Linearization of planar involutions in $\mathcal{C}^1$

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Abstract. The celebrated Kerékjártó Theorem asserts that planar continuous periodic maps can be continuously linearized. We prove that $\mathcal{C}^1$-planar involutions can be $\mathcal{C}^1$-linearized.

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1 Introduction and statement of the main result

A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *m-periodic* if $F^m = \mathrm{Id}$, where $F^m = F \circ F^{m-1}$, and m is the smallest positive natural number with this property. When $m = 2$ then it is said that F is an *involution.*

When there exists a $\mathcal{C}^k$-diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\psi \circ F \circ \psi^{-1}$ is a linear map then it is said that F is $\mathcal{C}^k$*-linearizable.* In this case, the map ψ is called a *linearization* of F. This property is very important because it is not difficult to describe the dynamics of the discrete dynamical system generated by linearizable maps. For instance, planar m-periodic linearizable maps behave as planar m-periodic linear maps: they are either symmetries with respect to a "line" or "rotations".

There is a strong relationship between periodic maps and linearizable maps. For instance, it is well-known that when $n = 1$ every $\mathcal{C}^k$ periodic map is either the identity, or it is 2-periodic and $\mathcal{C}^k$-conjugated to the involution $-\mathrm{Id}$, see for instance [8]. When $n = 2$ the following result holds, see [4] for a simple and nice proof.

Theorem 1.1. *(Kerékjártó Theorem) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous m-periodic map. Then F is $\mathcal{C}^0$-linearizable.*

The situation changes for $n \geq 3$. In [1, 2], Bing shows that for any $m \geq 2$ there are continuous m-periodic maps in $\mathbb{R}^3$ which are not linearizable. Nevertheless, Montgomery and Bochner give a positive local result proving that for $\mathcal{C}^k, k \geq 1$, m-periodic maps having a fixed point are always locally $\mathcal{C}^k$-linearizable in a neighborhood of this point, see [9] or Theorem 3.1 below. In any case, in [3, 5, 7] it is shown that for $n \geq 7$ there are continuous and also differentiable periodic maps on $\mathbb{R}^n$ without fixed points.

The aim of this paper is to prove the following improvement for planar involutions of the result of Kerékjártó.

Theorem A. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$-differentiable involution. Then F is $\mathcal{C}^1$-linearizable.*

As we will see, our proof uses classical ideas of differential topology together with some ad hoc tricks for extending and gluing non-global diffeomorphisms. The authors thank Professor Sánchez Gabites for suggesting the use of the classification theorem of surfaces for the proof of Lemma 2.5.

2 Preliminary results on differential topology

In this paper, unless it is explicitly stated, a differentiable map will mean a map of class $\mathcal{C}^1$. Also a diffeomorphism will be a $\mathcal{C}^1$- diffeomorphism.

2.1 Results in dimension n

We state two results that we will use afterwards when $n = 2$. The first one asserts that any local diffeomorphism can be extended to be a global diffeomorphism, see [10].

Theorem 2.1. *Let M be a differentiable manifold and let $g : V \to g(V) \subset M$ be a diffeomorphism defined on a neighborhood V of a point $p \in M$. Then there exists a diffeomorphism $f : M \to M$ such that $f|_W = g|_W$ for some neighborhood $W \subset V$ of p.*

The second one is given in [6] for $\mathcal{C}^\infty$- manifolds. Here we state a slightly modified version of the theorem for $\mathcal{C}^1$-manifolds. We leave the details of this generalization to the reader. Notice that it allows to glue diffeomorphisms that match as a global homeomorphism, only changing them in a neighborhood of the gluing set, but not on the gluing set itself.

Theorem 2.2. *For each $i = 0, 1$, let W_i be an n-dimensional $\mathcal{C}^1$-manifold without boundary which is the union of two closed n-dimensional submanifolds M_i, N_i such that*

$$M_i \cap N_i = \partial M_i = \partial N_i = V_i.$$

Let $f : W_0 \to W_1$ be a homeomorphism which maps M_0 and N_0 diffeomorphically onto M_1 and N_1 respectively. Then there is a diffeomorphism $\tilde{f} : W_0 \to W_1$ such that $f(M_0) = M_1$, $f(N_0) = N_1$ and $\tilde{f}|_{V_0} = f|_{V_0}$. Moreover $\tilde{f}$ can be chosen such that it coincides with f outside a given neighborhood Q of V_0.

2.2 Results in the plane

The aim of this subsection is to prove the following local result, that will play a key role in our proof of Theorem A.

Lemma 2.3. *Let $D \subset \mathbb{R}^2$ be an open and simply connected set such that $\{0\} \times \mathbb{R} \subset D$. Then there exist a open set V such that $\{0\} \times \mathbb{R} \subset V \subset D$ and a diffeomorphism $\psi : D \to \mathbb{R}^2$ such that $\psi|_V = \mathrm{Id}$.*

To prove Lemma 2.3 we introduce two more results. The first one is a direct corollary of the natural generalization for non-compact $\mathcal{C}^1$-surfaces of the theorem of classification of $\mathcal{C}^\infty$-compact surfaces given in [6].

Theorem 2.4. *Let M be a simply connected and non-compact $\mathcal{C}^1$- surface such that ∂M is connected and non-empty. Then M is diffeomorphic to $H = \{(x, y) \in \mathbb{R}^2 : x \geq 1\}$.*

The second result is a lemma that allows to transform by a diffeomorphism any $\mathcal{C}^1$-curve "going from infinity to infinity" into a straight line.

Lemma 2.5. *Let C be a closed, connected and non-compact $\mathcal{C}^1$-submanifold of $\mathbb{R}^2$. Then there exists a diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(C) = \{0\} \times \mathbb{R}$.*

Proof. First of all note that $\mathbb{R}^2 \setminus C$ has two connected components that we will denote by C^+ and C^-. Denote also by C_1 and C_2 the simply connected and non compact differentiable surfaces obtained by adding C to C^+ and C^-. Applying Theorem 2.4 to C_1 and C_2 we obtain diffeomorphisms $\phi_1 : C_1 \longrightarrow H_1$ and $\phi_2 : C_2 \longrightarrow H_2$ where $H_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ and $H_2 = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$. Clearly the map $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism of $\{0\} \times \mathbb{R}$ into itself. Thus $(\phi_2 \circ \phi_1^{-1})(0, y) = (0, \lambda(y))$ for a certain diffeomorphism $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$. Consider the diffeomorphism $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $h(x, y) = (x, \lambda(y))$ and define $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ as

$$G(x, y) = \begin{cases} (h \circ \phi_1)(x, y), & \text{if } (x, y) \in C_1; \\ \phi_2(x, y), & \text{if } (x, y) \in C_2. \end{cases}$$

Thus applying Theorem 2.2 with $W_0 = W_1 = \mathbb{R}^2$, $M_0 = C_1, N_0 = C_2, M_1 = H_1, N_1 = H_2$ and $f = G$ we obtain the desired diffeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. $\square$

We are ready to prove the main result of this subsection.

Proof of Lemma 2.3. We consider first the case when there exists $\epsilon > 0$ such that $[-\epsilon, \epsilon] \times \mathbb{R} \subset D$. In this particular case denote by

$$D_+ = \{(x, y) \in D : x > 0\} \quad \text{and} \quad D_\epsilon = \{(x, y) \in D : x \geq \epsilon\}.$$

Since D is an open and simply connected set, by the Riemann Theorem there exists a diffeomorphism $G : D \to \mathbb{R}^2$. Set

$$C_+ = G(\{\epsilon\} \times \mathbb{R}).$$

Clearly we have that C_+ is a closed, connected and non-compact submanifold of $\mathbb{R}^2$. Thus by Lemma 2.5 there exists a diffeomorphism

$$\Phi_+ : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{such that} \quad \Phi_+(C_+) = \{\epsilon\} \times \mathbb{R}.$$

Composing Φ_+ with an appropriate involution, if necessary, we can assume that $(\Phi_+ \circ G)(D_\epsilon) = \{(x, y) \in \mathbb{R}^2 : x \geq \epsilon\} \doteq H_\epsilon$. Set

$$\psi_+ = \Phi_+ \circ G.$$

Thus we have that $\psi_+(D_\epsilon) = H_\epsilon$ and $\psi_+(\{\epsilon\} \times \mathbb{R}) = \{\epsilon\} \times \mathbb{R}$. Therefore $\psi_+(\epsilon, y) = (\epsilon, h(y))$ for some diffeomorphism h of $\mathbb{R}$. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by $H(x, y) = (x, h^{-1}(y))$.

Lastly if we denote by $\Upsilon_+ = H \circ \psi_+$ we get that Υ_+ is a diffeomorphism between D_ϵ and H_ϵ such that $\Upsilon_+|_{\{\epsilon\}\times\mathbb{R}} = \mathrm{Id}$. As before, denote by $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and consider the map $T_+ : D_+ \to \mathbb{R}^2_+$ defined by

$$T_+(z) = \begin{cases} \Upsilon_+(z) & \text{if } x \in D_\epsilon, \\ z & \text{otherwise} \end{cases}$$

Applying Theorem 2.2 with $W_\epsilon = D_+$, $W_1 = \mathbb{R}^2_+$, $M_0 = M_1 = (0,\epsilon] \times \mathbb{R}$, $N_0 = D_\epsilon$, $N_1 = H_\epsilon$ and $f = T_+$ we obtain a diffeomorphism $g_+ : D_+ \to \mathbb{R}^2_+$ such that $g|_{(0,\epsilon/2)\times\mathbb{R}} = \mathrm{Id}$.

In a similar way if we denote by $D_- = \{(x,y) \in D; x < 0\}$, and $\mathbb{R}^2_- = \{(x,y) \in \mathbb{R}^2 : x < 0\}$ we can construct a diffeomorphism $g_- : D_- \to \mathbb{R}^2_-$ such that $g_+|_{(-\epsilon/2,0)\times\mathbb{R}} = \mathrm{Id}$. Clearly the map $g : D \to \mathbb{R}^2$ defined by

$$g(z) = \begin{cases} g_+(z) & \text{if } x \in D_+, \\ g_-(z) & \text{if } x \in D_-, \\ z & \text{otherwise.} \end{cases}$$

is a diffeomorphism and $g|_{(-\epsilon/2,\epsilon/2)\times\mathbb{R}} = \mathrm{Id}$. This ends the proof in this particular case.

Next we will see how to reduce the general case to one that we have already solved.

Consider a differentiable map $\sigma : \mathbb{R} \to (0,1)$ such that $D_\sigma \doteq \{(x,y) \in \mathbb{R}^2; |x| < \sigma(y)\} \subset D$. Denote by $D_{\sigma/3} \doteq \{(x,y) \in \mathbb{R}^2; |x| < \sigma(y)/3\}$. We want to transform with a diffeomorphism the set D_σ into the vertical strip $(-1,1)\times\mathbb{R}$. Moreover, we want that this diffeomorphism is the identity on $D_{\sigma/3}$. To this end we construct a diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ of the type $h(x,y) = (h_y(x), y)$ where $h_y : \mathbb{R} \to \mathbb{R}$ is an odd diffeomorphism satisfying $h_y(x) = x$ if $0 \le x \le \frac{\sigma(y)}{3}$ and $h_y(\sigma(y)) = 1$. Then h maps diffeomorphically D onto $h(D)$. Moreover, $h|_{D_{\sigma/3}} = \mathrm{Id}$ and $h(D) \supset h(D_\sigma) = (-1,1) \times \mathbb{R}$. Using the first part of the proof with any $\epsilon < 1$ we can assert that there exist a diffeomorphism $g : h(D) \to \mathbb{R}^2$ and a neighborhood V of $\{0\} \times \mathbb{R}$ such that $g|_V = \mathrm{Id}$. We obtain the desired result by considering the diffeomorphism $g \circ h$ and the neighborhood $V \cap D_{\sigma/3}$. $\square$

The last preliminary result is given in next lemma.

Lemma 2.6. *Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be continuous maps, such that $\alpha(y) \neq 0$ for all $y \in \mathbb{R}$. Then, there exists a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F|_{\{0\}\times\mathbb{R}} = \mathrm{Id}$ and*

$$(dF)_{(0,y)} = \begin{pmatrix} \alpha(y) & 0 \\ \beta(y) & 1 \end{pmatrix}$$

for all $y \in \mathbb{R}$.

Proof. Set $R(x,y) = 1 + \beta(x+y) - \beta(y)$ and $S(x,y) = \alpha(x+y) - \frac{\beta(x+y)(\alpha(x+y)-\alpha(y))}{R(x,y)}$. We have that $R(0,y) = 1$ and $S(0,y) = \alpha(y) \neq 0$ for all $y \in \mathbb{R}$. By continuity, there exists an open neighborhood V of $\{0\} \times \mathbb{R}$ such that $R(x,y) \neq 0$ and $S(x,y) \neq 0$ for all $(x,y) \in V$. Moreover we can choose V simply connected and satisfying the following property: If (x,y_1) and (x,y_2) belong to V then $(x,y) \in V$ for all $y \in (y_1,y_2)$. Now consider $H : V \to \mathbb{R}^2$ defined as

$$H(x,y) = (H_1(x,y), H_2(x,y)) = \left(\int_y^{y+x} \alpha(s)\,ds\,,\; y + \int_y^{y+x} \beta(s)\,ds \right).$$

Clearly H is $\mathcal{C}^1$ and $H(0,y) = (0,y)$ for all $y \in \mathbb{R}$.

We claim that H restricted to an appropriate open subset of V is an embedding. To prove this fact, note first that $\det((dH)_{(x,y)}) = R(x,y)S(x,y) \neq 0$ for all $(x,y) \in V$. Then H is a local diffeomorphism. Moreover, by the Implicit Function Theorem, since $\frac{\partial H_2}{\partial y}(0,b) \neq 0$ it follows that for any $b \in \mathbb{R}$ there exist an open interval I_b containing 0 and a differentiable map $\phi_b : I_b \to \mathbb{R}$ satisfying the following property: For all $x \in I_b$, $(x, \phi_b(x)) \in V$ and $H_2(x, \phi_b(x)) = b$. We can choose I_b maximal with respect this property. Since $\frac{\partial H_2}{\partial y}(x,y) \neq 0$ for all $(x,y) \in V$ it follows that I_b and ϕ_b are uniquely determined and the graph of $\phi_b(x)$ tends to the boundary of V when x tends to the boundary of I_b.

For any $b \in \mathbb{R}$ denote by J_b the graph of ϕ_b and set $\tilde{W} = \cup_{b\in\mathbb{R}} J_b$. Now we claim that H restricted to $\tilde{W}$ is globally one-to-one. To do this note that the equation $H(x,y) = (a,b)$ with $(x,y) \in \tilde{W}$ implies that $(x,y) \in J_b$. Then calling $L_b(s) = H_1(s, \phi_b(s))$ we need to solve the equation $L_b(s) = a$. Since

$$
\begin{aligned}
L_b'(s) &= \frac{\partial H_1}{\partial x}(s, \phi_b(s)) + \frac{\partial H_1}{\partial y}(s, \phi_b(s))\phi_b'(s) \\
&= \frac{\partial H_1}{\partial x}(s, \phi_b(s)) - \frac{\partial H_1}{\partial y} \frac{\frac{\partial H_2}{\partial x}}{\frac{\partial H_2}{\partial y}}(s, \phi_b(s)) = S(s, \phi_b(s)) \neq 0,
\end{aligned}
$$

it follows that L_b is monotone and consequently $H(x,y) = (a,b)$ has at most one solution in $\tilde{W}$.

Lastly, we claim that there exists an open neighborhood W of $\{0\}\times\mathbb{R}$ contained in $\tilde{W}$. For $b \in \mathbb{R}$, let $\bar{W}_b$ be an open neighborhood of $(0,b)$ in V such that $H|_{\bar{W}_b}$ is a diffeomorphism onto $H(\bar{W}_b)$ and let $\epsilon > 0$ be such that $(-\epsilon, \epsilon) \times (b-\epsilon, b+\epsilon) \subset H(\bar{W}_b)$. Then $W_b = H^{-1}((-\epsilon,\epsilon) \times (b-\epsilon, b+\epsilon))$ is open. Note that

$$
W_b = \bigcup_{s\in(-\epsilon,\epsilon)} H^{-1}((-\epsilon,\epsilon) \times \{s\}) \subset \bigcup_{s\in(-\epsilon,\epsilon)} J_s \subset \tilde{W}.
$$

Therefore the claim is proved by selecting $W \subset \cup_{b\in\mathbb{R}} W_b$ with the following properties: W is open, connected, simply connected and contains $\{0\}\times\mathbb{R}$. Thus we will have that $H|_W$ is a diffeomorphism onto $H(W)$. Therefore $H(W)$ is also connected and simply connected. By Lemma 2.3 there exist open sets $V_1 \subset W$, $V_2 \subset H(W)$ and diffeomorphisms $\varphi_1 : W \to \mathbb{R}^2$ and $\varphi_2 : H(W) \to \mathbb{R}^2$ such that $\varphi_1|_{V_1} = \mathrm{Id}$ and $\varphi_2|_{V_2} = \mathrm{Id}$. Then $F = \varphi_2 \circ H \circ \varphi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism and for any $(x,y) \in V_1 \cap H^{-1}(V_2)$ we have

$$
d(F)_{(x,y)} = d(\varphi_2)_{H\circ\varphi^{-1}(x,y)} \circ d(H)_{\varphi^{-1}(x,y)} \circ d(\varphi^{-1})_{(x,y)} = \mathrm{Id} \circ d(H)_{(x,y)} \circ \mathrm{Id}\,.
$$

In particular, we obtain that

$$
d(F)_{(0,y)} = d(H)_{(0,y)} = \begin{pmatrix} \alpha(y) & 0 \\ \beta(y) & 1 \end{pmatrix},
$$

for all $y \in \mathbb{R}$, as we wanted to prove. □

3 Proof of Theorem A

We will use the classical Kerékjártó Theorem and the Montgomery-Bochner Theorem, see [9]. We also include the proof of the second result because it is very simple and explains what is understood by a locally linearizable map.

Theorem 3.1. *(Montgomery-Bochner Theorem, see [9]). Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set and let $F : \mathcal{U} \to \mathcal{U}$ be a class $\mathcal{C}^r, r \geq 1$, m-periodic map, having a fixed point $p \in \mathcal{U}$. Then, there is a a neighborhood of p, where F is $\mathcal{C}^r$-linearizable and conjugated to the linear map $L(x) := d(F)_p\, x$.*

Proof. Consider the map from $\mathcal{U}$ into $\mathbb{R}^n$, $\psi = \sum_{i=0}^{m-1} L^{-i} \circ F^i$. Since both, F and L, are m-periodic it holds that $L \circ \psi = \psi \circ F$. Moreover, since $d(\psi)_p = m\,\mathrm{Id}$, by applying the Inverse Function Theorem we get that ψ is locally invertible and has the same regularity as F. □

Proof of Theorem A. By the Kerékjártó Theorem the map F is $\mathcal{C}^0$ conjugated to a linear involution. Hence it is conjugated either to $S(x, y) = (-x, y)$ or to $-\mathrm{Id}$. First we consider the case when F is $\mathcal{C}^0$-conjugated to S. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the homeomorphism such that $F \circ g = g \circ S$. Then, since g is a homeomorphism, we know that $L := g(\{0\} \times \mathbb{R})$ is a non-compact, closed and connected topological submanifold of $\mathbb{R}^2$ which is fixed by F. We claim that L is a differentiable submanifold of $\mathbb{R}^2$. To do this we will show that L is locally the graph of a $\mathcal{C}^1$ function.

Let $(a, b) \in L$. Then (a, b) is a fixed point of F and by the Montgomery-Bochner theorem $d(F)_{(a,b)}$ is conjugated to S. Then $d(F)_{(a,b)} - \mathrm{Id} \neq 0$. If we write $F = (F_1, F_2)$ this implies that at least one of the functions $F_1(x, y) - x$ and $F_2(x, y) - y$ has non-zero gradient at (a, b). Assume for instance that $\frac{\partial(F_1(x,y)-x)}{\partial x}(a, b) \neq 0$. By the Implicit Function Theorem there exist neighborhoods V of (a, b) and W of b and a $\mathcal{C}^1$- map $\psi : W \to \mathbb{R}$ such that $L \cap W = \{(\psi(t), t) : t \in W\}$. This proves the claim.

By Lemma 2.5 there exists a diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(L) = \{0\} \times \mathbb{R}$. Therefore $\tilde{F} = \varphi \circ\, F \circ \varphi^{-1}$ is a $\mathcal{C}^1$- involution that has $\{0\} \times \mathbb{R}$ as a line of fixed points. Then $\tilde{F}(0, y) = (0, y)$. Thus $d(\tilde{F})_{(0,y)} = \begin{pmatrix} A(y) & 0 \\ B(y) & 1 \end{pmatrix}$ for some $A, B : \mathbb{R} \to \mathbb{R}$ continuous. Moreover since $d(\tilde{F})_{(0,y)}$ must be conjugated to S it follows that $A(y) = -1$ for all $y \in \mathbb{R}$.

Now using Lemma 2.6 we choose $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ a diffeomorphism such that $\phi|_{\{0\}\times\mathbb{R}^2} = \mathrm{Id}$ and

$$d(\phi)_{(0,y)} = \begin{pmatrix} 1 & 0 \\ -B(y)/2 & 1 \end{pmatrix}.$$

Lastly define

$$\Phi(x, y) = \begin{cases} \phi(x, y) & \text{if } x \geq 0, \\ \tilde{F}(\phi(S(x, y)) & \text{otherwise.} \end{cases}$$

which is $\mathcal{C}^1$ because

$$\begin{aligned} \lim_{x\to 0^+} d(\Phi)_{(x,y)} &= \begin{pmatrix} 1 & 0 \\ -B(y)/2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ B(y) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B(y)/2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \lim_{x\to 0^-} d(\Phi)_{(x,y)}. \end{aligned}$$

Since $\det(d(\phi)_{(0,y)}) = 1$ it follows that ϕ preserves orientation. In addition we know that all points on the line $x = 0$ are fixed and then $\phi(\{x, y) \in \mathbb{R}^2 : x \geq 0\}) = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. Thus we obtain that Φ is a diffeomorphism. Computing directly Φ^{-1} we have

$$\Phi^{-1}(x, y) = \begin{cases} \phi^{-1}(x, y) & \text{if } x \geq 0, \\ S(\phi^{-1}(\tilde{F}(x, y)) & \text{otherwise.} \end{cases}$$

Finally, again a direct computation gives that $\Phi^{-1} \circ \tilde{F} \circ \Phi = S$. Since $\tilde{F} = \varphi \circ \ F \circ \varphi^{-1}$ the map $\Phi^{-1} \circ \varphi$ is the desired $\mathcal{C}^1$-conjugation. This ends the proof for this case.

Now we consider the case when F is $\mathcal{C}^0$-conjugated to $-\operatorname{Id}$. Then F has a unique fixed point p. By the proof of Theorem 3.1 the map $\operatorname{Id} - F$ conjugates F to $-\operatorname{Id}$ in a neighborhood W of p. By Theorem 2.1 the embedding $(\operatorname{Id} - F)|_V$ can be extended to be a global diffeomorphism $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\pi|_V = (\operatorname{Id} - F)|_V$ for some neighborhood $V \subset W$ of p. Since F is topologically conjugated to $-\operatorname{Id}$ we can select V so that $F(V) \subset V$. Consider now $\tilde{F} = \pi \circ F \circ \pi^{-1}$. The map $\tilde{F}$ has 0 as a fixed point and $\tilde{F}|_{\pi(V)} = -\operatorname{Id}$. Let $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be the homeomorphism such that $\gamma^{-1} \circ \tilde{F} \circ \gamma = -\operatorname{Id}$ and consider $L = \gamma(\{0\} \times \mathbb{R})$. Then L is a connected, closed and non-compact topological submanifold of $\mathbb{R}^2$ invariant by $\tilde{F}$. Our next objective will be to modify L for obtaining a $\mathcal{C}^1$ submanifold with the same properties.

Let $r > 0$ be such that $B_r = \{x \in \mathbb{R}^2 : |x| < r\} \subset \pi(V)$ and set $t_0 = \max\{t \in \mathbb{R} : |\gamma(0,t)| = r\}$. Then $L_1 = \gamma(\{0\} \times (t_0, \infty))$ does not intersect B_r. Since $\tilde{F}|_{B_r} = -\operatorname{Id}$ it follows that $\tilde{F}(L_1) = \gamma(\{0\} \times (-t_0, -\infty))$ neither cuts B_r. Set $L_0 = \{t\gamma(0,t_0); t \in [-1,1]\}$ and

$$\tilde{L} = L_1 \cup L_0 \cup \tilde{F}(L_1).$$

Clearly $\tilde{L}$ is also a connected closed and non-compact topological submanifold of $\mathbb{R}^2$ invariant by $\tilde{F}$. Hence it divides $\mathbb{R}^2$ in two connected and open regions A and B that are permuted by $\tilde{F}$. Consider now a differentiable map $f : (0, \infty) \to \mathbb{R}^2$ satisfying the following properties:

1. $f(t) = t\gamma(0, t_0)$ if $t \leq 1/2$,
2. $f(t) \in A$ for all $t > 1/2$,
3. $\lim_{t \to \infty} |f(t)| = \infty$,
4. f is one to one.

Denote by $M_0 = f((0, \infty))$. By construction, M_0 is a connected and differentiable submanifold of $\mathbb{R}^2$ and $M_0 \cap \tilde{F}(M_0) = \emptyset$. Thus $M = M_0 \cup \tilde{F}(M_0) \cup \{(0,0)\}$ is a connected, closed and non-compact differentiable submanifold of $\mathbb{R}^2$ which is invariant by $\tilde{F}$. By Lemma 2.5 there exists a diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(M) = \{0\} \times \mathbb{R}$. Therefore the map

$$\hat{F} = \varphi \circ \tilde{F} \circ \varphi^{-1}$$

is a differentiable involution that has $\{0\} \times \mathbb{R}$ as an invariant line. Thus $\hat{F}(0,y) = (0, g(y))$ for a certain one dimensional differentiable involution $g : \mathbb{R} \to \mathbb{R}$. In this case the map $h(y) = y - g(y)$ is a global diffeomorphism that conjugates g with $-\operatorname{Id}$. Therefore the map $\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tilde{\varphi}(x,y) = (x, h(y))$ is a diffeomorphism that conjugates $\hat{F}$ with an involution $\bar{F}$ that satisfies that $\bar{F}|_{\{0\} \times \mathbb{R}} = -\operatorname{Id}$. Therefore

$$d(\bar{F})_{(0,y)} = \begin{pmatrix} A(y) & 0 \\ B(y) & -1 \end{pmatrix},$$

for some continuous functions A and B with $A(0) = -1$ and $B(0) = 0$. Note that since $A(0) = -1$ and $\bar{F}$ is a diffeomorphism, it follows that $A(y) < 0$ for all $y \in \mathbb{R}$. On the other hand since $\bar{F}^2 = \operatorname{Id}$ we will have

$$d(\bar{F})_{(0,-y)} \circ d(\bar{F})_{(0,y)} = \operatorname{Id},$$

which implies that

$$A(-y) = \frac{1}{A(y)} \quad \text{and} \quad B(-y) = \frac{B(y)}{A(y)}$$

for all $y \in \mathbb{R}$.

Consider now the continuous maps $a, b : \mathbb{R} \to \mathbb{R}$ defined as:

$$a(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ -\frac{1}{A(y)} & \text{otherwise,} \end{cases} \quad \text{and} \quad b(y) = \begin{cases} 0 & \text{if } y \geq 0, \\ -\frac{B(y)}{A(y)} & \text{otherwise.} \end{cases}$$

Direct computations show that

$$a(y) = -A(-y)a(-y) \text{ and } b(y) = b(-y) - B(-y)a(-y),$$

for all $y \in \mathbb{R}$.

Since $a(y) \neq 0$ for all $y \in \mathbb{R}$, by Lemma 2.6 we can choose a diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying that $\phi|_{\{0\}\times\mathbb{R}} = \mathrm{Id}$ and

$$d(\tilde{\phi})_{(0,y)} = \begin{pmatrix} a(y) & 0 \\ b(y) & 1 \end{pmatrix}.$$

As in the previous case we define the map

$$\Phi(x, y) = \begin{cases} \phi(x, y) & \text{if } x \geq 0, \\ \overline{F}(\phi(-x, -y)) & \text{otherwise,} \end{cases}$$

satisfying

$$\begin{aligned} \lim_{x\to 0^+} d(\Phi)_{(x,y)} &= \begin{pmatrix} a(y) & 0 \\ b(y) & 1 \end{pmatrix} = \begin{pmatrix} -A(-y)a(-y) & 0 \\ b(-y) - B(-y)a(-y) & 1 \end{pmatrix} \\ &= \begin{pmatrix} A(-y) & 0 \\ B(-y) & -1 \end{pmatrix} \begin{pmatrix} a(-y) & 0 \\ b(-y) & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \lim_{x\to 0^-} d(\Phi)_{(x,y)}. \end{aligned}$$

The same considerations as in the previous case show that Φ is a $\mathcal{C}^1$-diffeomorphism that conjugates $\bar{F}$ and $-\mathrm{Id}$. Since $\bar{F}$ and F are $\mathcal{C}^1$-conjugated this fact ends the proof of the theorem. □

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