Functions with average and bounded motions of a forced discontinuous oscillator

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Abstract

In this paper we prove the existence of bounded solutions in the real line for the equation $\ddot{u} + \mathrm{sign}(u) = p(t)$, where p is a function with average. Some useful density results for the space of functions with zero average are also obtained.

Key words: Forced oscillator; Landesman-Lazer condition; generating functions; twist maps; functions with average.

To Professor George Sell, in memoriam

1 Introduction

Consider the differential equation

$$\ddot{u} + c\dot{u} + g(u) = p(t) \tag{1}$$

where $c \geq 0$ is a parameter, the function $g(u)$ models saturation and the forcing term $p(t)$ is bounded, continuous and has an average $\bar{p} \in \mathbb{R}$. The notion of average is classical for almost periodic functions but, as noticed in [1], it can be extended to the class of functions such that the limit

$$\bar{p} = \lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} p(s)\, ds$$

exists and it is uniform with respect to $t \in \mathbb{R}$. Note that $\overline{p}$ has to be independent of t. We are interested in conditions on the average implying the existence of bounded solutions of the equation (1), that is, solutions $u(t)$ satisfying

$$\sup_{t \in \mathbb{R}} \{u(t)^2 + \dot{u}(t)^2\} < \infty.$$

Motivated by the periodic case and the so-called Landesman-Lazer conditions, Ahmad considered this question in [1] when $c > 0$ and the function $g : \mathbb{R} \to \mathbb{R}$ is continuous and has finite limits $g(\pm\infty) = \lim_{\xi \to \pm\infty} g(\xi)$ with $g(-\infty) \leq g(\xi) \leq g(+\infty)$ for each $\xi \in \mathbb{R}$. He proved that the condition

$$g(-\infty) < \bar{p} < g(+\infty) \tag{2}$$

is sufficient for the existence of a bounded solution. This is a sharp result because the condition with non-strict inequality is necessary. Extensions to other classes of equations appeared after [1]. See in particular the references [6, 7, 5, 2] where different methods based on guiding functions and upper and lower solutions were developed. Going back to the equation (1) we observe that these techniques do not seem to apply to the non-friction case $c = 0$. The recent paper by Soave and Verzini [9] employs a variational technique to prove that Ahmad's result can be extended to the non-friction case if the function g satisfies some additional conditions. Namely, g is of class C^2, increasing and has a unique inflection point. In these assumptions the condition (2) implies the existence of infinitely many bounded solutions. This is a remarkable result inspired by the model example $g(u) = \arctan u$, but the problem remains open for a general nonlinearity. The purpose of the present paper is to introduce an alternative approach that is also useful in the case $c = 0$. To illustrate our method we consider the equation

$$\ddot{u} + \operatorname{sign}(u) = p(t)$$

where sign denotes the discontinuous function $\operatorname{sign}(u) = \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0. \end{cases}$

The value of this function at $u = 0$ will be irrelevant. In agreement with the condition (2) it will be proved that this equation has infinitely many bounded solutions if $-1 < \overline{p} < 1$. The nonlinearity sign has been chosen because it is one of the accepted models for saturation but also because it does not satisfy the technical conditions imposed in [9]. The extension of our methodology to more general classes of nonlinearities is a topic for future work.

In [9] Soave and Verzini considered the action functional associated to the equation (1) with $c = 0$ and applied the so-called Dual Nehari method. In contrast our approach will be based on the use of generating functions and the results on twist maps developped in [4]. A key ingredient in the proof will be a property of the average of a function that is perhaps of independent interest and we describe now. Given an integer $N \geq 1$ and periodic function $f(t)$ it is well known that the linear differential equation

$$y^{(N)} = f(t) \tag{3}$$

has a bounded solution if and only if $\overline{f} = 0$. For non-periodic functions this equivalence is no longer true but we will prove that it is close to be true. More precisely, if we consider the spaces $A_0 = \{f : \overline{f} = 0\}$ and $BP^N = \{f : \text{(3) has a bounded solution}\}$, then BP^N is a dense subspace of A_0. Here the topology is induced by uniform convergence.

The rest of the paper is divided in three sections. Section 2 is devoted to the properties of the average. These properties are employed in Section 3 for the study of a simple linear equation. Finally the main result is proved in Section 4, the results of the previous section are useful to prove that our problem is in the framework of the theory of twist maps with non-periodic angles.

2 Average and bounded primitives

Let BC denote the class of bounded and continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. It becomes a Banach space with the norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|\,.$$

Following [1] we say that a function $f \in BC$ has average if the limit below exists

$$\bar{f} = \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} f(s)\,ds$$

uniformly in $t \in \mathbb{R}$.

Almost periodic functions are basic examples of functions with this property (see [3], page 44). Another example is the function $f(t) = \sin t + \frac{1}{1+t^2}$, in this case $\overline{f} = 0$. The class of all functions having average will be denoted by A. It is not hard to prove that A is a closed subspace of BC and the linear form $f \in A \mapsto \overline{f} \in \mathbb{R}$ is continuous. In particular, the space of functions with zero average,

$$A_0 = \left\{f \in A\colon \bar{f} = 0\right\}$$

is a Banach space.

Given $N \geq 1$ we consider the space BP^N composed by those functions $f \in BC$ such that the equation

$$y^{(N)} = f(t) \tag{4}$$

has a bounded solution $y \in BC$.

When $f \in BP^1$ we observe that

$$\left|\frac{1}{T}\int_t^{t+T} f(s)\,ds\right| = \left|\frac{F(t+T)-F(t)}{T}\right| \leq \frac{2}{T}\|F\|_\infty$$

where $F(t)$ is a bounded primitive of $f(t)$. This implies that $\bar{f} = 0$ and so $BP^1 \subset A_0$. Using the mean value theorem, it is not hard to prove that if $f \in BC$ and $y \in BC$ is a solution of (4), then all the intermediate derivatives $y^{(k)}$, $1 \leq k \leq N-1$, are also in BC. This fact leads to the chain

$$\cdots \subset BP^N \subset \cdots \subset BP^2 \subset BP^1 \subset A_0$$

and we claim that all the inclusions are proper even when they are restricted to the class of almost periodic functions. Indeed the function

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin\left(\frac{t}{3^n}\right)$$

is almost periodic and has zero average, in particular $\varphi \in A_0$. The primitive

$$\Phi(t) = \int_0^t \varphi(t)dt = 2\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n \sin^2\left(\frac{t}{2\cdot 3^n}\right)$$

satisfies an estimate of the type $ct^{1-\sigma} \leq \Phi(t) \leq Ct^{1-\sigma}$ if $t \geq 1$ for some positive constants c, C and $\sigma = \frac{\log 2}{\log 3}$. See the appendix in [8]. In consequence Φ is unbounded and so φ is not in BP^1. For each $N \geq 1$ we can construct an almost periodic function f_N in BP^N but not in BP^{N+1}. The succesive derivatives of φ are also almost periodic and we define $f_N = \varphi^{(N)}$. By construction f_N is in BP^N and the solutions of $y^{(N+1)} = f_N$ are of the type $y = \Phi + p_N$ where p_N is a polynomial of degree at most N. Clearly all these solutions are unbounded.

Proposition 2.1. *For each $N \geq 1$, the space BP^N is dense in A_0.*

For $N = 1$, this result is exactly Lemma 2.1 in [6]. To extend it to an arbitrary N we recall some facts coming from Finite Differences Calculus.

Given a function $f\colon \mathbb{R} \to \mathbb{R}$ and a number $T > 0$, we define

$$\Delta_T f(t) = f(t+T) - f(t).$$

The higher order difference operator is defined by composition, $\Delta_T^2 = \Delta_T \circ \Delta_T$, $\Delta_T^k = \Delta_T \circ \Delta_T^{k-1}$. If the function f is of class C^N then

$$\frac{1}{T^N}\Delta_T^N f(t) = f[t, t+T, ..., T+NT] = f^{(N)}(\xi) \tag{5}$$

for some ξ lying between t and $t + NT$. Here $f[t_0, t_1, ..., t_N]$ is the divided difference.

The translation operator is defined by the formula

$$\mathcal{T}_T f(t) = f(t+T).$$

From $\mathcal{T}_T^k = \mathcal{T}_{kT}$ and $\Delta_T = \mathcal{T}_T - Id$ we deduce that

$$\Delta_T^N = \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} \mathcal{T}_{kT}\,. \tag{6}$$

Finally we introduce the operators associated to primitives and derivatives

$$\mathcal{I}\, f(t) = \int_0^t f(s)\, ds, \quad \mathcal{D}\, f(t) = f'(t)$$

whenever they are defined. Given a continuous function f we observe that $\mathcal{D}\mathcal{I}f = f$. Also,

$$\left(\mathcal{I}^N \Delta_T^N\right) f = \left(\Delta_T^N \mathcal{I}^N\right) f + p_{N-1} \tag{7}$$

where p_{N-1} is some polynomial of degree at most $N-1$. To prove this identity we note that

$$\mathcal{D}^N \left(\Delta_T^N \mathcal{I}^N\right) f = \Delta_T^N \left(\mathcal{D}^N \mathcal{I}^N\right) f = \Delta_T^N f$$

because $\mathcal{D}$ and Δ_T commute. Also,

$$\mathcal{D}^N \left(\mathcal{I}^N \Delta_T^N\right) f = \left(\mathcal{D}^N \mathcal{I}^N\right) \Delta_T^N f = \Delta_T^N f$$

and so $\left(\Delta_T^N \mathcal{I}^N\right) f$ and $\left(\mathcal{I}^N \Delta_T^N\right) f$ are both solutions of the equation $y^{(N)}(t) = \Delta_T^N f(t)$.

Proof of Proposition 2.1. As mentioned above, the density of BP^1 in A_0 is proved in [6]. It will be sufficient to prove that for each $N \geq 2$ and $\epsilon > 0$, given $f \in BP^{N-1}$ there exist functions f^* and f^{**} satisfying $f = f^* + f^{**}$, $f^* \in BP^N$, $\|f^{**}\|_\infty < \epsilon$.

To prove this claim we start with the function $f \in BP^{N-1}$ and denote by $F(t)$ a bounded solution of $y^{(N-1)} = f(t)$. Define

$$f^{**} = \frac{1}{T^N} \Delta_T^N \mathcal{I}(F).$$

From the identity (6) we deduce that

$$f^{**} = \frac{1}{T^N} \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} \int_0^{t+kT} F(s)\, ds\,.$$

After splitting the integral in the form $\int_0^{t+kT} = \int_0^t + \int_t^{t+kT}$ and observing that

$$\sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} = 0$$

we obtain the formula

$$f^{**} = \frac{1}{T^N} \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} \int_t^{t+kT} F(s)\, ds\,.$$

Then

$$\|f^{**}\|_\infty \leq \frac{1}{T^{N-1}} \left[\sum_{k=0}^{N} \binom{N}{k} k\right] \|F\|_\infty$$

and we obtain $\|f^{**}\|_\infty < \epsilon$ by letting $T \to \infty$.

Next we define $f^* = f - f^{**}$. It remains to prove that $f^* \in BP^N$. The functions $\mathcal{I}^N(f)$ and $\mathcal{I}(F)$ are solutions of $y^{(N)} = f(t)$ and so they must differ in a polynomial of degree at most $N-1$, say

$$\mathcal{I}^N(f) - \mathcal{I}(F) = q_{N-1}.$$

From the definitions of f^* and f^{**},

$$\mathcal{I}^N(f^*) = \mathcal{I}^N(f) - \mathcal{I}^N(f^{**}) = \mathcal{I}(F) - \frac{1}{T^N}\mathcal{I}^N \Delta_T^N \mathcal{I}(F) + q_{N-1}.$$

The identity (7) leads to

$$\mathcal{I}^N(f^*) = \mathcal{I}(F) - \frac{1}{T^N}\Delta_T^N \mathcal{I}^{N+1}(F) + p_{N-1} + q_{N-1}.$$

Define $\Psi = \mathcal{I}^{N+1}(F)$. The function $y(t) = \mathcal{I}(F) - \frac{1}{T^N}\Delta_T^N \mathcal{I}^{N+1}(F)$ differs from $\mathcal{I}^N(f^*)$ in a polynomial of degree at most $N-1$. Hence $y(t)$ is a solution of $y^{(N)} = f^*(t)$ and we are going to prove that $y \in BC$. This will complete the proof. From (5) we deduce that

$$\frac{1}{T^N}\Delta_T^N \Psi(t) = \Psi^{(N)}(\xi) = \mathcal{I}(F)(\xi)$$

for some $\xi \in [t, t+NT]$. Finally we apply the mean value theorem to the function $\mathcal{I}(F)$,

$$|y(t)| = |\mathcal{I}F(t) - \mathcal{I}F(\xi)| = |F(\eta)|\, |t-\xi|$$

where η lies between t and ξ. Then $y \in BC$ with

$$\|y\|_\infty \leq \|F\|_\infty NT.$$

□

3 The linear equation

Consider the equation

$$\ddot{y} = f(t) \tag{8}$$

under the assumption $f \in A$, $\bar{f} < 0$.

In this section we obtain some properties of the solutions of the Cauchy and the Dirichlet problems. Let us start with the initial value problem and let $y(t; t_0, v)$ be the solution of (8) with initial conditions

$$y(t_0) = 0,\ \dot{y}(t_0) = v.$$

Lemma 3.1. *There exists $v_* > 0$ such that for $v > v_*$ the solution $y(t; t_0, v)$ has exactly one zero to the right of t_0, say $\tau > t_0$. Moreover the function $\tau = \tau(t_0, v)$ belongs to $C^1(\mathbb{R}\times]v_*, +\infty[)$ and*

$$\lim_{v\to+\infty} \frac{\tau(t_0, v) - t_0}{v} = -\frac{2}{\bar{f}} \tag{9}$$

uniformly in $t_0 \in \mathbb{R}$.

Proof. Throughout the proof, ϵ will be a fixed but arbitrary number satisfying $0 < 3\epsilon < \left|\bar{f}\right|$. The notation c_ϵ will be employed for different constants depending only on ϵ. We apply the results of the previous section to split f in the form

$$f = \ddot{F} + f^{**} + \bar{f}$$

where F is a C^2 function with $F, \dot{F}, \ddot{F} \in BC$ and $\|f^{**}\|_\infty < \epsilon$. The solution of the initial value problem can be expressed as

$$y(t; t_0, v) = F(t) - F(t_0) + \left(v - \dot{F}(t_0)\right)(t - t_0) + \frac{\bar{f}}{2}(t - t_0)^2 + \int_{t_0}^{t} (t - s) f^{**}(s)\, ds. \tag{10}$$

After applying the mean value theorem to F, this identity leads to the estimates

$$\left[-2\left\|\dot{F}\right\|_\infty + v + \frac{\bar{f} - \epsilon}{2}(t - t_0)\right](t - t_0) \leq y(t; t_0, v) \leq \left[2\left\|\dot{F}\right\|_\infty + v + \frac{\bar{f} + \epsilon}{2}(t - t_0)\right](t - t_0).$$

Here we are assuming that $v > 0$ and $t \geq t_0$ and it is easy to deduce that any zero to the right of t_0 must lie on the interval I_ϵ defined by the inequalities

$$\frac{2v}{\left|\bar{f}\right| + \epsilon} - c_\epsilon \leq t - t_0 \leq \frac{2v}{\left|\bar{f}\right| - \epsilon} + c_\epsilon \tag{11}$$

with $c_\epsilon = \frac{4\left\|\dot{F}\right\|_\infty}{\left|\bar{f}\right| - \epsilon}$.

Since $y(t; t_0, v) \to -\infty$ as $t \to +\infty$, at least one zero must be contained in I_ϵ. Next we shall prove that this zero is unique. To this end we shall find $v_\epsilon > 0$ such that

$$\dot{y}(t; t_0, v) < 0 \text{ if } t \in I_\epsilon \text{ and } v > v_\epsilon.$$

To prove this, we differentiate the formula (10) with respect to t and derive a formula for $\dot{y}(t; t_0, v)$. From there we obtain

$$\dot{y}(t; t_0, v) \leq 2\left\|\dot{F}\right\|_\infty + v + (\bar{f} + \epsilon)(t - t_0) \text{ if } t \geq t_0.$$

The number v_ϵ is obtained by combining this inequality with (11).

At this point it is convenient to sum up our conclusions. The function $\tau = \tau(t_0, v)$ is well defined for $v > v_\epsilon$ and satisfies

$$\frac{2v}{\left|\bar{f}\right| + \epsilon} - c_\epsilon \leq \tau(t_0, v) - t_0 \leq \frac{2v}{\left|\bar{f}\right| - \epsilon} + c_\epsilon. \tag{12}$$

Then

$$\frac{2}{\left|\bar{f}\right| + \epsilon} \leq \liminf_{v \to +\infty} \frac{\tau(t_0, v) - t_0}{v} \leq \limsup_{v \to +\infty} \frac{\tau(t_0, v) - t_0}{v} \leq \frac{2}{\left|\bar{f}\right| - \epsilon}$$

uniformly in $t_0 \in \mathbb{R}$.

Since ϵ is arbitrarily small, we conclude that the limit (9) exists.

To complete the proof of the Lemma, we must show that τ is C^1 on $\mathbb{R}\times]\tilde{v}_\epsilon, +\infty[$, for some $\tilde{v}_\epsilon > v_\epsilon$. To do this, we go back to the formula (10) and consider the C^1 function

$$\Phi(\tau, t_0, v) = \frac{y(\tau; t_0, v)}{\tau - t_0} = F[\tau, t_0] - \dot{F}(t_0) + v + \frac{\bar{f}}{2}(\tau - t_0) + \frac{1}{\tau - t_0}\int_{t_0}^{\tau}(\tau - s)f^{**}(s)\, ds,$$

defined on

$$G_\epsilon = \left\{(\tau, t_0, v) \in \mathbb{R}^3 \colon \tau > t_0, v > v_\epsilon\right\}.$$

From the previous discussions we know that $\tau(t_0, v)$ is the unique solution of the implicit function problem

$$\Phi(\tau, t_0, v) = 0.$$

We will now show that the implicit function theorem can be applied, because $\frac{\partial\Phi}{\partial\tau}(\tau(t_0, v), t_0, v) < 0$ if v is large enough. To prove that this condition holds, we differentiate Φ with respect to τ and observe that

$$\left|\frac{\partial\Phi}{\partial\tau}(\tau, t_0, v) - \frac{\bar{f}}{2}\right| \leq \frac{2\|\dot{F}\|_\infty}{\tau - t_0} + \frac{\epsilon}{2}. \tag{13}$$

Then we can invoke (9) to conclude that

$$-\frac{\partial\Phi}{\partial\tau}(\tau(t_0, v), t_0, v) \geq \frac{|\bar{f}|}{4} \text{ if } v > \tilde{v}_\epsilon.$$

□

Next we consider the Dirichlet problem on the interval $[l, r]$ with $l < r$. The solution of (8) satisfying

$$y(l) = y(r) = 0$$

will be denoted by $y_D(t; l, r)$. This solution can be expressed in terms of the Green function, and this explicit formula shows that the function

$$(t, l, r) \in \mathcal{D} \subset \mathbb{R}^3 \mapsto (y_D(t; l, r), \dot{y}_D(t; l, r)) \in \mathbb{R}^2$$

is C^1. The domain $\mathcal{D}$ is defined by

$$\mathcal{D} = \left\{(t, l, r) \in \mathbb{R}^3 \colon l < r, t \in [l, r]\right\}.$$

Lemma 3.2. *In the previous notations the statements below hold,*

(a) There exists $T > 0$ such that

$$y_D(t; l, r) > 0$$

if $t \in]l, r[$ and $r - l > T$.

(b)

$$\lim_{r-l\to+\infty} \frac{1}{r-l}\left[\dot{y}_D(t; l, r) + \bar{f}\left(\frac{r+l}{2} - t\right)\right] = 0$$

(c)

$$\lim_{r-l\to+\infty} \frac{\partial}{\partial l}\, \dot{y}_D(t;l,r) = \lim_{r-l\to+\infty} \frac{\partial}{\partial r}\, \dot{y}_D(t;l,r) = -\frac{\bar{f}}{2}\,.$$

The limits in (b) and (c) are uniform in $t \in [l,r]$, $r,l \in \mathbb{R}$, $l < r$.

Proof. (a) We go back to the proof of the previous Lemma and consider the function $\tau(t_0, v)$. If we fix some $\hat{v} > v_*$, then (12) implies that $\tau(t_0,\hat{v}) - t_0$ has an upper bound independent of $t_0 \in \mathbb{R}$, say M. On the other hand, $\tau(t_0, v) - t_0 \to +\infty$ as $v \to +\infty$, so the function $\tau(t_0,\cdot) - t_0$ maps the interval $[\hat{v}, +\infty[$ onto another interval containing $[M, +\infty[$. By implicit differentiation of $\Phi(\tau(t_0, v), t_0, v) = 0$, we obtain

$$\frac{\partial \Phi}{\partial \tau}(\tau(t_0,v),t_0,v)\, \frac{\partial \tau}{\partial v}(t_0,v) + 1 = 0.$$

The definition of v_* was adjusted so that $\frac{\partial\Phi}{\partial\tau}(\tau(t_0,v),t_0,v) < 0$ if $v > v_*$ and so $\frac{\partial\tau}{\partial v}(t_0,v) > 0$. This implies that $\tau(t_0,\cdot)$ has a smooth inverse $v = v(t_0,\tau)$, defined on an interval containing $[M, +\infty[$. The uniqueness for the Dirichlet problem leads to the identity

$$y_D(t;l,r) = y\,(t;l,v\,(l,r))$$

if $r - l \geq M$. Then (a) is a consequence of the definition of the function $v(l,r)$.

(b) Once again we employ the formula (10) to deduce the new formula

$$\dot{y}_D(t;l,r) = \dot{F}(t) - \dot{F}(l) + v(l,r) + \bar{f}(t-l) + \int_l^t f^{**}(s)\,ds. \tag{14}$$

On the other hand, the limit (9) implies that

$$\frac{v(l,r)}{r-l} \to -\frac{\bar{f}}{2} \text{ as } r - l \to +\infty$$

in a uniform fashion. The conclusion follows from these two facts.

(c) From the original equation (8) we deduce that

$$\dot{y}_D(t;l,r) = v(l,r) + \int_l^t f(s)ds.$$

Then

$$\frac{\partial \dot{y}_D}{\partial l}(t;l,r) = \frac{\partial v}{\partial l}(l,r) - f(l), \quad \frac{\partial \dot{y}_D}{\partial r}(t;l,r) = \frac{\partial v}{\partial r}(l,r).$$

To estimate the partial derivatives of v we differentiate the equation $\Phi(r,l,v(l,r)) = 0$ to obtain

$$\frac{\partial \Phi}{\partial \tau}(r,l,v(l,r)) + \frac{\partial v}{\partial r}(l,r) = 0, \quad \frac{\partial \Phi}{\partial t_0}(r,l,v(l,r)) + \frac{\partial v}{\partial l}(l,r) = 0.$$

From the estimate (13) it is easy to conclude that the second limit holds. To prove the validity of the first limit we go back to the definition of Φ to find the estimate

$$|\frac{\partial \Phi}{\partial t_0}(\tau, t_0, v) + \ddot{F}(t_0) + \frac{\overline{f}}{2}| \leq \frac{2\|\dot{F}\|_\infty}{\tau - t_0} + \frac{3\epsilon}{2}.$$

In consequence

$$|\frac{\partial \dot{y}_D}{\partial l}(t; l, r) - \ddot{F}(l) - \frac{\overline{f}}{2} + f(l)| \leq \frac{2\|\dot{F}\|_\infty}{r - l} + \frac{3\epsilon}{2}.$$

Finally we observe that

$$-\ddot{F}(l) - \frac{\overline{f}}{2} + f(l) = f^{**}(l) + \frac{\overline{f}}{2}.$$

□

4 A discontinuous equation

Consider the equation

$$\ddot{u} + \mathrm{sign}(u) = p(t) \tag{15}$$

with $p \in BC$.

Given an interval $I \subset \mathbb{R}$, a function $u \in C^1(I)$ is a solution of (15) if it satisfies the two conditions below,

(i) $u(t)^2 + \dot{u}(t)^2 > 0$ for each $t \in I$;

(ii) Let $Z = \{t \in I \colon u(t) = 0\}$ be the set of zeros of u. Then, for each interval $J \subset I \setminus Z$, the function $u|_J$ belongs to $C^2(J)$ and satisfies (15) on this interval.

A bounded solution is a solution defined on $I = \mathbb{R}$ such that

$$\sup_{t \in \mathbb{R}} \left[u(t)^2 + \dot{u}(t)^2 \right] < \infty.$$

Theorem 4.1. *Assume that $p \in A$ and $|\bar{p}| < 1$. Then there exists a sequence (u_n) of bounded solutions satisfying*

$$\inf_{t \in \mathbb{R}} \left[u_n(t)^2 + \dot{u}_n(t)^2 \right] \to \infty \text{ as } n \to \infty.$$

The first step of the proof will be the construction of solutions defined on $[t_0, t_1]$ and having exactly one change of sign in $]t_0, t_1[$. With this goal, we apply Lemma 3.2 on the interval $[l, r] = [t_0, \tau]$ with $f_+ = -1 + p$ and on the interval $[l, r] = [\tau, t_1]$ with $f_- = -1 - p$. We find associated numbers T_+ and T_- such that if $\tau - t_0 > T_+$ then $y_{D_+}(t; t_0, \tau)$ is positive on $]t_0, \tau[$ and a similar statement holds for $y_{D_-}(t; \tau, t_1)$.

A first attempt to construct solutions of (15) could be to juxtapose the two solutions of Dirichlet problems,

$$U(t; t_0, t_1, \tau) = \begin{cases} y_{D_+}(t; t_0, \tau), & t \in [t_0, \tau] \\ -y_{D_-}(t; \tau, t_1), & t \in [\tau, t_1] \end{cases}$$

where $t_0 + T_+ < \tau < t_1 - T_-$. In general $U(\cdot, t_0, t_1, \tau)$ will not be a solution, because the derivatives at τ will not match. Then we must adjust τ so that the equation below holds,

$$F(t_0, t_1, \tau) := \dot{y}_{D_+}(\tau; t_0, \tau) + \dot{y}_{D_-}(\tau; \tau, t_1) = 0.$$

In order to analyse this equation, we first compute the derivative with respect to τ,

$$\frac{\partial F}{\partial \tau}(t_0, t_1, \tau) = \ddot{y}_{D_+}(\tau; t_0, \tau) + \ddot{y}_{D_-}(\tau; \tau, t_1) + \frac{\partial \dot{y}_{D_+}}{\partial \tau}(t; t_0, \tau)\bigg|_{t=\tau} + \frac{\partial \dot{y}_{D_-}}{\partial \tau}(t; \tau, t_1)\bigg|_{t=\tau}.$$

From the equations $\ddot{y} = f_\pm$ we deduce that the first two terms in the sum are precisely $-1+p(\tau)$ and $-1 - p(\tau)$. To deal with the last two terms we employ Lemma 3.2 (c). After fixing $\epsilon > 0$ small enough, we find $T_1 = T_1(\epsilon) \geq \max(T_+, T_-)$ such that, if $\tau - t_0 > T_1$ and $t_1 - \tau > T_1$, then

$$\left| \frac{\partial F}{\partial \tau}(t_0, t_1, \tau) + 2 + \frac{\bar{f}_+}{2} + \frac{\bar{f}_-}{2} \right| < \epsilon$$

or equivalently,

$$\left| \frac{\partial F}{\partial \tau}(t_0, t_1, \tau) + 1 \right| < \epsilon$$

and so $\frac{\partial F}{\partial \tau}(t_0, t_1, \tau)$ is negative when the two intervals are large enough. Next we are going to find $T_2 > T_1$ such that if $t_1 - t_0 > T_2$ then $F(t_0, t_1, \cdot)$ has a change of sign at $\hat{\tau} = \hat{\tau}(t_0, t_1)$, with $\hat{\tau} - t_0 > T_1$ and $t_1 - \hat{\tau} > T_1$. To prove that this is possible, we apply Lemma 3.2 (b) and find $T_2 > T_1$ such that if $\tau - t_0 > T_2$ then

$$\left| \dot{y}_{D_+}(t; t_0, \tau) + \bar{f}_+ \left(\frac{t_0 + \tau}{2} - t \right) \right| < \epsilon(\tau - t_0), \tag{16}$$

and similarly for $\dot{y}_{D_-}$.

Evaluating these expressions at $t = \tau$ we observe that $\dot{y}_{D_+}(\tau; t_0, \tau)$ remains bounded if $\tau = t_0 + T_2$ with arbitrary $t_0 \in \mathbb{R}$. However, $\dot{y}_{D_+}(\tau; t_0, \tau) \to -\infty$ if $\tau - t_0 \to \infty$. The solution y_{D_-} has an analogous behavior excepting that $\dot{y}_{D_-}(\tau; t_0, \tau) \to +\infty$ if $t_1 - \tau \to \infty$. Hence $F(\tau; t_0, t_1)$ is positive when $t_1 - \tau$ is large compared to $\tau - t_0$ and negative in the opposite case. Let $\hat{\tau}$ be a zero of $F(\cdot; t_0, t_1)$. From the definition of T_1 and T_2 we know that

$$\frac{\partial F}{\partial \tau}(\hat{\tau}; t_0, t_1) < 0.$$

Thus $\hat{\tau}$ is unique and the function $\hat{\tau} = \hat{\tau}(t_0, t_1)$ is well defined and C^1 when $t_1 - t_0$ is large enough.

We are ready to construct solutions of (15). Define

$$u(t; t_0, t_1) = U\left(t; t_0, t_1, \hat{\tau}(t_0, t_1)\right).$$

It is not hard to show that $u(\cdot; t_0, t_1)$ satisfies all the conditions for a solution of (15) defined on $I = [t_0, t_1]$. In particular $Z = \{t_0, \hat{\tau}(t_0, t_1), t_1\}$ and the condition (i) is verified via Lemma 3.2 (b).

Associated to (15) we consider the Lagrangian function

$$L(t,u,\dot{u}) = \frac{1}{2}\dot{u}^2 - |u| + p(t)u$$

and define the function

$$h(t_0,t_1) = \int_{t_0}^{t_1} L\left(t, u(t;t_0,t_1), \dot{u}(t;t_0,t_1)\right)\, dt.$$

In the next proposition we summarize the properties of this function.

Proposition 4.2. *Given $\delta > 0$ there exists $T_\delta > 0$ such that the function h belongs to $C^1(\Omega_\delta)$, where*

$$\Omega_\delta = \left\{(t_0,t_1) \in \mathbb{R}^2 \colon t_1 - t_0 > T_\delta\right\}.$$

Moreover

$$\partial_{t_0} h(t_0,t_1) = \frac{1}{2}\dot{u}(t_0;t_0,t_1)^2, \quad \partial_{t_1} h(t_0,t_1) = -\frac{1}{2}\dot{u}(t_1;t_0,t_1)^2 \tag{17}$$

and

$$-(\mu+\delta)(t_1-t_0)^3 \le h(t_0,t_1) \le -(\mu-\delta)(t_1-t_0)^3, \tag{18}$$

with $\mu = \frac{1}{96}(1-\overline{p}^2)^2$.

Proof. For some $T > 0$ the function h is well defined on $t_1 - t_0 > T$. We shall check first the identities (17) and later we shall check the estimate (18) when $t_1 - t_0 > T_\delta$ for some appropriate $T_\delta > T$. The computation of the derivatives of h follows along the lines of Section 7.2 of [4] but some additional care is needed. In our case L is not C^1 but we can split the integral in the form

$$h(t_0,t_1) = \int_{t_0}^{\hat{\tau}} L + \int_{\hat{\tau}}^{t_1} L$$

and deduce easily that h is C^1. Moreover,

$$\partial_{t_0} h(t_0,t_1) = \frac{1}{2}\dot{u}\left(\hat{\tau}\right)^2 \frac{\partial\hat{\tau}}{\partial t_0} - \frac{1}{2}\dot{u}(t_0)^2 + \int_{t_0}^{\hat{\tau}} \frac{d}{d\,t_0} L dt - \frac{1}{2}\dot{u}\left(\hat{\tau}\right)^2 \frac{\partial\hat{\tau}}{\partial t_0} + \int_{\hat{\tau}}^{t_1} \frac{d}{d\,t_0} L dt$$

where $u(t)$ is an abreviation for $u(t;t_0,t_1)$. Before computing the two integrals it is useful to observe that the function

$$\hat{\Omega}_\delta \to \mathbb{R}, \quad (t;t_0,t_1) \mapsto u(t;t_0,t_1)$$

is C^1. The domain of this map is defined by

$$\hat{\Omega}_\delta = \{(t;t_0,t_1) :\ (t_0,t_1) \in \Omega_\delta,\ t_0 \le t \le t_1\}$$

and, in principle, it is clear that this function is C^1 in each of the subsets

$$\hat{\Omega}_\delta^- = \{(t;t_0,t_1) \in \hat{\Omega}_\delta :\ t_0 \le t \le \hat{\tau}(t_0,t_1)\}, \ \ \hat{\Omega}_\delta^+ = \{(t;t_0,t_1) \in \hat{\Omega}_\delta :\ \hat{\tau}(t_0,t_1) \le t \le t_1\}$$

but the derivatives of u with respect to t_0 and t_1 could have a jump discontinuity along the common boundary $t = \hat{\tau}(t_0, t_1)$. To show that this is not the case we differentiate the identity $u(\hat{\tau}(t_0, t_1); t_0, t_1) = 0$ on $\Omega_\delta^\pm$ to obtain

$$\dot{u}(\hat{\tau}; t_0, t_1)\frac{\partial \hat{\tau}}{\partial t_0} + \frac{\partial u}{\partial t_0}(\hat{\tau} \pm 0; t_0, t_1) = 0$$

and so the left and right derivatives coincide. An integration by parts shows that

$$\int_{t_0}^{\hat{\tau}} \frac{d}{d\,t_0} L dt = \int_{t_0}^{\hat{\tau}} \{\frac{\partial L}{\partial u}\frac{\partial u}{\partial t_0} + \frac{\partial L}{\partial \dot{u}}\frac{\partial \dot{u}}{\partial t_0}\} dt = [\frac{\partial L}{\partial \dot{u}}\frac{\partial u}{\partial t_0}]_{t=t_0}^{t=\hat{\tau}} + \int_{t_0}^{\hat{\tau}} \{\frac{\partial L}{\partial u} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{u}})\}\frac{\partial \dot{u}}{\partial t_0} dt.$$

The equation (15) together with the definition of L lead to

$$\int_{t_0}^{\hat{\tau}} \frac{d}{d\,t_0} L dt = \dot{u}(\hat{\tau})\frac{\partial u}{\partial t_0}(\hat{\tau}) - \dot{u}(t_0)\frac{\partial u}{\partial t_0}(t_0).$$

Finally we differentiate $u(t_0; t_0, t_1) = 0$ with respect to t_0 to deduce that $\dot{u}(t_0) + \frac{\partial u}{\partial t_0}(t_0) = 0$. Similarly

$$\int_{\hat{\tau}}^{t_1} \frac{d}{d\,t_0} L dt = -\dot{u}(\hat{\tau})\frac{\partial u}{\partial t_0}(\hat{\tau})$$

and we obtain the first identity in (17). The second is obtained in the same way.

To prove the estimate for h it is convenient to simplify its expression. Multiplying the equation (15) by $u = u(t; t_0, t_1)$ and integrating by parts,

$$-\int_{t_0}^{t_1} \dot{u}^2 + \int_{t_0}^{t_1} |u| = \int_{t_0}^{t_1} p\, u$$

and so

$$h(t_0, t_1) = -\frac{1}{2}\int_{t_0}^{t_1} \dot{u}^2 = -\frac{1}{2}\int_{t_0}^{\hat{\tau}} {\dot{y}_{D_+}}^2 - \frac{1}{2}\int_{\hat{\tau}}^{t_1} {\dot{y}_{D_-}}^2.$$

From (16),

$$\int_{t_0}^{\hat{\tau}} {\dot{y}_{D_+}}^2 = {\bar{f}_+}^2 \int_{t_0}^{\hat{\tau}} \left(\frac{t_0 + \hat{\tau}}{2} - t\right)^2 dt + \gamma_\epsilon(\hat{\tau} - t_0)^3$$

with $\gamma_\epsilon \to 0$ as $\epsilon \to 0$. Hence

$$\int_{t_0}^{\hat{\tau}} {\dot{y}_{D_+}}^2 = {\bar{f}_+}^2 \frac{(\hat{\tau} - t_0)^3}{12} + \gamma_\epsilon(\hat{\tau} - t_0)^3$$

and similarly

$$\int_{\hat{\tau}}^{t_1} {\dot{y}_{D_-}}^2 = {\bar{f}_-}^2 \frac{(t_1 - \hat{\tau})^3}{12} + \Gamma_\epsilon(t_1 - \hat{\tau})^3$$

with $\Gamma_\epsilon \to 0$ as $\epsilon \to 0$.

To complete the proof we must estimate the lengths $\hat{\tau} - t_0$ and $t_1 - \hat{\tau}$ in terms of $t_1 - t_0$. Working as in the proof of Lemma 3.1 we obtain the estimate

$$\frac{2v}{|\bar{f}_+| + \epsilon} - c_\epsilon \leq \hat{\tau} - t_0 \leq \frac{2v}{|\bar{f}_+| - \epsilon} + c_\epsilon,$$

where $v = -\dot{y}_{D_+}(\hat{\tau}; t_0, \hat{\tau}) = \dot{y}_{D_-}(\hat{\tau}; \hat{\tau}, t_1)$. A similar estimate holds for $t_1 - \hat{\tau}$. Then

$$a_\epsilon v - c_\epsilon \leq t_1 - t_0 \leq A_\epsilon v + c_\epsilon \tag{19}$$

with A_ϵ, $a_\epsilon \to \frac{4}{1-\bar{p}^2}$ as $\epsilon \to 0$. From here,

$$b_\epsilon(t_1 - t_0) - c_\epsilon \leq \hat{\tau} - t_0 \leq B_\epsilon(t_1 - t_0) + c_\epsilon \tag{20}$$

with B_ϵ, $b_\epsilon \to \frac{1+\bar{p}}{2}$ as $\epsilon \to 0$ and

$$d_\epsilon(t_1 - t_0) - c_\epsilon \leq t_1 - \hat{\tau} \leq D_\epsilon(t_1 - t_0) + c_\epsilon \tag{21}$$

with D_ϵ, $d_\epsilon \to \frac{1-\bar{p}}{2}$ as $\epsilon \to 0$. Finally

$$\begin{aligned} -2h(t_0, t_1) \leq & \frac{(1-\bar{p})^2}{12} {B_\epsilon}^3 (t_1 - t_0)^3 + \gamma_\epsilon^*(t_1 - t_0)^3 + \\ & + \frac{(1+\bar{p})^2}{12} {D_\epsilon}^3 (t_1 - t_0)^3 + \Gamma_\epsilon^*(t_1 - t_0)^3 + E_\epsilon(t_1 - t_0)^2 \end{aligned}$$

with $B_\epsilon^3 + \gamma_\epsilon^* \to \frac{(1+\bar{p})^3}{8}$, $D_\epsilon^3 + \Gamma_\epsilon^* \to \frac{(1-\bar{p})^3}{8}$ and E_ϵ bounded independently of ϵ.

Similar lower estimates also hold. The proof is completed by adjusting ϵ with δ and T_δ so that (18) holds. $\square$

We are ready to prove Theorem 4.1. According to the previous Proposition the function h satisfies the same conditions as the generating function in Section 7.3 of [4]. We can repeat the arguments there to find numbers $\sigma > 1$ and $d^* > 0$ such that if $d > d^*$ then there exists a sequence $(t_k^d)_{k \in \mathbb{Z}}$ satisfying

$$\partial_1 h(t_k^d, t_{k+1}^d) + \partial_2 h(t_{k-1}^d, t_k^d) = 0 \tag{22}$$

and

$$d \leq t_{k+1}^d - t_k^d \leq \sigma d \tag{23}$$

for each $k \in \mathbb{Z}$. In view of (22) and (17) we deduce that the function

$$u_d : \mathbb{R} \to \mathbb{R}, \;\; u_d(t) = u(t; t_k^d, t_{k+1}^d) \text{ if } t \in [t_k^d, t_{k+1}^d]$$

is a solution of (4.1). To verify the condition (i) in the definition of solution we can employ (16). We are going to prove that this solution is bounded, more precisely

$$\|u_d\|_\infty = O(d^2), \;\; \|\dot{u}_d\|_\infty = O(d) \quad \text{as } d \to \infty,$$

To prove this estimate we first observe that $|\ddot{u}_d(t)| \leq 1 + \|p\|_\infty$ for almost every $t \in]t_k^d, t_{k+1}^d[$. Then we define the sequence $v_k^d = |\dot{u}_d(t_k^d)|$ and combine the inequality (19) with the condition (23) to deduce that $v_k^d \leq \sigma_1 d + c_1$ where σ_1 and c_1 are positive constants independent of $k \in \mathbb{Z}$ and $d > d^*$. Then, if $t \in [t_k^d, t_{k+1}^d]$,

$$|\dot{u}_d(t)| \leq v_k^d + \int_{t_k^d}^{t} |\ddot{u}_d(s)| ds \leq v_k^d + (1 + \|p\|_\infty)(t_{k+1}^d - t_k^d).$$

Using again (23) we conclude that

$$\|\dot{u}_d\|_\infty \leq (\sigma_1 + (1 + \|p\|_\infty)\sigma)d + c_1.$$

The estimate for u_d is a consequence of

$$|u_d(t)| = |u_d(t) - u_d(t_k^d)| \leq \|\dot{u}_d\|_\infty (t - t_k^d) \ \text{ if } t \in [t_k^d, t_{k+1}^d].$$

The proof of Theorem 4.1 is almost complete. It remains to prove that if we take a sequence $d_n \to \infty$ then

$$\inf_{t \in \mathbb{R}} [u_{d_n}(t)^2 + \dot{u}_{d_n}(t)^2] \to \infty.$$

This can be easily proved by combining Lemma 3.2 (b) and the inequalities (20) and (21).

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