Dynamics of Kepler problem with linear drag

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Abstract

We study the dynamics of Kepler problem with linear drag. We prove that motions with nonzero angular momentum have no collisions and travel from infinity to the singularity. In the process the energy

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takes all real values and the angular velocity becomes unbounded. We also prove that there are two types of linear motions: capture-collision and ejection-collision. The behaviour of solutions at collisions is the same as in the conservative case. Proofs are obtained using the geometric theory of ODEs and two regularizations for the singularity of Kepler equation. The first, already considered in Diacu (1999), is mainly used for the study of the linear motions. The second, the well known Levi-Civita transformation, allows to complete the study of the asymptotic values of the energy and to prove the existence of collision solutions with arbitrary energy.

Keywords: Kepler equation, linear drag, collision, Levi-Civita transformation

1 Introduction

Recently, there has been new interest in modeling and analyzing the effect of dissipative forces on the dynamics of small bodies (see, for example the recent European programme Stardust, http://www.strath.ac.uk/stardust/). In this connection, the study of the restricted three body problem with dissipation has been considered in Celletti et al. (2011), where some numerical results covering many aspects of its dynamics are given. In Margheri et al. (2012), for the same problem, some analytical results are obtained about the existence or not of periodic orbits for general drag forces. The results in Margheri et al. (2012) are of local nature and we would also like to describe some aspects of the dynamics of the system from a more global point of view. A natural starting point for this task is to set the mass μ of one of the primaries equal to zero and to address first the corresponding Kepler problem with drag. The corresponding equation in the inertial frame is

$$\ddot{z} + D(z, \dot{z})\dot{z} = -\frac{z}{|z|^3}$$

where $z \in \mathbb{C}$ and $D(z, \dot{z})$ is a real-valued function describing the drag force. This force can model different non-gravitational effects such as particle collisions, solar radiation or atmospheric resistance (Celletti et al. (2011), Leach (1987)). This explains why different types of drag forces have been considered in the literature, including, in particular, Stokes dissipation and the Poynting-Robertson force. Several authors (see Danby (1962), Mittleman & Jezewski (1982), Mavraganis & Michalakis (1994)) have worked with the choice

$$D(z, \dot{z}) = \frac{\alpha}{|z|^2}, \tag{1}$$

where $\alpha > 0$ is a parameter. This formula seems particularly meaningful when modeling collisions of particles or photons. Moreover, from a mathematical point of view, it has an exceptional feature: orbits can be obtained in a closed form. This is shared by the larger family of drag forces, which also follow an inverse square law in the distance to the singularity, considered in Breiter & Jackson (1998), Diacu (1999). All these drags correspond to integrable equations.

To develop a *qualitative* theory of Kepler problem for general families of drag forces seems a natural task. As far as we know the first contributions in these directions are Corne & Rouche (1973) and Diacu (1999). The first paper is concerned with a general family of drag forces while the second deals with the generalized Stokes force also considered in Breiter & Jackson (1998). In Diacu (1999) a global description of the rectilinear trajectories is obtained using the integrability of the corresponding system and some properties of the motions near collision are given, taking advantage of the presence of a singular point on the collision manifold of a suitably regularized problem.

To continue the program of the qualitative study of dissipative Kepler problems, in this paper we will consider the case of a linear dissipation, corresponding to

$$D(z, \dot{z}) = \epsilon, \tag{2}$$

where ϵ is a positive constant. At first sight this might seem the simplest choice for the drag force. However, orbits cannot be obtained explicitly in this case, and different techniques must be used to analyze the corresponding dynamics. We expect that the methods developed in this paper will give some hints for the study of the general non-integrable case. We find an additional reason to analyze the linear drag (2): the corresponding force is simultaneously a member of two important families of drag forces (Stokes and Jacobi). As already mentioned we refer to Celletti et al. (2011) for more details on Stokes forces. The second family can be extracted directly from Jacobi's textbook on Mechanics Jacobi (2009). We find somehow surprising that Jacobi already dedicated a chapter of his course to the study of Kepler problem with dissipation. Indeed he assumed

$$D(z, \dot{z}) = \epsilon |\dot{z}|^{p-1},$$

where p is a parameter, and the linear case corresponds to $p = 1$.

The Kepler equation corresponding to the linear drag is

$$\ddot{z} + \epsilon \dot{z} = -\frac{z}{|z|^3}. \tag{3}$$

The results in Corne & Rouche (1973) are applicable to equation (3) and the main conclusion which follows is that the set $\{(z, \dot{z}) \in \mathbb{C}^2 : \ z = 0\}$

is a global attractor. The notion of attractor must be understood in an appropriate sense. Essentially, it says that all solutions satisfy

$$|z(t)| \to 0 \ \text{ as } t \uparrow t_1$$

where the two cases $t_1 < \infty$ and $t_1 = \infty$ are possible. Note that the first case corresponds to collisions. Having this as a starting point, in our paper we give a detailed description of the qualitative dynamics of (3).

More precisely, in Section 2 we discuss some qualitative properties of solutions with non-zero angular momentum. By a direct analysis of the differential equation which determines the evolution of the radial component of the solutions, we prove that the particles travel from infinity to the singularity along trajectories that are well defined for all times, see Propositions 2.1 and 2.2. This means that there are no collisions at finite time. Moreover, using a regularization of Kepler problem already presented in Diacu (1999), we prove that, as the particles approach the singularity, their angular velocity becomes unbounded, see Proposition 2.5.

In Section 3 we analyze the linear motions (that is the motions with zero angular momentum) of (3). They will be classified in two families: ejection-collision and capture-collision. We will prove that ejections and collisions occur with finite energy, see Proposition 3.1. The unique separatrix will be characterized in terms of the asymptotic values of the energy. The proofs use mainly techniques coming from phase-portrait analysis. Finally, following the lines of Sperling (1969/1970) and Ortega (2011), we will show that the behavior at collisions is the same as in the conservative case.

In order to study the asymptotic behaviour of the energy on the solutions of equation (3), in Section 4 we adapt to the dissipative setting the classical Levi-Civita regularization. It is well known[1] that in the conservative setting the Levi-Civita change of variables transforms Kepler problem into a *linear equation* of the form $w'' - \frac{E}{2}w = 0$, where the energy E takes constant values along the motions w. It must be noticed that only those solutions lying on a suitable manifold do correspond to physically meaningful motions. In the dissipative case the energy is not constant along the motions and so it has to be added as an additional unknown. Moreover in the dissipative framework, we get a regularized *nonlinear system* of polynomial type. The flow on the corresponding invariant manifold can be interpreted as the Levi-Civita regularized system. We have used this system to prove that the energy along a non rectilinear motion tends to $\pm\infty$ as t tends to $\mp\infty$, see Proposition 4.1, and that ejection and collisions can occur with any prescribed value

[1] Besides the classical works by Levi-Civita it is interesting to mention Goursat's paper Goursat (1889). The authors thank Dr. Lei Zhao for calling their attention to this paper.

of the energy, see Proposition 4.2. We note that a regularization of the restricted three body problem via the Levi-Civita change of variables has been already considered in Celletti et al. (2011) to investigate numerically the qualitative behaviour of the solutions of a dissipative restricted three body problem. Ours is a more analytical approach, which stresses the geometrical interpretation of the regularization for the Kepler problem with linear drag. Perhaps the regularized system which we present here has some independent interest since it could be an useful tool in the study of the dynamics of more complex systems in dissipative celestial mechanics.

We gather below some straightforward facts about equation (3) which will be useful in what follows.

If we introduce the 'energy function'

$$E(z, \dot{z}) = \frac{|\dot{z}|^2}{2} - \frac{1}{|z|}, \tag{4}$$

then along the solutions of (3) it is

$$\dot{E}(t) := \frac{dE}{dt}(z(t), \dot{z}(t)) = -\epsilon|\dot{z}(t)|^2. \tag{5}$$

This fact rules out the existence of periodic orbits. Moreover, it is easily checked that the angular momentum satisfies

$$z \wedge \dot{z} = \mathcal{C}\mathrm{e}^{-\epsilon t}, \quad \mathcal{C} := z(0) \wedge \dot{z}(0), \tag{6}$$

where the symbol '$\wedge$' denotes the vector product.

We rewrite now equation (3) using polar coordinates. If we consider the change of variables $z = re^{i\theta}$, the new coordinates satisfy the following differential system:

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 + \epsilon\dot{r} = -\frac{1}{r^2} \\ \frac{d}{dt}r^2\dot{\theta} = -\epsilon r^2\dot{\theta}. \end{cases} \tag{7}$$

Recalling that $|z \wedge \dot{z}| = \pm r^2\dot{\theta}$, by (6) we get that the radial component of the solutions of (3) satisfies

$$\ddot{r} - \alpha^2 \, \frac{\mathrm{e}^{-2\epsilon t}}{r^3} + \epsilon\dot{r} = -\frac{1}{r^2} \tag{8}$$

where $\alpha = |\mathcal{C}|$.

In the next section we will study the non rectilinear motions of (3), whose radial component satisfies (8) with $\alpha \neq 0$.

2 Motions with nonzero angular momentum

Let $z(t)$ be a solution of (3) defined on the maximal interval $I_z =]t_0, t_1[$ with $0 \in I_z$. The results in Corne & Rouche (1973) imply that

$$|z(t)| \to 0 \quad \text{as } t \uparrow t_1. \tag{9}$$

In particular $z(t)$ is bounded in $[0, t_1[$. This result is valid for arbitrary solutions, including those with zero angular momentum. In this section we will study the non rectilinear motions, whose radial component satisfies (8) with $\alpha \neq 0$. In our first result we prove that these solutions are defined everywhere.

Proposition 2.1 *Let $z = re^{i\theta}$ be a solution of (3) with $\alpha \neq 0$. Then $z(t)$ is well defined for all times, that is $I_z =]-\infty, +\infty[$.*

Proof. It is enough to prove that $r(t)$, understood as a solution of (8), is defined on the whole real line. Let us first prove that $t_1 = +\infty$. Assume by contradiction that $t_1 < +\infty$. By (5) we know that the energy function

$$E(t) = E(z(t), \dot{z}(t)) = \frac{1}{2}(\dot{r}(t))^2 + \frac{\alpha^2 e^{-2\epsilon t}}{2r^2(t)} - \frac{1}{r(t)}$$

is decreasing along the solutions of (3). Since by (9) it is

$$\lim_{t \uparrow t_1} \left(\frac{\alpha^2 e^{-2\epsilon t}}{2r(t)^2} - \frac{1}{r(t)} \right) = +\infty$$

we get a contradiction with the inequality $E(t) < E(0)$, $t \in [0, t_1[$. Then, we conclude that $r(t)$ is defined in $[0, +\infty[$.

To prove that $t_0 = -\infty$ we will employ the following inequality, valid for an arbitrary parameter $A > 0$,

$$\frac{A}{x^3} - \frac{1}{x^2} \geq -\frac{4}{27A^2} \quad \text{for each } x > 0. \tag{10}$$

Again we proceed by contradiction and assume that $t_0 > -\infty$. Then, by the general theory of continuation of solutions of ODEs either

$$\limsup_{t \to t_0^+} [r(t) + |\dot{r}(t)|] = +\infty$$

or

$$\liminf_{t \to t_0^+} r(t) = 0.$$

Assume first that there exists a sequence $t_n \downarrow t_0$ such that $r(t_n) \to 0$. If we define

$$b(t) := \frac{\alpha^2 e^{-2\epsilon t}}{r(t)^3} - \frac{1}{r(t)^2}, \tag{11}$$

from the differential equation (8) and the inequality (10),

$$\ddot{r}(t) + \epsilon \dot{r}(t) = b(t) \geq -\frac{4e^{4\epsilon t}}{27\alpha^4} \geq -\frac{4}{27\alpha^4} =: \gamma$$

if $t \in]t_0, 0]$. The differential inequality $\ddot{r} + \epsilon \dot{r} \geq \gamma$ is equivalent to $\frac{d}{dt}(e^{\epsilon t}\dot{r}) \geq \gamma e^{\epsilon t}$ and it implies that

$$\dot{r}(t) \leq (\dot{r}(0) - \frac{\gamma}{\epsilon})e^{-\epsilon t} + \frac{\gamma}{\epsilon}, \;\; t \in]t_0, 0].$$

We have found the upper estimate

$$\dot{r}(t) \leq |\dot{r}(0) - \frac{\gamma}{\epsilon}|e^{-\epsilon t_0} + \frac{|\gamma|}{\epsilon} =: \beta, \;\; t \in]t_0, 0],$$

which has several consequences. First we observe that for each $t \in]t_0, 0]$ we can select N large enough so that $t_n < t$ if $n \geq N$. Hence

$$r(t) = r(t_n) + \int_{t_n}^{t} \dot{r}(\xi) d\xi \leq r(t_n) + \beta(t - t_0).$$

Letting $n \to \infty$ we deduce that

$$r(t) \leq \beta(t - t_0), \;\; \text{if } t \in]t_0, 0]. \tag{12}$$

In particular,

$$\lim_{t \to t_0^+} r(t) = 0. \tag{13}$$

Our next step will be to prove the existence of a number $\delta > 0$ such that

$$\dot{r}(t) \geq 0 \;\; \text{if } t \in]t_0, t_0 + \delta].$$

Together with a previous estimate this will imply that $0 \leq \dot{r} \leq \beta$ and so $\dot{r}(t)$ remains bounded on $]t_0, t_0 + \delta]$. To prove the positivity of $\dot{r}$ near t_0 we observe that it is possible to find a sequence $t_n^* \downarrow t_0$ satisfying $\dot{r}(t_n^*) \geq 0$. This is a consequence of (13). Also, if $b(t)$ is defined by (11), we have that the inequality

$$b(t) > 0 \;\; \text{if } t \in]t_0, t_0 + \delta] \tag{14}$$

will hold for appropriate δ. Going back to (8) we deduce that $\ddot{r} + \epsilon\dot{r} > 0$ on the interval $]t_0, t_0 + \delta]$. Hence the function $e^{\epsilon t}\dot{r}(t)$ is increasing and

$$e^{\epsilon t_n^*}\dot{r}(t_n^*) < e^{\epsilon t}\dot{r}(t) \;\text{ if } t \in]t_0, t_0 + \delta] \text{ and } n \geq N^*,$$

where N^* is such that $t_n^* < t$ if $n \geq N^*$. We easily deduce that $\dot{r}$ is non-negative near t_0. Once we know that the limit (13) holds and $\dot{r}(t)$ is bounded we observe that the function $b(t)$ is integrable (in the Lebesgue sense) on the interval $]t_0, t_0 + \delta]$. Indeed, if we integrate (8) between t and $t_0 + \delta$, we obtain

$$\dot{r}(t_0 + \delta) - \dot{r}(t) + \epsilon(r(t_0 + \delta) - r(t)) = \int_t^{t_0+\delta} b(\tau)d\tau$$

and so the integral remains finite as $t \downarrow t_0$. Once again we apply (13) to find numbers $\mu > 0$ and $\delta_1 \in]0, \delta]$ such that

$$b(t) \geq \frac{\mu}{r(t)^3}, \;\; t \in]t_0, t_0 + \delta_1].$$

We have arrived at a contradiction because the above inequality and (12) imply that $b(t)$ has a divergent integral. This discussion shows that the solutions cannot go to zero. In particular the function $b(t)$ remains bounded on $]t_0, 0]$. It remains to discuss the possibility of a blow-up, that is $r(t) + |\dot{r}(t)| \to \infty$ as $t \to t_0^+ > -\infty$. However, this is excluded because $r(t)$ is a solution of $\ddot{r} + \epsilon\dot{r} = b(t)$, and we conclude that $t_0 = -\infty$.

□

As a corollary we get:

Proposition 2.2 *Let $z(t) = r(t)e^{i\theta(t)}$ be a non collinear solution of* (3)*. Then*

$$|z(t)| \to \infty, \; |\dot{z}(t)| \to \infty \;\; \textit{as } t \to -\infty \quad \textit{and} \quad |z(t)| \to 0 \;\; \textit{as } t \to +\infty.$$

Before the proof of this result we present two lemmas. The first is a simple observation on differential inequalities. The second is concerned with the solutions of (8).

Lemma 2.3 *Assume that $\gamma, \tau \in \mathbb{R}$ and $R(t)$ is a function in $C^2(]-\infty, \tau])$ satisfying*

$$\ddot{R}(t) + \epsilon\dot{R}(t) \geq \gamma, \;\; t \in]-\infty, \tau], \; \dot{R}(\tau) < \frac{\gamma}{\epsilon}.$$

Then $\dot{R}(t) \to -\infty$ as $t \to -\infty$.

The proof of this lemma is obtained via a straightforward integration of the inequality $\frac{d}{dt}(e^{\epsilon t}\dot{R}(t)) \geq \gamma e^{\epsilon t}$.

Lemma 2.4 *Let $r(t)$ be a solution of* (8) *with $\alpha \neq 0$. Then*

$$\dot{r}(\tau) < -\frac{4}{27\epsilon\alpha^4}e^{4\epsilon\tau}$$

for some $\tau \in \mathbb{R}$.

Proof. By a contradiction argument assume that the inequality

$$\dot{r}(t) \geq -\frac{4}{27\epsilon\alpha^4}e^{4\epsilon t}$$

holds for every $t \in \mathbb{R}$. Then the function

$$f(t) = r(t) + \frac{1}{27\epsilon^2\alpha^4}e^{4\epsilon t}$$

should be increasing and the limit $r(-\infty)$ should exist and could not be $+\infty$. Note that $r(-\infty) = f(-\infty) \leq f(t)$ for each $t \in \mathbb{R}$. In consequence the function $b(t)$ defined by (11) would satisfy $b(t) \to +\infty$ as $t \to -\infty$. From $\ddot{r} + \epsilon\dot{r} = b(t)$ we could deduce the existence of a number $\hat{\tau} \in \mathbb{R}$ such that

$$\ddot{r}(t) + \epsilon\dot{r}(t) \geq 1 \ \text{ if } t \in]-\infty, \hat{\tau}].$$

As another consequence of the existence of the finite limit $r(-\infty)$ we could select a sequence $t_n \to -\infty$ with $\dot{r}(t_n) \to 0$. In particular there should exist n large enough so that $t_n < \hat{\tau}$ and $\dot{r}(t_n) < \frac{1}{\epsilon}$. If we apply Lemma 2.3 with $R = r$, $\gamma = 1$ and $\tau = t_n$ we conclude that $\dot{r}(-\infty) = -\infty$. We have arrived at the searched contradiction because $r(-\infty) < +\infty$ and $\dot{r}(-\infty) = -\infty$ cannot hold simultaneously.

□

Proof of Proposition 2.2. The limit as $t \to +\infty$ is a consequence of the results in Corne & Rouche (1973) together with Proposition 2.1. The limits as $t \to -\infty$ are a consequence of

$$\lim_{t \to -\infty} \dot{r}(t) = -\infty. \tag{15}$$

Note that $|z(t)| = r(t)$ and $|\dot{z}(t)| \geq |\dot{r}(t)|$. To prove (15) we fix the number τ given by Lemma 2.4. Then, taking also into account (10), we apply again Lemma 2.3 with $R = r$ and $\gamma = -\frac{4}{27\alpha^4}e^{4\epsilon\tau}$.

□

Remark 2.1 An obvious consequence of Proposition 2.2 which will be useful in what follows is that there exists a sequence $t_n \to +\infty$ such that $\dot{r}(t_n) \leq 0$ and $\dot{r}(t_n) \to 0$ as $n \to +\infty$.

To prove our next result, namely the unboundedness of the angular velocity of non rectilinear motions, we introduce the regularization of (3) presented in Diacu (1999) to study the Kepler problem for a generalized drag force. This framework will be also used in the next section to give a fairly complete description of the linear motions.

As in Diacu (1999), we set $u = \dot{r}$, $\phi = \dot{\theta}$ and use the rescaling of time $d\tau = r^{-2}dt$. Then, taking into account system (7), we have that (3) is equivalent to the following first order system in the new time τ :

$$\begin{cases} r' = ur^2 \\ u' = -\epsilon ur^2 + r^3\phi^2 - 1 \\ \phi' = -2u\phi r - \epsilon\phi r^2. \end{cases} \tag{16}$$

Since by the previous proposition we know that $0 < r(t) \leq M,\ t \geq 0$ for a suitable M, it is $\tau = \int_0^t \frac{1}{r^2(\sigma)}\,d\sigma \geq \frac{t}{M^2}$, so that $\tau \to +\infty$ when $t \to +\infty$.

Proposition 2.5 *There exists a sequence* $t_n \to +\infty$ *such that*

$$|\dot{\theta}(t_n)| \to +\infty, \quad n \to +\infty.$$

Proof. By contradiction, assume that there exists a constant $M > 0$ such that

$$|\phi(\tau)| \leq M, \quad \tau \in [0, +\infty[. \tag{17}$$

Summing up the first and second equations of system (16) and integrating we get

$$u(\tau) + \epsilon r(\tau) = u(0) + \epsilon r(0) + \int_0^\tau r^3(\sigma)\phi^2(\sigma)\,d\sigma - \tau. \tag{18}$$

By Proposition 2.2 and subsequent remark, we can consider a sequence $\tau_n \to +\infty$ such that $r_n := r(\tau_n) \to 0$ and $u_n := u(\tau_n) \to 0$ as $n \to +\infty$. Moreover, since $r(\tau) \to 0$ as $\tau \to +\infty$ and (17) holds, we can find $\bar{\tau} > 0$ such that

$$r^3(\sigma)\phi^2(\sigma) \leq \frac{1}{2}, \quad \sigma \in [\bar{\tau}, +\infty[.$$

Then, by (18) for any $\tau_n > \bar{\tau}$ we obtain

$$u_n + \epsilon r_n \leq u(0) + \epsilon r(0) + \int_0^{\bar{\tau}} r^3(\sigma)\phi^2(\sigma)\,d\sigma - \frac{\tau_n + \bar{\tau}}{2}.$$

Taking the limit as $n \to \infty$ we get a contradiction, since the first member of the inequality tends to zero and the second to $-\infty$. Therefore, $|\phi(\tau)|$ is unbounded in $[0, +\infty[$ and the same is true for $\dot{\theta}(t)$.

□

Notice that, since the plane $\phi = 0$ is invariant, if $\phi(0) = \dot{\theta}(0) \neq 0$, then the sign of $\dot{\theta}(t)$ is constant, so that, in the statement above, $|\dot{\theta}(t_n)| = \dot{\theta}(t_n)$ if $\dot{\theta}(0) > 0$ and $|\dot{\theta}(t_n)| = -\dot{\theta}(t_n)$ if $\dot{\theta}(0) < 0$.

Remark 2.2 System (16) is of polynomial type and so it is well defined on the whole space $\mathbb{R}^3$. It has no equilibria and there are two invariant planes, namely the plane $r = 0$ (which is the collision manifold) and the plane $\phi = 0$ (which corresponds to rectilinear motions), see Figure 1 below. The study of the near collisions solutions presented in Diacu (1999) for the generalized Stokes drag takes advantage of the presence of the equilibrium $(r, u, \phi) = (0, 1, 0)$ on the collision manifold $r = 0$ for the regularized problem. In the case of the linear drag, there are no equilibria on the collision manifold. Moreover, although we know by Proposition 2.2 and Remark 2.1 that $r(t) \to 0$ as $t \to +\infty$, and that $u(\tau_n) = \dot{r}(\tau_n) \to 0^-$ for a suitable $\tau_n \to +\infty$, by Proposition 2.5 we get that the non rectilinear orbits eventually escape from any compact set of the (r, u, ϕ) phase space.

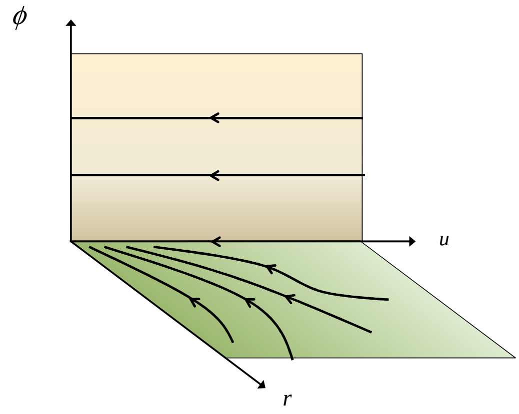

Figure 1: Flow in the invariant planes of system (16)

Many asymptotic behaviours of the non rectilinear orbits are compatible with these facts, and the configuration of the flow on the invariant manifolds $r = 0$ and $\phi = 0$, together with the continuous dependence of the solutions on the initial conditions, are not sufficient to choose between different alternatives.

3 Study of the linear motions

In this section we use the framework provided by the regularized system (16) to study the rectilinear motions of (3).

Setting $\alpha = 0$ in (8) we see that the linear motions of the Kepler equation with linear drag are governed by the scalar equation

$$\ddot{r} + \varepsilon \dot{r} = -\frac{1}{r^2} \tag{19}$$

where $r > 0$. The energy function along the rectilinear motions is given by

$$E(r, \dot{r}) = \frac{\dot{r}^2}{2} - \frac{1}{r} \tag{20}$$

and along these solutions it is

$$\dot{E}(t) = -\varepsilon \dot{r}^2(t). \tag{21}$$

We recall that a solution of (19) is said an ejection-collision solution if it is defined in a bounded interval $]t_0, t_1[$ and satisfies

$$\lim_{t \to t_0^+} r(t) = \lim_{t \to t_1^-} r(t) = 0, \quad \lim_{t \to t_0^+} \dot{r}(t) = +\infty, \ \lim_{t \to t_1^-} \dot{r}(t) = -\infty.$$

and is said a capture-collision solution if it is defined on an interval of the form $] - \infty, t_1[$ with $t_1 \in \mathbb{R}$ and satisfies

$$\lim_{t \to -\infty} r(t) = +\infty, \lim_{t \to t_1^-} r(t) = 0, \quad \lim_{t \to t_1^-} \dot{r}(t) = -\infty.$$

In the setting of (16), the linear motions correspond to the motions on the invariant plane $\phi = 0$, which are governed by the system

$$\begin{cases} r' = ur^2 \\ u' = -\epsilon ur^2 - 1. \end{cases} \tag{22}$$

In what follows we will focus on system (22) and give a fairly complete description of its solutions and of the behaviour of the corresponding energy.

Proposition 3.1 *The solutions of* (19) *are either ejection-collision or capture-collision solutions, and ejections and collisions occur with finite energy.*

Moreover, in the phase-plane $(r, \dot{r})$ *associated to* (19) *there exists a capture-collision orbit* γ *whose energy tends to zero as* $t \to -\infty$ *and which separates the ejection-collision orbits from the remaining capture-collision orbits, whose energy tend to* $+\infty$ *as* $t \to -\infty$.

Proof. We will first determine the asymptotic behaviour of the solutions of (22) in their domain $]\tau_0, \tau_1[$, showing as well that $\tau_1 = +\infty$ and that, depending on the initial condition, τ_0 may be finite or not. Then, we will study the behaviour of the energy on the solutions. Finally, in order to classify properly the solutions of (19) as ejection-collision or capture-collision solutions, we will go back to the original time t to address the finiteness or not of the endpoints t_0 and t_1 of the domain of the solutions of (19).

The proof is divided in several steps. The more evident facts we use are just stated. From now on we assume $\tau_0 < 0 < \tau_1$.

i) The equality

$$H := u + \epsilon r + \tau = constant \tag{23}$$

holds along the solutions of (22), that is H is a constant of motion.

This follows immediately summing and integrating the two equations of system (22).

ii) The line $r = 0$ is an orbit so that $\Omega := \{(r, u) : r > 0\}$ is invariant.

In what follows we will work in Ω.

iii) The isoclines $r' = 0$ and $u' = 0$ determine the regions (see Figure 2)

$$A_0 = \{(r, u) \in \Omega : u \geq 0\}, \ \ A_1 = \{(r, u) \in \Omega : 0 > u > -\frac{1}{\epsilon r^2}\},$$

$$A_2 = \{(r, u) \in \Omega : u \leq -\frac{1}{\epsilon r^2}\}.$$

The set A_1 is positively invariant while A_0 and A_2 are negatively invariant.

iv) all the orbits eventually enter into A_1.

By contradiction, assume that an orbit remains in A_0 or A_2. If the orbit remains in A_0, then $u(\tau) \geq 0$ if $\tau \in J =]\tau_0, \tau_1[$. Then, $r' \geq 0$ and $u' < 0$ for every $\tau \in J$. In particular $0 \leq u(\tau) \leq u(0)$ if $\tau \in [0, \tau_1[$. From (23) it follows that

$$\epsilon r(\tau) = u(0) + \epsilon r(0) - u(\tau) - \tau \leq u(0) + \epsilon r(0).$$

We conclude that the positive orbit remains in the rectangle $[0, \frac{u(0)}{\epsilon} + r(0)] \times [0, u(0)]$. This implies that $\tau_1 = +\infty$ and the limits $r(+\infty)$, $u(+\infty)$ exist and

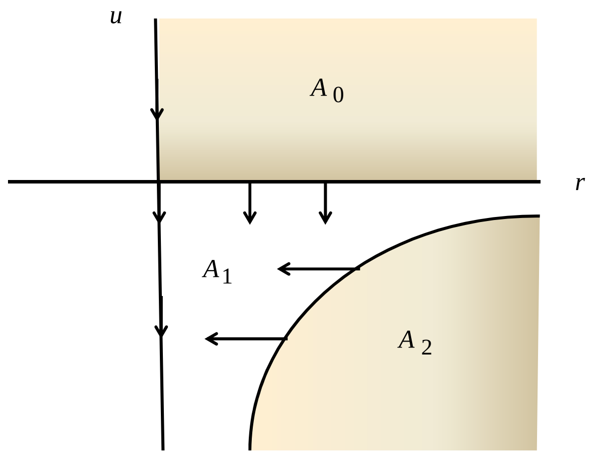

Figure 2: Regions defined by the isoclines $r' = 0$ and $u' = 0$ of system (22)

are finite. This is absurd since $(r(+\infty), u(+\infty))$ should be an equilibrium. If the orbit remains in A_2 then

$$u(\tau) \leq -\frac{1}{\epsilon r^2(\tau)}, \quad \tau \in]\tau_0, \tau_1[.$$

This case is easier. From $u' > 0$, $r' < 0$ we deduce that the positive orbit should remain in $[0, r(0)] \times [u(0), 0]$ and converge to an equilibrium, and again we arrive at a contradiction, ending the proof that all the orbits enter eventually in A_1.

v) for every orbit, it is $\tau_1 = +\infty$ and, moreover, $r(+\infty) = 0$, $u(+\infty) = -\infty$.

It is not restrictive to assume $(r(\tau), u(\tau)) \in A_1$ if $\tau \in [0, \tau_1[$. Hence $r' < 0$, $u' < 0$ and, in particular, $0 < r(\tau) \leq r(0)$, $\tau \in [0, \tau_1[$. Next we prove that $\tau_1 = +\infty$. Otherwise $r(\tau)$ and

$$u(\tau) = u(0) + \epsilon r(0) - \epsilon r(\tau) - \tau$$

would remain bounded for $\tau \in [0, \tau_1[$ and the solution could be extended to a larger interval. Once we know that $\tau_1 = +\infty$ we deduce that the limits $r(+\infty) \in [0, r(0)[$ and $u(+\infty) \in [-\infty, u(0)[$ exist. Then

$$\lim_{\tau \to +\infty} [u(\tau) + \tau] = u(0) + \epsilon r(0) - \epsilon r(+\infty)$$

is finite and it must be $u(+\infty) = -\infty$. The existence of finite $r(+\infty)$ implies that, for some sequence $\tau_n \to +\infty$, $r'(\tau_n) = u(\tau_n)r^2(\tau_n) \to 0$ as $n \to +\infty$. As a consequence $r(\tau_n) \to 0$ and therefore $r(+\infty) = 0$.

The behaviour of the solutions in the past and the finiteness or not of τ_0 depend on the initial condition.

More precisely, we have the following three cases:

vi)-1) if $(r(0), u(0)) \in A_0$, then $\tau_0 = -\infty$, and $r(-\infty) = 0$, $u(-\infty) = +\infty$.

We know that $r' > 0$ and $u' < 0$ if $\tau \in]\tau_0, 0[$, and so $0 < r(\tau) < r(0)$. By using (23) and repeating a previous argument we deduce that $\tau_0 = -\infty$. Since $r(-\infty)$ is finite we deduce from (23) that $u(-\infty) = +\infty$. Finally, an argument completely analogous to the the last one in v) shows that $r(-\infty) = 0$.

vi)-2) if $(r(0), u(0)) \in A_2$, then $\tau_0 > -\infty$, and $r(\tau_0^+) = +\infty$, $u(\tau_0^+) = -\infty$.

We know that $u' > 0$, $r' < 0$ on $]\tau_0, 0[$. Let us prove that $\tau_0 > -\infty$ by a contradiction argument. If $\tau_0 = -\infty$, then $r(-\infty) \in]r(0), +\infty]$, $u(-\infty) \in [-\infty, u(0)[$ and from (23) we deduce that $r(-\infty) = +\infty$. From the first equation of system (22) we get

$$\int_\tau^0 u(\sigma)\, d\sigma = \frac{1}{r(\tau)} - \frac{1}{r(0)}$$

implying that $u(\tau)$ is integrable in $]-\infty, 0]$. This is absurd because $u(\tau) \leq u(0) < 0$. Once we know that $\tau_0 > -\infty$, we can say that $r(\tau_0^+) \in]r(0), +\infty]$, $u(\tau_0^+) \in [-\infty, u(0)[$ and at least one of these two numbers is not finite. Since

$$\lim_{\tau \to \tau_0^+} [u(\tau) + \epsilon r(\tau)] = u(0) + \epsilon r(0) - \tau_0 \in \mathbb{R},$$

we deduce that none of them can be finite.

vi)-3) The remaining case: the parabolic orbit in A_1.

We note that the orbits of (22) with initial conditions in A_0 or in A_2 do not fill completely the phase-space Ω. In fact, since the set A_1 is connected, there exists at least one orbit remaining in A_1 for all τ. For such orbits it is $r(\tau) \to +\infty$ and $u(\tau) \to 0$ as $\tau \to \tau_0 = -\infty$ (the fact that $\tau_0 = -\infty$ follows immediately from (23)). To show that there is only one of such trajectories we consider the change of variables $(r, u, \tau) \to (s, u, \eta)$ where $r = \frac{1}{s}$, $d\eta = \frac{d\tau}{s^2}$ which transforms system (22) in the following

$$\begin{cases} \dfrac{ds}{d\eta} = -us^2 \\ \dfrac{du}{d\eta} = -\epsilon u - s^2. \end{cases} \tag{24}$$

Any trajectory of (22) which remains in A_1 for all time τ is transformed in a trajectory of (24) approaching $(s, u) = (0, 0)$ as $\eta \to -\infty$. Note that $\int_{-\infty}^{0} r(\sigma)^2 d\sigma = +\infty$. By theorem 7.1 of chapter 2 in Zhang et al. (1992) the origin is a saddle-node for (24). This means that, up to a homeomorphism, the phase portrait around the origin is equivalent to the one illustrated in the following figure:

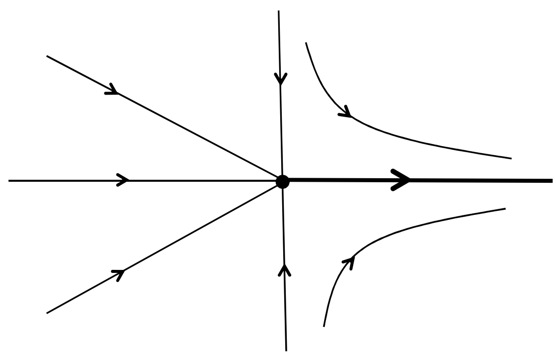

Figure 3: Topological representation of the phase plane of system (24) around the saddle-node

In consequence there is a unique orbit emanating from the origin. This implies that the orbit γ of (22) which never leaves A_1 is unique. This is a separatrix between the orbits which, in the past, eventually enter A_0 from the ones that eventually enter A_2. The proof in Zhang et al. (1992) is long because it gives a complete picture of the dynamics. Actually, we only need the uniqueness of the orbit emanating from the origin. In an appendix at the end of the paper we have extracted from Zhang et al. (1992) the geometric argument proving this uniqueness.

We pass now to the study of the behaviour of the energy on the solutions. We show first that the limit of the energy as $\tau \to +\infty$ is finite for all the orbits of (22). The energy $E(\tau) = E(r(\tau), u(\tau))$ satisfies $E'(\tau) = -\epsilon r^2(\tau) u^2(\tau)$. We have proved that, eventually, any solution $(r(\tau), u(\tau)) \in \Omega$ of (22) enters A_1 so that, without loss of generality, we may assume that $r(\tau) \leq r(0),\ u(\tau) < 0$ for any $\tau \in [0, +\infty[$. From (23) we get

$$\tau - \epsilon r(0) - u(0) \leq -u(\tau) = |u(\tau)| = \epsilon r(\tau) - \epsilon r(0) - u(0) + \tau \leq \tau - u(0) \quad (25)$$

and so

$$\lim_{\tau \to +\infty} \frac{u(\tau)}{\tau} = -1.$$

Then, by the first equation of (22) we obtain

$$\frac{1}{r(\tau)} - \frac{1}{r(0)} = \int_0^\tau |u(\sigma)|\, d\sigma$$

and, after integrating the inequality (25), we deduce that

$$\lim_{\tau \to +\infty} \tau^2 r(\tau) = 2. \quad (26)$$

It follows immediately that r^2u^2 behaves like $\frac{4}{\tau^2}$ as $\tau \to +\infty$. As a consequence, $E'(\tau)$ is integrable in $[0, +\infty[$ so that

$$\lim_{\tau \to +\infty} E(\tau) = E(0) + \int_0^{+\infty} E'(\sigma)\, d\sigma \in \mathbb{R}$$

and the energy has a finite limit.

A similar argument shows that for all orbits above γ, the ones which eventually enter A_0 in the past, it is

$$\lim_{\tau \to -\infty} E(\tau) \in \mathbb{R}.$$

For trajectories which lie below γ, which eventually enter A_2 in the past, since $r(\tau_0^+) = +\infty$, $u(\tau_0^+) = -\infty$ it is $E(\tau_0^+) = \frac{u^2(\tau_0^+)}{2} - \frac{1}{r(\tau_0^+)} = +\infty$. Finally, if $(r(0), u(0)) \in \gamma$ from $r(-\infty) = +\infty$, $u(-\infty) = 0$ it follows immediately $E(-\infty) = 0$.

To end the proof of the proposition, it remains only to study the finiteness or not of t_0 and t_1, the endpoints of the maximal intervals of definition of the solutions of (19). Consider a solution $r(t)$ of (19) defined in $]t_0, t_1[$ and let $]\tau_0, \tau_1[$ be the domain of the corresponding solution of (22), $(r(\tau), u(\tau)) := (r(T(\tau)), \dot{r}(T(\tau)))$ of (22). If $T(\tau)$ is the inverse function of $\tau(t) = \int_0^t \frac{1}{r^2(\sigma)}\, d\sigma$, we investigate the finiteness or not of

$$t_0 = \lim_{\tau \to \tau_0^+} T(\tau),\ t_1 = \lim_{\tau \to \tau_1^-} T(\tau).$$

We start by showing that it is always $t_1 < +\infty$. In fact, $T'(\tau) = r^2(\tau)$, and from (26) it follows that $r^2(\tau)$ is integrable in $[0, +\infty[$, since it behaves like $\frac{4}{\tau^4}$ for large τ. We conclude that

$$t_1 = \int_0^{+\infty} T'(\sigma)\, d\sigma \in \mathbb{R}.$$

As to the time t_0, we have to distinguish three cases. If $(r(0), \dot{r}(0)) = (r(0), u(0))$ is above γ, an argument similar to the one used above shows that $t_0 > -\infty$.

If $(r(0), \dot{r}(0))$ is lies on γ, we know that $\tau_0 = -\infty$, $r(\tau_0^+) = +\infty$ and therefore

$$t_0 = -\int_{\tau_0}^{0} r^2(\sigma)\, d\sigma = -\infty.$$

If $(r(0), \dot{r}(0))$ is below γ we know that $u(\tau) < 0$ for each $\tau \in]\tau_0, 0]$. After integrating the second equation in (22) we are lead to the identity

$$\ln\left|\frac{u(0)}{u(\tau)}\right| = \int_\tau^0 \frac{u'(\sigma)}{u(\sigma)} d\sigma = -\epsilon \int_\tau^0 r(\sigma)^2 d\sigma - \int_\tau^0 \frac{d\sigma}{u(\sigma)}.$$

From $\tau_0 > -\infty$ and $u(\tau_0^+) = -\infty$ we deduce that the integral $\int_{\tau_0}^0 \frac{d\sigma}{u(\sigma)}$ is finite and

$$t_0 = -\int_{\tau_0}^{0} r^2(\sigma)\, d\sigma = -\infty.$$

Our proof is concluded.

□

A consequence of Proposition 3.1 is that the behaviour of solutions at collisions or ejections is the same as in the conservative case. Indeed, given a solution $r(t)$ of (19) defined on a maximal interval I having t_* as a finite end point, the expansions below hold

$$r(t) = (\frac{9}{2})^{1/3}(t - t_*)^{2/3} + O((t - t_*)^{4/3}) \quad (27)$$

$$\dot{r}(t) = \pm\frac{2}{3}(\frac{9}{2})^{1/3}(t - t_*)^{-1/3} + O((t - t_*)^{1/3}) \quad (28)$$

as $t \to t_*$, $t \in I$. The sign + corresponds to ejections and − is for collisions.

These estimates were obtained by Sperling in Sperling (1969/1970) in the context of a general perturbed Kepler problem of the type

$$\ddot{z} = -\frac{z}{|z|^3} + P(t, z, \dot{z}).$$

It was assumed in Sperling (1969/1970) that the perturbation P is bounded and this is not true in our case, since we have $P = -\epsilon \dot{z}$. Fortunately we know that the energy is finite at t_* and this will be sufficient to justify the estimates. From $\dot{E} = -\epsilon \dot{r}^2$ we deduce that the function

$$K(t) = \frac{1}{2} \int_{\hat{t}}^{t} \dot{r}(\tau)^2 d\tau$$

is finite and continuous at $t = t_*$. Here $\hat{t}$ is any fixed instant lying on the interval I. From the previous formulas we deduce that

$$\frac{1}{2}\dot{r}(t)^2 - \frac{1}{r(t)} = E(\hat{t}) - 2\epsilon K(t)$$

and, since $\dot{r}$ does not change sign in a neighborhood of t_*, we deduce that $r(t)$ satisfies one of the two differential equations

$$\dot{r} = \pm\sqrt{b(t) + \frac{2}{r}},$$

where $b(t)$ is continuous on $I \cup \{t_*\}$. Now it is possible to repeat the proof in Sperling (1969/1970) (see also Ortega (2011)) to verify that (27) and (28) hold in our case.

4 Asymptotic values of the energy through the Levi-Civita regularization

The Levi-Civita transformation is defined by the change of variables

$$z = w^2, \; ds = \frac{dt}{|z|}, \; z \in \mathbb{C} \setminus \{0\}. \tag{29}$$

We recall that the map $z = w^2$ is not a diffeomorphism of $\mathbb{C} \setminus \{0\}$ and so this change is not one-to-one. The crucial property of $z = w^2$ is that it is a covering map and every solution $z = re^{i\theta}$ of (3) produces two lifts $w_1 = r^{1/2}e^{i\frac{\theta}{2}}$, $w_2 = r^{1/2}e^{i(\frac{\theta}{2}+\pi)}$.

Some computations show that equation (3) is transformed in the following equation

$$w'' - \frac{|w'|^2}{\bar{w}} + \epsilon|w|^2 w' + \frac{w}{2|w|^2} = 0 \tag{30}$$

where the derivatives are taken with respect to time s. In order to remove the singularity from the equation, we express the energy and its derivative in terms of the new variables, getting

$$E = 2\frac{|w'|^2}{|w|^2} - \frac{1}{|w|^2}, \;\; E' = -2\epsilon(E|w|^2 + 1). \tag{31}$$

Substituting the first equality above in (30) we arrive to the following system

$$\begin{cases} w' = v \\ v' = \frac{Ew}{2} - \epsilon|w|^2 v \\ E' = -2\epsilon(E|w|^2 + 1) \end{cases} \tag{32}$$

which defines a polynomial vector field in $\mathbb{R}^5$. Not all orbits of this system are in correspondence with orbits of the original Kepler problem. As an example consider the solution of (32) given by $w = v = 0, E(s) = -2\epsilon s$, whose orbit is a line in $\mathbb{R}^5$ that is unrelated to (3). From the formula for the energy (31) we deduce that physically meaningful solutions should satisfy

$$E|w|^2 + 1 - 2|w'|^2 = 0. \tag{33}$$

This implies that the system (32) has to be considered on the manifold $\mathcal{M}$ defined by this equation.[2] Note that if we define the function

$$\mathcal{J}(E, w, w') := E|w|^2 + 1 - 2|w'|^2$$

then the derivative along the vector field satisfies

$$\frac{d\mathcal{J}}{ds} = -2\epsilon|w|^2\mathcal{J}.$$

In particular $\mathcal{M} = \mathcal{J}^{-1}(0)$ is invariant under the flow. To establish the precise correspondence between orbits of (3) and (32)-(33) we start with a solution $z(t) = r(t)e^{i\theta(t)}$ of (3) defined on a maximal interval I with $0 \in I$. Define Sundman's integral

$$S(t) = \int_0^t \frac{d\tau}{r(\tau)}$$

and consider the energy along the solution

$$E(t) = \frac{1}{2}|\dot{z}(t)|^2 - \frac{1}{|z(t)|}.$$

[2] It can be proved that $\mathcal{M}$ is a connected manifold of dimension four that is not compact and has the same type of homotopy of a 3-sphere $\mathbb{S}^3$.

The function S is an increasing diffeomorphism between I and some open interval J. The inverse function $T = T(s)$, $s \in J$, allows to define the solution of (32)-(33),

$$w(s) = r(T(s))^{1/2} e^{i\frac{\theta(T(s))}{2}}, \quad v = w', \quad E(s) = E(T(s)).$$

When $z(t)$ has non-zero angular momentum we know that $I =]-\infty, +\infty[$ and $r(t) \to 0$ as $t \to +\infty$. From here we deduce that $S(t) \to +\infty$ as $t \to +\infty$ and so $J =]s_0, +\infty[$ with $-\infty \leq s_0 < 0$. We observe that the interval J is maximal for the solution of (32) because $|w(s)| \to \infty$ as $s \to s_0^+$. At this point we are ready to describe the asymptotic behaviour of the energy.

Proposition 4.1 *If $z(t)$ is a solution of* (3) *with non zero angular momentum, then*

$$\lim_{t \to \pm\infty} E(t) = \mp\infty.$$

Proof. Let us prove first that $E(t) \to +\infty$ as $t \to -\infty$. From (15) of Proposition 2.2 we know that $\dot{r}(t) \to -\infty$ as $t \to -\infty$. Hence we can find a time t_* such that $|\dot{z}(t)| \geq |\dot{r}(t)| \geq 1$ if $t \leq t_*$. In consequence

$$E(t) = E(t_*) + \epsilon \int_t^{t_*} |\dot{z}(\tau)|^2 d\tau \geq E(t_*) + \epsilon(t_* - t) \to +\infty \text{ as } t \to -\infty.$$

We use Levi-Civita regularization to analyze the behaviour at $+\infty$. Let $E(s) = E(T(s))$. Assume by contradiction that $\lim_{s\to+\infty} E(s) = E_0 \in \mathbb{R}$ (recall that this limit exists because the energy is decreasing.) Since $w(s) \to 0$ as $s \to +\infty$, from (32) we have $E'(s) \leq -\epsilon$ for sufficiently large s, and we deduce $E(s) \to -\infty$ as $s \to +\infty$, a contradiction. Going back to the variable t, we get our result.

□

Let us return to the discussion on how to go from (3) to (32)-(33), now for rectilinear solutions. If the angular momentum vanishes and the solution $z(t) = r(t)$ has a maximal interval $I =]t_0, t_1[$, the corresponding interval $J =]s_0, s_1[$ will not be maximal for (32). The reason is that a collision solution of (3) can be produced by the rule $r(t_1^+) = r(t_1^-) = 0$, $E(t_1^+) = E(t_1^-)$ (see Ortega (2011)) and this generalized solution will lead to a classical solution of (32)-(33). Note that the number s_1 is finite as a consequence of the expansion (27). The same can be said about s_0 if $t_0 > -\infty$. At this point it may be interesting to observe that the transformed of a linear motion must satisfy very particular initial conditions at collisions. From $r(t_1) = 0$ and the identity (33) we obtain

$$w(s_1) = 0, \quad v(s_1) = -\frac{1}{\sqrt{2}}, \quad E(s_1) = E_1 \in \mathbb{R}.$$

The same conclusion is obtained in the past if $t_0 > -\infty$, now $v(s_0) = \frac{1}{\sqrt{2}}$.

The proof of our last result will suggest how to go (locally in time) from solutions of (32)-(33) to solutions of (3).

Proposition 4.2 *For any time $t_1 \in \mathbb{R}$ and any energy value $E_1 \in \mathbb{R}$, there exists a solution $u(t)$ of (19) defined in a maximal interval of the form $I =]t_0, t_1[$ for which*

$$\lim_{t \to t_1^-} E(r(t), \dot{r}(t)) = E_1. \tag{34}$$

An analogous statement holds considering the initial time t_0 of an ejection trajectory and any fixed energy value $E_0 \in \mathbb{R}$.

Proof. Fix any time $t_1 \in \mathbb{R}$ and any energy value $E_1 \in \mathbb{R}$, and consider the solution $s \to (w(s), v(s), E(s))$ of the system (32) with initial conditions

$$w(0) = 0,\ v(0) = -\frac{1}{\sqrt{2}},\ E(0) = E_1.$$

The solution of the previous Cauchy problem is unique and well defined in a neighborhood of $s = 0$. Moreover, there exists a left neighborhood of $s = 0$, say $I_\delta := [-\delta, 0]$ for a suitable positive δ, such that $w(s) > 0$ in I_δ. Setting $T(s) = t_1 - \int_s^0 w^2(\sigma)\, d\sigma$, we see that the restriction to I_δ of $T(s)$ admits an inverse function $T^{-1} : [t_1 - \eta, t_1] \to I_\delta,\ s = S(t),$ which is continuous on $J_\eta := [t_1 - \eta, t_1]$ and C^2 on $J_\eta \setminus \{t_1\}$. Then, on $J_\eta \setminus \{t_1\}$ we can define the C^2 function $r(t) := w^2(S(t))$. A computation shows that $r(t)$ satisfies the equation

$$\ddot{r} + \varepsilon \dot{r} = \frac{1}{r^2}\mathcal{J} - \frac{1}{r^2}. \tag{35}$$

Since the initial conditions lie on $\mathcal{M}$, the function $\mathcal{J}$ vanishes along the solution of (32) and we conclude that $r(t)$ satisfies (19). By the definition of $r(t)$, it is clear that $r(t) > 0$ in J_η and that $\lim_{t \to t_1^-} r(t) = 0$, and since by Proposition 3.1 all the linear motions are of collision type it follows also that $\lim_{t \to t_1^-} \dot{r}(t) = -\infty$. Moreover, from $\mathcal{J} \equiv 0$ we obtain

$$E(S(t)) = 2\frac{(w')^2(S(t))}{w^2(S(t))} - \frac{1}{w^2(S(t))} = \frac{\dot{r}^2(t)}{2} - \frac{1}{r(t)} = E(r(t), \dot{r}(t)) \to E_1$$

as $t \to t_1^-$, and also (34) is proved. The statement about the ejection trajectories can be proved adapting in a straightforward manner the above proof. $\square$

5 Conclusions

In this paper we have studied the dynamics of a Kepler problem with linear drag. Our results about the solutions $z(t)$ of this problem can be classified in two categories. We summarize them below. Recall that $z = re^{i\theta}$ and $E = \frac{1}{2}|\dot{z}|^2 - \frac{1}{|z|}$.

1. Non-vanishing angular momentum ($\dot{\theta} \neq 0$)

- There are no collisions
- $\lim_{t\to-\infty} |z(t)| = \infty, \quad \lim_{t\to+\infty} z(t) = 0$
- $\lim_{t\to\pm\infty} E(t) = \mp\infty$
- $\liminf_{t\to\infty} |\dot{r}(t)| = 0, \quad \limsup_{t\to\infty} |\dot{\theta}(t)| = \infty.$

It would be interesting to obtain a more precise description of the asymptotic behaviour of the velocity $\dot{z}(t)$ but this seems a delicate question.

2. Collinear solutions ($\dot{\theta} = 0, z = r$)

We have shown that the linear orbits are of one of the following two types

- ejection-collision, $-\infty < t_0 < t_1 < +\infty$,

$$\lim_{t\to t_0^+} r(t) = \lim_{t\to t_1^-} r(t) = 0, \quad \lim_{t\to t_0^+} \dot{r}(t) = +\infty, \ \lim_{t\to t_1^-} \dot{r}(t) = -\infty$$

- capture-collision, $-\infty = t_0 < t_1 < +\infty$,

$$\lim_{t\to-\infty} r(t) = +\infty, \lim_{t\to t_1^-} r(t) = 0, \quad \lim_{t\to t_1^-} \dot{r}(t) = -\infty.$$

In both cases,

- the energy has a finite limit at collisions.

Moreover,

- the energy has a finite limit at ejections. Ejection and collisions may occur with any (finite) value of the energy.

At a capture the behaviour of velocity/energy is more delicate. There exists a unique capture-collision orbit acting as a separatrix between ejection-collision and capture-collision solutions. This orbit is important because it can be seen as an analogue of parabolic orbits in the dissipative world. It satisfies

- $\lim_{t\to-\infty} E(t) = 0$ (separatrix).

For the remaining capture-collision orbits

- $\lim_{t\to-\infty} E(t) = +\infty$.

Finally we proved that the behaviour at collisions is the same as in the conservative case

- $r(t) \sim (\frac{9}{2})^{1/3}(t-t_*)^{2/3}$ as $t \to t_*$, where $t_* = t_1$ or $t_* = t_0 > -\infty$.

A fundamental tool to study the asymptotic behaviour of the energy on the motions of equation (3) has been the Levi-Civita regularization, adapted to the dissipative framework. Perhaps the regularized system has some independent interest since, it could be an useful tool in the qualitative study of the dynamics of more complex systems in dissipative celestial mechanics.

Overall, we expect that the methods developed in this paper will give us some hints for the study of the general non-integrable case, allowing us to give some rigorous results about the topology of the phase space for the dissipative restricted three body problem, starting maybe by considering more general drags for the Kepler problem.

6 Appendix: uniqueness of the parabolic orbit

We consider the system (24) and prove that there exists at most one non-trivial orbit γ whose α-limit is the origin. The geometry of the vector field X associated to the system is sketched below in Figure 4,
and the qualitative analysis of the flow shows that for any such orbit γ there exists a sequence of instants $\eta_n \to -\infty$ such that the orbit lies on the region $\mathcal{R}$ at these instants. Since $\mathcal{R}$ is positively invariant the whole orbit γ must be contained in $\mathcal{R}$. The divergence of the vector field X is negative on a small disk around the origin, namely

$$\text{div } X = -2su - \epsilon < 0 \quad \text{if } u^2 + s^2 < \epsilon.$$

Let us assume by contradiction that γ_1 and γ_2 are two orbits whose α-limit set is the origin. We can select a small disk D such that both orbits get out of it (see Figure 5). Let p_i, $i = 1, 2$, be the first point of γ_i lying on ∂D. Since the two orbits are contained in the region $\mathcal{R}$ we know that the points

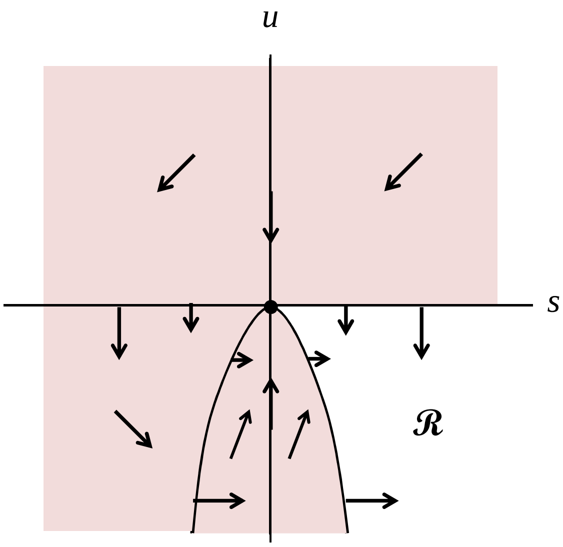

Figure 4: Vector field associated to system (24)

p_1 and p_2 are in the fourth quadrant. Assume for instance that p_2 is below p_1, then we draw a vertical segment emanating from p_2 and touching γ_1 only at the end point q. Consider the Jordan curve Γ composed by the segment σ from q to p_2 and the arcs $\alpha_1 = \widehat{0q}$ contained in γ_1 and $\alpha_2 = \widehat{0p_2}$ contained in γ_2. The bounded domain determined by Γ will be called Ω.

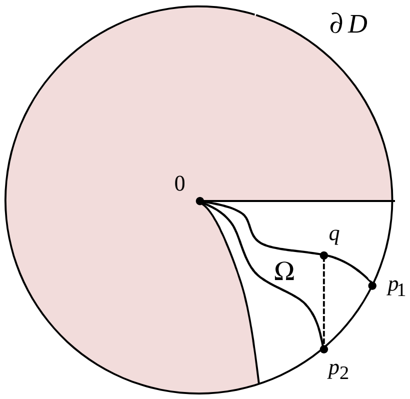

Figure 5: Construction used to prove the uniqueness of the parabolic orbit

The outward normal unit vector n is the horizontal vector $(1, 0)$ in σ and it is perpendicular to the vector field in $\alpha_1 \cup \alpha_2$. Then

$$\langle X, n\rangle = -us^2 > 0 \text{ on } \sigma \text{ and } \langle X, n\rangle = 0 \text{ on } \alpha_1 \cup \alpha_2.$$

The divergence theorem implies that $\int\int_\Omega \text{div } X = \int_\sigma \langle X, n\rangle$, and we have a contradiction because these integrals have opposite signs.

References

Breiter, S. & Jackson, A. (1998), 'Unified analytical solutions to two-body problems with drag', *Mon. Not. R. Astron. Soc* **299**, 237–243.

Celletti, A., Stefanelli, L., Lega, E. & Froeschlé, C. (2011), 'Some results on the global dynamics of the regularized restricted three-body problem with dissipation', *Celestial Mech. Dynam. Astronom.* **109**(3), 265–284.

Corne, J. & Rouche, N. (1973), 'Attractivity of closed sets proved by using a family of lyapunov functions', *J. Diff. Equs.* **13**, 231–246.

Danby, J. M. A. (1962), *Fundamentals of celestial mechanics*, The Macmillan Company, New York.

Diacu, F. (1999), 'Two body problems with drag or thrust: qualitative results', *Celest. Mech. Dyn. Astr.* **75**, 1–15.

Goursat, E. (1889), 'Les transformations isogonales en mécanique', *C. R. Acad. Sci. Paris* **CVIII**, 446–448.

Jacobi, C. G. J. (2009), *Jacobi's lectures on dynamics*, Vol. 51 of *Texts and Readings in Mathematics*, revised edn, Hindustan Book Agency, New Delhi. Delivered at the University of Königsberg in the winter semester 1842–1843 and according to the notes prepared by C. W. Brockardt, Edited by A. Clebsch, Translated from the original German by K. Balagangadharan, Translation edited by Biswarup Banerjee.

Leach, P. G. L. (1987), 'The first integrals and orbit equation for the Kepler problem with drag', *J. Phys. A* **20**(8), 1997–2002.

Margheri, A., Ortega, R. & Rebelo, C. (2012), 'Some analytical results about periodic orbits in the restricted three body problem with dissipation', *Celest. Mech. Dyn. Astr.* **113**, 279–290.

Mavraganis, A. G. & Michalakis, D. G. (1994), 'The two-body problem with drag and radiation pressure', *Celestial Mech. Dynam. Astronom.* **58**(4), 393–403.

Mittleman, D. & Jezewski, D. (1982), 'An analytic solution to the classical two-body problem with drag', *Celestial Mech.* **28**(4), 401–413.

Ortega, R. (2011), 'Linear motions in a periodically forced Kepler problem', *Port. Math.* **68**(2), 149–176.

Sperling, H. J. (1969/1970), 'The collision singularity in a perturbed two-body problem', *Celestial Mech.* **1**, 213–221.

Zhang, Z. F., Ding, T. R., Huang, W. Z. & Dong, Z. X. (1992), *Qualitative theory of differential equations*, Vol. 101 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI. Translated from the Chinese by Anthony Wing Kwok Leung.