PERIODIC SOLUTIONS TO SINGULAR SECOND ORDER DIFFERENTIAL EQUATIONS: THE REPULSIVE CASE

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Abstract

This paper is devoted to study the existence of periodic solutions to the second–order differential equation $u'' + f(u)u' + g(u) = h(t, u)$, where h is a Carathéodory function and f, g are continuous functions on $(0, +\infty)$ which may have singularities at zero. The repulsive case is considered. By using Schaefer's fixed point theorem, new conditions for existence of periodic solutions are obtained. Such conditions are compared with those existent in the related literature and applied to the Rayleigh–Plesset equation, a physical model for the oscillations of a spherical bubble in a liquid under the influence of a periodic acoustic field. Such a model has been the main motivation of this work.

1 Introduction

In this paper, we are concerned with the periodic problem

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h(t, u(t)) \qquad \text{for a. e. } t \in [0, \omega], \tag{1.1}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{1.2}$$

where $f, g \in C\big(\mathbb{R}^+; \mathbb{R}\big)$ may have singularities at zero, and $h \in \mathrm{Car}\,\big([0, \omega] \times \mathbb{R}_+; \mathbb{R}\big)$. By a positive solution to (1.1), (1.2) we understand a function $u \in AC^1\big(\mathbb{R}/\omega\mathbb{Z}; \mathbb{R}^+\big)$ verifying (1.1). A special case which may serve as a model is

$$u''(t) + f(u(t))u'(t) + \frac{g_1}{u^\nu(t)} - \frac{g_2}{u^\gamma(t)} = h_0(t)u^\delta(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1.3}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{1.4}$$

where $\nu, \gamma \in \mathbb{R}^+$, $g_1, g_2, \delta \in \mathbb{R}_+$, $h_0 \in L\big([0, \omega]; \mathbb{R}\big)$ and $f \in C\big(\mathbb{R}^+; \mathbb{R}\big)$. In the related literature, it is said that the nonlinearity g has an attractive singularity (resp. repulsive singularity) at zero if $\lim\limits_{x\to 0^+} g(x) = +\infty$ (resp. $\lim\limits_{x\to 0^+} g(x) = -\infty$). This paper is devoted

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to the repulsive case, which in the model equation (1.3) means $\gamma > \nu$ or else $\gamma = \nu$ and $g_1 < g_2$.

Generally speaking, differential equations with singularities have been considered from the very beginning of the discipline. The main reason is that singular forces are ubiquitous in applications, being gravitational and electromagnetic forces the most obvious examples. Even if we restrict our attention to the model equation (1.3), it has a long and rich history. It seems that the first reference goes back to Nagumo in 1943 [15]. After some works in the sixties [6–8, 11], the paper of Lazer and Solimini [12] is acknowledged as a major milestone and the origin of a fruitful line of research. Without any intention of being exhaustive, one can cite for instance [1,2,4,10,19,22–24,27] and their references. Also, the monographs [17,18] contain a whole section dedicated to the periodic problem and a quite complete bibliography up to 2008. Beginning with the paper of Habets-Sanchez [10], many of this references have considered the inclusion of a friction term of Liénard type $f(u)u'$, but up to our knowledge none of them have considered the possibility of a singularity also in $f(u)$.

We have been compelled to consider the case of a possible singularity in $f(u)$ motivated by the following physical model. In Physics of Fluids, the Rayleigh–Plesset equation

$$\rho\left[R\ddot{R} + \frac{3}{2}\dot{R}^2\right] = [P_v - P_\infty(t)] + P_{g_0}\left(\frac{R_0}{R}\right)^{3k} - \frac{2S}{R} - \frac{4\mu\dot{R}}{R}$$

is a largely studied model for the oscillations of the radius $R(t)$ of a spherical bubble immersed in a fluid under the influence of a periodic acoustic field P_∞ (see, e.g., [9]). The rest of constants are physical parameters which are described with more detail in Section 3. The change variable $R = u^{\frac{2}{5}}$ leads to

$$\ddot{u} = \frac{5\,[P_v - P_\infty(t)]}{2\rho}u^{\frac{1}{5}} + \left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)\frac{1}{u^{\frac{6k-1}{5}}} - \frac{5S}{u^{\frac{1}{5}}} - 4\mu\frac{\dot{u}}{u^{\frac{4}{5}}},$$

which is an equation like (1.3) with $f(u) = 4\mu u^{-\frac{4}{5}}$. Up to our knowledge, the existing results about singular equations do not fit adequately this case.

By using a combination of Schaefer's fixed point theorem with techniques of a priori estimates, we have proved a result which is interesting in two aspects: first, it covers the physical application which was our initial motivation; second, it has independent interest from a theoretical point of view as a complement of the existing literature.

The structure of the paper is as follows: after Introduction, in Section 2 the main result is presented and compared with other mathematical results on singular equation available in the literature. Afterwards, the main result is applied to the Rayleigh-Plesset equation in Section 3. The rest of the paper is devoted to the proof of the main result. We have organised the proof into three sections. In Section 4 the Schaefer's fixed point theorem is presented. Section 5 includes the fixed point formulation of the problem and some auxiliary results. Finally, in Section 6 we perform the required a priori estimates in order to finish the proof.

For convenience, we finish this introduction with a list of notation which is used throughout the paper:

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $[x]_+ = \max\{x, 0\}$, $[x]_- = \max\{-x, 0\}$.

$C\big([0,\omega];\mathbb{R}\big)$ is the Banach space of continuous functions $u : [0,\omega] \to \mathbb{R}$ with the norm

$$\|u\|_\infty = \max\{|u(t)| : t \in [0,\omega]\}.$$

$C(D_1; D_2)$, where $D_1, D_2 \subseteq \mathbb{R}$, is the set of continuous functions $u : D_1 \to D_2$.

$C^1\big([0,\omega];\mathbb{R}\big)$ is the Banach space of continuous functions $u : [0,\omega] \to \mathbb{R}$ with continuous derivative, with the norm $\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty$.

$AC\big([0,\omega];\mathbb{R}\big)$ is a set of all absolutely continuous functions.

$AC^1\big([0,\omega];\mathbb{R}\big)$ is a set of all functions $u : [0,\omega] \to \mathbb{R}$ such that u and u' are absolutely continuous.

$L\big([0,\omega];\mathbb{R}\big)$ is the Banach space of Lebesgue integrable functions $p : [0,\omega] \to \mathbb{R}$ with the norm

$$\|p\|_1 = \int_0^\omega |p(s)|ds.$$

$L\big([0,\omega];\mathbb{R}_+\big) = \big\{p \in L\big([0,\omega];\mathbb{R}\big) : p(t) \geq 0 \text{ for a.e. } t \in [0,\omega]\big\}$.

For a given $p \in L\big([0,\omega];\mathbb{R}\big)$, its mean value is defined by

$$\overline{p} = \frac{1}{\omega}\int_0^\omega p(s)ds.$$

Finally, a function $f : [0,\omega] \times D_1 \to D_2$ belongs to the set of Carathéodory functions $\mathrm{Car}\,\big([0,\omega] \times D_1; D_2\big)$ if and only if $f(\cdot, x) : [0,\omega] \to D_2$ is measurable for all $x \in D_1$, $f(t, \cdot) : D_1 \to D_2$ is continuous for a.e. $t \in [0,\omega]$, and for each compact set $K \subset D_1$, there exists $m_K \in L\big([0,\omega];\mathbb{R}_+\big)$ such that $|f(t,x)| \leq m_K(t)$ for a.e. $t \in [0,\omega]$ and all $x \in K$.

Throughout the paper, speaking about periodic function u we mean that both u and u' are periodic functions; i.e.,

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega).$$

2 Main result and comparison with previously known results

In this section we present the main result of the paper and discuss some consequences in order to compare it with related results.

Theorem 2.1. *Let $\eta \in \mathrm{Car}\,\big([0,\omega] \times \mathbb{R}_+;\mathbb{R}_+\big)$ be a non–decreasing function with respect to the second variable, $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be non–decreasing and $r > 0$ be such that the following items are fulfilled:*

1. $-\eta(t,x) \leq h(t,x) \leq h_0(t)\rho(x)$ *for a. e.* $t \in [0,\omega]$, $x \geq r$,
2. $g(x) \geq \overline{h}_0\rho(x)$ *for* $x \geq r$,

3. $\lim_{x\to 0_+} g(x) = -\infty, \qquad \int_0^1 g(x)dx = -\infty,$

4. $g^* \overset{def}{=} \limsup_{x\to+\infty} \frac{[g(x)]_+}{x} < \left(\frac{\pi}{\omega}\right)^2,$

5. $\limsup_{x\to+\infty} \frac{\int_0^\omega \eta(t,x)dt}{x} < \frac{4}{\omega}\left(1 - g^*\left(\frac{\omega}{\pi}\right)^2\right),$

6. $\int_0^1 [f(s)]_+ ds < +\infty \quad$ *or* $\quad \int_0^1 [f(s)]_- ds < +\infty.$

Then there exists at least one positive solution to the problem (1.1), (1.2).

The proof will be performed later in Section 6. Such a result finds a direct application to Eq. (1.3) in the sublinear case $\delta < 1$.

Corollary 2.1. *Let us assume* $0 \le \delta < 1$, $\gamma > \nu$, $\gamma \ge 1$, $g_2 > 0$ *and*

$$\int_0^1 [f(s)]_+ ds < +\infty \qquad \text{or} \qquad \int_0^1 [f(s)]_- ds < +\infty. \tag{2.1}$$

If $\overline{h}_0 \le 0$ *and* $g_1 + |\overline{h}_0| > 0$, *then there exists at least one positive solution to the problem* (1.3), (1.4).

Proof. It can be proved by applying Theorem 2.1 with $\eta(t,x) = [h_0(t)]_- x^\delta$, $\rho(x) = x^\delta$ and $h(t,x) = h_0(t)x^\delta$. Indeed, hypotheses 1, 3, 4, 5, and 6 of Theorem 2.1 are straightforward. Finally, hypothesis 2 one can easily prove by using the inequality $g_1 + \left|\overline{h}_0\right| > 0$. □

The linear case $\delta = 1$ is also covered by Theorem 2.1 as follows.

Corollary 2.2. *Let us assume* $\delta = 1$, $\gamma > \nu$, $\gamma \ge 1$, $g_2 > 0$ *and suppose that* (2.1) *holds. If* $\overline{h}_0 \le 0$, $g_1 + |\overline{h}_0| > 0$ *and*

$$\int_0^\omega [h_0(s)]_- ds < \frac{4}{\omega},$$

then there exists at least one positive solution to the problem (1.3), (1.4).

Proof. It can be proved by applying Theorem 2.1 with $\eta(t,x) = [h_0(t)]_- x$, $\rho(x) = x$, $h(t,x) = h_0(t)x$ and reasoning as we did in Corollary 2.1. □

Corollary 2.2 can be compared with [1, Theorem 3.1]. Although both results are independent, our result imposes a weaker condition over f since it may have a singularity at zero, and also the condition over h_0 is of integral type, while in [1] a uniform bound is needed.

Many classical papers consider the case where the right-hand side only depends on t and f is continuous at zero, that is, $f \in C\big(\mathbb{R}_+;\mathbb{R}\big)$ and $\delta = 0$. We consider this case in a separated corollary.

Corollary 2.3. *Let us consider the problem*

$$u''(t) + f(u(t))u'(t) + g(u(t)) = h_0(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{2.2}$$
$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2.3}$$

where $f \in C\big(\mathbb{R}_+; \mathbb{R}\big)$, $h_0 \in L\big([0, \omega]; \mathbb{R}\big)$, *and* $g \in C\big(\mathbb{R}^+; \mathbb{R}\big)$ *verifies the conditions*

1. $\lim\limits_{x \to 0+} g(x) = -\infty$,
2. $\int_0^1 g(x)dx = -\infty$,
3. $\limsup\limits_{x \to +\infty} \dfrac{g(x)}{x} < \left(\dfrac{\pi}{\omega}\right)^2$,
4. *there exists* $r > 0$ *such that* $g(x) \geq \overline{h}_0$ *for every* $x \geq r$.

Then there exists at least one positive solution to the problem (2.2), (2.3).

Proof. It is enough to apply Theorem 2.1 with $h(t,x) = h_0(t)$, $\eta(t,x) = [h_0(t)]_-$ and $\rho \equiv 1$. □

Let us observe that the condition 3 is in some sense optimal, since in [2] the authors have constructed an example of $h \in C\big([0, \omega]; \mathbb{R}\big)$ such that the equation

$$u'' + \left(\frac{\pi}{\omega}\right)^2 u - \frac{1}{u^3} = h(t)$$

has no periodic solution. Corollary 2.3 covers the classical model equation of Lazer-Solimini [12]. It also improves the following result by Mawhin.

Theorem 2.2 (see [14])**.** *Let us assume that* $f(x) \equiv c \in \mathbb{R}$. *Fix* $0 < a < \dfrac{1}{2\omega^2 e^{2|c|\omega}}$ *and* $b \geq 0$ *such that*

1. $g(x) \leq ax + b \qquad$ *for* $x > 0$,
2. $\lim\limits_{x \to 0+} g(x) = -\infty$,
3. $\int_0^1 g(x)dx = -\infty$,
4. $\liminf\limits_{x \to +\infty} g(x) > \overline{h}_0$.

Then there exists at least one positive solution to the problem (2.2), (2.3).

Another related result was proved by Habets and Sanchez.

Theorem 2.3 (see [10])**.** *Let* $f \in C\big(\mathbb{R}_+; \mathbb{R}\big)$ *and let*

1. $g(x) - h_0(t) \leq c \qquad$ *for* $t \in [0, \omega], \quad x > 0$,

2. $g(x) < \overline{h}_0$ *for all* $x < r_0$,

3. $\int_0^1 g(x)dx = -\infty$,

4. $g(x) > \overline{h}_0$ *for all* $x > r_1$,

5. $\int_0^\omega h_0^2(s)ds < +\infty$

be fulfilled with suitable constants $c > 0$ *and* $0 < r_0 < 1 < r_1 < +\infty$. *Then the problem* (2.2), (2.3) *has at least one positive solution.*

One can easily verify that Corollary 2.3 improves Theorem 2.3 in a certain way.

3 Application to a physical model: the Rayleigh-Plesset equation

In this section we will use our main mathematical result to study the Rayleigh-Plesset equation, which models the oscillations of a spherical bubble in a liquid subjected to a periodic acoustic field. The Rayleigh-Plesset equation plays a prominent role in Dynamics of Fluids. It can be derived by taking spherical coordinates in Euler equations and assuming some physically admissible simplifications, as shown in many reviews and monographs (see for instance [3, 5, 9, 16, 25]). A variety of physical, biological and medical models rely on this equation (see bibliographies of the cited references), in connection with the physical phenomena of cavitation and sonoluminescence.

Following [9], the evolution in time of the radius $R(t)$ of the bubble is ruled by

$$\rho\left[R\ddot{R}+\frac{3}{2}\dot{R}^2\right] = [P_v - P_\infty(t)] + P_{g_0}\left(\frac{R_0}{R}\right)^{3k} - \frac{2S}{R} - \frac{4\mu\dot{R}}{R}. \tag{3.1}$$

Here, at the left-hand side $\dot{R}$ and $\ddot{R}$ are the first and second derivatives of the bubble radius with respect to time and ρ is the density of the liquid. At the right-hand side we have four different terms. The first one is $P_v - P_\infty(t)$, which measures the difference between the vapour pressure P_v inside the bubble and the applied pressure, which is time-periodic. The second term is related with the non-condensability of the gas. More exactly, P_{g_0} and R_0 correspond, respectively, to the gas pressure and initial radius of the bubble, while k is the polytropic coefficient, which contents information about thermic transmission behaviour of the system liquid–gas. If the behaviour is isothermal then the coefficient k is equal to one. The most usual case considered in the cited references is when polytropic coefficient is greater than or equal to one, but possibly it is any real number. In this paper, we consider the adiabatic case (when $k \geq 1$). The third terms corresponds to surface tension, i.e., the energy which is needed to increase the surface of a liquid by area unit. Finally, the last term corresponds to the viscosity of liquid.

When the surface tension and viscosity effects are neglected (a physically admissible simplification for bubbles of big radius), we may obtain the classical Rayleigh equation

$$\rho\left[R\ddot{R}+\frac{3}{2}\dot{R}^2\right]=P_v-P_\infty(t),$$

which was proposed in 1907 by Rayleigh. Furthermore, we observe that when the applied pressure is constant, the Rayleigh equation has a first integral

$$\dot{R}^2=\frac{2}{3}\frac{P_v-P_\infty}{\rho}\left[1-\left(\frac{R_0}{R}\right)^3\right].$$

Nevertheless, when the applied pressure $P_\infty(t)$ is time-varying, most of the present knowledge about the dynamics of this models is based on numerical computations.

If the change of variables $R=u^{\frac{2}{5}}$ is introduced in the Rayleigh-Plesset equation, we obtain

$$\ddot{u}=\frac{5\left[P_v-P_\infty(t)\right]}{2\rho}u^{\frac{1}{5}}+\left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)\frac{1}{u^{\frac{6k-1}{5}}}-\frac{5S}{u^{\frac{1}{5}}}-4\mu\frac{\dot{u}}{u^{\frac{4}{5}}},$$

which corresponds to a Liénard equation, more exactly, it is an equation of the type (1.3), where $h_0(t)=\frac{5[P_v-P_\infty(t)]}{2\rho}$, $g_1=5S$, $g_2=\frac{5P_{g_0}R_0^{3k}}{2\rho}$, $\delta=\nu=\frac{1}{5}$, $\gamma=\frac{6k-1}{5}$ and $f(x)=\frac{4\mu}{x^{\frac{4}{5}}}$. If $k\geq 1$, then $\gamma\geq 1$. A direct application of Corollary 2.1 gives the following result.

Theorem 3.1. *Let us assume $k\geq 1$ and $P_v\leq\overline{P}_\infty$. Then there exists at least one positive periodic solution to the equation* (3.1).

As far as we know, this is the first analytical proof of a well-known numerical evidence exposed in many related works, see for instance [9]. In a subsequent paper, we will consider the case when the polytropic coefficient k is any real number and also the case $P_v>\overline{P}_\infty$.

4 Compact operators and Schaefer's theorem

Throughout the paper we are going to consider the Banach space $X=C^1\big([0,\omega];\mathbb{R}\big)\times\mathbb{R}$ with the norm $\|(u,a)\|=\|u\|_{C^1}+|a|$. The following result is known as a Schaefer's fixed point theorem and it is a direct consequence of the Schauder's fixed point theorem (see [20], or more recent books [21,26]). We formulate it in a suitable for us form.

Theorem 4.1 (see [20]). *Let $F:X\to X$ be a continuous operator which is compact on each bounded subset of X. If there exists $r>0$ such that every solution to*

$$(u,a)=\lambda F(u,a) \tag{4.1}$$

for $\lambda\in(0,1)$ verifies

$$\|(u,a)\|\leq r, \tag{4.2}$$

then (4.1) *has a solution for $\lambda=1$.*

Our aim is to apply this result to a given operator whose fixed points correspond to periodic solutions of our differential equation. In order to define such operator and prove its compactness the following definition is needed.

Definition 4.1. An operator $H : X \to L\big([0,\omega];\mathbb{R}\big)$, resp. $A : X \to \mathbb{R}$ is called *Carathéodory* if it is continuous and for every $r > 0$ there exists a function $q_r \in L\big([0,\omega];\mathbb{R}_+\big)$, resp. a constant $M_r \in \mathbb{R}_+$ such that

$$\big|H(u,a)(t)\big| \le q_r(t) \qquad \text{for a. e. } t \in [0,\omega], \quad \|(u,a)\| \le r,$$

resp.

$$\big|A(u,a)\big| \le M_r \qquad \text{for } \|(u,a)\| \le r.$$

Lemma 4.1. *Let $H : X \to L\big([0,\omega];\mathbb{R}\big)$ and $A : X \to \mathbb{R}$ be Carathéodory operators. Define an operator $\Omega : X \to C^1\big([0,\omega];\mathbb{R}\big)$ by*

$$\Omega(u,a)(t) = -\frac{1}{\omega}\left[(\omega - t)\int_0^t sH(u,a)(s)ds + t\int_t^\omega (\omega - s)H(u,a)(s)ds\right] \quad \textit{for } t \in [0,\omega].$$

Then the operator $F : X \to X$ given by $F = (\Omega, A)$ is compact on each bounded subset of X.

Proof. It is sufficient to prove that both Ω and A transform each bounded subset of X into a precompact set. First, note that the image of each bounded subset of X by A is in fact a precompact set since $\mathbb{R}$ is a finite-dimensional space and A is a Carathéodory operator.

On the other hand, the definition of Ω implies

$$|\Omega(u,a)(t)| \le \frac{\omega}{4}\int_0^\omega |H(u,a)(s)|\,ds \qquad \text{for } t \in [0,\omega], \tag{4.3}$$

$$\left|\frac{d}{dt}\Omega(u,a)(t)\right| \le \int_0^\omega |H(u,a)(s)|\,ds \qquad \text{for } t \in [0,\omega], \tag{4.4}$$

$$\left|\frac{d^2}{dt^2}\Omega(u,a)(t)\right| \le |H(u,a)(t)| \qquad \text{for a. e. } t \in [0,\omega]. \tag{4.5}$$

Furthermore, since H is a Carathéodory operator, for every $r > 0$ there exists a function $q_r \in L\big([0,\omega];\mathbb{R}_+\big)$ such that

$$|H(u,a)(t)| \le q_r(t) \qquad \text{for a. e. } t \in [0,\omega], \quad \|(u,a)\| \le r. \tag{4.6}$$

Now let $M \subset X$ be a bounded set. Obviously, there exists $r > 0$ such that $\|(u,a)\| \le r$ for every $(u,a) \in M$. Then, from (4.3)–(4.6), for $(u,a) \in M$, we obtain

$$\|\Omega(u,a)\|_\infty \le \frac{\omega}{4}\|q_r\|_1\,,$$

$$\left\|\frac{d}{dt}\Omega(u,a)\right\|_\infty \le \|q_r\|_1\,,$$

$$\left|\frac{d^2}{dt^2}\Omega(u,a)(t)\right| \le q_r(t) \qquad \text{for a. e. } t \in [0,\omega].$$

By Arzelà–Ascoli theorem, the set $\Omega(M)$ is precompact. $\square$

The following corollary is an immediate consequence of Theorem 4.1 and Lemma 4.1.

Corollary 4.1. *Let $H : X \to L\big([0,\omega];\mathbb{R}\big)$ and $A : X \to \mathbb{R}$ be Carathéodory operators. If there exists $r > 0$ (not depending on λ) such that every solution to the problem*

$$u''(t) = \lambda H(u,a)(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{4.7}$$

$$u(0) = 0, \qquad u(\omega) = 0, \tag{4.8}$$

$$a = \lambda A(u,a) \tag{4.9}$$

for $\lambda \in (0,1)$ verifies (4.2)*, then* (4.7)–(4.9) *has a solution for $\lambda = 1$.*

5 Auxiliary results

In this section we will develop some preliminaries in order to prove the main theorem. The first aim is to rewrite the problem (1.1), (1.2) as a fixed point problem.

Let us define the continuous operator $T : X \to C^1\big([0,\omega];\mathbb{R}\big)$ by

$$T(u,a)(t) = e^a + u(t) - \min\big\{u(s) : s \in [0,\omega]\big\}.$$

For $\lambda \in (0,1)$ we consider the problem

$$\begin{aligned} u''(t) + \lambda f(T(u,a)(t))u'(t) + \lambda g(T(u,a)(t)) = \lambda h(t, T(u,a)(t)) \\ + \frac{\lambda}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right] \quad \text{for a. e. } t \in [0,\omega], \end{aligned} \tag{5.1}$$

$$u(0) = 0, \qquad u(\omega) = 0, \tag{5.2}$$

$$a = \lambda a - \frac{\lambda}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right]. \tag{5.3}$$

Remark 5.1. It can be easily seen that if $(u,a) \in X$ is a solution to (5.1)–(5.3), then the function u is periodic.

Lemma 5.1. *If there exists $r > 0$ such that for each solution (u,a) to* (5.1)–(5.3) *with $\lambda \in (0,1)$ the estimate* (4.2) *holds, then there exists at least one positive solution to* (1.1)*,* (1.2)*.*

Proof. We define the operators $H : X \to L\big([0,\omega];\mathbb{R}\big)$ and $A : X \to \mathbb{R}$ as follows:

$$\begin{aligned} H(u,a)(t) = -f(T(u,a)(t))u'(t) - g(T(u,a)(t)) + h(t, T(u,a)(t)) \\ + \frac{1}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s, T(u,a)(s))ds\right] \qquad \text{for a. e. } t \in [0,\omega], \end{aligned}$$

$$A(u,a)=a-\frac{1}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds-\int_0^\omega h(s,T(u,a)(s))ds\right].$$

It is clear that both H and A are Carathéodory operators. By Corollary 4.1, the problem (5.1)–(5.3) with $\lambda=1$ has at least one solution (u,a). Furthermore, from (5.3) (with $\lambda=1$) we obtain that

$$\int_0^\omega g(T(u,a)(s))ds=\int_0^\omega h(s,T(u,a)(s))ds, \tag{5.4}$$

and, consequently, from (5.1) with $\lambda=1$, (5.2) and (5.4) we conclude that u is a periodic function satisfying

$$u''(t)+f(T(u,a)(t))u'(t)+g(T(u,a)(t))=h(t,T(u,a)(t)) \qquad \text{for a. e. } t\in[0,\omega].$$

Now we define v by

$$v(t)=T(u,a)(t) \qquad \text{for } t\in[0,\omega].$$

Then v is a positive solution to (1.1), (1.2). □

The section is completed by lemmas presenting some useful inequalities.

Lemma 5.2. *Let $u\in AC\big([0,\omega];\mathbb{R}\big)$ be such that*

$$u(0)=u(\omega). \tag{5.5}$$

Then the inequality

$$(M-m)^2\leq\frac{\omega}{4}\int_0^\omega {u'}^2(s)ds \tag{5.6}$$

holds where

$$M=\max\big\{u(t):t\in[0,\omega]\big\}, \qquad m=\min\big\{u(t):t\in[0,\omega]\big\}.$$

Proof. Let us define $\widetilde{u}:[0,2\omega]\to\mathbb{R}$ by

$$\widetilde{u}(t)=\begin{cases}u(t) & \text{if } t\in[0,\omega],\\ u(t-\omega) & \text{if } t\in(\omega,2\omega].\end{cases} \tag{5.7}$$

Evidently, (5.5) implies that $\widetilde{u}\in AC\big([0,2\omega];\mathbb{R}\big)$ and also there exist $t_0\in[0,\omega]$ and $t_1\in(t_0,t_0+\omega)$ such that

$$\widetilde{u}(t_0)=m, \qquad \widetilde{u}(t_1)=M, \qquad \widetilde{u}(t_0+\omega)=m.$$

Then

$$M-m=\int_{t_0}^{t_1}\widetilde{u}'(s)ds, \qquad m-M=\int_{t_1}^{t_0+\omega}\widetilde{u}'(s)ds.$$

Using the Cauchy–Bunyakovskii–Schwarz inequality we prove that

$$M - m \le \sqrt{(t_1 - t_0)\left(\int_{t_0}^{t_1} \widetilde{u}'^2(s)ds\right)},$$

$$M - m \le \sqrt{(t_0 + \omega - t_1)\left(\int_{t_1}^{t_0+\omega} \widetilde{u}'^2(s)ds\right)}.$$

Multiplying both inequalities and using that $AB \le \frac{1}{4}(A+B)^2$ for each $A, B \in \mathbb{R}_+$ we can prove

$$(M-m)^2 \le \frac{\omega}{4}\int_{t_0}^{t_0+\omega} \widetilde{u}'^2(s)ds.$$

Finally, from the last inequality, in virtue of (5.7), we obtain (5.6). □

Lemma 5.3. *Let $\rho \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function and let $v \in AC^1\big([0,\omega];\mathbb{R}\big)$ be a positive function such that*

$$v(0) = v(\omega), \qquad v'(0) = v'(\omega).$$

Then

$$\int_0^\omega \frac{v''(t)}{\rho(v(t))}\,dt \ge 0. \tag{5.8}$$

Proof. There exists a sequence $\rho_n \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ of non–decreasing functions which are absolutely continuous such that

$$\lim_{n\to+\infty} \|\rho_n \circ v - \rho \circ v\|_\infty = 0, \tag{5.9}$$

$$\rho_n(m_v) = \rho(m_v) \qquad \text{where} \qquad m_v = \min\big\{v(s) : s \in [0,\omega]\big\}.$$

Then,

$$\int_0^\omega \frac{v''(t)}{\rho_n(v(t))}\,dt = \int_0^\omega \frac{\rho_n'(v(t))v'^2(t)}{\rho_n^2(v(t))}\,dt \ge 0 \tag{5.10}$$

and

$$\left|\int_0^\omega \left[\frac{v''(t)}{\rho_n(v(t))} - \frac{v''(t)}{\rho(v(t))}\right]dt\right| \le \frac{\|\rho_n \circ v - \rho\circ v\|_\infty}{\rho^2(m_v)}\int_0^\omega |v''(t)|\,dt. \tag{5.11}$$

Now from (5.9)–(5.11) we obtain (5.8). □

Lemma 5.4. *Let $v \in AC^1\big([0,\omega];\mathbb{R}\big)$ be such that*

$$v(0) = v(\omega), \qquad v'(0) = v'(\omega). \tag{5.12}$$

Then

$$\int_0^\omega v^2(t)dt \le \left(\frac{\omega}{\pi}\right)^2 \int_0^\omega v'^2(t)dt + 2m\int_0^\omega v(t)dt \tag{5.13}$$

where

$$m = \min\big\{v(t) : t \in [0,\omega]\big\}.$$

Proof. Let $t_m \in [0,\omega]$ be a point such that

$$v(t_m) = m, \tag{5.14}$$

and define

$$w(t) = \begin{cases} v(t) - m & \text{for } t \in [0,\omega], \\ v(t-\omega) - m & \text{for } t \in (\omega, 2\omega]. \end{cases} \tag{5.15}$$

Obviously, in accordance with (5.12) and (5.14) we have

$$w \in AC^1\big([0,2\omega];\mathbb{R}\big), \tag{5.16}$$

$$w(t_m) = 0, \qquad w(t_m+\omega) = 0. \tag{5.17}$$

Using Wirtinger's inequality, by virtue of (5.15)–(5.17), we obtain

$$\int_{t_m}^{t_m+\omega} w^2(t)dt \le \Big(\frac{\omega}{\pi}\Big)^2 \int_0^\omega {v'}^2(t)dt. \tag{5.18}$$

On the other hand,

$$\int_{t_m}^{t_m+\omega} w^2(t)dt = \int_0^\omega (v(t)-m)^2 dt \ge \int_0^\omega v^2(t)dt - 2m\int_0^\omega v(t)dt. \tag{5.19}$$

From (5.18) and (5.19) we get (5.13). □

6 A priori estimates and proof of the main result

A priori estimates of possible solutions to the problem (5.1)–(5.3) with $\lambda \in (0,1)$ are established in this section. This will lead to a direct proof of Theorem 2.1.

Lemma 6.1. *Let $h_0 \in L\big([0,\omega];\mathbb{R}\big)$, $\rho \in C\big(\mathbb{R}^+;\mathbb{R}^+\big)$ be a non–decreasing function such that*

$$h(t,x) \le h_0(t)\rho(x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \ge r, \tag{6.1}$$

for some $r > 0$, and let us assume that

$$g(x) \ge \overline{h}_0\rho(x) \qquad \text{for } x \ge r. \tag{6.2}$$

Then for each solution (u,a) to (5.1)–(5.3), we have

$$a \le \ln(1+r). \tag{6.3}$$

Proof. Let us suppose that (6.3) is false. Then

$$a > \ln(1+r) > 0, \tag{6.4}$$

$$T(u,a)(t) > 1+r \qquad \text{for } t \in [0,\omega]. \tag{6.5}$$

Using (6.4) in (5.3), we get

$$\frac{\lambda}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s,T(u,a)(s))ds\right] < 0. \tag{6.6}$$

From (5.1) using (6.1), (6.5) and (6.6) we obtain

$$u''(t) + \lambda f(T(u,a)(t))u'(t) + \lambda g(T(u,a)(t)) < \lambda h_0(t)\rho(T(u,a)(t)) \qquad \text{for a. e. } t \in [0,\omega]. \tag{6.7}$$

Dividing by $\rho(T(u,a)(t))$ the equation (6.7), integrating in $[0,\omega]$, and using (5.2), one gets

$$\int_0^\omega \frac{u''(t)}{\rho(T(u,a)(t))}dt + \lambda\int_0^\omega \frac{g(T(u,a)(t))}{\rho(T(u,a)(t))}dt < \lambda\omega\overline{h}_0.$$

According to Lemma 5.3, Remark 5.1 and $\lambda > 0$, it follows that

$$\int_0^\omega \frac{g(T(u,a)(t))}{\rho(T(u,a)(t))}dt < \omega\overline{h}_0. \tag{6.8}$$

On the other hand, applying (6.5) and the hypothesis (6.2) we obtain

$$\omega\overline{h}_0 \le \int_0^\omega \frac{g(T(u,a)(t))}{\rho(T(u,a)(t))}dt$$

which, however, contradicts (6.8). □

Lemma 6.2. *Let $r > 0$, and let $\eta \in \mathrm{Car}\big([0,\omega]\times\mathbb{R}_+;\mathbb{R}_+\big)$ be a function non–decreasing in the second variable such that*

$$-\eta(t,x) \le h(t,x) \qquad \textit{for a. e. } t \in [0,\omega], \quad x \ge r. \tag{6.9}$$

Furthermore, let us assume that

$$\limsup_{x\to 0_+} g(x) < +\infty, \tag{6.10}$$

$$g^* \overset{def}{=} \limsup_{x\to+\infty} \frac{[g(x)]_+}{x} < \left(\frac{\pi}{\omega}\right)^2, \tag{6.11}$$

$$\limsup_{x\to+\infty} \frac{\int_0^\omega \eta(s,x)ds}{x} < \frac{4}{\omega}\left(1 - g^*\left(\frac{\omega}{\pi}\right)^2\right). \tag{6.12}$$

Then for each $a_0 > 0$ there exists a constant $K > 0$ such that any solution (u, a) of (5.1)–(5.3) with $a \le a_0$ verifies

$$M - m \le K \tag{6.13}$$

where

$$M = \max\big\{u(s) : s \in [0, \omega]\big\}, \qquad m = \min\big\{u(s) : s \in [0, \omega]\big\}.$$

Proof. Define the truncated function

$$\widetilde{\eta}(t, x) = \begin{cases} \eta(t, x) & \text{if } x \ge r, \\ \eta(t, r) & \text{if } x < r \end{cases} \tag{6.14}$$

and

$$\xi(t, x) = \widetilde{\eta}(t, x) + \varphi_r(t) \tag{6.15}$$

where

$$\varphi_r(t) = \sup\big\{\, |h(t, x)| : 0 \le x \le r\big\} \qquad \text{for a. e. } t \in [0, \omega]. \tag{6.16}$$

Obviously, ξ is a function non–decreasing in the second variable. Using (6.9) and (6.14)–(6.16), we obtain the inequality

$$-\xi(t, x) \le h(t, x) \qquad \text{for a. e. } t \in [0, \omega], \quad x \in \mathbb{R}_+. \tag{6.17}$$

Furthermore,

$$\limsup_{x \to +\infty} \frac{\int_0^\omega \xi(s, x)ds}{x} = \limsup_{x \to +\infty} \left(\frac{\int_0^\omega \widetilde{\eta}(s, x)ds}{x} + \frac{\|\varphi_r\|_1}{x} \right) = \limsup_{x \to +\infty} \frac{\int_0^\omega \eta(s, x)ds}{x}. \tag{6.18}$$

According to (5.3) we can rewrite (5.1) as

$$u''(t) + \lambda f(T(u, a)(t))u'(t) + \lambda g(T(u, a)(t)) = \lambda h(t, T(u, a)(t)) - (1 - \lambda)a. \tag{6.19}$$

Multiplying (6.19) by $T(u, a)(t)$ and integrating on $[0, \omega]$, we obtain, with respect to Remark 5.1,

$$\begin{aligned} -\int_0^\omega u'^2(s)ds + \lambda \int_0^\omega g(T(u, a)(s))T(u, a)(s)ds = \lambda \int_0^\omega h(s, T(u, a)(s))T(u, a)(s)ds \\ - (1 - \lambda)a \int_0^\omega T(u, a)(s)ds. \end{aligned}$$

Then

$$\begin{aligned} \int_0^\omega u'^2(s)ds = \lambda \int_0^\omega g(T(u, a)(s))T(u, a)(s)ds \\ - \lambda \int_0^\omega h(s, T(u, a)(s))T(u, a)(s)ds + (1 - \lambda)a \int_0^\omega T(u, a)(s)ds \end{aligned} \tag{6.20}$$

is fulfilled.

On the other hand, from (6.11) and (6.12) it follows the existence of $\varepsilon_0 > 0$ and $r_0 > 0$ such that

$$\frac{g(x)}{x} \le g^* + \varepsilon_0 < \left(\frac{\pi}{\omega}\right)^2 \qquad \text{for } x \ge r_0 \tag{6.21}$$

and

$$\limsup_{x\to+\infty} \frac{\int_0^\omega \eta(s,x)ds}{x} < \frac{4}{\omega}\left(1 - (g^* + \varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right). \tag{6.22}$$

Moreover, (6.10) implies that

$$M_g = \sup\big\{g(x) : x \in (0, r_0]\big\} < +\infty. \tag{6.23}$$

Hence, from (6.21) and (6.23) we obtain

$$g(x) \le (g^* + \varepsilon_0)x + M_g \qquad \text{for } x > 0. \tag{6.24}$$

Now, (6.24) implies

$$\begin{aligned}\int_0^\omega g(T(u,a)(s))T(u,a)(s)ds \le (g^* + \varepsilon_0)\int_0^\omega \big(T(u,a)(s)\big)^2 ds& \\ + M_g \int_0^\omega T(u,a)(s)ds.&\end{aligned} \tag{6.25}$$

Using Lemma 5.4 in (6.25) we arrive at

$$\begin{aligned}\int_0^\omega g(T(u,a)(s))T(u,a)(s)ds \le (g^* + \varepsilon_0)\left(\frac{\omega}{\pi}\right)^2 \int_0^\omega {u'}^2(s)ds& \\ + \big((g^* + \varepsilon_0)2e^a + M_g\big)\int_0^\omega T(u,a)(s)ds.&\end{aligned} \tag{6.26}$$

If we use the inequalities (6.17), (6.26) and the hypothesis $a \le a_0$ in (6.20) we prove

$$\begin{aligned}\left[1 - (g^* + \varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right]\int_0^\omega {u'}^2(s)ds \le \int_0^\omega \xi(s, e^{a_0} + M - m)T(u,a)(s)ds& \\ + K_0\int_0^\omega T(u,a)(s)ds&\end{aligned} \tag{6.27}$$

where

$$K_0 = (g^* + \varepsilon_0)2e^{a_0} + M_g + a_0.$$

Obviously, possible constant solution (zero solution) to (5.1)–(5.3) satisfy (6.13) by itself. Therefore, in what follows we can assume, without loss of generality, that $M \ne m$, i.e., $M - m > 0$. Thus, let $\varepsilon = \dfrac{e^{a_0}}{M - m}$. In addition,

$$\varepsilon \to 0 \qquad \text{as} \qquad M - m \to +\infty, \tag{6.28}$$

and

$$T(u,a)(t) \le (1+\varepsilon)(M-m) \qquad \text{for } t \in [0,\omega].$$

Consequently, (6.27) implies

$$\left[1-(g^*+\varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right]\int_0^\omega {u'}^2(s)ds \le \left(K_0\omega + \int_0^\omega \xi(s,(1+\varepsilon)(M-m))ds\right)(1+\varepsilon)(M-m).$$

According to Lemma 5.2, from the last inequality we obtain

$$\frac{4}{\omega}\left[1-(g^*+\varepsilon_0)\left(\frac{\omega}{\pi}\right)^2\right] \le \frac{(1+\varepsilon)^2\left(K_0\omega+\int_0^\omega \xi(s,y)ds\right)}{y} \tag{6.29}$$

where $y=(1+\varepsilon)(M-m)$. Finally, (6.18), (6.22), (6.28) and (6.29) imply the existence of a constant K such that (6.13) is verified. □

Remark 6.1. Note that from the inequality (6.13), in view of (5.2), it also follows that

$$\|u\|_\infty \le K.$$

Lemma 6.3. *Let us assume that*

$$\int_0^1 [f(s)]_+\, ds < +\infty \tag{6.30}$$

or

$$\int_0^1 [f(s)]_-\, ds < +\infty. \tag{6.31}$$

Furthermore, assume that (6.10) *is verified. Then, for each* $a_0 \ge 0$ *and* $K>0$ *there exists a constant* $K_1>0$ *such that every solution* (u,a) *to* (5.1)–(5.3) *with*

$$\|u\|_\infty \le K \qquad \text{and} \qquad a \le a_0 \tag{6.32}$$

verifies the boundary

$$\|u'\|_\infty \le \lambda K_1 + a_0\omega. \tag{6.33}$$

Proof. Assume that the condition (6.30) is fulfilled. Let (u,a) be a solution of (5.1)–(5.3), then u is a periodic function and, in addition, there exist $t_0, t_1 \in [0,\omega]$ such that

$$u(t_0)=m, \qquad u(t_1)=M \tag{6.34}$$

where

$$M=\max\{u(t) : t\in[0,\omega]\}, \qquad m=\min\{u(t) : t\in[0,\omega]\}.$$

By integrating (6.19) on the interval $[t_0, t] \subseteq [t_0, t_0 + \omega]$, we obtain

$$\vartheta(u')(t) + \lambda \int_{t_0}^{t} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_{t_0}^{t} g(\vartheta(T(u,a))(s))ds$$
$$= \lambda \int_{t_0}^{t} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds - (1-\lambda)a(t-t_0)$$

where $\vartheta : C\big([0,\omega];\mathbb{R}\big) \to C\big([0,2\omega];\mathbb{R}\big)$ and $\vartheta_1 : \mathrm{Car}\,\big([0,\omega] \times \mathbb{R}_+;\mathbb{R}\big) \to \mathrm{Car}\,\big([0,2\omega] \times \mathbb{R}_+;\mathbb{R}\big)$, respectively, are operators of the periodical extension, i.e.,

$$\vartheta(v)(t) = \begin{cases} v(t) & \text{if } t \in [0,\omega], \\ v(t-\omega) & \text{if } t \in (\omega, 2\omega], \end{cases} \tag{6.35}$$

$$\vartheta_1(h)(t,x) = \begin{cases} h(t,x) & \text{if } t \in [0,\omega], \\ h(t-\omega,x) & \text{if } t \in (\omega, 2\omega]. \end{cases} \tag{6.36}$$

Obviously,

$$-\vartheta(u')(t) = \lambda \int_{t_0}^{t} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_{t_0}^{t} g(\vartheta(T(u,a))(s))ds$$
$$- \lambda \int_{t_0}^{t} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds + (1-\lambda)a(t-t_0). \tag{6.37}$$

Using (6.32) and (6.34) we get

$$0 < T(u,a)(t_0) \le T(u,a)(t) \le T(u,a)(t_1) \le e^{a_0} + 2K \qquad \text{for } t \in [0,\omega]. \tag{6.38}$$

Then, by (6.10) and the fact that $h \in \mathrm{Car}\,\big([0,\omega] \times \mathbb{R}_+;\mathbb{R}\big)$, the number μ and the function σ defined by

$$\mu = \sup\big\{[g(s)]_+ : s \in (0, e^{a_0} + 2K]\big\}, \quad \sigma(s) = \sup\big\{\,|h(s,x)| : x \in [0, e^{a_0} + 2K]\big\}, \tag{6.39}$$

satisfy

$$0 \le \mu < +\infty, \qquad \sigma \in L\big([0,\omega];\mathbb{R}_+\big). \tag{6.40}$$

Using (6.32), (6.38)–(6.40) and $t_0 \le t \le t_0 + \omega$ in the equation (6.37), we obtain

$$-\vartheta(u')(t) \le \lambda \int_0^{e^{a_0}+2K} [f(s)]_+ \, ds + \lambda\omega\mu + \lambda \|\sigma\|_1 + \omega a_0. \tag{6.41}$$

Put $K_1 = \int_0^{e^{a_0}+2K} [f(s)]_+ \, ds + \omega\mu + \|\sigma\|_1$. Then, from (6.41), we have

$$-\vartheta(u')(t) \le \lambda K_1 + \omega a_0 \qquad \text{for } t \in [t_0, t_0 + \omega]. \tag{6.42}$$

On the other hand, if we integrate on the interval $[t, t_1 + \omega] \subseteq [t_1, t_1 + \omega]$ the equation (6.19), we obtain

$$\begin{aligned}(6.43)\quad \vartheta(u')(t) = \lambda \int_t^{t_1+\omega} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_t^{t_1+\omega} g(\vartheta(T(u,a))(s))ds \\ - \lambda \int_t^{t_1+\omega} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds + (1-\lambda)a(t_1 + \omega - t).\end{aligned}$$

Using (6.32), (6.38)–(6.40) and $t_1 \le t \le t_1 + \omega$ in the equation (6.43), we have

$$(6.44)\qquad \vartheta(u')(t) \le \lambda K_1 + \omega a_0 \qquad \text{for } t \in [t_1, t_1 + \omega].$$

From (6.42) and (6.44) we conclude that (6.33) is verified. Therefore the proof is finished for this case.

Now we suppose that (6.31) is fulfilled. By defining

$$(6.45)\qquad v(t) = u(\omega - t) \qquad \text{for } t \in [0, \omega]$$

we obtain that

$$v''(t) - \lambda f(T(v,a)(t))v'(t) + \lambda g(T(v,a)(t)) = \lambda \widetilde{h}(t, T(v,a)(t)) - (1-\lambda)a \quad \text{for a. e. } t \in [0, \omega],$$

where

$$\widetilde{h}(t,x) = h(\omega - t, x) \qquad \text{for a. e. } t \in [0,\omega], \quad x \in \mathbb{R}_+.$$

If we follow analogical steps as above, using (6.31) instead of (6.30), we arrive at

$$(6.46)\qquad \|v'\|_\infty \le \lambda K_1 + a_0 \omega$$

with

$$K_1 = \int_0^{e^{a_0} + 2K} [f(s)]_- ds + \omega\mu + \|\sigma\|_1 .$$

Now, (6.45) and (6.46) imply (6.33). □

Remark 6.2. If we take $a_0 = 0$ in Lemma 6.3, we obtain that

$$(6.47)\qquad \|u'\|_\infty \le \lambda K_1$$

whenever (u, a) is a solution to (5.1)–(5.3) with $a \le 0$.

Lemma 6.4. *We suppose that*

$$(6.48)\qquad \lim_{x \to 0_+} g(x) = -\infty, \qquad \int_0^1 g(s)ds = -\infty,$$

and (6.30) *or* (6.31) *is satisfied. Then for each* $K > 0$ *there exists a constant* $a_1 > 0$ *such that every solution* (u, a) *to* (5.1)*–*(5.3) *with*

$$(6.49)\qquad \|u\|_\infty \le K \qquad \textit{and} \qquad a \le 0$$

admits the estimate

$$(6.50)\qquad -a_1 \le a.$$

Proof. We define σ as in (6.39) with $a_0 = 0$. Obviously, because $h \in \mathrm{Car}\big([0,\omega]\times\mathbb{R}_+;\mathbb{R}\big)$, we have $\sigma \in L\big([0,\omega];\mathbb{R}_+\big)$. Let (u,a) be a solution to (5.1)–(5.3). From (5.3), by virtue of (6.39) and (6.49), it follows that

$$\begin{aligned}\frac{a(1-\lambda)}{\lambda} &= -\frac{1}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s,T(u,a)(s))ds\right]\\ &\geq -\frac{1}{\omega}\int_0^\omega g(T(u,a)(s))ds - \frac{1}{\omega}\left\|\sigma\right\|_1,\end{aligned}$$

and consequently,

$$-\frac{1}{\omega}\int_0^\omega g(T(u,a)(s))ds \leq \frac{a(1-\lambda)}{\lambda} + \frac{1}{\omega}\left\|\sigma\right\|_1.$$

Hence, according to (6.49) we obtain

$$-\int_0^\omega g(T(u,a)(s))ds \leq \left\|\sigma\right\|_1. \tag{6.51}$$

On the other hand, (6.48) implies that there exists $s_0 > 0$ such that

$$g(s) < -\frac{\left\|\sigma\right\|_1}{\omega} \leq 0 \qquad \text{for } s \in (0,s_0). \tag{6.52}$$

We denote by $t_m \in [0,\omega]$ the point where $u(t_m) = \min\big\{u(t) : t \in [0,\omega]\big\}$. Obviously, either

$$T(u,a)(t_m) = e^a \geq s_0, \tag{6.53}$$

or

$$T(u,a)(t_m) = e^a < s_0. \tag{6.54}$$

Clearly, if we get an estimate (6.50) in the case (6.54), the same estimate will be valid also for every solution (u,a) to (5.1)–(5.3) verifying (6.53). Hence, without loss of generality, we can suppose that (6.54) is fulfilled.

If $T(u,a)(t) < s_0$ for every $t \in [0,\omega]$, from (6.51) and (6.52) we obtain a contradiction. Therefore, there exist points $t_1, t_2 \in (t_m, t_m+\omega)$ such that

$$\vartheta(T(u,a))(t) < s_0 \qquad \text{for } t \in [t_m,t_1), \qquad \vartheta(T(u,a))(t_1) = s_0, \tag{6.55}$$

$$\vartheta(T(u,a))(t) < s_0 \qquad \text{for } t \in (t_2,t_m+\omega], \qquad \vartheta(T(u,a))(t_2) = s_0, \tag{6.56}$$

where ϑ is an operator defined by (6.35). Since $a \leq 0$, we have

$$\frac{\lambda}{\omega}\left[\int_0^\omega g(T(u,a)(s))ds - \int_0^\omega h(s,T(u,a)(s))ds\right] \geq 0,$$

and thus

$$u''(t) + \lambda f(T(u,a)(t))u'(t) + \lambda g(T(u,a)(t)) \geq \lambda h(t, T(u,a)(t)) \qquad \text{for a. e. } t \in [0,\omega].$$

Obviously,

$$\begin{aligned}(6.57)\quad [\vartheta(u')(t)]' + \lambda f(\vartheta(T(u,a))(t))\vartheta(u')(t) + \lambda g(\vartheta(T(u,a))(t)) \\ \geq \lambda \vartheta_1(h)(t, \vartheta(T(u,a))(t)) \quad \text{for a. e. } t \in [0,2\omega]\end{aligned}$$

where ϑ and ϑ_1 are operators defined by (6.35) and (6.36), respectively.

First, let us assume that (6.30) is verified. Integrating on $[t_m, t_1]$ the inequality (6.57) we obtain

$$\begin{aligned}\vartheta(u')(t_1) + \lambda \int_{t_m}^{t_1} f(\vartheta(T(u,a))(s))\vartheta(u')(s)ds + \lambda \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds \\ \geq \lambda \int_{t_m}^{t_1} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds.\end{aligned}$$

By a change of variables and using (6.54) and (6.55) we get

$$\vartheta(u')(t_1) + \lambda \int_{e^a}^{s_0} f(s)ds - \lambda \int_{t_m}^{t_1} \vartheta_1(h)(s, \vartheta(T(u,a))(s))ds \geq -\lambda \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds.$$

According to Lemma 6.3, Remark 6.2, and the conditions (6.48) and (6.49) we obtain that there exists a constant $K_1 > 0$ such that (6.47) is fulfilled. Using (6.47), (6.54), the inequality $\lambda > 0$ and the fact that $x \leq [x]_+$ for any $x \in \mathbb{R}$ we obtain

$$(6.58)\qquad - \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds \leq K_2$$

where

$$K_2 = K_1 + \int_0^{s_0} [f(s)]_+ ds + \|\sigma\|_1 .$$

Multiplying by K_1 in the inequality (6.58), we find

$$-K_1 \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))ds \leq K_2 K_1.$$

Using (6.47), (6.52), and (6.55) we obtain

$$- \int_{t_m}^{t_1} g(\vartheta(T(u,a))(s))\vartheta(u')(s)ds \leq K_2 K_1.$$

After a simple change of variables and using (6.54) and (6.55) we arrive at

$$(6.59)\qquad - \int_{e^a}^{s_0} g(s)ds \leq K_2 K_1.$$

Using (6.48) we ensure the existence of $a_1 > 0$ such that (6.50) is fulfilled.

Now assume that (6.31) holds true. Integrating on $[t_2, t_m + \omega]$ the inequality (6.57) and following analogous steps as above, using (6.56) instead of (6.55), we arrive at (6.59) with

$$K_2 = K_1 + \int_0^{s_0} [f(s)]_- ds + \|\sigma\|_1 \, .$$

Then, the condition (6.48) implies the existence of a constant $a_1 > 0$ such that (6.50) is fulfilled. □

Proof of Theorem 2.1. The result immediately follows from Lemma 5.1, Lemmas 6.1–6.4, and Remark 6.1. □

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