Unchanged direction motion in General Relativity: the problems of prescribing acceleration and the extensibility of trajectories

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Abstract

The notion of unchanged direction (UD) motion in General Relativity is introduced, extending widely the concept of uniformly accelerated motion. An observer which obeys a UD motion is characterized as a pointing future unit timelike curve with all its curvatures identically zero up to the first one. The initial value problem when the acceleration of the motion is prescribed is analysed. It is also studied the completeness of inextensible UD motions, that can be physically interpreted saying that observers which obey a UD motion live forever. For certain spacetimes with relevant symmetries that includes the Generalized Robertson-Walker spacetimes, a geometric approach leads to the completeness. On the other hand, a more analytical approach permits to prove completeness of inextensible UD motions in a plane wave spacetime.

PACS: 04.25.-g, 02.30.Hq, 02.40.Ky, 02.40.Vh.
Keywords: Unchanged direction motion, General Relativity, Fermi-Walker connection, Third-order differential equation, Unchanged direction motion with prescribed acceleration, Completeness of inextensible trajectories, Generalized Robertson-Walker spacetime, Plane Wave spacetime.

1 Introduction

In a non-Relativistic setting, a particle may be detected as accelerated by using an *accelerometer*. An accelerometer may be intuitively thought as a sphere in whose center there is a small

[*]The first and third authors are partially supported by Spanish MICINN Grant with FEDER funds MTM2014-52232-P. The second one by Spanish MICINN Grant with FEDER funds MTM2013-47828-C2-1-P and by the Junta de Andalucía Regional Grant P09-FQM-4496.

ball which is supported on elastic radii of the sphere surface. If a free falling observer carries such a accelerometer, then it will notice that the small ball remains just at the center. On the other hand, the ball is displaced at rest if the observer obeys an accelerated motion. If the ball moves along a radius, the observer may though that its motion obeys a rectilinear trajectory. Through this intuitive idea, an observer is able to detect its proper acceleration, independently if the underlying spacetime is relativistic or not. The definition of rectilinear motion in Relativity has been discussed a few times as far as we know [9], [10]. Sometimes motivated by aspects of intergalactic rocket travels in the same way that the use of the special relativistic formulas for hyperbolic motion dealt in [12].

The first aim of this paper is to provide a rigorous notion for the assertion "the proper acceleration does not change its direction", according with a gyroscope or an accelerometer as above. Thus, unchanged direction motion (UD motion in brief) in General Relativity is introduced and studied. Our approach lies in the realm of modern Lorentzian geometry and, as far as we know, is new. In order to do that, recall that a *particle* of mass $m > 0$ in a spacetime $(M, \langle\, ,\, \rangle)$ is a curve $\gamma : I \longrightarrow M$, such that its velocity $\gamma\,'$ satisfies $\langle \gamma\,', \gamma\,' \rangle = -m^2$ and points to future. A particle with $m = 1$ is called an *observer*. In this case, the covariant derivative of $\gamma\,'$, $\dfrac{D\gamma\,'}{dt}$, is its (proper) acceleration, which may be seen as a mathematical translation of the values which measures an accelerometer, [18, Sect. 3.1]. To be rigorous, we will use a *connection* along γ which permits to compare spatial directions at different instants of the life of γ. In General Relativity this connection is known as the *Fermi-Walker* connection of γ [18, Prop. 2.2.2] (see Section 2 for more details). Using the corresponding Fermi-Walker covariant derivative $\dfrac{\widehat{D}}{dt}$, we will say that a particle with nowhere zero acceleration obeys a *unchanged direction* (UD) motion if the following third-order differential equation is fulfilled (Definition 1),

$$\frac{\widehat{D}}{dt}\left(\left|\frac{D\gamma\,'}{dt}\right|^{-1}\frac{D\gamma\,'}{dt}\right) = 0, \tag{1}$$

i.e., if the normalized acceleration vector field of the observer γ is Fermi-Walker parallel along its world line (Section 2). If the acceleration vector field of an observer is Fermi-Walker parallel then it is said that it obeys a uniformly accelerated motion [6]. Note that an observer γ obeys a uniformly accelerated motion and the constant $\left|\dfrac{D\gamma\,'}{dt}\right|$ is positive then it obeys a UD motion, however the class of UD observers is much bigger that the one of uniformly accelerated observers. In fact, if an observer in $n(\geq 2)$-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^n$ lies in a (totally geodesic) Lorentzian plane, then it obeys a UD motion. Conversely, each UD observer γ in $\mathbb{L}^n$ is contained in a Lorentzian plane determined by the point $\gamma(0)$, the initial acceleration and the initial 4-velocity. More generally, every UD observer in a spacetime of constant sectional curvature must be contained in a 2-dimensional totally geodesic Lorentzian submanifold (Section 3).

More generally, we will introduce in Section 2 the notion of piecewise UD motion (Definition 2). A piecewise UD observer is essentially an observer which may change its direction only at the instant when its accelerometer marks zero. Each piecewise UD observer naturally appears as a solution of an ODE, (6), much more general than formula (1). Now we can assert that a free falling observer obeys trivially a piecewise UD motion although it cannot be considered as a UD observer in general. Also in Section 2, the problem of finding a UD observer γ prescribing its scalar acceleration $\left|\dfrac{D\gamma'}{dt}\right|$ is suitably stated and completely solved, obtaining an explicit first integral of a UD observer with prescribed acceleration (Theorem 2.1). Piecewise UD observers will be geometrically characterized later (Proposition 3.2(c)). Note that our approach fits within the mean curvature prescription problem for a definite submanifold in spacetime (Proposition 3.2(e)). Among the geometric characterizations of piecewise UD motions in Proposition 3.2 (widely extending [6], [16]), it is remarkable that an observer obeys a piecewise UD motion if and only if its development, in the sense of [8, Sect. III.4], in the tangent space to spacetime at any of its points is a piecewise planar curve (Proposition 3.2(c)).

In Section 4, UD observers are characterized as the curves obtained by projection on the spacetime of the integral curves of a certain vector field defined on a certain fiber bundle over the spacetime (Lemma 4.2). Using this vector field, the completeness of inextensible UD motions is analysed in the search of geometric assumptions which assure that inextensible UD observers do not disappear in a finite proper time, in particular, the absence of this type of singularities. It is shown that a UD observer in spacetime which admits a conformal and closed timelike vector field (in particular, in a Generalized Robertson-Walker spacetime) can be extended if it is contained in a compact subset of spacetime (Theorem 4.5). Finally, Section 5 deals with the problem of extensibility of UD observers but this time in a plane wave spacetime. It should be pointed out that the technique used here is different to the one used in previous section. In fact, now the key tool is the explicit first integral of a UD observer obtained in (Theorem 2.1). Thus, it is proved that every inextensible UD trajectory with prescribed acceleration a in a plane wave spacetime must be complete (Theorem 5.3).

2 The notion of unchanged direction motion

Consider a spacetime M, i.e., a time orientable $n(\geq 2)-$dimensional manifold endowed with a Lorentzian metric $\langle\, ,\, \rangle$ which we agree to have signature $(-,+,...,+)$., and with a fixed time orientation. The points of M as called events and an observer in M as a (smooth) curve $\gamma : I \longrightarrow M$, I an open interval of the real line $\mathbb{R}$, such that $\langle \gamma'(t), \gamma'(t)\rangle = -1$ and $\gamma'(t)$ lies in the future time cone given by the time orientation in $T_{\gamma(t)}M$, in brief, $\gamma'(t)$ is future pointing for any proper time t of γ.

At each event $\gamma(t)$, the tangent space $T_{\gamma(t)}M$ splits as follows

$$T_{\gamma(t)}M = T_t \oplus R_t,$$

where $T_t = \mathrm{Span}\{\gamma'(t)\}$ and $R_t = T_t^{\perp}$. Clearly, T_t is a negative definite line in $T_{\gamma(t)}M$ and R_t is a spacelike hyperplane of $T_{\gamma(t)}M$. For $n = 4$ the 3-dimensional subspace R_t is interpreted as the instantaneous physical space observed by γ at the instant t in its clock.

In order that γ does compare $v_1 \in R_{t_1}$ with $v_2 \in R_{t_2}$, for $t_1 < t_2$ and $|v_1| = |v_2|$, it could think to use the parallel transport defined by the Levi-Civita covariant derivative along γ,

$$P^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M.$$

However, this linear isometry does not satisfy $P^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$ in general. This fact is a serious difficulty. In order to avoid it we will recall that γ possesses a (private) connection, called the Fermi-Walker connection of γ which is defined as follows. First consider the Levi-Civita connection ∇ associated to the Lorentzian metric of spacetime. The Levi-Civita connection induces a connection along $\gamma : I \longrightarrow M$ such that the corresponding covariant derivative is the well-known covariant derivative of vector fields $Y \in \mathfrak{X}(\gamma)$, namely, $\dfrac{DY}{dt}(t) = \nabla_{\gamma'(t)}\widetilde{Y}$, where $\widetilde{Y}$ is a local extension of Y in an open neighbourhood of $\gamma(t)$ in M.

For each $Y \in \mathfrak{X}(\gamma)$ denote by Y_t^T, Y_t^R the orthogonal projections of Y_t on T_t and R_t, respectively, i.e.,

$$Y_t^T = -\langle Y_t, \gamma'(t)\rangle\, \gamma'(t) \qquad \text{and} \qquad Y_t^R = Y_t - Y_t^T.$$

Clearly, we have $Y^T, Y^R \in \mathfrak{X}(\gamma)$. According to [18, Prop. 2.2.1] the Fermi-Walker connection of γ is the unique unique connection $\widehat{\nabla}$ along γ which satisfies

$$\widehat{\nabla}_X Y = \left(\nabla_X Y^T\right)^T + \left(\nabla_X Y^R\right)^R,$$

for any $\;X \in \mathfrak{X}(I)$ and $Y \in \mathfrak{X}(\gamma)$.

Now denote by $\widehat{D}/dt$ the covariant derivative corresponding to $\widehat{\nabla}$. Then, it is not difficult to prove the following relationship with the Levi-Civita covariant derivative [18, Prop. 2.2.2],

$$\frac{\widehat{D}Y}{dt} = \frac{DY}{dt} + \langle \gamma', Y\rangle \frac{D\gamma'}{dt} - \left\langle \frac{D\gamma'}{dt}, Y \right\rangle \gamma', \tag{2}$$

for any $Y \in \mathfrak{X}(\gamma)$. Clearly, we have $\dfrac{\widehat{D}}{dt} = \dfrac{D}{dt}$ if and only if γ is a geodesic, i.e., the observer is free falling.

Associated to the Fermi-Walker covariant derivative along γ there exist a parallel transport

$$\widehat{P}^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_1)}M,$$

which is a lineal isometry and satisfies $\widehat{P}^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$. Therefore, given $v_1 \in R_{t_1}$ and $v_2 \in R_{t_2}$, with $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ may consider $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ instead v_1, with the advantage that $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ may be compared with v_2 (see also [13, Sec. 6.5]). Note that the acceleration vector field $\dfrac{D\gamma'}{dt}$ satisfies $\dfrac{D\gamma'}{dt}(t) \in R_t$, for any t. In fact, it may be observed by γ whereas the velocity vector field γ' is not observable by γ.

Now, we are in a position to give accurately the notion of unchanged direction observer.

Definition 1 An observer $\gamma : I \longrightarrow M$ is said to obey an unchanged direction (UD) motion if

$$\widehat{P}^{\gamma}_{t_1,t_2}\left(\frac{D\gamma'}{dt}(t_1)\right) = \lambda(t_1,t_2)\,\frac{D\gamma'}{dt}(t_2), \tag{3}$$

for a certain proportional factor λ and for any $t_1, t_2 \in I$ with $t_1 < t_2$.

Clearly, if an observer γ is a free falling then it obeys a UD motion. More generally, a uniformly accelerated (UA) observer [6] satisfies (3) with $\lambda = 0$. Thus, it obeys a UD motion. Of course, the family of UD observers is much bigger than the one of the UA observers.

Note that, if $\dfrac{D\gamma'}{dt}(t) \neq 0$ for all $t \in I$, Definition 1 is equivalent to say that the normalized acceleration, $\left|\dfrac{D\gamma'}{dt}\right|^{-1} \dfrac{D\gamma'}{dt}$, is Fermi-Walker parallel along γ. Taking into account that the Leibniz rule holds true for the Fermi-Walker covariant derivative,

$$\frac{\widehat{D}}{dt}\langle Y_1\,, Y_2\rangle = \Big\langle \frac{\widehat{D}Y_1}{dt}, \,Y_2\Big\rangle + \Big\langle Y_1\,, \frac{\widehat{D}Y_2}{dt}\Big\rangle\,, \tag{4}$$

for any $Y_1, Y_2 \in \mathfrak{X}(\gamma)$. From (3) we arrive to the following expression,

$$\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{\widehat{D}}{dt}\Big(\frac{D\gamma'}{dt}\Big) = \Big\langle \frac{\widehat{D}}{dt}\Big(\frac{D\gamma'}{dt}\Big), \frac{D\gamma'}{dt}\Big\rangle \frac{D\gamma'}{dt}\,. \tag{5}$$

We observe that this equation is well defined for every observer, not only for those with acceleration nonzero everywhere. By using (2), last formula can be equivalently expressed as follows,

$$\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{D^2\gamma'}{dt^2} = \frac{1}{2}\frac{d}{dt}\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{D\gamma'}{dt} + \Big|\frac{D\gamma'}{dt}\Big|^4 \gamma'\,. \tag{6}$$

Note that if γ is a UA observer, then $\Big|\dfrac{D\gamma'}{dt}\Big|^2 = a^2$. If γ is not free falling, then a a positive constant, and (6) reduces to

$$\frac{D^2\gamma'}{dt^2} = a^2\,\gamma',$$

which is just the equation defining a UA motion [6].

However, a solution of equation (6) does not describe a UD observer in general. In fact, a solution γ of equation (6) is a UD observer whenever $\dfrac{D\gamma'}{dt} \neq 0$ everywhere on the domain I of γ. On the other hand, when the acceleration vector field vanishes identically on a subinterval J of I, then equation (6) is automatically satisfied on J and γ is a free falling on J until it eventually returns to be accelerated out of J in a possibly different direction. Thus we introduce the following notion.

Definition 2 An observer $\gamma : I \longrightarrow M$ is said to obey a piecewise unchanged direction motion if γ satisfies equation (6).

From an analytical point of view, the Cauchy problem associated to equation (6) does not have a unique solution in general. If fact, in a local coordinate system, equation (6) gives a system of ordinary differential equations which cannot be written in normal form. Hence, the classical Picard-Lindelöf theorem cannot be applied, and the existence and uniqueness of the solutions are not a priori guaranteed. We will see before that although existence is true, there is not uniqueness in general.

Now we will state the prescription acceleration problem as follows. Let $a : I \longrightarrow \mathbb{R}$ be a smooth function (the prescribed acceleration function) and consider the initial value problem

$$\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{D^2\gamma'}{dt^2} = \frac{1}{2}\frac{d}{dt}\Big|\frac{D\gamma'}{dt}\Big|^2 \frac{D\gamma'}{dt} + \Big|\frac{D\gamma'}{dt}\Big|^4 \gamma'\,,$$

$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = a(0)\, w,$$

where $p \in M$ and $v, w \in T_pM$ such that $|v|^2 = -1$, $\langle v, w\rangle = 0$ and $|w|^2 = 1$. Thus, $a^2(t)$ will prescribe the square modulus of the proper acceleration vector field. The sign of $a(t)$ indicates if the sense of the acceleration is the same or the opposite respect to the initial one, i.e., if $a(t)$ has and $a(0)$ have the same sign then γ observes that its accelerometer points at the proper time t in the same sense that in the initial instant.

Using $a^2(t) = \left|\frac{D\gamma'}{dt}\right|^2(t)$ in equation (6), we get,

$$a^2(t)\, \frac{D^2\gamma'}{dt^2} = \frac{1}{2}\frac{d}{dt}(a^2(t))\frac{D\gamma'}{dt} + a^4(t)\gamma'. \tag{7}$$

If the observer is always accelerated, i.e., if the prescription function $a(t) \neq 0$ for all $t \in I$, the last equation reduces to

$$\frac{D^2\gamma'}{dt^2} = \frac{a'(t)}{a(t)}\frac{D\gamma'}{dt} + a^2(t)\gamma'. \tag{8}$$

Conversely, if an observer γ satisfies (8) with $a(t) \neq 0$ for all $t \in I$, then its acceleration satisfies $\left|\frac{D\gamma'}{dt}\right|^2(t) = a^2(t)$. In order to prove this, if we multiply both members of equation (8) by γ', then

$$\left\langle \gamma'(t)\,, \frac{D^2\gamma'}{dt^2}(t)\right\rangle = -a^2(t),$$

and, since γ is an observer, we get the announced result.

Note that if $a(t) \neq 0$ everywhere on I, the initial value problem associated to equation (8) has a unique local solution. The lack of uniqueness of the initial value problem associated to equation (6) is now clear. In fact, take two different prescription functions with the same initial value. The solutions of the Cauchy problem corresponding to (8) are two different solutions of (6).

However, if the prescribed acceleration $a(t)$ vanishes at some instant, the uniqueness of solutions of (7) is not guaranteed. Moreover, as commented before, γ can be a solution of (7), even although it is not a UD observer (only a piecewise UD observer). It is necessary to add some additional assumption in (7) to assure that each solution is a unique UD observer.

Let $U(t)$ be the unitary acceleration, defined only at the instant t at which $a(t) \neq 0$,

$$U(t) = \left|\frac{D\gamma'}{dt}\right|^{-1}\frac{D\gamma'}{dt}.$$

Put $J = \{t \in I \ : \ a(t) \neq 0\}$ and for each $t_0 \in I$ let us consider

$$t_0^* = \sup\{t \in I, \ \ t < t_0, \ \ t \in J\},$$

and the "extended unitary acceleration", U^*, defined on the whole interval I as follows,

$$U^*(t) = \begin{cases} U(t) & \text{if} \quad t \in J, \\ P^{\gamma}_{t^*,\, t}\left(\lim\limits_{s\in J,\, s\to t^{*-}} U(s)\right) & \text{if} \quad t \notin J. \end{cases}$$

Definition 3 An observer $\gamma : I \longrightarrow M$ is said to be an *UD observer with prescribed acceleration* $a : I \longrightarrow \mathbb{R}$, if it satisfies (7) and

$$\lim_{t\to t_0^-} U^*(t) = \pm \left(\lim_{t\to t_0^+} U^*(t) \right), \tag{9}$$

for all $t_0 \in I$ such that $a(t_0) = 0$.

In particular, for a prescription function $a(t)$ which only vanishes in a discrete subset $T \subset I$, a solution γ of (7) is a UD observer which prescribed acceleration a if and only if

$$\lim_{t\to t_0^-} U(t) = \pm \lim_{t\to t_0^+} U(t),$$

where $t \in I \setminus T$ and $t_0 \in T$.

Note that at any instant where the limits have opposite sign, the observer changes the sense of his acceleration, but not the direction.

The following result will be useful tool in order to study the completeness of the inextensible trajectories in Section 5.

Theorem 2.1 *Let $a : I \longrightarrow \mathbb{R}$ be a smooth function and $v, w \in T_pM$ such that $|v|^2 = -1$, $|w|^2 = 1$ and $\langle v, w\rangle = 0$. The 4-velocity of the unique UD observer γ satisfying the initial conditions*

$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = a(0)\, w, \tag{10}$$

is given by

$$\gamma'(t) = \cosh\big(V(t)\big)\, L(t) + \sinh\big(V(t)\big)\, M(t), \tag{11}$$

where

$$V(t) = \int_0^t a(s)ds,$$

and L, M are two Levi-Civita parallel vector fields along γ with $L(0) = v$ and $M(0) = a(0)\, w$.

Proof. First, if γ satisfies (11) and $a(t) \neq 0$ everywhere in I, then it is a solution of (8). From assumptions on L and M, after easy computations we get

$$\frac{\widehat{D}L}{dt}(t) = -a(t)\, M(t) \qquad \text{and} \qquad \frac{\widehat{D}M}{dt}(t) = -a(t)\, L(t).$$

Also note

$$U(t) = \left|\frac{D\gamma'}{dt}\right|^{-1} \frac{D\gamma'}{dt} = \operatorname{sign}\big(a(t)\big)\, \big(\sinh(V)L + \cosh(V)M\big), \tag{12}$$

Therefore, from these identities, we conclude $\dfrac{\widehat{D}U}{dt} = 0$.

We analyse now the case when $a(t)$ vanishes at some instant. Let $t_0 \in I$ be such that $a(t_0) = 0$. We have,

$$U^*(t_0) = \operatorname{sign}\left(\lim_{t\to t_0^*-} \frac{a(t)}{|a(t)|}\right) \left[\sinh\big(V(t_0^*)\big)L(t_0) + \cosh\big(V(t_0^*)\big)M(t_0)\right].$$

Since $V(t_0^*) = V(t_0)$, condition (9) is satisfied, and γ is an UD observer with prescribed acceleration $a(t)$.

$\square$

By using the Levi-Civita parallel transport, we can express (11) as the following first order integro-differential equation,

$$\gamma'(t) = \cosh\big(V(t)\big)\, P^{\gamma}_{0,t}(v) + \sinh\big(V(t)\big)\, a(0)\, P^{\gamma}_{0,t}(w), \qquad |v|^2 = -1, \quad |w|^2 = 1, \quad \langle v, w\rangle = 0. \tag{13}$$

3 UD motion from a geometric viewpoint

Now we proceed to find the Frenet equations which satisfies (and in fact redefines) a UD observer in the particular case of nowhere zero acceleration.

Consider a UD observer $\gamma : I \longrightarrow M$ with $\left|\frac{D\gamma'}{dt}\right| > 0$ everywhere on I. In this case, the two following vector fields along γ are well-defined,

$$e_1(t) = \gamma'(t) \qquad \text{and} \qquad e_2(t) = \left|\frac{D\gamma'}{dt}\right|^{-1} \frac{D\gamma'}{dt}(t).$$

Then, from (2) and (3) we have

$$\frac{De_1}{dt} = \left|\frac{D\gamma'}{dt}\right| e_2(t), \tag{14}$$

$$\frac{De_2}{dt} = \left|\frac{D\gamma'}{dt}\right| e_1(t). \tag{15}$$

In particular, if $\left|\frac{D\gamma'}{dt}\right|^2 = a^2$, with a constant, the observer γ obeys a uniformly accelerated motion [6] and these equations define a Lorentzian circle [11], [14].

Conversely, assume this system holds true for an observer γ. Then, equation (3) also is satisfied. In other words, a UD observer may be seen as a unit timelike curve with (first) curvature $\left|\frac{D\gamma'}{dt}\right|$ and identically zero torsion and the rest of curvatures. From the reduction of codimension Erbacher theorem (see [7]), we conclude that if the spacetime has constant sectional curvature, then a UD observer is contained in a 2-dimensional totally geodesic Lorentzian submanifold.

We next characterize a piecewise UD observer from the point of view of its development curve [8, Sect. III.4]. We will say that a curve in an affine space is *piecewise planar* if its torsion, whenever is defined, is zero. Thus, inspired from [8, Prop. III.6.2], we get,

Proposition 3.1 *An observer $\gamma : I \longrightarrow M$ obeys a piecewise UD motion if and only if its development $\overline{\gamma}$ in the tangent space $T_{\gamma(t_0)}M$ is a piecewise planar curve for any $t_0 \in I$.*

Proof. Put

$$\overline{X}(t) = P^{\gamma}_{t,0}(\gamma'(t)).$$

where $P^{\gamma}_{0,t}$ is the Levi-Civita parallel displacement of tangent vectors along γ from $\gamma(t)$ to $p = \gamma(0)$. Recall that the development $\overline{\gamma}$ is the unique curve in T_pM starting in the origin of T_pM such that its tangent vector $\overline{X}(t)$ is parallel to $\gamma'(t)$ (in the usual sense).

By simplicity of notation, we suppose that $t_0 = 0$. First, we assume that γ is an UD observer and let $\overline{\gamma}(t)$ its development. Since $P^{\gamma}_{t,0} : T_{\gamma(t)}M \longrightarrow T_pM$ is a linear isometry, we have

$$\frac{d\overline{X}(t)}{dt} = \lim_{h\to 0}\frac{\overline{X}(t+h) - X^*(t)}{h} = \lim_{h\to 0}\frac{\overline{X}(t+h) - \overline{X}(t)}{h} =$$

$$= P^{\gamma}_{t,0}\left(\lim_{h\to 0}\frac{P^{\gamma}_{t+h,t}X(t+h) - X(t)}{h}\right) = P^{\gamma}_{t,0}\big(\frac{D\gamma'}{dt}\big).$$

Making use of this identity, (14) implies that $\left|\frac{D\gamma'}{dt}\right|(t)$ is the first curvature of the development. Thus, at any instant where the acceleration of γ does not vanish, the normal vector of $\overline{\gamma}$ is $\overline{Y}(t) = P^{\gamma}_{t,0}\big(U(t)\big)$ and therefore

$$\frac{d\overline{Y}}{dt}(t) = P^{\gamma}_{t,0}\big(\frac{DU}{dt}(t)\big).$$

From (15), we deduce that the torsion of $\overline{\gamma}$ is zero. Therefore, γ is a piecewise planar curve.

Conversely, assume the development $\overline{\gamma}$ is a planar curve in the tangent of a point p. Then

$$\frac{d\overline{X}}{dt}(t) = \left|\frac{D\gamma'}{dt}\right|(t)\,\overline{Y}(t) \qquad \text{and} \qquad \frac{d\overline{Y}}{dt}(t) = \left|\frac{D\gamma'}{dt}\right|(t)\,\overline{X}(t),$$

are satisfied. Since $P^{\gamma}_{t,0}$ is an isometry between $T_{\gamma(t)}M$ and T_pM, from these equations we obtain (14) and (15).

□

The previous results can be summarized as follows,

Proposition 3.2 *For any observer $\gamma : I \longrightarrow M$ with nowhere zero acceleration the following assertions are equivalent:*

(*a*) *γ is a piecewise UD observer.*

(*b*) *γ is a solution of third-order differential equation* (6)*.*

(*c*) *The development of γ is a piecewise planar curve in the tangent space of every point.*

(*d*) *γ has all the curvatures equal to zero except (possibly) the first one.*

(*e*) *γ, viewed as an isometric immersion from $(I, -dt^2)$ to M, is (totally umbilical) with parallel normalized mean curvature vector whenever it is defined.*

4 Completeness of the inextensible UD trajectories in spacetimes with some symmetries

This section is devoted to the study of the completeness of the inextensible solutions of equation (8), i.e., the UD equation with never zero prescribed acceleration. Here we assume the prescription function a is smooth, positive and defined on $\mathbb{R}^+$.

First of all, we are going to relate the solutions of equation (8) with the integral curves of a certain vector field on a Stiefel type bundle on M (compare with [8, p. 6]).

Given a Lorentzian linear space E, denote by $V_{n,2}(E)$ the (n,2)-Stiefel manifold over E, defined by

$$V_{n,2}(E) = \big\{(v, w) \in E \times E \;:\; |v|^2 = -1, \;\; \langle v, w\rangle = 0\big\}.$$

The (n,2)-Stiefel bundle over the spacetime M is then defined as follows,

$$V_{n,2}(M) = \bigcup_{\mathrm{p}\in \mathrm{M}} \{p\} \times V_{n,2}(T_pM).$$

Note that $V_{n,2}(M)$ is a bundle on M with dimension 3n-2 and fiber diffeomorphic to the tangent fiber bundle of the $(n-1)$-dimensional hyperbolic space.

Now we construct a vector field $G \in \mathfrak{X}\big(V_{n,2}(M)\big)$, which is the key tool in the study of completeness that follows,

Lemma 4.1 *Let $\sigma : I \longrightarrow M$ be a curve satisfying the following initial value problem*

$$\frac{D^2\sigma'}{dt^2} = \left[\frac{a'(t)}{a(t)} - \big\langle \sigma', \frac{D\sigma'}{dt}\big\rangle\right]\frac{D\sigma'}{dt} + a^2(t)\,\sigma'. \tag{16}$$

$$\sigma(0) = p, \quad \sigma'(0) = v, \quad \frac{D\sigma'}{dt}(0) = w,$$

where v is a future pointing unit timelike tangent vector and w is orthogonal to v. Then σ is an observer, and $\Big|\dfrac{D\sigma'(t)}{dt}\Big|^2 = a^2(t)$ holds everywhere on I.

Proof. Multiplying (16) by σ' and $\dfrac{D\sigma'}{dt}$ we obtain two ordinary differential equations which, after easy computations, can written as follows,

$$\frac{1}{2}\,x''(t) - y(t) = a^2(t)x(t) + \frac{1}{2}\left[\frac{a'(t)}{a(t)} - \frac{1}{2}x'(t)\right]x'(t),$$

$$\frac{1}{2}\,y'(t) = \frac{1}{2}\big(a^2(t) - y(t)\big)\,x'(t) + \frac{a'(t)}{a(t)}\,y(t),$$

where $x(t) := |\sigma'(t)|^2$ and $y(t) := \Big|\dfrac{D\sigma'(t)}{dt}\Big|^2$. From the assumption, we know that $x(t)$ and $y(t)$ satisfy the initial conditions

$$x(0) = -1,$$

$$x'(0) = 2\left\langle \sigma'(0), \frac{D\sigma'}{dt}(0)\right\rangle = 2\langle v, w\rangle = 0,$$

$$y(0) = |w|^2 = a^2(0).$$

Since $x(t) = -1$ and $y(t) = a^2(t)$ are solutions of the previous initial value problem, the result is a direct consequence of the existence and uniqueness of solutions to second order differential equations. $\square$

Observe that solutions of (8) under the initial conditions (10) are obviously solutions of the problem (16). In the previous result we have proved that the converse is true. The advantage now is that (16) is a real initial value problem.

Now, we are in a position to construct the announced vector field G. Let (p, v, w) be a point of $V_{n,2}(M)$, and $f \in C^\infty\big(V_{n,2}(M)\big)$. Let σ be the unique inextensible curve solution of (16) satisfying the initial conditions

$$\sigma(0) = p, \quad \sigma'(0) = v, \quad \frac{D\sigma'}{dt}(0) = w,$$

for $(p, v, w) \in V_{n,2}(M)$. Define

$$G_{(p,v,w)}(f) := \frac{d}{dt}\Big|_{t=0} f\big(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t)\big).$$

From Lemma 4.1, we know $\big(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t)\big) \in V_{n,2}(M)$ and the following result easily follows,

Lemma 4.2 *There exists a unique vector field G on $V_{n,2}(M)$ such that its integral curves are $t \longmapsto \big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}(t)\big)$, where γ is a solution γ of equation* (8).

Once defined G, we will look for assumptions which assert its completeness (as a vector field). Recall that an integral curve α of a vector field defined on some interval $[0, b)$, $b < +\infty$, can be extended to b (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$, such that $\{\alpha(t_n)\}_n$ converges (see for instance [15, Lemma 1.56]). The following technical result directly follows from this fact and Lemma 4.2,

Lemma 4.3 *Let $\gamma : [0, b) \longrightarrow M$ be a solution of equation* (8) *with $0 < b < \infty$. The curve γ can be extended to b as a solution of* (16) *if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$ such that $\big\{\gamma(t_n), \gamma'(t_n), \frac{D\gamma'}{dt}(t_n)\big\}_n$ is convergent in $V^a_{n,2}(M)$.*

Although we know that $|\gamma'(t)|^2 = -1$ from Lemma 4.1, this is not enough to apply Lemma 4.3 even in the geometrically relevant case of M compact. The reason is similar to the possible geodesic incompleteness of a compact Lorentzian manifold (see for instance [15, Example 7.16]). However, it is relevant that if a compact Lorentzian manifold admits a timelike conformal vector field, then it must be geodesically complete [17]. On the other hand, a (non-compact or not) Lorentzian manifold which admits a timelike conformal vector field is called conformally stationary (CS) spacetime [1]. Any CS spacetime is globally pointwise conformally equivalent to a stationary spacetime (i.e., a spacetime which admits a timelike Killing vector field) [1]. It is well-know that this kind of infinitesimal symmetries have played an important role to the construction of exact solutions to the Einstein equation. Thus, both from mathematical and physical viewpoint it is natural to assume the existence of such infinitesimal conformal symmetry to deal with the extensibility of the solutions of (8).

Recall that a vector field K on M is called conformal if any (local) flow of K consists of (local) conformal transformations, i.e., the Lie derivative of the metric with respect to K satisfies

$$L_K\langle\,,\rangle = 2h\langle\,,\rangle, \tag{17}$$

for some $h \in C^\infty(M)$. In particular, if any (local) flow of K consists of (local) isometries, i.e., formula (17) holds with $h = 0$, then K is called a Killing vector field. If K is conformal then formula (17) implies

$$\frac{d}{dt}\langle K, \gamma'\rangle = \Big\langle K, \frac{D\gamma'}{dt}\Big\rangle + h(\gamma)|\,\gamma'|^2, \tag{18}$$

for any curve $\gamma : I \longrightarrow M$.

On the other hand, if a vector field K satisfies

$$\nabla_X K = hX \qquad \text{for} \quad \text{all} \qquad X \in \mathfrak{X}(M), \tag{19}$$

then clearly we get (17). Moreover, for the 1-form K^b metrically equivalent to K, we have

$$dK^b(X,Y) = \langle \nabla_X K, Y\rangle - \langle \nabla_Y K, X\rangle = 0,$$

for all $X, Y \in \mathfrak{X}(M)$, i.e., K^b is closed. A vector field K which satisfies (19) is call conformal and closed. A Lorentzian manifold which admits a timelike conformal and closed vector field is locally a Generalized Roberson-Walker spacetime [5], [19].

The following result, inspired from [3, Lemma 9], will be decisive to assure that the image of the curve in $V^a_{n,2}(M)$, associated to a UD observer γ, is contained in a compact subset.

Lemma 4.4 *Let M be a spacetime and let Q be a unit timelike vector field on M. If $\gamma : I \longrightarrow M$ is a solution of* (8) *such that $\gamma(I)$ lies in a compact subset of M and $\langle Q, \gamma'\rangle$ is bounded on I, then the image of $t \longmapsto \big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}\big)$ is contained in a compact subset of $V_{n,2}(M)$.*

Proof. Consider the 1-form Q^b metrically equivalent to Q. Now we can construct on M a Riemannian metric $g_R := \langle\,,\,\rangle + 2\,Q^b \otimes Q^b$. We have,

$$g_R(\gamma', \gamma') = \langle \gamma', \gamma'\rangle + 2\,\langle Q, \gamma'\rangle^2,$$

which, from the assumptions, is bounded on I. Hence, taking into account that

$$\Big|\frac{D\gamma'}{dt}\Big|^2 \leq \max_{t\in[0,b]} a^2(t),$$

there exists a constant $c > 0$ such that

$$\Big(\gamma, \gamma', \frac{D\gamma'}{dt}\Big)(I) \subset C, \qquad C := \big\{(p, v, w) \in V_{n,2}(M) \,:\, p \in C_1, \quad g_R(v, v) \leq c\big\},$$

where C_1 is a compact set on M such that $\gamma(I) \subset C_1$. Hence, C is a compact in $V_{n,2}(M)$. □

Now, we are in a position to state the following extensibility result (compare with [3, Th. 1] and [4, Th. 1]),

Theorem 4.5 *Let M be a spacetime which admits a timelike conformal and closed vector field K. Suppose that* $\mathrm{Inf}_M\sqrt{-\langle K,K\rangle}>0$ *and a positive prescription function defined on* $\mathbb{R}^+$ *is given. Then, each solution* $\gamma : I \longrightarrow M$ *of* (8) *such that* $\gamma(I)$ *lies in a compact subset of* M *can be extended.*

Proof. Let $I=[0,b)$, $0<b<+\infty$, be the domain of a solution γ of equation (8). From (18), it follows

$$\frac{d^2}{dt^2}\langle K,\gamma'\rangle=\Big\langle\frac{DK}{dt},\frac{D\gamma'}{dt}\Big\rangle+\Big\langle K,\frac{D^2\gamma'}{dt^2}\Big\rangle-\frac{d}{dt}(h\circ\gamma).$$

Now, note that the first right term vanishes because K is conformal and closed,

$$\Big\langle\frac{DK}{dt},\frac{D\gamma'}{dt}\Big\rangle=h(\gamma)\,\Big\langle\gamma',\frac{D\gamma'}{dt}\Big\rangle=0.$$

On the other hand, for the second right term we have,

$$\begin{aligned}\Big\langle K,\frac{D^2\gamma'}{dt^2}\Big\rangle &= a^2(t)\langle K,\gamma'\rangle+\frac{a'(t)}{a(t)}\Big\langle\frac{D\gamma'}{dt},K\Big\rangle\\ &= a^2(t)\langle K,\gamma'\rangle+\frac{a'(t)}{a(t)}\left(\frac{d}{dt}\langle K,\gamma'\rangle+h\circ\gamma\right).\end{aligned}$$

Thus, the function $t\mapsto\langle K,\gamma'\rangle$ satisfies the following differential equation,

$$\frac{d^2}{dt^2}\langle K,\gamma'\rangle-\frac{a'(t)}{a(t)}\frac{d}{dt}\langle K,\gamma'\rangle-a^2(t)\langle K,\gamma'\rangle=\frac{a'(t)}{a(t)}(h\circ\gamma)-(h\circ\gamma)'(t). \tag{20}$$

Using now that $\gamma(I)$ is contained in a compact of M, the function $h\circ\gamma$ is bounded on I. Moreover, since I is assumed bounded, from (20) we have a constant $c_1>0$ such that

$$|\langle K,\gamma'\rangle|\le c_1. \tag{21}$$

Now, if we put $Q:=\dfrac{K}{|K|}$, where $|K|^2=-\langle K,K\rangle>0$, then Q is a unit timelike vector field. Now, from (21) we obtain,

$$|\langle Q,\gamma'\rangle|\le m\,c_1 \qquad \text{on} \quad I,$$

where $m=\mathrm{Sup}_M|K|^{-1}<\infty$. The proof ends making use of Lemmas 4.3 and 4.4.

□

Remark 4.6 The previous result gives the following result of mathematical interest: *Let M be a compact spacetime which admits a timelike conformal and closed vector field K. Then, each inextensible solution of* (8) *must be complete.*

Example 4.7 Let $f\in C^\infty(\mathbb{R})$ be a periodic positive function and let (N,g) be a compact Riemannian manifold. Consider the Generalized Robertson-Walker spacetime $\mathbb{R}\times_f N$, i.e., the warped product with base $(\mathbb{R},-dt^2)$, fiber (N,g) and warping function f. The Lorentzian manifold $\mathbb{R}\times_f N$ is a Lorentzian covering of the compact spacetime $\mathbb{S}^1\times_{\widetilde{f}}N$, where $\widetilde{f}$ denotes the induced function from f on $\mathbb{S}^1$. The result in [17] may be applied to $\mathbb{S}^1\times_{\widetilde{f}}N$ with $K=\widetilde{f}Q$, which is timelike, conformal and closed, where Q is the vector field on $\mathbb{S}^1$ naturally induced from ∂_θ. As a practical application of Theorem 4.5, we get that any inextensible UD observer with prescribed acceleration $a:\mathbb{R}\to\mathbb{R}^+$ in the spacetime $\mathbb{R}\times_f N$ must be complete.

5 Completeness of the inextensible UD trajectories in a Plane Wave spacetime

Let us consider a spacetime M which admits a global chart with coordinates $\big(x_1, \cdots, x_n\big)$. In these coordinates, we can write equation (13) as follows,

$$\begin{aligned}
\gamma_k'(t) = &\quad \cosh\big(V(t)\big)\, L_k(t) + \sinh\big(V(t)\big)\, M_k(t), \\
L_k'(t) = &\quad -\textstyle\sum_{i,j}\Big[\Gamma_{ij}^k \cosh\big(V(t)\big)\, L_i(t)\, L_j(t) + \Gamma_{ij}^k \sinh\big(V(t)\big)\, M_i(t)\, L_j(t)\Big], \\
M_k'(t) = &\quad -\textstyle\sum_{i,j}\Big[\Gamma_{ij}^k \sinh\big(V(t)\big)\, M_i(t)\, M_j(t) + \Gamma_{ij}^k \cosh\big(V(t)\big)\, M_i(t)\, L_j(t)\Big], \\
&\gamma_k(0) = p_k, \quad L_k(0) = v_k \quad M_k(0) = w_k.
\end{aligned} \tag{22}$$

Here, v_k and w_k are the coordinates of the vectors v and w respectively, and satisfy

$$\sum_{i,j} v_i v_j g_{ij}(0) = -1, \qquad \text{and} \qquad \sum_{i,j} v_i w_j g_{ij}(0) = 0,$$

being $g_{ij}(0)$ the coefficients of the metric in the point $\gamma(0)$ in these coordinates. Moreover, all the Christoffel symbols are evaluated on γ, i.e., $\Gamma_{ij}^k(t) := \Gamma_{ij}^k(\gamma(t))$.

A (four dimensional) *plane wave* is a spacetime $(\mathbb{R}^4, g)$ which admits a coordinate system (u, v, x, y) such that the Lorentzian metric may be expressed as follows,

$$g = H(u, x, y)\, du^2 + 2dudv + dx^2 + dy^2,$$

where $H(u, x, y)$ is a quadratic function in the coordinates x and y with coefficients depending on u (see [2] and references therein), that is,

$$H(u, x, y) = A(u)x^2 + B(u)y^2 + C(u)xy + D(u)x + E(u)y + F(u). \tag{23}$$

The coordinates are known as a Brinkmann coordinate system of $(\mathbb{R}^4, g)$. In these coordinates, the Christoffel symbols of g are easily computed as follows,

$$\Gamma_{i,j}^1 = 0 \qquad \text{for all } i, j = 1, \ldots, 4, \tag{24}$$

$$\Gamma_{1,1}^2 = \frac{1}{2}\frac{\partial H}{\partial u},\ \Gamma_{1,3}^2 = \Gamma_{3,1}^2 = \frac{1}{2}\frac{\partial H}{\partial x},\ \Gamma_{1,4}^2 = \Gamma_{4,1}^2 = \frac{1}{2}\frac{\partial H}{\partial y}, \tag{25}$$

$$\Gamma_{1,1}^3 = -\frac{1}{2}\frac{\partial H}{\partial x},\ \Gamma_{1,1}^4 = -\frac{1}{2}\frac{\partial H}{\partial y}, \tag{26}$$

and the remaining symbols are zero.

Now, let us consider a UD observer $\gamma : I \to \mathbb{R}^4$ satisfying the initial conditions as previously,

$$\gamma(0) = p, \quad \gamma\,'(0) = v, \quad \frac{D\gamma\,'}{dt}(0) = a(0)\, w.$$

Our objective is to prove that such trajectory is extensible to the whole real line, i.e., that the maximal interval of definition of γ is $I = \mathbb{R}$. Making use of Proposition 2.1, we can write

$$\gamma'(t) = \cosh\big(V(t)\big)\, L(t) + \sinh\big(V(t)\big)\, M(t),$$

where $L, M : I \to \mathbb{R}^4$ are solutions of system (22) with initial conditions $L(0) = v$ and $M(0) = a(0)w$. Denote by (L_1, L_2, L_3, L_4) and (M_1, M_2, M_3, M_4) the respective coordinates of L and M. We have the following simple but important fact,

Lemma 5.1 *The first components of L and M satisfy*

$$L_1(t) = v_1, \quad M_1(t) = a(0)w_1, \quad \textit{for all} \quad t.$$

Proof. It trivially follows from (24) and (22) that $L_1' = M_1' = 0$. Therefore, L_1 and M_1 are constants and equal to the respective initial condition.

□

Of course, a direct consequence of the latter result is

$$\gamma_1'(t) = v_1 \cosh\big(V(t)\big) + a(0)w_1 \sinh\big(V(t)\big),$$

which provides with the following explicit expression for the first component of γ,

$$\gamma_1(t) = v_1 \int_0^t \cosh\big(V(s)\big)ds + a(0)w_1 \int_0^t \sinh\big(V(t)\big)ds + p_1. \tag{27}$$

Lemma 5.2 *The functions L_3, M_3, L_4, M_4 are prolongable to the whole real line as solutions of system* (22).

Proof. A first observation is that from (26), the equations from (22) for $k = 3, 4$ are

$$\begin{aligned} L_k'(t) = &\quad -\Gamma_{11}^k(\gamma(t))\left[\cosh\big(V(t)\big)\, L_1(t)^2 + \sinh\big(V(t)\big)\, M_1(t)\, L_1(t)\right], \\ M_k'(t) = &\quad -\Gamma_{11}^k(\gamma(t))\left[\sinh\big(V(t)\big)\, M_1(t)^2 + \cosh\big(V(t)\big)\, M_1(t)\, L_1(t)\right], \end{aligned}$$

and as a consequence of Lemma 5.1,

$$\begin{aligned} L_k'(t) = &\quad -\Gamma_{11}^k(\gamma(t))\left[v_1^2 \cosh\big(V(t)\big) + a(0)v_1 w_1 \sinh\big(V(t)\big)\right], \\ M_k'(t) = &\quad -\Gamma_{11}^k(\gamma(t))\left[a(0)^2 w_1^2 \sinh\big(V(t)\big) + a(0)v_1 w_1 \cosh\big(V(t)\big)\right], \end{aligned}$$

for $k = 3, 4$. To simplify the writing, we define the functions

$$\begin{aligned} f(t) = &\quad v_1^2 \cosh\big(V(t)\big) + a(0)v_1 w_1 \sinh\big(V(t)\big), \\ g(t) = &\quad a(0)^2 w_1^2 \sinh\big(V(t)\big) + a(0)v_1 w_1 \cosh\big(V(t)\big). \end{aligned}$$

Thus,

$$\begin{aligned} L_k'(t) = &\quad -f(t)\Gamma_{11}^k(\gamma(t)), \\ M_k'(t) = &\quad -g(t)\Gamma_{11}^k(\gamma(t)). \end{aligned} \tag{28}$$

Considering that H is defined by (23), we have

$$\Gamma_{11}^3(\gamma) = -\frac{1}{2}\frac{\partial H}{\partial x}(\gamma(t)) = 2A(\gamma_1)\gamma_3 + C(\gamma_1)\gamma_4 + D(\gamma_1),$$

and

$$\Gamma^4_{11}(\gamma) = -\frac{\partial H}{\partial y}(\gamma(t)) = 2B(\gamma_1)\gamma_4 + C(\gamma_1)\gamma_3 + D(\gamma_1),$$

where $\gamma_1(t)$ is explicitly given by (27). Since

$$\gamma_k(t) = \int_0^t \big[\cosh\big(V(s)\big)\, L_3(s) + \sinh\big(V(s)\big)\, M_k(s)\big]\, ds + p_k,$$

then system (28) (with $k = 3, 4$) can be seen as a integro-differential system of four equations. To pass to a standard system of differential equations, we define the new variables

$$\mathcal{L}_k(t) = \int_0^t \cosh\big(V(s)\big)\, L_3(s) ds, \qquad \mathcal{M}_k(t) = \int_0^t \sinh\big(V(s)\big)\, M_k(s) ds.$$

For the new variables,

$$\mathcal{L}''_k = a(t)\tanh\big(V(s)\big)\mathcal{L}'_k - f(t)\cosh\big(V(s)\big)\Gamma^k_{11}(\gamma(t))$$
$$\mathcal{M}''_k = a(t)\tanh\big(V(s)\big)\mathcal{M}'_k - g(t)\sinh\big(V(s)\big)\Gamma^k_{11}(\gamma(t))$$

and introducing the specific formulas for $\Gamma^3_{11}, \Gamma^4_{11}$ computed before, we arrive to

$$\begin{aligned}\mathcal{L}''_3 &= a(t)\tanh\big(V(t)\big)\mathcal{L}'_3 + \tfrac{1}{2}f(t)\cosh\big(V(t)\big)\,[2A(\gamma_1)\,[\mathcal{L}_3 + \mathcal{M}_3 + p_3] \\ &\quad + C(\gamma_1)\,[\mathcal{L}_4 + \mathcal{M}_4 + p_4] + D(\gamma_1)]\end{aligned}$$

$$\begin{aligned}\mathcal{M}''_3 &= a(t)\coth\big(V(t)\big)\mathcal{M}'_k + \tfrac{1}{2}g(t)\sinh\big(V(t)\big)\,[2A(\gamma_1)\,[\mathcal{L}_3 + \mathcal{M}_3 + p_3] \\ &\quad + C(\gamma_1)\,[\mathcal{L}_4 + \mathcal{M}_4 + p_4] + D(\gamma_1)]\end{aligned}$$

$$\begin{aligned}\mathcal{L}''_4 &= a(t)\tanh\big(V(t)\big)\mathcal{L}'_4 + \tfrac{1}{2}f(t)\cosh\big(V(t)\big)\,[2B(\gamma_1)\,[\mathcal{L}_4 + \mathcal{M}_4 + p_4] \\ &\quad + C(\gamma_1)\,[\mathcal{L}_3 + \mathcal{M}_3 + p_3] + E(\gamma_1)]\end{aligned}$$

$$\begin{aligned}\mathcal{M}''_4 &= a(t)\coth\big(V(t)\big)\mathcal{M}'_4 + \tfrac{1}{2}g(t)\sinh\big(V(t)\big)\,[2B(\gamma_1)\,[\mathcal{L}_4 + \mathcal{M}_4 + p_4] \\ &\quad + C(\gamma_1)\,[\mathcal{L}_3 + \mathcal{M}_3 + p_3] + E(\gamma_1)]\end{aligned}$$

with $\gamma_1(t)$ given by (27). This is a linear system of second order differential equations on the involved variables, and can be easily transformed into a first order system $x' = A(t)x$ of order 8. Now, the basic theory of linear systems states that every solution has the whole real line as a maximal interval of definition, closing the proof.

□

Up to now, we have that L_1, L_3, L_4 (resp. M_1, M_3, M_4) are defined on the whole $\mathbb{R}$. It remains to prove the completeness of $L_3(t)$ (resp. $M_3(t)$). The equations (22) for L_2 is

$$L'_2(t) = -\sum_{i,j}\big[\Gamma^2_{ij}\cosh\big(V(t)\big)\, L_i(t)\, L_j(t) + \Gamma^2_{ij}\sinh\big(V(t)\big)\, M_i(t)\, L_j(t)\big]$$

but note that $\Gamma^2_{ij} = 0$ if $i = 2$ or $j = 2$, and moreover H does not depend on the second variable. This implies that the right-hand side part of the latter equation depends on functions

$L_k(t), M_k(t)$ (k=1,3,4), which we have proved that are globally defined, but not on L_2, M_2. Thus, $L_2'(t)$ is defined for every t, and a simple integration leads to the conclusion. An analogous argument serves for $M_2(t)$.

Previous results are picked up in the following theorem.

Theorem 5.3 *Let M be a plane wave spacetime and $a : \mathbb{R} \longrightarrow M$ a positive smooth function. Every inextensible UD trajectory with prescribed acceleration a is complete.*

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