leq h(t,\beta(t))\qquad\text{for a. e. } t\in[0,\omega],\\ \beta(0)=\beta(\omega),\qquad \beta'(0)\leq\beta'(\omega).\end{gathered}$$

Next theorem is well-known in the related literature (see, e.g., [1] or more general case in [6, Theorem 8.12]).

Proposition 2.1. *Let α and β be lower and upper functions to the problem* (1.1), (1.2) *such that*

$$\alpha(t) \leq \beta(t) \qquad \text{for } t \in [0,\omega].$$

Then there exists a positive solution u to the problem (1.1), (1.2) *such that*

$$\alpha(t) \leq u(t) \leq \beta(t) \qquad \text{for } t \in [0,\omega].$$

The objective of this section is to develop a new technique for construction of upper and lower functions.

2.1 Auxiliary results

Given $x_1 \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}_+$ fixed constants, let us define the operator $T : C^1\big([0,\omega];\mathbb{R}\big) \to C^1\big([0,\omega];\mathbb{R}\big)$ by

$$T(u)(t) = x_1 + x_0\left(u(t) - \min\big\{u(s) : s \in [0,\omega]\big\}\right) \qquad \text{for } t \in [0,\omega] \tag{2.1}$$

and consider the auxiliar problem

$$u''(t) + f(T(u)(t))u'(t) = q(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{2.2}$$

$$u(0) = 0, \qquad u(\omega) = 0, \tag{2.3}$$

where $f \in C\big(\mathbb{R}^+;\mathbb{R}\big)$ and $q \in L\big([0,\omega];\mathbb{R}\big)$. By a solution to the problem (2.2), (2.3) we understand a function $u \in AC^1\big([0,\omega];\mathbb{R}\big)$ which satisfies (2.2) almost everywhere on $[0,\omega]$, and verifies (2.3).

Lemma 2.1. *Let $u \in AC\big([0,\omega];\mathbb{R}\big)$ be such that*

$$u(0) = u(\omega). \tag{2.4}$$

Then the inequality

$$(M-m)^2 \leq \frac{\omega}{4}\int_0^\omega {u'}^2(s)ds \tag{2.5}$$

holds where

$$M = \max\big\{u(t) : t \in [0,\omega]\big\}, \qquad m = \min\big\{u(t) : t \in [0,\omega]\big\}. \tag{2.6}$$

Proof. Let us define $\widetilde{u} : [0,2\omega] \to \mathbb{R}$ by

$$\widetilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [0,\omega], \\ u(t-\omega) & \text{if } t \in (\omega,2\omega]. \end{cases} \tag{2.7}$$

Evidently, (2.4) implies that $\widetilde{u} \in AC\big([0,2\omega];\mathbb{R}\big)$ and also there exist $t_0 \in [0,\omega]$ and $t_1 \in (t_0, t_0+\omega)$ such that

$$\widetilde{u}(t_0) = m, \qquad \widetilde{u}(t_1) = M, \qquad \widetilde{u}(t_0+\omega) = m.$$

Then

$$M - m = \int_{t_0}^{t_1} \widetilde{u}'(s)ds, \qquad m - M = \int_{t_1}^{t_0+\omega} \widetilde{u}'(s)ds.$$

Using the Cauchy–Bunyakovskii–Schwarz inequality we obtain

$$M - m \le \sqrt{(t_1 - t_0)\left(\int_{t_0}^{t_1} \widetilde{u}'^2(s)ds\right)},$$

$$M - m \le \sqrt{(t_0 + \omega - t_1)\left(\int_{t_1}^{t_0+\omega} \widetilde{u}'^2(s)ds\right)}.$$

Multiplying both inequalities and using that $AB \le \frac{1}{4}(A+B)^2$ for each $A, B \in \mathbb{R}_+$ we can prove that

$$(M - m)^2 \le \frac{\omega}{4} \int_{t_0}^{t_0+\omega} \widetilde{u}'^2(s)ds.$$

Finally, from the last inequality, in virtue of (2.7), we obtain (2.5). $\square$

Lemma 2.2. *For every possible solution u to the problem*

$$u''(t) + \lambda f(T(u)(t))u'(t) = \lambda q(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{2.8}$$

$$u(0) = 0, \qquad u(\omega) = 0 \tag{2.9}$$

with $\lambda \in (0, 1]$, the estimate

$$M - m \le \frac{\omega}{4} \max\left\{\int_0^\omega [q(s)]_+ ds, \int_0^\omega [q(s)]_- ds\right\} \tag{2.10}$$

holds, where the numbers M and m are given by (2.6).

Proof. Multiplying (2.8) by u and integrating on $[0, \omega]$, we get

$$-\int_0^\omega u'^2(s)ds = \lambda \int_0^\omega q(s)u(s)ds.$$

Hence,

$$\int_0^\omega u'^2(s)ds \le \lambda \left(M \int_0^\omega [q(s)]_- ds - m \int_0^\omega [q(s)]_+ ds\right). \tag{2.11}$$

Note that (2.9) implies $M \ge 0$, $m \le 0$. Therefore, from (2.11) we obtain

$$\int_0^\omega u'^2(s)ds \le \max\left\{\int_0^\omega [q(s)]_+ ds, \int_0^\omega [q(s)]_- ds\right\}(M - m). \tag{2.12}$$

Now, (2.10) is a direct consequence of Lemma 2.1 and (2.12). $\square$

The following result is known as a Schaefer's fixed point theorem (see [8], or more recent books [9, 10]). We formulate it here in a suitable for us form.

Lemma 2.3 (see [8])**.** *Let* $F : C^1([0,\omega];\mathbb{R}) \to C^1([0,\omega];\mathbb{R})$ *be a continuous operator which is compact on each bounded subset of* $C^1([0,\omega];\mathbb{R})$[3]*. If there exists* $r > 0$ *such that every solution to*

$$u = \lambda F(u) \tag{2.13}$$

for $\lambda \in (0,1)$ *verifies*

$$\|u\|_{C^1} \le r, \tag{2.14}$$

then (2.13) *has a solution for* $\lambda = 1$*.*

Next lemma is a generalized version of a lemma proved by Mawhin in [5, Lemma 6.2.].

Lemma 2.4. *For every* $x_1 \in \mathbb{R}^+$*,* $x_0 \in \mathbb{R}_+$ *and* $q \in L([0,\omega];\mathbb{R})$ *there exists a solution* u *to the problem* (2.2)*,* (2.3)*. Furthermore,*

$$u'(\omega) - u'(0) = \int_0^\omega q(s)ds \tag{2.15}$$

and (2.10) *is fulfilled, where the constants* M *and* m *are defined by* (2.6)*.*

Proof. Let u be a possible solution to (2.8), (2.9) with $\lambda \in (0,1)$. According to Lemma 2.2 we have

$$\|u\|_\infty \le \frac{\omega}{4}\|q\|_1. \tag{2.16}$$

On the other hand, it is obvious that there exists $t_0 \in [0,\omega]$ such that

$$u'(t_0) = 0. \tag{2.17}$$

The integration of (2.8) from t_0 to t with respect to (2.1), (2.17), (2.10), and the inclusion $\lambda \in (0,1)$, yields

$$|u'(t)| \le \left| \int_{t_0}^t f(T(u)(s))u'(s)ds - \int_{t_0}^t q(s)ds \right| \le \int_{x_1}^{x_1 + x_0\frac{\omega}{4}\|q\|_1} |f(s)|ds + \|q\|_1 \quad \text{for } t \in [0,\omega],$$

whence we obtain

$$\|u'\|_\infty \le \left(M_f x_0 \frac{\omega}{4} + 1\right)\|q\|_1, \tag{2.18}$$

where

$$M_f = \max\left\{ |f(x)| : x_1 \le x \le x_1 + x_0\frac{\omega}{4}\|q\|_1 \right\}.$$

Therefore, in view of (2.16) and (2.18), u satisfies (2.14) with

$$r = \left[(1 + x_0 M_f)\frac{\omega}{4} + 1\right]\|q\|_1.$$

[3] i.e., it transforms every bounded set into a relatively compact set.

Define $F : C^1([0,\omega];\mathbb{R}) \to C^1([0,\omega];\mathbb{R})$ by

$$F(v)(t) = \frac{1}{\omega}\left[(\omega - t)\int_0^t s\big(f(T(v)(s))v'(s) - q(s)\big)ds \right. \\ \left. + \, t\int_t^\omega (\omega - s)\big(f(T(v)(s))v'(s) - q(s)\big)ds\right] \qquad \text{for } t \in [0,\omega].$$

Then, every solution to (2.13) with $\lambda \in (0,1)$ is a solution to (2.8), (2.9) and thus according to Lemma 2.3 the problem (2.2), (2.3) has at least one solution u. Integrating (2.2) from 0 to ω we obtain (2.15). The estimate (2.10) immediately follows from Lemma 2.2. □

Lemma 2.5. *Let $h \in L([0,\omega];\mathbb{R})$. Then,*

$$\lim_{n\to+\infty}\int_0^\omega [h(s) - n]_+ ds = 0 \tag{2.19}$$

and

$$\lim_{n\to+\infty}\int_0^\omega [h(s) + n]_- ds = 0. \tag{2.20}$$

Proof. Let us define

$$h_n(t) = \begin{cases} n & \text{if } h(t) > n, \\ h(t) & \text{if } h(t) \le n, \end{cases} \qquad \text{for a. e. } t \in [0,\omega], \quad n \in \mathbb{N}. \tag{2.21}$$

Then,

$$h(t) = h_n(t) + [h(t) - n]_+ \qquad \text{for a. e. } t \in [0,\omega], \quad n \in \mathbb{N}. \tag{2.22}$$

Integrating (2.22) over a period,

$$\int_0^\omega h(s)ds = \int_0^\omega h_n(s)ds + \int_0^\omega [h(s) - n]_+ ds \qquad \text{for } n \in \mathbb{N}. \tag{2.23}$$

On the other hand, from (2.21) and (2.22) we get

$$-[h(t)]_- \le h_n(t) \le h(t) \qquad \text{for a. e. } t \in [0,\omega], \quad n \in \mathbb{N}$$

and

$$\lim_{n\to+\infty} h_n(t) = h(t) \qquad \text{for a. e. } t \in [0,\omega].$$

Thus, according to Lebesgue Theorem we have

$$\lim_{n\to+\infty}\int_0^\omega h_n(s)ds = \int_0^\omega h(s)ds. \tag{2.24}$$

Now, from (2.23) and (2.24) we get (2.19). The identity (2.20) can be proved by similar arguments. □

2.2 Construction of lower functions

Along this subsection, we will use the notation

$$\Phi_+ = \int_0^\omega [\varphi(t)]_+ dt \qquad \Phi_- = \int_0^\omega [\varphi(t)]_- dt.$$

where $\varphi \in L\big([0,\omega];\mathbb{R}\big)$ is a function defined below (see (2.28)). The first result of this section gives sufficient conditions for the construction of a lower function.

Proposition 2.2. *Let $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $0 < x_1 \le x_2 < +\infty$, and $c \in \mathbb{R}$ be such that*

$$h(t,x) \le h_0(t)\rho_0(x) \qquad \textit{for a. e. } t \in [0,\omega], \quad x \in [x_1,x_2], \tag{2.25}$$

$$\frac{g(x)}{\rho_0(x)} \ge c \ge \overline{h}_0 \qquad \textit{for } x \in [x_1,x_2], \tag{2.26}$$

and

$$\rho_0(x_2)\frac{\omega}{4}\Phi_+ \le x_2 - x_1 \le \rho_0(x_1)\frac{\omega}{4}\Phi_-, \tag{2.27}$$

where

$$\varphi(t) = h_0(t) - c \qquad \textit{for a. e. } t \in [0,\omega]. \tag{2.28}$$

Then there exists a lower function α to the problem (1.1), (1.2) *such that*

$$x_1 \le \alpha(t) \le x_2 \qquad \textit{for } t \in [0,\omega].$$

Proof. By the definition of φ and (2.26), we obtain $\Phi_- \ge \Phi_+ \ge 0$. As a first case we suppose that

$$\Phi_+ > 0.$$

Put

$$x_0 = \frac{4(x_2 - x_1)}{\omega\Phi_-\Phi_+}, \tag{2.29}$$

$$q(t) = \Phi_-[\varphi(t)]_+ - \Phi_+[\varphi(t)]_- \qquad \text{for a. e. } t \in [0,\omega], \tag{2.30}$$

and let $T : C^1\big([0,\omega];\mathbb{R}\big) \to C^1\big([0,\omega];\mathbb{R}\big)$ be the operator defined by (2.1). Note that

$$\int_0^\omega q(s)ds = 0. \tag{2.31}$$

According to Lemma 2.4 there exists a solution u to (2.2), (2.3) such that (2.10) and (2.15) hold. By using (2.30) and (2.31), we obtain

$$M - m \le \frac{\omega}{4}\Phi_+\Phi_-, \tag{2.32}$$

$$u'(0) = u'(\omega), \tag{2.33}$$

where the constants M and m are defined by (2.6). Put

$$\alpha(t) = T(u)(t) \qquad \text{for } t \in [0, \omega]. \tag{2.34}$$

Then, according to (2.1)–(2.3), (2.29), (2.30) and (2.32)–(2.34) we arrive at

$$\alpha''(t) + f(\alpha(t))\alpha'(t) = x_0\Phi_-[\varphi(t)]_+ - x_0\Phi_+[\varphi(t)]_- \qquad \text{for a. e. } t \in [0, \omega], \tag{2.35}$$

$$\alpha(0) = \alpha(\omega), \qquad \alpha'(0) = \alpha'(\omega), \tag{2.36}$$

$$x_1 \le \alpha(t) \le x_2 \qquad \text{for } t \in [0, \omega]. \tag{2.37}$$

Using that ρ_0 is a non–decreasing function, from the inequality (2.37) we obtain

$$\rho_0(x_1) \le \rho_0(\alpha(t)) \le \rho_0(x_2) \qquad \text{for } t \in [0, \omega]. \tag{2.38}$$

From the inequality (2.27), by virtue of (2.29), we get

$$x_0\Phi_+ \le \rho_0(x_1), \qquad \rho_0(x_2) \le x_0\Phi_-. \tag{2.39}$$

Now (2.38) and (2.39) imply

$$x_0\Phi_+ \le \rho_0(\alpha(t)) \le x_0\Phi_- \qquad \text{for } t \in [0, \omega]. \tag{2.40}$$

Using (2.40) in (2.35) we get

$$\alpha''(t) + f(\alpha(t))\alpha'(t) \ge \rho_0(\alpha(t))\varphi(t) \qquad \text{for a. e. } t \in [0, \omega]. \tag{2.41}$$

On the other hand, we can prove, using (2.26), (2.28) and (2.37), that

$$\varphi(t) \ge h_0(t) - \frac{g(\alpha(t))}{\rho_0(\alpha(t))} \qquad \text{for a. e. } t \in [0, \omega]. \tag{2.42}$$

From (2.41), on account of (2.25), (2.37) and (2.42), it follows that

$$\alpha''(t) + f(\alpha(t))\alpha'(t) + g(\alpha(t)) \ge h(t, \alpha(t)) \qquad \text{for a. e. } t \in [0, \omega]. \tag{2.43}$$

Consequently, (2.36), (2.37) and (2.43) ensure us that α is a lower function to the problem (1.1), (1.2).

Now, we consider the remaining case

$$\Phi_+ = 0.$$

Of course, in this case

$$\varphi(t) \le 0 \qquad \text{for a. e. } t \in [0, \omega].$$

Then, defining α by

$$\alpha(t) = x_1 \qquad \text{for } t \in [0, \omega] \tag{2.44}$$

we can prove easily that α is a positive function which fulfils (2.36) and (2.41). Again, from (2.26), (2.28) and (2.44) we obtain (2.42) and using (2.25), (2.42) and (2.44) in (2.41) we arrive at (2.43). Finally, also in this case, (2.36), (2.43) and (2.44) imply that α is a lower function to the problem (1.1), (1.2). □

A simplified version of the latter proposition is presented below.

Proposition 2.3. *Let $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that*

$$h(t,x) \leq h_0(t)\rho_0(x) \qquad \text{for a. e. } t \in [0,\omega], \quad 0 < x \leq x_0, \tag{2.45}$$

$$\frac{g(x)}{\rho_0(x)} \geq c \geq \overline{h}_0 \qquad \text{for } 0 < x \leq x_0, \tag{2.46}$$

and, in addition, let there exist $x_2 \in (0, x_0]$ such that

$$x_2 - \rho_0(x_2)\frac{\omega}{4}\Phi_+ > 0, \tag{2.47}$$

$$\rho_0(x_2)\Phi_+ \leq \rho_0\left(x_2 - \rho_0(x_2)\frac{\omega}{4}\Phi_+\right)\Phi_- \tag{2.48}$$

where $\varphi(t) = h_0(t) - c$ for almost every $t \in [0,\omega]$. Then there exists a lower function α to the problem (1.1), (1.2) *with*

$$0 < \alpha(t) \leq x_2 \qquad \text{for } t \in [0,\omega].$$

Proof. In order to apply Proposition 2.2, we define

$$x_1 = x_2 - \rho_0(x_2)\frac{\omega}{4}\Phi_+. \tag{2.49}$$

By (2.47), $x_1 > 0$. Then, it is clear that (2.45) and (2.46) imply (2.25) and (2.26). It remains to show that (2.27) holds. Indeed, by the definition of x_1 we have

$$x_2 - x_1 = \frac{\omega}{4}\rho_0(x_2)\Phi_+. \tag{2.50}$$

On the other hand, using (2.49) in (2.48) we get

$$\frac{\omega}{4}\rho_0(x_2)\Phi_+ \leq \frac{\omega}{4}\rho_0(x_1)\Phi_-. \tag{2.51}$$

Therefore, (2.50) and (2.51) imply (2.27). □

The following corollaries are direct consequences of Proposition 2.3.

Corollary 2.1. *Let $x_0 > \frac{\omega}{8}\|h_0 - \overline{h}_0\|_1$ be such that*

$$g(x) \geq \overline{h}_0 \qquad \text{for } 0 < x \leq x_0.$$

Then there exists a lower function α to the problem (1.4), (1.2) *with*

$$0 < \alpha(t) \leq x_0 \qquad \text{for } t \in [0,\omega]. \tag{2.52}$$

Proof. The assertion immediately follows from Proposition 2.3 with $h(t,x) \equiv h_0(t)$ if we put $c = \overline{h}_0$, $\rho_0 \equiv 1$, and $x_2 = x_0$. Note also that in this case $\|h_0 - \overline{h}_0\|_1 = \Phi_+ + \Phi_- = 2\Phi_+$. $\square$

Corollary 2.2. *Let $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that* (2.45) *and* (2.46) *are fulfilled. Let, moreover, there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that*

$$\lim_{n\to+\infty} y_n = 0, \tag{2.53}$$

$$\lim_{n\to+\infty} \frac{\rho_0(y_n)}{y_n} = 0, \tag{2.54}$$

and let there exist $\varepsilon \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\rho_0(y_n)}{\rho_0(y_n(1-\varepsilon))}\Phi_+ \leq \Phi_- \qquad \textit{for } n \geq n_0 \tag{2.55}$$

where $\varphi(t) = h_0(t) - c$ for almost every $t \in [0,\omega]$. Then there exists a lower function α to the problem (1.1), (1.2) *satisfying* (2.52).

Proof. According to Proposition 2.3, it is sufficient to prove that (2.47) and (2.48) are fulfilled for some $x_2 \in (0, x_0]$. According to (2.53) and (2.54), there exists $n_1 \geq n_0$ such that

$$y_n \leq x_0 \qquad \text{for } n \geq n_1,$$

$$-\frac{\omega}{4}\Phi_+\rho_0(y_n) \geq -\varepsilon y_n \qquad \text{for } n \geq n_1, \tag{2.56}$$

Adding y_n to both sides of the inequality (2.56) and applying that ρ_0 is a non–decreasing function, we obtain

$$\rho_0\left(y_n - \frac{\omega}{4}\Phi_+\rho_0(y_n)\right) \geq \rho_0\left(y_n(1-\varepsilon)\right) \qquad \text{for } n \geq n_1. \tag{2.57}$$

Now, if we put $x_2 = y_{n_1}$ we obtain, on account of (2.55)–(2.57) that (2.47) and (2.48) are fulfilled. $\square$

Corollary 2.3. *Let $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_0 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that* (2.45) *and* (2.46) *are fulfilled. If $\dfrac{\rho_0(x)}{x}$ is a non–increasing function and*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0(x_0)}{x_0} \leq \Phi_- - \Phi_+ \tag{2.58}$$

where $\varphi(t) = h_0(t) - c$ for almost every $t \in [0,\omega]$, then there exists a lower function α to the problem (1.1), (1.2) *satisfying* (2.52).

Proof. According to Proposition 2.3, it is sufficient to prove that (2.47) and (2.48) are satisfied with $x_2 = x_0$. From (2.58) we easily obtain (2.47). On the other hand, since the function $\frac{\rho_0(x)}{x}$ is non–increasing,

$$\frac{\rho_0\left(x_0 - \rho_0(x_0)\frac{\omega}{4}\Phi_+\right)}{x_0 - \rho_0(x_0)\frac{\omega}{4}\Phi_+} \geq \frac{\rho_0(x_0)}{x_0}.$$

Consequently,

$$\rho_0\left(x_0 - \rho_0(x_0)\frac{\omega}{4}\Phi_+\right) \geq \rho_0(x_0)\left(1 - \frac{\omega}{4}\Phi_+\frac{\rho_0(x_0)}{x_0}\right). \tag{2.59}$$

Multiplying both sides of (2.59) by Φ_- and using the inequality (2.58) we get (2.48). □

2.3 Construction of upper functions

The following assertions dealing with the existence of an upper function to the problem considered can be proved analogously to the results formulated in Subsection 2.2, therefore, their proofs are omitted.

Proposition 2.4. *Let $h_1 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $0 < x_1 \leq x_2 < +\infty$, and $c \in \mathbb{R}$ be such that*

$$h(t,x) \geq h_1(t)\rho_1(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \in [x_1,x_2], \tag{2.60}$$

$$\frac{g(x)}{\rho_1(x)} \leq c \leq \overline{h}_1 \qquad \text{for } x \in [x_1,x_2], \tag{2.61}$$

and

$$\rho_1(x_2)\frac{\omega}{4}\Phi_- \leq x_2 - x_1 \leq \rho_1(x_1)\frac{\omega}{4}\Phi_+ \tag{2.62}$$

where

$$\varphi(t) = h_1(t) - c \qquad \text{for a. e. } t \in [0,\omega]. \tag{2.63}$$

Then there exists an upper function β to the problem (1.1), (1.2) *such that*

$$x_1 \leq \beta(t) \leq x_2 \qquad \text{for } t \in [0,\omega].$$

Proposition 2.5. *Let $h_1 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that*

$$h(t,x) \geq h_1(t)\rho_1(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \geq x_0, \tag{2.64}$$

$$\frac{g(x)}{\rho_1(x)} \leq c \leq \overline{h}_1 \qquad \text{for } x \geq x_0, \tag{2.65}$$

and, in addition, let there exist $x_1 \geq x_0$ such that

$$\rho_1\left(x_1 + \rho_1(x_1)\frac{\omega}{4}\Phi_+\right)\Phi_- \leq \rho_1(x_1)\Phi_+ \tag{2.66}$$

where $\varphi(t) = h_1(t) - c$ for almost every $t \in [0,\omega]$. Then there exists an upper function β to the problem (1.1), (1.2) *with*

$$x_1 \leq \beta(t) \qquad \text{for } t \in [0,\omega]. \tag{2.67}$$

Corollary 2.4. *Let there exists $x_0 > 0$ such that*

$$g(x) \leq \overline{h}_0 \qquad \text{for } x \geq x_0.$$

Then there exists an upper function β to the problem (1.4), (1.2) *with*

$$\beta(t) \geq x_0 \qquad \text{for } t \in [0, \omega]. \tag{2.68}$$

Corollary 2.5. *Let $h_1 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that* (2.64) *and* (2.65) *hold, and let there exist a sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that*

$$\lim_{n\to+\infty} y_n = +\infty, \tag{2.69}$$

$$\lim_{n\to+\infty} \frac{\rho_1(y_n)}{y_n} = 0. \tag{2.70}$$

Furthermore, let there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\rho_1(y_n(1+\varepsilon))}{\rho_1(y_n)}\Phi_- \leq \Phi_+ \qquad \text{for } n \geq n_0 \tag{2.71}$$

where $\varphi(t) = h_1(t) - c$ for almost every $t \in [0,\omega]$. Then there exists an upper function β to the problem (1.1), (1.2) *satisfying* (2.68).

Corollary 2.6. *Let $h_1 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho_1 \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function, $x_0 > 0$, and $c \in \mathbb{R}$ be such that* (2.64) *and* (2.65) *are fulfilled. If $\frac{\rho_1(x)}{x}$ is a non–increasing function such that*

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_1(x_0)}{x_0} \leq \Phi_+ - \Phi_- \tag{2.72}$$

where $\varphi(t) = h_1(t) - c$ for almost every $t \in [0,\omega]$, then there exists an upper function β to the problem (1.1), (1.2) *satisfying* (2.68).

3 Main results

3.1 The general equation

Throughout this subsection, we will use the following notation:

$$\Phi_+ = \int_0^\omega [\varphi(t)]_+ dt, \quad \Phi_- = \int_0^\omega [\varphi(t)]_- dt, \quad \Psi_+ = \int_0^\omega [\psi(t)]_+ dt, \quad \Psi_- = \int_0^\omega [\psi(t)]_- dt,$$

where $\varphi, \psi \in L\big([0,\omega];\mathbb{R}\big)$ are functions defined below.

Theorem 3.1. *Let $\rho_0, \rho_1 \in C\big(\mathbb{R}^+; \mathbb{R}^+\big)$ be non–decreasing functions, $h_0, h_1 \in L\big([0,\omega]; \mathbb{R}\big)$, and $0 < r_0 \le r_1 < +\infty$ be such that*

$$h(t,x) \le h_0(t)\rho_0(x) \qquad \text{for a. e. } t \in [0,\omega], \quad 0 < x \le r_0, \tag{3.1}$$

$$h(t,x) \ge h_1(t)\rho_1(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \ge r_1, \tag{3.2}$$

and let there exist $c_0, c_1 \in \mathbb{R}$ such that

$$\frac{g(x)}{\rho_0(x)} \ge c_0 \ge \overline{h}_0 \qquad \text{for } 0 < x \le r_0, \tag{3.3}$$

$$\frac{g(x)}{\rho_1(x)} \le c_1 \le \overline{h}_1 \qquad \text{for } x \ge r_1. \tag{3.4}$$

Furthermore, let us suppose that ρ_0 fulfils at least one of the following conditions:

a) there exists a sequence $\{x_n\}_{n=1}^{+\infty}$ of positive numbers such that

$$\lim_{n\to+\infty} x_n = 0, \qquad \lim_{n\to+\infty} \frac{\rho_0(x_n)}{x_n} = 0,$$

and there exist $\varepsilon_0 \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\rho_0(x_n)}{\rho_0(x_n(1-\varepsilon_0))}\Phi_+ \le \Phi_- \qquad \text{for } n \ge n_0,$$

where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$;

b) the function $\frac{\rho_0(x)}{x}$ is non–increasing and

$$\frac{\omega}{4}\Phi_+\Phi_-\frac{\rho_0(r_0)}{r_0} \le \Phi_- - \Phi_+,$$

where $\varphi(t) = h_0(t) - c_0$ for almost every $t \in [0,\omega]$.

Besides, let us suppose that ρ_1 fulfils at least one of the following conditions:

c) there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ of positives numbers such that

$$\lim_{n\to+\infty} y_n = +\infty, \qquad \lim_{n\to+\infty} \frac{\rho_1(y_n)}{y_n} = 0,$$

and there exist $\varepsilon_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$\frac{\rho_1(y_n(1+\varepsilon_1))}{\rho_1(y_n)}\Psi_- \le \Psi_+ \qquad \text{for } n \ge n_1,$$

where $\psi(t) = h_1(t) - c_1$ for almost every $t \in [0,\omega]$;

d) the function $\frac{\rho_1(x)}{x}$ *is non–increasing and*

$$\frac{\omega}{4}\Psi_+\Psi_-\frac{\rho_1(r_1)}{r_1} \leq \Psi_+ - \Psi_-,$$

where $\psi(t) = h_1(t) - c_1$ *for almost every* $t \in [0, \omega]$.

Then there exists at least one positive solution to the problem (1.1), (1.2).

Proof. According to Corollaries 2.2, 2.3, 2.5, and 2.6, the conditions of the theorem guarantee a well–ordered couple of lower and upper functions, therefore the result is a direct consequence of Proposition 2.1. □

Remark 3.1. Note that (3.3) (resp. (3.4)) implies $\Phi_- \geq \Phi_+$ (resp. $\Psi_+ \geq \Psi_-$). In addition, the conditions $a)$ and $c)$ are verified if, for instance, $\Phi_- \neq \Phi_+$ and $\Psi_- \neq \Psi_+$, $\rho_i(x) = x^{\mu_i}$ $(i = 0, 1)$ with $\mu_0 > 1 > \mu_1 \geq 0$. On the other hand, conditions $b)$ and $d)$ are fulfilled if, for instance, $\rho_i(x) = x$ $(i = 0, 1)$ and

$$\frac{\omega}{4}\Phi_+\Phi_- \leq \Phi_- - \Phi_+, \qquad \frac{\omega}{4}\Psi_+\Psi_- \leq \Psi_+ - \Psi_-. \tag{3.5}$$

Next, we formulate some corollaries which can be obtained immediately from Theorem 3.1 and Remark 3.1.

Corollary 3.1. *Let* $h_0, h_1 \in L\big([0, \omega]; \mathbb{R}\big)$, $\mu_0 > 1 > \mu_1 \geq 0$ *and* $0 < r_0 \leq r_1 < +\infty$ *be such that*

$$\begin{gathered} h(t,x) \leq h_0(t)x^{\mu_0} \qquad \textit{for a. e. } t \in [0,\omega], \quad 0 < x \leq r_0, \\ h(t,x) \geq h_1(t)x^{\mu_1} \qquad \textit{for a. e. } t \in [0,\omega], \quad x \geq r_1, \\ \liminf_{x \to 0_+} \frac{g(x)}{x^{\mu_0}} > \overline{h}_0, \qquad \limsup_{x \to +\infty} \frac{g(x)}{x^{\mu_1}} < \overline{h}_1. \end{gathered}$$

Then there exists at least one positive solution to the problem (1.1), (1.2).

Proof. According to Remark 3.1, one can apply Theorem 3.1 with $\rho_0(x) = x^{\mu_0}$, $\rho_1(x) = x^{\mu_1}$. □

Corollary 3.2. *Let* $h_0, h_1 \in L\big([0, \omega]; \mathbb{R}\big)$ *and* $0 < r_0 \leq r_1 < +\infty$ *be such that*

$$\begin{gathered} h(t,x) \leq h_0(t)x \qquad \textit{for a. e. } t \in [0,\omega], \quad 0 < x \leq r_0, \\ h(t,x) \geq h_1(t)x \qquad \textit{for a. e. } t \in [0,\omega], \quad x \geq r_1, \\ \overline{h}_0 < \liminf_{x \to 0_+} \frac{g(x)}{x} < +\infty, \qquad \overline{h}_1 > \limsup_{x \to +\infty} \frac{g(x)}{x} > -\infty. \end{gathered} \tag{3.6}$$

In addition, we suppose that

$$\frac{\omega}{4}H_0^+H_0^- < H_0^- - H_0^+, \qquad \frac{\omega}{4}H_1^+H_1^- < H_1^+ - H_1^-, \tag{3.7}$$

where

$$H_0^+ = \int_0^\omega [h_0(t) - g_*]_+ dt, \qquad H_0^- = \int_0^\omega [h_0(t) - g_*]_- dt,$$
$$H_1^+ = \int_0^\omega [h_1(t) - g^*]_+ dt, \qquad H_1^- = \int_0^\omega [h_1(t) - g^*]_- dt,$$

and

$$g_* = \liminf_{x\to 0_+} \frac{g(x)}{x}, \qquad g^* = \limsup_{x\to +\infty} \frac{g(x)}{x}.$$

Then there exists at least one positive solution to the problem (1.1)*,* (1.2)*.*

Proof. From (3.6) and (3.7) we obtain that there exists $\varepsilon > 0$ small enough such that $\varepsilon < \min\{g_* - \overline{h}_0, \overline{h}_1 - g^*\}$ and (3.5) is verified, where

$$\varphi(t) = h_0(t) - g_* + \varepsilon \quad \text{for a. e. } t \in [0,\omega], \qquad \psi(t) = h_1(t) - g^* - \varepsilon \quad \text{for a. e. } t \in [0,\omega].$$

Hence, setting $c_0 = g_* - \varepsilon$, $c_1 = g^* + \varepsilon$ and $\rho_i(x) = x$ $(i = 0, 1)$, the corollary follows from Theorem 3.1. $\square$

Corollary 3.3. *Let $h_0, h_1 \in L([0,\omega];\mathbb{R})$ and $0 < r_0 \le r_1 < +\infty$ be such that*

$$h(t,x) \le h_0(t)x \qquad \textit{for a. e. } t \in [0,\omega], \quad 0 < x \le r_0,$$
$$h(t,x) \ge h_1(t)x \qquad \textit{for a. e. } t \in [0,\omega], \quad x \ge r_1,$$
$$\lim_{x\to 0_+} \frac{g(x)}{x} = +\infty, \qquad \lim_{x\to +\infty} \frac{g(x)}{x} = -\infty. \tag{3.8}$$

Then there exists at least one positive solution to the problem (1.1)*,* (1.2)*.*

Proof. Using (3.8) and Lemma 2.5, we can find $c_0 > \overline{h}_0$ and $c_1 < \overline{h}_1$ such that (3.5) is fulfilled where $\varphi(t) = h_0(t) - c_0$, $\psi(t) = h_1(t) - c_1$. Moreover, $g(x) \ge c_0 x$ nearby zero and $g(x) \le c_1 x$ nearby $+\infty$. Hence, taking $\rho_i(x) = x$ $(i = 0, 1)$, the corollary follows from Theorem 3.1. $\square$

In conclusion, the conditions nearby zero guarantee the existence of a positive lower function, whereas the conditions nearby infinite guarantee the existence of an upper function. Both ideas can be combined in order to get a wide variety of results.

We finish the section with two results dealing with the classical singular Liénard equation (1.4).

Theorem 3.2. *Let $\frac{\omega}{8}\|h_0 - \overline{h}_0\|_1 < r_0 \le r_1 < +\infty$ be such that*

$$g(x) \ge \overline{h}_0 \qquad \textit{for } 0 < x \le r_0,$$
$$g(x) \le \overline{h}_0 \qquad \textit{for } x \ge r_1.$$

Then there exists at least one positive solution to the problem (1.4)*,* (1.2)*.*

Proof. It is a direct consequence of Corollaries 2.1 and 2.4. □

Theorem 3.3. *Let*

$$\limsup_{x \to 0_+} g(x) = +\infty$$

and let $r_1 > 0$ be such that

$$g(x) \le \overline{h}_0 \qquad \textit{for } x \ge r_1.$$

If

$$\text{ess sup}\,\big\{h_0(t) : t \in [0,\omega]\big\} < +\infty,$$

then there exists at least one positive solution to the problem (1.4), (1.2).

Proof. The existence of a lower function follows from Proposition 2.2 with $h(t,x) = h_0(t)$, $\rho_0 \equiv 1$,

$$c = \text{ess sup}\,\big\{h_0(t) : t \in [0,\omega]\big\},$$

and $x_1 = x_2 > 0$ sufficiently small such that $g(x_1) \ge c$.

The existence of an upper function follows from Proposition 2.5 with $h(t,x) = h_0(t)$, $h_1 \equiv h_0$, $\rho_1 \equiv 1$, $c = \overline{h}_0$, and $x_0 = x_1 = r_1$.

Consequently, the assertion follows from Proposition 2.1. □

It is interesting to compare our results with the existing ones in the related literature. For instance, it is easy to verify that Theorem 3.3 generalises in some sense the result of Lazer and Solimini [4, Theorem 2.1] for the equation

$$u'' + g(u) = h(t) \tag{3.9}$$

with attractive singularity and without friction.

On the other hand, if [6, Lemma 8.19] is applied to (3.9), the following result is obtained.

Theorem 3.4. *Assume that there exist $0 < r_0 < r_1 < +\infty$ such that*

1. $r_0 > 2\omega\|h - \overline{h}\|_1$,
2. $g(x) \ge \overline{h}$ *if* $0 < x \le r_0$,
3. $g(x) \le \overline{h}$ *if* $x \ge r_1$.

Then the problem (3.9), (1.2) *has at least one positive solution.*

Note that Theorem 3.2 is more general. First, it works for the equation with friction. Besides, since $\frac{\omega}{8}\left\|h - \bar{h}\right\|_1 < 2\omega\left\|h - \bar{h}\right\|_1$, it is evident that the assumption of Theorem 3.2 is better.

A related interesting result can be found in [7].

Theorem 3.5 (see [7, Corollary 3.3])**.** *Assume that*

1. $\limsup_{x\to+\infty} g(x) < \overline{h}_0$*,*

2. there exists $r > 0$ *such that* $h_0(t) \le g(r)$ *for a.e.* $t \in [0,\omega]$*.*

Then the problem (3.9)*,* (1.2) *has at least one positive solution.*

Theorem 3.3 shows that Theorem 3.5 is still valid also in the case when the term $f(x)x'$ is incorporated to the equation, and even in the case when f has a singularity at zero.

3.2 The model equation

This subsection is devoted to the model equation (1.3). We distinguish two different cases depending on the type of the singularity of the term $g_1 u^{-\nu} - g_2 u^{-\gamma}$, i.e., if it is attractive or repulsive singularity. All the proofs of the results obtained below relies on the construction of an ordered pair of lower and upper functions and a direct application of Proposition 2.1.

3.2.1 The repulsive case

Theorem 3.6. *Let* $0 \le \delta < 1$*,* $\gamma > \nu$*,* $g_1 > 0$ *and* $g_2 > 0$*. If* $\overline{h}_0 > 0$ *and*

$$h_0(t) \le \sup\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\} \qquad \text{for a. e. } t \in [0,\omega], \tag{3.10}$$

then there exists at least one positive solution to the problem (1.3)*,* (1.2)*.*

Proof. We note that

$$\sup\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\} < +\infty$$

and, in fact, there exists $r_0 \in \mathbb{R}^+$ such that

$$\frac{g_1}{r_0^{\nu+\delta}} - \frac{g_2}{r_0^{\gamma+\delta}} = \sup\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\}.$$

According to (3.10), it can be easily verified that the function $\alpha(t) = r_0$ for $t \in [0,\omega]$ is a lower function to the problem (1.3), (1.2).

To prove the existence of an upper function, we apply Corollary 2.5 taking $\{y_n\}_{n=1}^{+\infty}$ an arbitrary sequence of positive numbers satisfying (2.69), $c \in (0, \overline{h}_0)$, $\rho_1(x) = x^\delta$, $h_1 \equiv h_0$, $x_0 > r_0$ large enough, and $\varepsilon > 0$ small enough such that

$$(1+\varepsilon)^\delta \Phi_- \le \Phi_+. \tag{3.11}$$

Consequently, the assertion follows from Proposition 2.1. $\square$

Theorem 3.7. *Let $\delta = 1$, $\gamma > \nu$, $g_1 > 0$ and $g_2 > 0$. Assume that $\overline{h}_0 > 0$,*

$$h_0(t) \le \sup\left\{\frac{g_1}{x^{\nu+1}} - \frac{g_2}{x^{\gamma+1}} : x \in \mathbb{R}^+\right\} \qquad \textit{for a. e. } t \in [0,\omega], \tag{3.12}$$

and

$$\frac{\omega}{4}\int_0^\omega [h_0(s)]_+ ds \int_0^\omega [h_0(s)]_- ds < \int_0^\omega [h_0(s)]_+ ds - \int_0^\omega [h_0(s)]_- ds. \tag{3.13}$$

Then there exists at least one positive solution to the problem (1.3), (1.2).

Proof. As in the proof of Theorem 3.6, we can check that there exists a constant $r_0 \in \mathbb{R}^+$ such that $\alpha(t) = r_0$ for $t \in [0,\omega]$ is a lower function to the problem (1.3), (1.2).

On the other hand, from (3.13) it follows that there exists a sufficiently small constant $c > 0$ such that $c < \overline{h}_0$ and

$$\frac{\omega}{4}\Phi_+\Phi_- \le \Phi_+ - \Phi_-,$$

where

$$\varphi(t) = h_0(t) - c \quad \text{for a. e. } t \in [0,\omega].$$

Therefore, if we put $\rho_1(x) = x$ and $h_1 \equiv h_0$, taking into account that

$$\lim_{x\to+\infty} \frac{g_1}{x^{1+\nu}} - \frac{g_2}{x^{1+\gamma}} = 0,$$

the existence of an upper function large enough follows from Corollary 2.6.

Consequently, the assertion follows from Proposition 2.1. □

Theorem 3.8. *Let $\delta > 1$, $\gamma > \nu$, $g_1 > 0$ and $g_2 > 0$. If*

$$0 \le h_0(t) \le \sup\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\} \qquad \textit{for a. e. } t \in [0,\omega] \tag{3.14}$$

and $\overline{h}_0 > 0$, then there exists at least one positive solution to the problem (1.3), (1.2).

Proof. Analogously to the previous proofs, there exists a constant $r_0 \in \mathbb{R}^+$ such that the function $\alpha(t) = r_0$ for $t \in [0,\omega]$ is a lower function to the problem (1.3), (1.2).

On the other hand, from the first inequality of (3.14) it follows that $h_0(t)x^\delta \ge h_0(t)$ for almost every $t \in [0,\omega]$ and $x \ge 1$. Thus, the existence of an upper function to the problem (1.3), (1.2) follows from Corollary 2.5 by taking $h_1 \equiv h_0$, $\rho_1 \equiv 1$, an arbitrary sequence $\{y_n\}_{n=1}^{+\infty}$ of positive numbers such that (2.69) holds, $c \in (0, \overline{h}_0]$, $\varepsilon > 0$ arbitrary and $x_0 > r_0$ large enough.

Consequently, the assertion follows from Proposition 2.1. □

3.2.2 The attractive case

Theorem 3.9. *Let $0 \le \delta < 1$, $\gamma < \nu$ and $g_1 > 0$. If $\overline{h}_0 > 0$ and*

$$\text{ess sup}\left\{h_0(t) : t \in [0,\omega]\right\} < +\infty, \tag{3.15}$$

then there exists at least one positive solution to the problem (1.3), (1.2).

Proof. According to (3.15), we can choose $K > 0$ such that

$$K \geq h_0(t) \qquad \text{for a. e. } t \in [0, \omega].$$

As $\lim\limits_{x \to 0+} \frac{g_1}{x^\nu} - \frac{g_2}{x^\gamma} = +\infty$, there exits $x_1 > 0$ such that

$$\frac{g_1}{x_1^{\nu+\delta}} - \frac{g_2}{x_1^{\gamma+\delta}} \geq K.$$

Obviously, $\alpha \equiv x_1$ is a constant lower function to the problem (1.3), (1.2).

To prove the existence of an upper function we apply Corollary 2.5 taking $\{y_n\}_{n=1}^{+\infty}$ an arbitrary sequence of positive numbers satisfying (2.69), $c \in (0, \overline{h}_0)$, $\rho_1(x) = x^\delta$, $h_1 \equiv h_0$, $x_0 > K$ large enough, and $\varepsilon > 0$ small enough such that (3.11) holds.

Consequently, the assertion follows from Proposition 2.1. □

Theorem 3.10. *Let* $0 \leq \delta < 1$, $\gamma < \nu$, $g_1 > 0$ *and* $g_2 > 0$. *If* $\overline{h}_0 \leq 0$, (3.15) *is fulfilled and*

$$h_0(t) \geq \inf\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\} \qquad \text{for a. e. } t \in [0, \omega], \tag{3.16}$$

then there exists at least one positive solution to the problem (1.3), (1.2).

Proof. In this case,

$$\inf\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\} > -\infty$$

and there exists $x_0 > 0$ such that

$$\frac{g_1}{x_0^{\nu+\delta}} - \frac{g_2}{x_0^{\gamma+\delta}} = \inf\left\{\frac{g_1}{x^{\nu+\delta}} - \frac{g_2}{x^{\gamma+\delta}} : x \in \mathbb{R}^+\right\}.$$

According to (3.16), it can be easily verified that the function $\beta(t) = x_0$ for $t \in [0, \omega]$ is an upper function to the problem (1.3), (1.2).

To obtain a lower function, we proceed as in the proof of Theorem 3.9 choosing x_1 small enough.

Consequently, the assertion follows from Proposition 2.1. □

Remark 3.2. Note that the conditions guaranteeing solvability of the problem (1.3), (1.2) in the case where $0 \leq \delta < 1$, $\gamma = \nu$, and $g_1 > g_2$ can be derived from Theorem 3.9. Further, the case $0 \leq \delta < 1$, $\gamma = \nu \geq 1$, and $g_1 < g_2$ is investigated in [3].

However, in that case $\gamma = \nu$ and $g_1 < g_2$, only the conditions sufficient for the existence of non–ordered lower and upper functions are known to the authors. Thus our analysis is incomplete and the case $\gamma = \nu < 1$, $g_1 < g_2$ remains as an open problem.

Theorem 3.11. *Let* $\delta = 1$, $g_1 > 0$ *and* $g_2 = 0$. *If* $\overline{h}_0 > 0$ *and* (3.13) *is fulfilled, then there exists at least one positive solution to the problem* (1.3), (1.2).

Proof. Put $\rho_0(x) = x$. According to Lemma 2.5 we can choose $c_0 > \overline{h}_0$ large enough such that the condition b) of Theorem 3.1 is fulfilled. Obviously, also $r_0 > 0$ can be chosen such that (3.3) is satisfied.

On the other hand, put $h_1 \equiv h_0$ and $\rho_1(x) = x$. Then, in view of (3.13), there exists a constant $c_1 > 0$ such that $c_1 \leq \overline{h}_1$ and the condition d) of Theorem 3.1 and (3.4) are fulfilled with a suitable $r_1 > r_0$.

Consequently, the assertion follows from Theorem 3.1. □

Theorem 3.12. *Let $\delta = 1$, $\gamma < \nu$, $g_1 > 0$ and $g_2 > 0$. If $\overline{h}_0 \geq 0$ and*

$$\frac{\omega}{4} \int_0^\omega [h_0(s)]_+ ds \int_0^\omega [h_0(s)]_- ds \leq \int_0^\omega [h_0(s)]_+ ds - \int_0^\omega [h_0(s)]_- ds, \tag{3.17}$$

then there exists at least one positive solution to the problem (1.3), (1.2).

Proof. The proof is similar to that of Theorem 3.11. The only difference is that the inequality $g_1 x^{-(\nu+1)} - g_2 x^{-(\gamma+1)} < 0$ for x sufficiently large allows one to choose a constant c_1 equal to zero. □

Theorem 3.13. *Let $\delta = 1$, $\gamma < \nu$, $g_1 > 0$ and $g_2 > 0$. If $\overline{h}_0 \leq 0$ and*

$$h_0(t) \geq \inf \left\{ \frac{g_1}{x^{\nu+1}} - \frac{g_2}{x^{\gamma+1}} : x \in \mathbb{R}^+ \right\} \qquad \textit{for a. e. } t \in [0, \omega], \tag{3.18}$$

then there exists at least one positive solution to the problem (1.3), (1.2).

Proof. As in the proof of Theorem 3.10, we can verify that there exists a constant $x_1 > 0$ such that $\beta(t) = x_1$ for $t \in [0, \omega]$ is an upper function to the problem (1.3), (1.2).

On the other hand, put $\rho_0(x) = x$, and choose $c \geq \overline{h}_0$ and $x_0 > 0$ such that $x_0 < x_1$ and the conditions of Corollary 2.3 are fulfilled. Note that the existence of $c \geq \overline{h}_0$ large enough such that (2.58) holds follows from Lemma 2.5. Therefore, there exists a lower function.

Consequently, the assertion follows from Proposition 2.1. □

Theorem 3.14. *Let $1 < \delta$, $\gamma < \nu$ and $g_1 > 0$, $g_2 > 0$. If* (3.16) *is fulfilled, then there exists at least one positive solution to the problem* (1.3), (1.2).

Proof. The upper function is constructed as in the proof of Theorem 3.10.

On the other hand, put $\rho_0(x) = x^\delta$, and choose $\{y_n\}_{n=1}^{+\infty}$ a sequence of posivite numbers satisfying (2.53), $c > \overline{h}_0$, and $\varepsilon \in (0, 1)$ such that

$$(1 - \varepsilon)^{-\delta} \Phi_+ \leq \Phi_-.$$

Then there exists $x_0 > 0$ sufficiently small such that all the conditions of Corollary 2.2 are fulfilled. Consequently, there exists a lower function $\alpha(t) \leq x_0$.

Now the assertion follows from Proposition 2.1. □

3.2.3 The case $\gamma \leq 0$.

We finish the section with two results dealing with the problem (1.3), (1.2) in the case when the parameter γ is non–positive. This case is also interesting from the physical point of view.

Theorem 3.15. *Let $0 \leq \delta < 1$, $-\gamma > \delta$, $g_1 > 0$ and $g_2 > 0$. If* (3.15) *is fulfilled, then there exists at least one positive solution to the problem* (1.3), (1.2).

Proof. The assertion immediately follows from Theorem 3.1 b) and c) with $h(t,x) = h_0(t)x^\delta$, $h_1 \equiv h_0$, $\rho_i(x) = x^\delta$ $(i = 0,1)$, $c_0 = \text{ess sup}\big\{h_0(t) : t \in [0,\omega]\big\}$, $c_1 = \overline{h}_0 - 1$, and $g(x) = g_1 x^{-\nu} - g_2 x^{-\gamma}$. □

Theorem 3.16. *Let $0 \leq \delta < 1$, $\gamma \leq 0$, $|\gamma| \leq \delta$, $g_1 > 0$ and $g_2 > 0$, and $\overline{h}_0 > 0$. If* (3.15) *is fulfilled, then there exists at least one positive solution to the problem* (1.3), (1.2).

Proof. The assertion immediately follows from Theorem 3.1 b) and c) with $h(t,x) = h_0(t)x^\delta$, $h_1 \equiv h_0$, $\rho_i(x) = x^\delta$ $(i = 0,1)$, $c_0 = \text{ess sup}\big\{h_0(t) : t \in [0,\omega]\big\}$, $c_1 = 0$, and $g(x) = g_1 x^{-\nu} - g_2 x^{-\gamma}$. □

4 The Rayleigh–Plesset equation

In this section we will use our main mathematical results to make more complete the study of the Rayleigh–Plesset equation initiated in [3]. The physical background of this section was explained in [3]. Therefore, we only introduce some results which are direct consequences of our main results. We just remark that in the paper [3], the case where the polytropic coefficient k is greater than or equal to one is considered. Now, we are able to cover the whole range of values for this parameter.

Theorem 3.6 implies

Theorem 4.1. *Let $k > \frac{1}{3}$, $P_v > \overline{P}_\infty$ and*

$$\frac{5(P_v - P_\infty(t))}{2\rho} \leq \left(\frac{6k-2}{5}\right)\left[\frac{(\frac{2}{5})^{\frac{2}{5}}(5S)^{\frac{6k}{5}}}{(\frac{6k}{5})^{\frac{6k}{5}}\left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)^{\frac{2}{5}}}\right]^{\frac{5}{6k-2}} \qquad \text{for } t \in [0,\omega].$$

Then there exists at least one positive periodic solution to the Rayleigh–Plesset equation.

Theorems 3.9 and 3.10, respectively, lead to

Theorem 4.2. *Let $\frac{1}{6} < k < \frac{1}{3}$, $P_v > \overline{P}_\infty$, and*

$$\text{ess inf}\big\{P_\infty(t) : t \in [0,\omega]\big\} > -\infty. \tag{4.1}$$

Then there exists at least one positive periodic solution to the Rayleigh–Plesset equation.

Theorem 4.3. *Let* $\frac{1}{6} < k < \frac{1}{3}$*,* $P_v \le \overline{P}_\infty$*,* (4.1) *holds, and*

$$\frac{5(P_v - P_\infty(t))}{2\rho} \ge -\left(\frac{2-6k}{5}\right)\left[\frac{\left(\frac{6k}{5}\right)^{\frac{6k}{5}}\left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)^{\frac{2}{5}}}{\left(\frac{2}{5}\right)^{\frac{2}{5}}(5S)^{\frac{6k}{5}}}\right]^{\frac{5}{2-6k}} \qquad \textit{for } t \in [0,\omega].$$

Then there exists at least one positive periodic solution to the Rayleigh–Plesset equation.

Applying Theorem 3.9 with $g_1 = 5S - \frac{5P_{g_0}R_0^{3k}}{2\rho}$, $g_2 = 0$, and $\nu = 1/5$ we get

Theorem 4.4. *Let* $k = \frac{1}{3}$*,* $P_v > \overline{P}_\infty$*,* $2\rho S > P_{g_0}R_0^{3k}$*, and let* (4.1) *holds. Then there exists at least one positive periodic solution to the Rayleigh–Plesset equation.*

Remark 4.1. The open problem posed in Remark 3.2 corresponds to this last result when $2\rho S < P_{g_0}R_0^{3k}$.

Applying Theorems 3.15 and 3.16, we get, respectively,

Theorem 4.5. *Let* $k < 0$ *and let* (4.1) *holds. Then there exists at least one positive periodic solution to the Rayleigh–Plesset equation.*

Theorem 4.6. *Let* $0 \le k \le \frac{1}{6}$*,* $P_v > \overline{P}_\infty$*, and let* (4.1) *holds. Then there exists at least one positive periodic solution to the Rayleigh–Plesset equation.*

Let us finish by pointing out the presented results have a direct physical reading. For example, we can conclude that as a general rule a high density coefficient ρ of the liquid should benefit the presence of oscillating bubbles, an effect that seems physically plausible.

Bibliography

[1] C. De Coster, P. Habets, *The lower and upper solutions method for boundary value problems*, in Handbook of Differential Equations, ODE, Elsevier, 2004, pp. 69–160.

[2] J. P. Franc, *The Rayleigh–Plesset equation: A simple and powerful tool to understand various aspects of cavitation*, in Fluid Dynamics of Cavitation and Cavitating Turbopumps, Springer Vienna, 2008.

[3] R. Hakl, P. J. Torres and M. Zamora, *Periodic solutions of singular second order differential equations: the repulsive case*, preprint at http://www.ugr.es/ ecuadif/fuentenueva.htm.

[4] A. C. Lazer and S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc., **99** (1987), 109–114.

[5] J. Mawhin, *Topological degree and boundary value problems for nonlinear differential equations*, M. Furi (ed.) Topological Methods for Ordinary Differential Equations. Lect. Notes Math. 1537, Springer, Berlin, 1993, pp. 73–142.

[6] I. Rachůnková, S. Staněk, M. Tvrdý, *Solvability of Nonlinear Singular Problems for Ordinary Differential Equations*, Contemporary Mathematics and Its Applications 5. New York, NY: Hindawi Publishing Corporation, 2008, 268 pp.

[7] I. Rachůnková, M. Tvrdý and I. Vrkoč, *Existence of nonnegative and nonpositive solutions for second–order periodic boundary–value problems*, J. Differential Equations 176 (2001), No. 2, 445–469.

[8] H. Schaefer, *Über die Methode der a priori Schranken*, Math. Ann. **129** (1955), 415–416.

[9] D. R. Smart, *Fixed point theorems*, Cambridge Tracts in Mathematics, 66. London: Cambridge University Press (1974), 104 pp.

[10] E. Zeidler, *Nonlinear functional analysis and its applications. Volume I: Fixed-point theorems*, Springer–Verlag, 1985.