Existence of a solution to Dirichlet problem associated to the second-order differential equation with singularities: method of lower and upper functions

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Abstract

This paper is devoted to the study of the existence of a solution to the Dirichlet boundary value problem for the second-order differential equation

$$u'' = f(t,u) + g(t,u)u'; \qquad u(a+) = 0, \qquad u(b-) = 1,$$

where the functions $f, g : (a,b) \times (0,1) \to \mathbb{R}$ satisfy the local Carathéodory conditions and may have singularities both in the time (for $t = a$ and $t = b$) and the phase (for $u = 0$ and $u = 1$) variables. Sufficient conditions for the solvability of the above-mentioned problem are established.

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1 Statement of the Problem and Formulation of the Main Results

Consider the boundary value problem

$$u'' = f(t,u) + g(t,u)u', \tag{1.1}$$

$$u(a+) = 0, \qquad u(b-) = 1, \tag{1.2}$$

where $f, g \in \mathrm{Car}_{loc}\big((a,b)\times(0,1);\mathbb{R}\big)$. By a solution to the problem (1.1), (1.2) we understand a function $u \in AC^1_{loc}\big((a,b);\mathbb{R}\big)$ such that $0 < u(t) < 1$ for $t \in (a,b)$, satisfying Eq. (1.1) almost everywhere in (a,b) and verifying boundary conditions (1.2).

The aim of the present paper is to investigate the question on the solvability of the problem (1.1), (1.2) provided the functions f and g possess singularities both in the time (for $t = a$ and $t = b$) and the phase (for $u = 0$ and $u = 1$) variables. Singular problems of such a type arise in the applications (see, e.g., [2, 4, 9] and references therein). There are not so many papers dealing with the problem in question; maybe the closest works deal with Eq. (1.1) subjected to the boundary conditions

$$u(a+) = 0, \qquad u(b-) = 0 \tag{1.3}$$

(see, e.g., [1, 2, 3, 6, 7, 8, 10, 11]). However, in the mentioned papers the singularities of the right-hand side of Eq. (1.1) appear for $t = a$, $t = b$, and $u = 0$ only. The first step was made by S. Taliaferro in [11], where he established a necessary and sufficient condition for the solvability of the problem (1.1), (1.3) with $g \equiv 0$ and $f(t,x) = -h(t)/x^\lambda$, where $\lambda > 0$ and $h \in L_{loc}\big((a,b);\mathbb{R}_+\big)$. Most of the known results for the problem (1.1), (1.3) deals with the case when g does not depend on the second argument and $f(t,x) \le 0$ for $t \in (a,b)$, $x > 0$ (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 11]). The problem (1.1), (1.3) without sign restrictions on f was recently studied by A. Lomtatidze and P. J. Torres in [10].

We establish new criteria guaranteeing the solvability of the problem (1.1), (1.2) through the existence of a pair of well-ordered lower and upper functions. Our main results are used to obtain effective criteria for the particular case of the problem (1.1), (1.2), which often arises in applications, namely in the case when Eq. (1.1) has the form

$$u'' = g(u)u' + h(t)u^\nu - \frac{p(t)}{u^\lambda} + \frac{q(t)}{(1-u)^\mu} + \varphi(t), \tag{1.4}$$

where $g \in C\big([0,1];\mathbb{R}\big)$, $h, \varphi \in L_{loc}\big((a,b);\mathbb{R}\big)$, $p, q \in L_{loc}\big((a,b);\mathbb{R}_+\big)$, and ν, λ, μ are positive constants.

The paper is organised in the following way: After the formulation of the main results in Section 1, in Sections 2 and 3 we prove some a priori estimates and auxiliary propositions. Section 4 is devoted to the proofs of the main results.

The following notation is used throughout the paper:

$\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty)$.

$L\big([a,b];D\big)$, where $D \subseteq \mathbb{R}$, is the set of functions $p : [a,b] \to D$ which are Lebesgue integrable on the segment $[a,b]$.

$L_{loc}\big((a,b);D\big)$, where $D \subseteq \mathbb{R}$, is the set of functions $p : (a,b) \to D$ such that $p \in L\big([\alpha,\beta];D\big)$ whenever $[\alpha,\beta] \subset (a,b)$.

$L^\infty_{loc}\big((a,b);\mathbb{R}_+\big)$ is the set of functions $p : (a,b) \to \mathbb{R}_+$ which are essentially bounded on each segment contained in (a,b).

$C\big([a,b];D\big)$, where $D \subseteq \mathbb{R}$, is the set of continuous functions $u : [a,b] \to D$.

$C_{loc}\big((a,b);\mathbb{R}\big)$ is the set of functions $u : (a,b) \to \mathbb{R}$ such that $u \in C\big([\alpha,\beta];\mathbb{R}\big)$ whenever $[\alpha,\beta] \subset (a,b)$.

$AC^1\big([a,b];D\big)$, where $D \subseteq \mathbb{R}$, is the set of functions $u : [a,b] \to D$ which are absolutely continuous together with their first derivatives.

$AC^1_{loc}\big(I;D\big)$, where $I \subseteq (a,b)$, $D \subseteq \mathbb{R}$, is the set of functions $u : I \to D$ such that $u \in AC^1\big([\alpha,\beta];D\big)$ whenever $[\alpha,\beta] \subseteq I$.

$\mathrm{Car}_{loc}\big((a,b)\times D\times\mathbb{R};\mathbb{R}\big)$, where $D\subseteq\mathbb{R}$, is the Carathéodory class, i.e., the set of functions $f:(a,b)\times D\times\mathbb{R}\to\mathbb{R}$ such that $f(t,\cdot,\cdot):D\times\mathbb{R}\to\mathbb{R}$ is continuous for almost all $t\in(a,b)$, $f(\cdot,x,y):(a,b)\to\mathbb{R}$ is measurable for all $(x,y)\in D\times\mathbb{R}$, and

$$\sup\{|f(\cdot,x,y)|:(x,y)\in D_0\}\in L_{loc}\big((a,b);\mathbb{R}_+\big)$$

for any compact $D_0\subset D\times\mathbb{R}$.

$[p]_-=\frac{1}{2}(|p|-p)$.

$u(s+)$ and $u(s-)$ are one-sided limits of the function u at the point s from the right and from the left, respectively.

First we will recall the notion of lower and upper functions to the general equation

$$u''=h(t,u,u') \tag{1.5}$$

where $h\in\mathrm{Car}_{loc}\big((a,b)\times D\times\mathbb{R};\mathbb{R}\big)$. The following definition is a particular case of the definition of lower and upper functions introduced in [8] (see also [9]).

Definition 1.1 *The continuous function $\sigma:(a,b)\to D$ is said to be a lower (upper) function to Eq.* (1.5) *if $\sigma\in AC^1_{loc}((a,b)\setminus\{t_1,t_2,\dots,t_n\};D)$, where $a<t_1<t_2<\dots<t_n<b$, there exist finite limits $\sigma(a+)$, $\sigma(b-)$, $\sigma'(t_i+)$, $\sigma'(t_i-)$, $i=1,2,\dots,n$,*

$$\sigma'(t_i-)<\sigma'(t_i+)\qquad\Big(\sigma'(t_i-)>\sigma'(t_i+)\Big),\qquad i=1,2,\dots,n,$$

and

$$\sigma''(t)\ge h(t,\sigma(t),\sigma'(t))\qquad\Big(\sigma''(t)\le h(t,\sigma(t),\sigma'(t))\Big)\qquad\text{for a. e. } t\in(a,b).$$

Now we can formulate our main result:

Theorem 1.1 *Let σ_1 and σ_2 be lower and upper functions to the equation* (1.1) *such that*

$$0<\sigma_1(t)\le\sigma_2(t)<1\qquad\text{for } t\in(a,b), \tag{1.6}$$

$$\sigma_1(a+)=0,\qquad\sigma_2(a+)<1,\qquad\sigma_1(b-)>0,\qquad\sigma_2(b-)=1. \tag{1.7}$$

Further suppose that, for every $\eta\in(0,1/2)$,

$$|f(t,x)|\le p_\eta(t),\qquad g(t,x)\operatorname{sgn}(t-c)\le\frac{\lambda_\eta}{(b-t)(t-a)}+q_\eta(t)\qquad\text{for a. e. } t\in(a,b),$$
$$\sigma_{1\eta}(t)\le x\le\sigma_{2\eta}(t), \tag{1.8}$$

where $c\in(a,b)$, $\lambda_\eta\in[0,b-a)$, $q_\eta\in L\big([a,b];\mathbb{R}_+\big)$, $p_\eta\in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with

$$\int_a^b(b-s)(s-a)p_\eta(s)ds<+\infty, \tag{1.9}$$

and

$$\sigma_{1\eta}(t)=\max\{\eta,\sigma_1(t)\},\qquad\sigma_{2\eta}(t)=\min\{1-\eta,\sigma_2(t)\}\qquad\text{for } t\in(a,b). \tag{1.10}$$

Then the problem (1.1), (1.2) *has at least one solution u satisfying*

$$\sigma_1(t)\le u(t)\le\sigma_2(t)\qquad\text{for } t\in(a,b). \tag{1.11}$$

Condition (1.8) of Theorem 1.1 allows the function g to have strong singularities at the end-points, provided the one-sided bounds are fulfilled, as shown in Remark 1.1. Theorem 1.2 established afterwards can be understood as a complement to Theorem 1.1, applicable in the case when the function g can be bounded from both sides and f allows only one-sided restrictions.

Remark 1.1 *As an example, consider the problem*

$$u'' = \frac{\lambda}{t(1-t)(1-u)^3} - \frac{\lambda}{t(1-t)u^3} + \frac{(1-2t)u'}{t^2(1-t)^2(1-u)^2u^2}\,, \tag{1.12}$$

$$u(0+) = 0, \qquad u(1-) = 1 \tag{1.13}$$

with $\lambda > 0$. *The right-hand side function in* (1.12) *is singular in the phase variable and has non-integrable singularities with respect to the time variable (note that the singularities of the function* g *may be non-integrable even with the weight* $t(1-t)$*). However, according to Theorem 1.1, by choosing*

$$\sigma_1(t) = \begin{cases} \varepsilon t(1-t) & \text{for } t \in (0, 1/2), \\ \frac{\varepsilon}{4} & \text{for } t \in [1/2, 1), \end{cases} \qquad \sigma_2(t) = \begin{cases} 1 - \frac{\varepsilon}{4} & \text{for } t \in (0, 1/2], \\ 1 - \varepsilon t(1-t) & \text{for } t \in (1/2, 1), \end{cases}$$

with $\varepsilon \in (0, 2)$ *small enough, one can assure that there exists at least one solution to* (1.12), (1.13).

Theorem 1.2 *Let* σ_1 *and* σ_2 *be lower and upper functions to the equation* (1.1) *such that* (1.6) *and* (1.7) *hold. Further suppose that, for every* $\eta \in (0, 1/2)$,

$$f(t,x)\,\mathrm{sgn}(t-c) \le p_\eta(t), \qquad |g(t,x)| \le \frac{\lambda_\eta}{(b-t)(t-a)} + q_\eta(t) \qquad \text{for a. e. } t \in (a,b),$$
$$\sigma_{1\eta}(t) \le x \le \sigma_{2\eta}(t), \tag{1.14}$$

where $c \in (a,b)$, $\lambda_\eta \in [0, b-a)$, $q_\eta \in L\big([a,b];\mathbb{R}_+\big)$, $p_\eta \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ *with the property* (1.9), *and* $\sigma_{1\eta}$, $\sigma_{2\eta}$ *are given by* (1.10). *Then the problem* (1.1), (1.2) *has at least one solution* u *satisfying* (1.11).

For the problem (1.4), (1.2) we obtain the following assertion:

Corollary 1.1 *Let*

$$\int_a^b (b-s)(s-a)\big[|\psi(s)| + p(s) + q(s) + |\varphi(s)|\big]\,ds < +\infty,$$

and let there exist positive constants r, n, p_0, *and* q_0 *such that*

$$p(t) \ge p_0, \qquad q(t) \ge q_0 \qquad \text{for a. e. } t \in (a,b),$$
$$\big(|h(t)| + p(t) + q(t) + |\varphi(t)|\big)\big[(b-t)(t-a)\big]^n \le r \qquad \text{for a. e. } t \in (a,b).$$

Then there exists at least one solution to the problem (1.4), (1.2).

2 A Priori Estimates

In the following section we prove some a priori estimates.

Lemma 2.1 *Let $r_0 > 0$, $h_2 \in L\big([a,b];\mathbb{R}_+\big)$, and $h_0 \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with*

$$\int_a^b (b-s)(s-a)h_0(s)ds < +\infty. \tag{2.1}$$

Then there exists a constant $r^* > 0$ such that, for any closed interval $[t_1,t_2] \subset (a,b)$ and any function $u \in AC^1\big([t_1,t_2];\mathbb{R}\big)$, the inequalities

$$u''(t) \geq -h_0(t) - h_2(t)|u'(t)| \qquad \text{for a. e. } t \in (t_1,t_2), \tag{2.2}$$

$$|u(t)| \leq r_0 \qquad \text{for } t \in [t_1,t_2] \tag{2.3}$$

imply

$$(t_2-t)(t-t_1)|u'(t)| \leq r^* \qquad \text{for } t \in (t_1,t_2). \tag{2.4}$$

Let $[t_1,t_2] \subset (a,b)$, a function $u \in AC^1\big([t_1,t_2];\mathbb{R}\big)$ satisfy the conditions of the lemma, and let

$$\mu_1(t) = \int_{t_1}^{t} \exp\left(\int_s^t h_2(\xi)\operatorname{sgn} u'(\xi)d\xi\right) ds \qquad \text{for } t \in [t_1,t_2], \tag{2.5}$$

$$\mu_2(t) = \int_{t}^{t_2} \exp\left(-\int_t^s h_2(\xi)\operatorname{sgn} u'(\xi)d\xi\right) ds \qquad \text{for } t \in [t_1,t_2]. \tag{2.6}$$

Let $t_0 \in (t_1,t_2)$ be arbitrary but fixed such that $u'(t_0) \neq 0$. Then either

$$u'(t_0) < 0 \tag{2.7}$$

or

$$u'(t_0) > 0. \tag{2.8}$$

If (2.7) holds, then multiplying both sides of (2.2) by a function μ_1 and integrating such an inequality from t_1 to t_0, we obtain

$$|u'(t_0)|\mu_1(t_0) \leq u(t_1) - u(t_0) + \int_{t_1}^{t_0} h_0(s)\mu_1(s)ds. \tag{2.9}$$

If (2.8) is fulfilled, then multiplying both sides of (2.2) by a function μ_2 and integrating such an inequality from t_0 to t_2, we arrive at

$$|u'(t_0)|\mu_2(t_0) \leq u(t_2) - u(t_0) + \int_{t_0}^{t_2} h_0(s)\mu_2(s)ds. \tag{2.10}$$

Now, in view of (2.3), (2.5), and (2.6), from (2.9), resp. (2.10), we obtain

$$|u'(t_0)|(t_0-t_1) \leq \left[2r_0 + \int_{t_1}^{t_0}(s-a)h_0(s)ds\right] \exp\left(2\int_a^b h_2(s)ds\right), \tag{2.11}$$

resp.

$$|u'(t_0)|(t_2-t_0) \leq \left[2r_0 + \int_{t_0}^{t_2}(b-s)h_0(s)ds\right] \exp\left(2\int_a^b h_2(s)ds\right). \tag{2.12}$$

Multiplying (2.11) by $(t_2 - t_0)$, resp. (2.12) by $(t_0 - t_1)$, we get

$$|u'(t_0)|(t_2 - t_0)(t_0 - t_1) \le r^*$$

where

$$r^* = \left[2r_0(b-a) + \int_a^b (b-s)(s-a)h_0(s)ds\right] \exp\left(2\int_a^b h_2(s)ds\right). \tag{2.13}$$

Since $t_0 \in (t_1, t_2)$ was chosen arbitrarily, we conclude that (2.4) holds.

Lemma 2.2 *Let $c \in (a,b)$, $\lambda \in [0, b-a)$, $r_0 > 0$, $r_1 > 0$, $h_1 \in L\big([a,b];\mathbb{R}_+\big)$, and let $h_0 \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with the property (2.1). Then there exist positive constants r_a and r_b such that, for any closed interval $[t_1,t_2] \subset (a,b)$ with $c \in (t_1,t_2)$ and any function $u \in AC^1\big([t_1,t_2];\mathbb{R}\big)$, the inequalities (2.3),*

$$u''(t)\,\mathrm{sgn}(t-c) \le h_0(t) + \left[\frac{\lambda}{(b-t)(t-a)} + h_1(t)\right]|u'(t)| \quad \textit{for a. e. } t \in (t_1,t_2), \tag{2.14}$$

$$|u'(c)| \le r_1 \tag{2.15}$$

imply

$$(t-t_1)^{1+\lambda_0}|u'(t)| \le r_a \qquad \textit{for } t \in (t_1, c) \tag{2.16}$$

and

$$(t_2-t)^{1+\lambda_0}|u'(t)| \le r_b \qquad \textit{for } t \in (c, t_2), \tag{2.17}$$

where $\lambda_0 = \lambda/(b-a)$.

Let $[t_1,t_2] \subset (a,b)$ with $c \in (t_1,t_2)$, a function $u \in AC^1\big([t_1,t_2];\mathbb{R}\big)$ satisfy the conditions of the lemma, and let

$$h_2(t) = \frac{\lambda}{(b-t)(t-a)} + h_1(t) \qquad \text{for a. e. } t \in (a,b), \tag{2.18}$$

$$\rho_0 = \exp\left(\int_a^b h_1(s)ds\right), \qquad \lambda_0 = \frac{\lambda}{b-a}\,. \tag{2.19}$$

We will show that (2.16) holds, the estimate (2.17) can be proven analogously. Let, therefore, μ_1 be defined by (2.5) and let $t_0 \in (t_1, c)$ be arbitrary but fixed such that $u'(t_0) \ne 0$. Then either (2.7) or (2.8) holds. If (2.7) is fulfilled, then multiplying both sides of (2.14) by a function μ_1 and integrating such an inequality from t_1 to t_0, we obtain (2.9).

On the other hand, in view of (2.18) and (2.19) we have

$$\rho_0^{-1}\left[\frac{b-c}{(c-a)(b-a)}\right]^{\lambda_0} \frac{(t-t_1)^{1+\lambda_0}}{1+\lambda_0} \le \mu_1(t) \le \rho_0\left(\frac{b-a}{b-c}\right)^{\lambda_0} \frac{t-a}{1-\lambda_0} \qquad \text{for } t \in [t_1, c]. \tag{2.20}$$

Thus, in view of (2.3), (2.5), (2.18)–(2.20), from (2.9) we obtain

$$|u'(t_0)|(t_0-t_1)^{1+\lambda_0} \le \frac{\rho_0^2(1+\lambda_0)}{1-\lambda_0}\left(\frac{b-a}{b-c}\right)^{2\lambda_0}(c-a)^{\lambda_0}\left(2r_0 + \int_a^c h_0(s)(s-a)ds\right). \tag{2.21}$$

If (2.8) holds, then from (2.14), on account of (2.18), we obtain

$$-\left[u'(t)\exp\left(-\int_t^c h_2(s)\operatorname{sgn} u'(s)ds\right)\right]' \le h_0(t)\exp\left(-\int_t^c h_2(s)\operatorname{sgn} u'(s)ds\right)$$
$$\text{for a. e. } t \in (t_1, c). \tag{2.22}$$

The integration of (2.22) from t_0 to c, in view of (2.15), results in

$$|u'(t_0)| \le r_1 \exp\left(\int_{t_0}^c h_2(s)ds\right) + \int_{t_0}^c h_0(s)\exp\left(\int_{t_0}^s h_2(\xi)d\xi\right)ds,$$

whence by a direct calculation, with respect to (2.18) and (2.19), we obtain

$$|u'(t_0)| \le \rho_0 \left[\frac{b-a}{(t_0-a)(b-c)}\right]^{\lambda_0} \left(r_1(c-a)^{\lambda_0} + \int_{t_0}^c h_0(s)(s-a)^{\lambda_0}ds\right). \tag{2.23}$$

Multiplying both sides of (2.23) by $(t_0-t_1)^{1+\lambda_0}$ we find

$$|u'(t_0)|(t_0-t_1)^{1+\lambda_0} \le \rho_0 \left[\frac{(b-a)(c-a)}{b-c}\right]^{\lambda_0} \left(r_1(c-a) + \int_a^c h_0(s)(s-a)ds\right). \tag{2.24}$$

Therefore, on account of (2.21) and (2.24), in both cases (2.7) and (2.8) we have

$$|u'(t_0)|(t_0-t_1)^{1+\lambda_0} \le r_a$$

where

$$r_a = \frac{\rho_0^2(1+\lambda_0)}{1-\lambda_0}\left(\frac{b-a}{b-c}\right)^{2\lambda_0}(c-a)^{\lambda_0}\left(2r_0 + r_1(c-a) + \int_a^c h_0(s)(s-a)ds\right).$$

The estimate (2.17) can be proven analogously.

Lemma 2.3 *Let $r_0 > 0$, $\lambda \in [0, b-a)$, $h_1 \in L\big([a,b];\mathbb{R}_+\big)$, and let $h_0 \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ satisfy* (2.1). *Then there exist functions $H_1 \in C\big((a,c];\mathbb{R}_+\big)$ and $H_2 \in C\big([c,b);\mathbb{R}_+\big)$ satisfying*

$$H_1(a+) = 0, \qquad H_2(b-) = 0 \tag{2.25}$$

such that, for any closed interval $[t_1,t_2] \subset (a,b)$ with $c \in (t_1,t_2)$ and any function $u \in AC^1\big([t_1,t_2];\mathbb{R}\big)$, the inequalities (2.3) *and* (2.14) *imply the estimates*

$$u(t) \le u(t_1) + H_1(t) \qquad \textit{for } t \in [t_1, c], \tag{2.26}$$
$$u(t) \ge u(t_2) - H_2(t) \qquad \textit{for } t \in [c, t_2]. \tag{2.27}$$

First we show the existence of $H_1 \in C\big((a,c];\mathbb{R}_+\big)$ satisfying (2.25) and (2.26). Put $\lambda_0 = \lambda/(b-a)$ and define the operator $\sigma : L_{loc}\big((a,b);\mathbb{R}\big) \to C_{loc}\big((a,b);\mathbb{R}\big)$ by

$$\sigma(p)(t) \overset{def}{=} \exp\left(\int_c^t p(s)ds\right) \qquad \text{for } t \in (a,b), \qquad p \in L_{loc}\big((a,b);\mathbb{R}\big). \tag{2.28}$$

Further, let h_2 be defined by (2.18) and let

$$\psi(t) = h_2(t)\operatorname{sgn} u'(t) \qquad \text{for a. e. } t \in (a,b). \tag{2.29}$$

Note that both $\sigma(\psi)$ and $\sigma(-\psi)$ belong to $L\big([a,b];\mathbb{R}_+\big)$ and the following relations are fulfilled:

$$\frac{\int_{t_1}^{t}\sigma(-\psi)(s)ds}{\int_{t_1}^{c}\sigma(-\psi)(s)ds}\le\frac{\int_{a}^{t}\sigma(-\psi)(s)ds}{\int_{a}^{c}\sigma(-\psi)(s)ds}\qquad\text{for } a\le t_1\le t\le c, \tag{2.30}$$

$$\sigma(\psi)(s)\int_a^t\sigma(-\psi)(\xi)d\xi=\exp\left(\int_t^s\psi(\xi)d\xi\right)\int_a^t\exp\left(\int_\xi^t\psi(\eta)d\eta\right)d\xi\qquad\text{for } a\le t\le s\le c. \tag{2.31}$$

From (2.31) it follows that

$$\sigma(\psi)(s)\int_a^t\sigma(-\psi)(\xi)d\xi\le\sigma(h_2)(s)\int_a^t\sigma(-h_2)(\xi)d\xi\qquad\text{for } a\le t\le s\le c. \tag{2.32}$$

Put

$$\Phi(\psi)(t_1,t)\overset{def}{=}\int_t^c\sigma(-\psi)(s)ds\int_{t_1}^t h_0(s)\sigma(\psi)(s)\int_{t_1}^s\sigma(-\psi)(\xi)d\xi ds$$
$$+\int_{t_1}^t\sigma(-\psi)(s)ds\int_t^c h_0(s)\sigma(\psi)(s)\int_s^c\sigma(-\psi)(\xi)d\xi ds\qquad\text{for } a\le t_1\le t\le c,$$

Note that from (2.1), (2.28), and (2.29) it follows that $\Phi(\psi)$ is correctly defined also for $t_1=a$. Moreover, on account of (2.30) and (2.32) we have

$$\frac{\Phi(\psi)(t_1,t)}{\int_{t_1}^c\sigma(-\psi)(s)ds}\le\frac{\Phi(\psi)(a,t)}{\int_a^c\sigma(-\psi)(s)ds}\le\frac{\Phi(h_2)(a,t)}{\int_a^c\sigma(h_2)(s)ds}\qquad\text{for } a\le t_1\le t\le c. \tag{2.33}$$

Note that h_2 does not depend on u, and thus we can define

$$H_1(t)\overset{def}{=}2r_0\frac{\int_a^t\sigma(-h_2)(s)ds}{\int_a^c\sigma(h_2)(s)ds}+\frac{\Phi(h_2)(a,t)}{\int_a^c\sigma(h_2)(s)ds}\qquad\text{for } a<t\le c. \tag{2.34}$$

We can easily verified that $H_1\in C\big((a,c];\mathbb{R}_+\big)$ and it satisfies (2.25).

On the other hand, from (2.14) we have

$$u(t)\le u(t_1)+[u(c)-u(t_1)]\frac{\int_{t_1}^t\sigma(-\psi)(s)ds}{\int_{t_1}^c\sigma(-\psi)(s)ds}+\frac{\Phi(\psi)(t_1,t)}{\int_{t_1}^c\sigma(-\psi)(s)ds}\qquad\text{for } t\in[t_1,c]. \tag{2.35}$$

Now, using (2.3), (2.28)–(2.30), (2.33), and (2.34) in (2.35), we get (2.26).

The existence of a function $H_2\in C\big([c,b);\mathbb{R}_+\big)$ satisfying (2.25) and (2.27) can be proven analogously.

Lemma 2.4 *Let $c\in(a,b)$, $\lambda\in[0,b-a)$, $r_1>0$, $h_1\in L\big([a,b];\mathbb{R}_+\big)$, and let $h_0\in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with the property* (2.1). *Then there exists a function $\rho\in C\big((a,b);\mathbb{R}_+\big)\cap L\big([a,b];\mathbb{R}_+\big)$ such that the estimate*

$$|u'(t)|\le\rho(t)\qquad\textit{for } t\in[t_1,t_2]$$

holds whenever $t_1\in(a,c)$, $t_2\in(c,b)$, and the function $u\in AC^1\big([t_1,t_2];\mathbb{R}\big)$ satisfies the inequalities (2.15) *and*

$$|u'(t)|'\operatorname{sgn}(t-c)\le h_0(t)+\left[\frac{\lambda}{(b-t)(t-a)}+h_1(t)\right]|u'(t)|\qquad\textit{for a. e. } t\in(t_1,t_2). \tag{2.36}$$

Define ρ_0 and λ_0 by (2.19) and put

$$\rho_a(t) = \rho_0 \left(\frac{b-t}{t-a}\right)^{\lambda_0} \left[r_1 \left(\frac{c-a}{b-c}\right)^{\lambda_0} + \int_t^c \left(\frac{s-a}{b-s}\right)^{\lambda_0} h_0(s)ds \right] \qquad \text{for } t \in (a,c],$$

$$\rho_b(t) = \rho_0 \left(\frac{t-a}{b-t}\right)^{\lambda_0} \left[r_1 \left(\frac{b-c}{c-a}\right)^{\lambda_0} + \int_c^t \left(\frac{b-s}{s-a}\right)^{\lambda_0} h_0(s)ds \right] \qquad \text{for } t \in [c,b),$$

and

$$\rho(t) = \begin{cases} \rho_a(t) & \text{for } t \in (a,c], \\ \rho_b(t) & \text{for } t \in (c,b). \end{cases}$$

In view of the integrability of h_0 with the weight $(b-t)(t-a)$ it is clear that $\rho \in C\big((a,b);\mathbb{R}_+\big) \cap L\big([a,b];\mathbb{R}_+\big)$.

Suppose that the lemma is false. Then there exist $t_1 \in (a,c)$, $t_2 \in (c,b)$, a function $u \in AC^1\big([t_1,t_2];\mathbb{R}\big)$ satisfying (2.15) and (2.36), and a point $t_0 \in [t_1,t_2]$ such that

$$|u'(t_0)| > \rho(t_0). \tag{2.37}$$

Clearly, $t_0 \neq c$. First assume that $t_0 < c$. Define h_2 by (2.18). Then from (2.36) we get

$$|u'(t)|' \geq -h_0(t) - h_2(t)|u'(t)| \qquad \text{for a. e. } t \in (t_0,c),$$

and consequently,

$$|u'(t_0)| \leq |u'(c)| \exp\left(\int_{t_0}^c h_2(\xi)d\xi\right) + \int_{t_0}^c h_0(s) \exp\left(\int_{t_0}^s h_2(\xi)d\xi\right) ds. \tag{2.38}$$

According to (2.18) and (2.19) we have

$$\int_{t_0}^s h_2(\xi)d\xi = \lambda_0 \ln \frac{(b-t_0)(s-a)}{(b-s)(t_0-a)} + \int_{t_0}^s h_1(\xi)d\xi \leq \lambda_0 \ln \frac{(b-t_0)(s-a)}{(b-s)(t_0-a)} + \ln \rho_0 \qquad \text{for } s \in [t_0,c],$$

by virtue of which, using (2.15), we obtain $|u'(t_0)| \leq \rho(t_0)$ from (2.38). However, the latter inequality contradicts (2.37).

If $t_0 > c$, the contradiction can be obtained analogously.

3 Auxiliary Propositions

The following two lemmas deal with the existence of a solution to Eq. (1.5) satisfying the boundary conditions

$$u(a+) = c_1, \qquad u(b-) = c_2 \tag{3.1}$$

in the case where $h \in \mathrm{Car}_{loc}\big((a,b)\times\mathbb{R}^2;\mathbb{R}\big)$. The first one is a simple modification of Scorza Dragoni theorem and its proof can be found in [9].

Lemma 3.1 *Let σ_1 and σ_2 be, respectively, lower and upper functions to Eq.* (1.5) *such that*

$$\sigma_1(t) \leq \sigma_2(t) \qquad \text{for } t \in (a,b) \tag{3.2}$$

and

$$|h(t,x,y)| \leq q(t) \qquad \text{for a. e. } t \in (a,b), \quad \sigma_1(t) \leq x \leq \sigma_2(t), \quad y \in \mathbb{R},$$

where $q \in L\big([a,b];\mathbb{R}_+\big)$. *Then, for every* $c_1 \in [\sigma_1(a+),\sigma_2(a+)]$ *and* $c_2 \in [\sigma_1(b-),\sigma_2(b-)]$, *the problem* (1.5), (3.1) *has a solution* $u \in AC^1_{loc}\big((a,b);\mathbb{R}\big)$ *satisfying*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \qquad \text{for } t \in (a,b). \tag{3.3}$$

Lemma 3.2 *Let* σ_1 *and* σ_2 *be, respectively, lower and upper functions to Eq.* (1.5) *satisfying* (3.2). *Further suppose that*

$$h(t,x,y)\,\mathrm{sgn}\big[y(t-c)\big] \le h_0(t) + \left[\frac{\lambda}{(b-t)(t-a)} + h_1(t)\right]|y| \qquad \text{for a. e. } t \in (a,b),$$
$$\sigma_1(t) \le x \le \sigma_2(t), \quad y \in \mathbb{R}, \tag{3.4}$$

$$h(t,x,y) \ge -h_0(t) - h_2(t)|y| \qquad \text{for a. e. } t \in (\alpha,\beta), \quad \sigma_1(t) \le x \le \sigma_2(t), \quad y \in \mathbb{R}, \tag{3.5}$$

where $a < \alpha < c < \beta < b$, $\lambda \in [0, b-a)$, $h_1, h_2 \in L\big([a,b];\mathbb{R}_+\big)$, *and* $h_0 \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ *with the property* (2.1). *Then, for every* $c_1 \in [\sigma_1(a+),\sigma_2(a+)]$ *and* $c_2 \in [\sigma_1(b-),\sigma_2(b-)]$, *the problem* (1.5), (3.1) *has a solution* u *satisfying* (3.3).

Put

$$r_0 = \sup\big\{|\sigma_1(t)| + |\sigma_2(t)| + 1 : t \in (a,b)\big\}, \tag{3.6}$$

and define a number r^* by (2.13). Moreover, let

$$r_1 = \frac{r^*}{(\beta-c)(c-\alpha)} \tag{3.7}$$

and let $\rho \in C\big((a,b);\mathbb{R}_+\big) \cap L\big([a,b];\mathbb{R}_+\big)$ be such that the conclusion of Lemma 2.4 holds. Consider the equation

$$u'' = \chi(t,u')h(t,u,u'), \tag{3.8}$$

where

$$\chi(t,y) = \begin{cases} 1 & \text{for } t \in (a,b),\ |y| \le \rho_1(t), \\ 2 - \dfrac{|y|}{\rho_1(t)} & \text{for } t \in (a,b),\ \rho_1(t) < |y| < 2\rho_1(t), \\ 0 & \text{for } t \in (a,b),\ 2\rho_1(t) \le |y|, \end{cases} \tag{3.9}$$

and

$$\rho_1(t) = \rho(t) + |\sigma_1'(t)| + |\sigma_2'(t)| + 1 \qquad \text{for } t \in (a,b). \tag{3.10}$$

Let

$$t_{1n} \in (a,\alpha)\,, \quad t_{2n} \in (\beta,b)\,, \quad c_{jn} \in [\sigma_1(t_{jn}),\sigma_2(t_{jn})] \qquad \text{for } n \in \mathbb{N} \quad (j=1,2)$$

be such that

$$\lim_{n\to+\infty} t_{1n} = a, \qquad \lim_{n\to+\infty} t_{2n} = b, \qquad \lim_{n\to+\infty} c_{jn} = c_j, \qquad (j=1,2). \tag{3.11}$$

By Lemma 3.1, for any $n \in \mathbb{N}$ the equation (3.8) has a solution u_n defined on $[t_{1n}, t_{2n}]$ such that

$$u_n(t_{1n}) = c_{1n}, \qquad u_n(t_{2n}) = c_{2n}, \tag{3.12}$$
$$\sigma_1(t) \le u_n(t) \le \sigma_2(t) \qquad \text{for } t \in [t_{1n}, t_{2n}]. \tag{3.13}$$

In view of (3.5), (3.7), (3.8), and (3.13), for any $n \in \mathbb{N}$ the function $u \equiv u_n$ satisfies inequalities (2.2) and (2.3) with $t_1 = \alpha$, $t_2 = \beta$. Therefore, according to Lemma 2.1, with respect to (3.7), we have the estimate (2.15). Furthermore, (3.4), (3.8), and (3.13) imply (2.36) with $t_1 = t_{1n}$, $t_2 = t_{2n}$, and consequently, according to Lemma 2.4 we have

$$|u_n'(t)| \leq \rho(t) \qquad \text{for } t \in [t_{1n}, t_{2n}], \qquad n \in \mathbb{N}. \tag{3.14}$$

Thus every u_n is a solution to Eq. (1.5) on the closed interval $[t_{1n}, t_{2n}]$. On the other hand, with respect to (3.14), for every $n \in \mathbb{N}$ we have

$$|u_n(t) - c_{1n}| \leq \int_a^t \rho(s)ds, \qquad |u_n(t) - c_{2n}| \leq \int_t^b \rho(s)ds \qquad \text{for } t \in [t_{1n}, t_{2n}]. \tag{3.15}$$

According to Arzelá–Ascoli lemma, in view of (1.5), (3.13), and (3.14) we can assume without loss of generality that

$$\lim_{n\to+\infty} u_n = u \qquad \text{uniformly on every compact interval}$$

and u is a solution to (1.5) on (a, b). Moreover, from (3.11)–(3.13) and (3.15) we get (3.1) and (3.3).

Lemma 3.3 *Let σ_1 and σ_2 be, respectively, lower and upper functions to Eq.* (1.5) *satisfying* (3.2) *and*

$$\sigma_1(a+) = c_1, \qquad \sigma_2(b-) = c_2. \tag{3.16}$$

Further suppose that (3.5) *holds and*

$$h(t, x, y)\,\mathrm{sgn}(t - c) \leq h_0(t) + \left[\frac{\lambda}{(b - t)(t - a)} + h_1(t)\right]|y| \qquad \text{for a. e. } t \in (a, b),$$
$$\sigma_1(t) \leq x \leq \sigma_2(t), \quad y \in \mathbb{R}, \tag{3.17}$$

where $a < \alpha < c < \beta < b$, $\lambda \in [0, b - a)$, $h_1, h_2 \in L\big([a, b]; \mathbb{R}_+\big)$, and $h_0 \in L_{loc}\big((a, b); \mathbb{R}_+\big)$ with the property (2.1)*. Then the problem* (1.5), (3.1) *has a solution u satisfying* (3.3)*.*

Define a numbers r_0, r^*, and r_1 by (3.6), (2.13), and (3.7), respectively, and let r_a and r_b be positive constants such that the conclusion of Lemma 2.2 holds. Choose sequences $\{t_{jn}\}_{n=1}^{+\infty}$ and $\{s_{jn}\}_{n=0}^{+\infty}$ $(j = 1, 2)$ such that

$$a < t_{1n+1} < t_{1n} < \alpha < \beta < t_{2n} < t_{2n+1} < b \qquad \text{for } n \in \mathbb{N},$$
$$s_{1n+1} \in (t_{1n+1}, t_{1n}), \qquad s_{2n+1} \in (t_{2n}, t_{2n+1}) \qquad \text{for } n \in \mathbb{N},$$
$$s_{11} \in (t_{11}, \alpha), \qquad s_{10} \in (\alpha, c), \qquad s_{20} \in (c, \beta), \qquad s_{21} \in (\beta, t_{21}),$$
$$\lim_{n\to+\infty} t_{1n} = a, \qquad \lim_{n\to+\infty} t_{2n} = b. \tag{3.18}$$

Put

$$\rho(t) = \begin{cases} \dfrac{r_a}{(t - t_{1n})^{1+\lambda_0}} & \text{for } t \in [s_{1n}, s_{1n-1}), \\ \dfrac{r^*}{(\beta - t)(t - \alpha)} & \text{for } t \in [s_{10}, s_{20}], \\ \dfrac{r_b}{(t_{2n} - t)^{1+\lambda_0}} & \text{for } t \in (s_{2n-1}, s_{2n}], \end{cases} \qquad n \in \mathbb{N},$$

where $\lambda_0 = \lambda/(b-a)$, and define ρ_1 by (3.10). Obviously, $\rho,\ \rho_1 \in L^\infty_{loc}\big((a,b);\mathbb{R}_+\big)$. Consider the equation (3.8) with χ given by (3.9). By Lemma 3.1, for any $n \in \mathbb{N}$ the equation (3.8) has a solution u_n defined on $[t_{1n}, t_{2n}]$ such that (3.13) holds and

$$u_n(t_{1n}) = \sigma_1(t_{1n}), \qquad u_n(t_{2n}) = \sigma_2(t_{2n}), \tag{3.19}$$

In view of (3.5), (3.6), (3.8), and (3.13), for any $n \in \mathbb{N}$ the function $u \equiv u_n$ satisfies inequalities (2.2) and (2.3) with $t_1 = \alpha$, $t_2 = \beta$. Therefore, according to Lemma 2.1, with respect to (3.7), we have the estimate (2.15). Furthermore, (3.6), (3.8), (3.13), and (3.17) imply (2.3) and (2.14) with $t_1 = t_{1n}$, $t_2 = t_{2n}$, and consequently, according to Lemmas 2.1 and 2.2 we have

$$|u_n'(t)| \le \rho(t) \qquad \text{for } t \in [s_{1n}, s_{2n}], \qquad n \in \mathbb{N}. \tag{3.20}$$

Thus every u_n is a solution to Eq. (1.5) on the closed interval $[s_{1n}, s_{2n}]$. On the other hand, according to (3.19) and Lemma 2.3, for every $n \in \mathbb{N}$ we have

$$u_n(t) \le \sigma_1(t_{1n}) + H_1(t) \qquad \text{for } t \in [t_{1n}, c], \tag{3.21}$$
$$u_n(t) \ge \sigma_2(t_{2n}) - H_2(t) \qquad \text{for } t \in [c, t_{2n}] \tag{3.22}$$

with suitable $H_1 \in C\big((a,c];\mathbb{R}_+\big)$ and $H_2 \in C\big([c,b);\mathbb{R}_+\big)$ satisfying (2.25). According to Arzelá–Ascoli theorem, in view of (1.5), (3.13), and (3.20) we can assume without loss of generality that

$$\lim_{n\to+\infty} u_n = u \qquad \text{uniformly on every compact interval}$$

and u is a solution to (1.5) on (a,b). Moreover, from (2.25), (3.13), (3.16), (3.18), (3.21), and (3.22) we get (3.1) and (3.3).

Lemma 3.4 *Let $f, g \in \mathrm{Car}_{loc}\big((a,b)\times(0,1);\mathbb{R}\big)$, and let σ_1 and σ_2 be, respectively, lower and upper functions to Eq.* (1.1) *satisfying* (1.6) *and*

$$\sigma_1(a+) = 0, \qquad 0 < \sigma_2(a+) < 1. \tag{3.23}$$

Further suppose that, for every $\eta \in (0, 1/2)$, the relations (1.8) *hold, where $c \in (a,b)$, $\lambda_\eta \in [0, b-a)$, $q_\eta \in L\big([a,b];\mathbb{R}_+\big)$, $p_\eta \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with the property* (1.9), *and $\sigma_{1\eta}$ and $\sigma_{2\eta}$ are given by* (1.10). *Then, for every $b_0 \in (c,b)$, Eq.* (1.1) *has a solution u (defined on (a,b_0)) satisfying*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \qquad \textit{for } t \in (a, b_0), \tag{3.24}$$
$$u(a+) = 0, \qquad u(b_0-) = \sigma_2(b_0), \tag{3.25}$$

and possessing a finite limit $u'(b_0-)$.

Let $b_0 \in (c,b)$ be arbitrary but fixed, and let $\eta_0 \in (0, 1/2)$ and $t_{10} \in (a,c)$ be such that

$$\eta_0 < \sup\big\{\sigma_1(t) : t \in (a, b_0]\big\}, \qquad \eta_0 < \min\big\{\sigma_{2*}, 1 - \sigma_2^*\big\}, \tag{3.26}$$

where

$$\sigma_{2*} = \inf\big\{\sigma_2(t) : t \in (a, b_0]\big\}, \qquad \sigma_2^* = \sup\big\{\sigma_2(t) : t \in (a, b_0]\big\},$$

and

$$\sigma_1(t) < \sigma_1(t_{10}) \qquad \text{for } t \in (a, t_{10}), \qquad \sigma_1(t_{10}) = \eta_0. \tag{3.27}$$

Let $w \in AC^1_{loc}\big((a,t_{10}];\mathbb{R}\big)$ be a solution to

$$w''(t) = -p_{\eta_0}(t) - \left[\frac{\lambda_{\eta_0}}{(b-t)(t-a)} + q_{\eta_0}(t)\right] w'(t) \qquad \text{for a. e. } t \in (a,t_{10}), \tag{3.28}$$

$$w(a+) = \eta_0, \qquad w(t_{10}-) = 1. \tag{3.29}$$

Such a solution exists according to [9, Theorem 1.1]. Obviously, there exists $a_1 \in (a,t_{10})$ such that

$$w(t) < \sigma_2(t), \quad w'(t) > 0 \quad \text{for } t \in (a,a_1), \quad w(a_1) = \sigma_2(a_1). \tag{3.30}$$

Moreover, in view of (3.26), we have

$$\eta_0 \le w(t) \le 1-\eta_0 \qquad \text{for } t \in (a,a_1], \qquad w'(a_1) \ge \sigma_2'(a_1+). \tag{3.31}$$

Therefore, from (1.8), (3.27), and (3.28)–(3.31) it follows that

$$w''(t) \le f(t,w(t)) + g(t,w(t))w'(t) \qquad \text{for a. e. } t \in (a,a_1). \tag{3.32}$$

Moreover, from (3.28) and (3.30) it follows that $w''(t) \le 0$ for almost every $t \in (a,a_1)$. Thus (3.32) yields

$$w''(t) \le -\big[f(t,w(t)) + g(t,w(t))w'(t)\big]_- \qquad \text{for a. e. } t \in (a,a_1).$$

Let $t_{1n} \in (a,a_1]$, $n \in \mathbb{N}$, be such that

$$t_{1n+1} < t_{1n}, \qquad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} t_{1n} = a, \tag{3.33}$$
$$\sigma_1(t) < \sigma_1(t_{1n}) \qquad \text{for } t \in (a,t_{1n}), \quad n \in \mathbb{N},$$

and, for every $n \in \mathbb{N}$, put

$$\eta_n = \min\big\{\sigma_1(t) : t \in [t_{1n},b_0]\big\}, \qquad \chi_n(x) = \begin{cases} \eta_n & \text{for } x < \eta_n, \\ x & \text{for } \eta_n \le x \le 1-\eta_n, \\ 1-\eta_n & \text{for } 1-\eta_n < x, \end{cases}$$

$$F_n(t,x,y) = \begin{cases} -\big[f(t,\chi_n(x)) + g(t,\chi_n(x))y\big]_- & \text{for a. e. } t \in (a,t_{1n}), \ x,y \in \mathbb{R}, \\ f(t,\chi_n(x)) + g(t,\chi_n(x))y & \text{for a. e. } t \in (t_{1n},b_0), \ x,y \in \mathbb{R}, \end{cases}$$

$$\widetilde{\sigma}_{1n}(t) = \begin{cases} \sigma_1(t_{1n}) & \text{for } t \in (a,t_{1n}), \\ \sigma_1(t) & \text{for } t \in [t_{1n},b_0), \end{cases} \qquad \widetilde{\sigma}_{21}(t) = \begin{cases} w(t) & \text{for } t \in (a,a_1), \\ \sigma_2(t) & \text{for } t \in [a_1,b_0). \end{cases}$$

Then, obviously, $\widetilde{\sigma}_{1n}$ for $n \in \mathbb{N}$ are lower functions to the equation

$$u'' = F_n(t,u,u') \tag{3.34}$$

and $\widetilde{\sigma}_{21}$ is an upper function to the equation

$$u'' = F_1(t,u,u'). \tag{3.35}$$

Put

$$\alpha = \frac{a_1 + c}{2}, \qquad \beta = \frac{c + b_0}{2}, \qquad \lambda = \frac{\lambda_{\eta_1}(b_0 - a)}{b - a}, \tag{3.36}$$

$$h_2(t) = \begin{cases} \sup\big\{|g(t,x)| : \sigma_1(t) \le x \le \sigma_2(t)\big\} & \text{for a. e. } t \in (\alpha, \beta), \\ 0 & \text{for a. e. } t \in (a, \alpha) \cup (\beta, b_0), \end{cases} \tag{3.37}$$

$$h_0(t) = p_{\eta_1}(t), \qquad h_1(t) = q_{\eta_1}(t) \qquad \text{for a. e. } t \in (a, b_0), \tag{3.38}$$
$$h(t,x,y) = F_1(t,x,y) \qquad \text{for a. e. } t \in (a, b_0), \quad x, y \in \mathbb{R}.$$

Then the assumptions of Lemma 3.2 are fulfilled, and thus there exists a solution u_1 to Eq. (3.35) on the interval (a, b_0) such that

$$\widetilde{\sigma}_{11}(t) \le u_1(t) \le \widetilde{\sigma}_{21}(t) \qquad \text{for } t \in (a, b_0),$$
$$u_1(a+) = \sigma_1(t_{11}), \qquad u_1(b_0-) = \sigma_2(b_0).$$

Now, assuming the existence of a solution u_k for some $k \in \mathbb{N}$ to Eq. (3.34) with $n = k$ on the interval (a, b_0) with the properties

$$\widetilde{\sigma}_{1k}(t) \le u_k(t) \le \widetilde{\sigma}_{2k}(t) \qquad \text{for } t \in (a, b_0),$$
$$u_k(a+) = \sigma_1(t_{1k}), \qquad u_k(b_0-) = \sigma_2(b_0),$$

we define an upper function $\widetilde{\sigma}_{2k+1}$ to Eq. (3.34) with $n = k + 1$ by

$$\widetilde{\sigma}_{2k+1}(t) = u_k(t) \qquad \text{for } t \in (a, b_0).$$

Then we obtain, in view of Lemma 3.2 again with $\lambda = \lambda_{\eta_n}(b_0 - a)/(b - a)$,

$$h_0(t) = p_{\eta_n}(t), \qquad h_1(t) = q_{\eta_n}(t) \qquad \text{for a. e. } t \in (a, b_0), \tag{3.39}$$
$$h(t,x,y) = F_n(t,x,y) \qquad \text{for a. e. } t \in (a, b_0), \quad x, y \in \mathbb{R},$$

and α, β, and h_2 defined by (3.36) and (3.37), respectively, that there exists a solution u_{k+1} to Eq. (3.34) with $n = k + 1$ on the interval (a, b_0) such that

$$\widetilde{\sigma}_{1k+1}(t) \le u_{k+1}(t) \le \widetilde{\sigma}_{2k+1}(t) \qquad \text{for } t \in (a, b_0),$$
$$u_{k+1}(a+) = \sigma_1(t_{1k+1}), \qquad u_{k+1}(b_0-) = \sigma_2(b_0).$$

Thus we have a sequence $\{u_n\}_{n=1}^{+\infty}$ of solutions to Eq. (3.34) with

$$\sigma_1(t) \le u_{n+1}(t) \le u_n(t) \le \sigma_2(t) \qquad \text{for } t \in (a, b_0), \quad n \in \mathbb{N}, \tag{3.40}$$
$$u_n(a+) = \sigma_1(t_{1n}), \qquad u_n(b_0-) = \sigma_2(b_0), \quad n \in \mathbb{N}. \tag{3.41}$$

According to (3.34), (3.40), and Lemmas 2.1 and 2.4 with $b = b_0$, the sequences $\{u_n\}_{n=1}^{+\infty}$ and $\{u_n'\}_{n=1}^{+\infty}$ are uniformly bounded and equicontinuous on every compact interval of (a, b_0). Therefore, there exists $u_0 \in AC^1_{loc}\big((a, b_0); \mathbb{R}\big)$ such that

$$u_0 = \lim_{n \to +\infty} u_n \qquad \text{uniformly on every compact interval,}$$

u_0 is a solution to Eq. (1.1),

$$\sigma_1(t) \le u_0(t) \le \sigma_2(t) \qquad \text{for } t \in (a, b_0), \qquad u_0(b_0-) = \sigma_2(b_0), \tag{3.42}$$

and, in view of (3.40) and (3.41), we have

$$0 \le \liminf_{t\to a+} u_0(t) \le \limsup_{t\to a+} u_0(t) \le \sigma_1(t_{1n}) \qquad \text{for } n \in \mathbb{N}.$$

Now (3.23) and (3.33) yields $u_0(a+) = 0$.

Because u_0 is a solution to Eq. (1.1) also on (c, b_0) and the functions f and g are from the Carathéodory class, with respect to (3.42) and the continuity of u_0' on (c, b_0), the existence of a finite limit $u_0'(b_0-)$ is obvious.

Lemma 3.5 *Let $f, g \in \mathrm{Car}_{loc}\big((a,b)\times(0,1);\mathbb{R}\big)$, and let σ_1 and σ_2 be, respectively, lower and upper functions to Eq.* (1.1) *satisfying* (1.6) *and* (3.23)*. Further suppose that, for every $\eta \in (0,1/2)$, the relations* (1.14) *hold, where $c \in (a,b)$, $\lambda_\eta \in [0, b-a)$, $q_\eta \in L\big([a,b];\mathbb{R}_+\big)$, $p_\eta \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with the property* (1.9)*, and $\sigma_{1\eta}$ and $\sigma_{2\eta}$ are given by* (1.10)*. Then, for every $b_0 \in (c,b)$, Eq.* (1.1) *has a solution u (defined on (a,b_0)) satisfying* (3.24) *and* (3.25)*, and possessing a finite limit $u'(b_0-)$.*

The proof of the lemma is almost identical to the proof of the previous one. The only difference is the definition of the function h_0 in (3.38), resp. (3.39), when it is given by

$$h_0(t) = p_{\eta_1}(t) + p_0(t), \qquad \text{resp.} \qquad h_0(t) = p_{\eta_n}(t) + p_0(t) \qquad \text{for a. e. } t \in (a, b_0),$$

where

$$p_0(t) = \begin{cases} \sup\big\{|f(t,x)| : \sigma_1(t) \le x \le \sigma_2(t)\big\} & \text{for a. e. } t \in (c, b_0), \\ 0 & \text{for a. e. } t \in (a, c), \end{cases}$$

and that Lemmas 3.3 and 2.2 are used instead of Lemmas 3.2 and 2.4.

Analogously to the proofs of Lemmas 3.4 and 3.5, one can prove the following two assertions.

Lemma 3.6 *Let $f, g \in \mathrm{Car}_{loc}\big((a,b)\times(0,1);\mathbb{R}\big)$, and let σ_1 and σ_2 be, respectively, lower and upper functions to Eq.* (1.1) *satisfying* (1.6) *and*

$$\sigma_2(b-) = 1, \qquad 0 < \sigma_1(b-) < 1. \tag{3.43}$$

Further suppose that, for every $\eta \in (0,1/2)$, the inequalities (1.8) *are fulfilled, where $c \in (a,b)$, $\lambda_\eta \in [0, b-a)$, $q_\eta \in L\big([a,b];\mathbb{R}_+\big)$, $p_\eta \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with the property* (1.9)*, and $\sigma_{1\eta}$ and $\sigma_{2\eta}$ are given by* (1.10)*. Then, for every $a_0 \in (a,c)$, Eq.* (1.1) *has a solution u (defined on (a_0,b)) satisfying*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \qquad \text{for } t \in (a_0, b), \tag{3.44}$$

$$u(a_0+) = \sigma_1(a_0), \qquad u(b-) = 1, \tag{3.45}$$

and possessing a finite limit $u'(a_0+)$.

Lemma 3.7 *Let $f, g \in \mathrm{Car}_{loc}\big((a,b)\times(0,1);\mathbb{R}\big)$, and let σ_1 and σ_2 be, respectively, lower and upper functions to Eq.* (1.1) *satisfying* (1.6) *and* (3.43)*. Further suppose that, for every $\eta \in (0,1/2)$, the inequalities* (1.14) *are fulfilled, where $c \in (a,b)$, $\lambda_\eta \in [0, b-a)$, $q_\eta \in L\big([a,b];\mathbb{R}_+\big)$, $p_\eta \in L_{loc}\big((a,b);\mathbb{R}_+\big)$ with the property* (1.9)*, and $\sigma_{1\eta}$ and $\sigma_{2\eta}$ are given by* (1.10)*. Then, for every $a_0 \in (a,c)$, Eq.* (1.1) *has a solution u (defined on (a_0,b)) satisfying* (3.44) *and* (3.45)*, and possessing a finite limit $u'(a_0+)$.*

4 Proofs

[Proof of Theorem 1.1] If $\sigma_2(a+) > 0$ and/or $\sigma_1(b-) < 1$ then, according to Lemmas 3.4 and 3.6, there exist solutions v and w to Eq. (1.1) defined on (a, b_0) and (a_0, b), respectively, with

$$a_0 = \frac{a+c}{2}, \qquad b_0 = \frac{c+b}{2},$$
$$v(a+) = 0, \qquad v(b_0-) = \sigma_2(b_0), \qquad w(a_0+) = \sigma_1(a_0), \qquad w(b-) = 1,$$
$$v'(b_0-) \geq \sigma_2'(b_0+), \qquad w'(a_0+) \geq \sigma_1'(a_0-).$$

Therefore, without loss of generality we can assume that

$$\sigma_2(a+) = 0, \qquad \sigma_1(b-) = 1. \tag{4.1}$$

Let

$$h(t, x, y) = f(t, x) + g(t, x)y \qquad \text{for a. e. } t \in (a, b), \quad x, y \in \mathbb{R},$$

and let $t_{1n} \in (a, c)$ and $t_{2n} \in (c, b)$, $n \in \mathbb{N}$, be such that

$$t_{1n+1} < t_{1n}, \qquad t_{2n} < t_{2n+1} \qquad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} t_{1n} = a, \qquad \lim_{n\to+\infty} t_{2n} = b.$$

It is clear that there exists a sequence $\{\eta_n\}_{n=1}^{+\infty}$ of numbers from $(0, 1/2)$ such that

$$\eta_n \leq \sigma_1(t) \leq \sigma_2(t) \leq 1 - \eta_n \qquad \text{for } t \in [t_{1n}, t_{2n}], \quad n \in \mathbb{N}.$$

Therefore, for every $n \in \mathbb{N}$, the inequalities (1.8) imply the validity of the appropriate inequalities (3.4) and (3.5) on the interval $[t_{1n}, t_{2n}]$. Then, according to Lemma 3.2, for every $n \in \mathbb{N}$ there exists a solution u_n to the equation (1.1), defined on (t_{1n}, t_{2n}), such that

$$u_n(t_{1n}+) = \sigma_1(t_{1n}), \qquad u_n(t_{2n}-) = \sigma_2(t_{2n}),$$
$$\sigma_1(t) \leq u_n(t) \leq \sigma_2(t) \qquad \text{for } t \in (t_{1n}, t_{2n}). \tag{4.2}$$

According to (1.8) and Lemmas 2.1 and 2.4, for every compact interval $[s_1, s_2] \subset (a, b)$ with $c \in (s_1, s_2)$, there exists $n_0 \in \mathbb{N}$ such that the sequences $\{u_n\}_{n=n_0}^{+\infty}$ and $\{u_n'\}_{n=n_0}^{+\infty}$ are uniformly bounded and equicontinuous on $[s_1, s_2]$. Therefore, without loss of generality we can assume that there exists $u \in AC^1_{loc}\big((a, b); \mathbb{R}\big)$ such that

$$u = \lim_{n\to+\infty} u_n \qquad \text{uniformly on every compact interval,}$$

and u is a solution to (1.1). Moreover, from (1.7), (4.1), and (4.2) it follows that u satisfies (1.2) and (1.11).

Theorem 1.2 can be proven analogously. The only difference is that Lemmas 2.2, 3.3, 3.5, and 3.7 are used instead of Lemmas 2.4, 3.2, 3.4, and 3.6, respectively.

To prove Corollary 1.1 we need the following lemmas:

Lemma 4.1 *Let there exist positive constants r, n, p_0 such that*

$$p(t) \geq p_0 \qquad \textit{for a. e. } t \in (a, b), \tag{4.3}$$
$$\big(|h(t)| + q(t) + |\varphi(t)|\big)\big[(b-t)(t-a)\big]^n \leq r \qquad \textit{for a. e. } t \in (a, b). \tag{4.4}$$

Then there exists a lower function σ_1 to Eq. (1.4) satisfying

$$\sigma_1(a+) = 0, \qquad \sigma_1(b-) = 0, \qquad 0 < \sigma_1(t) \leq \frac{1}{2} \qquad \textit{for } t \in (a, b). \tag{4.5}$$

Let $\delta = \frac{(b-a)^2}{4}$, $k = 1 + \frac{n}{\lambda}$. According to (4.4) there exists $r_0 > 0$ such that

$$\big(\delta^{k\nu}|h(t)| + 2^{\mu}q(t) + |\varphi(t)|\big)\big[(b-t)(t-a)\big]^n \le r_0 \qquad \text{for a. e. } t \in (a,b). \tag{4.6}$$

Choose $\varepsilon \in (0,1)$ such that

$$\varepsilon\delta^k \le \frac{1}{2}, \qquad 2\varepsilon k\delta^{k-1} \le 1, \qquad \varepsilon^{\lambda}\delta^{\lambda}\left(\delta^n + g^*\varepsilon k\delta^{\frac{(1+\lambda)n}{\lambda}}(b-a) + r_0\right) \le p_0, \tag{4.7}$$

where $g^* = \max\big\{|g(s)| : s \in [0,1]\big\}$, and put $\sigma_1(t) = \varepsilon\big[(b-t)(t-a)\big]^k$. Then, obviously,

$$0 < \sigma_1(t) \le \frac{1}{2} \quad \text{for } t \in (a,b), \qquad |\sigma_1'(t)| \le \varepsilon k\delta^{k-1}(b-a) \quad \text{for } t \in (a,b), \tag{4.8}$$

and, with respect to (4.7),

$$\sigma_1''(t) \ge -1 \qquad \text{for } t \in (a,b). \tag{4.9}$$

On the other hand, from (4.7) we obtain

$$\varepsilon^{\lambda}\delta^{\lambda}\big[(b-t)(t-a)\big]^n\left(\frac{\delta^n}{\big[(b-t)(t-a)\big]^n}\big(1 + g^*\varepsilon k\delta^{k-1}(b-a)\big) + \frac{r_0}{\big[(b-t)(t-a)\big]^n}\right) \le p_0 \qquad \text{for } t \in (a,b),$$

whence, in view of (4.6) and (4.8), we get

$$\sigma_1^{\lambda}(t)\left(1 + g(\sigma_1(t))\sigma_1'(t) + h(t)\sigma_1^{\nu}(t) + \frac{q(t)}{(1-\sigma_1(t))^{\mu}} + \varphi(t)\right) \le p_0 \qquad \text{for a. e. } t \in (a,b).$$

Consequently, the latter inequality together with (4.3) and (4.9) implies that σ_1 is a lower function to Eq. (1.4) satisfying (4.5).

Analogously one can prove

Lemma 4.2 *Let there exist positive constants r, n, q_0 such that*

$$q(t) \ge q_0 \qquad \textit{for a. e. } t \in (a,b),$$
$$\big(|h(t)| + p(t) + |\varphi(t)|\big)\big[(b-t)(t-a)\big]^n \le r \qquad \textit{for a. e. } t \in (a,b).$$

Then there exists an upper function σ_2 to Eq. (1.4) satisfying

$$\sigma_2(a+) = 1, \qquad \sigma_2(b-) = 1, \qquad \frac{1}{2} \le \sigma_2(t) < 1 \qquad \textit{for } t \in (a,b).$$

[Proof of Corollary 1.1] According to Lemmas 4.1 and 4.2, there exist lower and upper functions v_1 and v_2, respectively, to Eq. (1.4) such that

$$v_1(a+) = 0, \qquad v_1(b-) = 0, \qquad v_2(a+) = 1, \qquad v_2(b-) = 1,$$
$$0 < v_1(t) \le \frac{1}{2} \le v_2(t) < 1 \qquad \text{for } t \in (a,b).$$

On the other hand, according to [9, Theorem 1.1] there exist solutions w_1 and w_2 to the problems

$$w'' = g^*w' + 2^{\mu}q(t) + |h(t)| + |\varphi(t)|,$$
$$w(c+) = 0, \qquad w(b-) = \frac{1}{2},$$

and

$$w'' = -g^* w' - 2^\lambda p(t) - |h(t)| - |\varphi(t)|,$$
$$w(a+) = \frac{1}{2}, \qquad w(c-) = 1,$$

respectively, where $g^* = \max\{|g(s)| : s \in [0,1]\}$ and $c \in (a,b)$ is arbitrary but fixed. Obviously, there exist $b_0 \in (c,b)$ and $a_0 \in (a,c)$ such that

$$\begin{gathered}
w_1(b_0) = v_1(b_0), \qquad \frac{1}{2} > w_1(t) > v_1(t) > 0 \qquad \text{for } t \in (b_0,b),\\
w_1'(t) \geq 0 \qquad \text{for } t \in (b_0,b), \qquad w_1'(b_0+) \geq v_1'(b_0-),\\
w_2(a_0) = v_2(a_0), \qquad \frac{1}{2} < w_2(t) < v_2(t) < 1 \qquad \text{for } t \in (a,a_0),\\
w_2'(t) \geq 0 \qquad \text{for } t \in (a,a_0), \qquad w_2'(a_0-) \geq v_2'(a_0+).
\end{gathered}$$

Moreover,

$$w_1''(t) \geq g(w_1(t))w_1'(t) + h(t)w_1^\nu(t) - \frac{p(t)}{w_1^\lambda(t)} + \frac{q(t)}{(1-w_1(t))^\mu} + \varphi(t) \qquad \text{for a. e. } t \in (b_0,b),$$
$$w_2''(t) \leq g(w_2(t))w_2'(t) + h(t)w_2^\nu(t) - \frac{p(t)}{w_2^\lambda(t)} + \frac{q(t)}{(1-w_2(t))^\mu} + \varphi(t) \qquad \text{for a. e. } t \in (a,a_0).$$

Therefore, if we put

$$\sigma_1(t) = \begin{cases} v_1(t) & \text{for } t \in (a,b_0),\\ w_1(t) & \text{for } t \in [b_0,b), \end{cases} \qquad \sigma_2(t) = \begin{cases} w_2(t) & \text{for } t \in (a,a_0),\\ v_2(t) & \text{for } t \in [a_0,b), \end{cases}$$

all the assumptions of Theorem 1.1 are fulfilled.

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