On the analysis of travelling waves to a nonlinear flux limited reaction–diffusion equation*

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Abstract

In this paper we study the existence and qualitative properties of travelling waves associated to a nonlinear flux limited partial differential equation coupled to a Fisher–Kolmogorov–Petrovskii–Piskunov type reaction term. We prove the existence and uniqueness of finite speed moving fronts of C^2 classical regularity, but also the existence of discontinuous entropy travelling wave solutions.

Key words. Flux limited, relativistic heat equation, singular travelling waves, nonlinear reaction-diffusion, KPP, travelling waves, optimal mass transportation, entropy solutions, complex systems, traffic flow, biomathematics.

AMS subject classifications. 35K55, 35B10, 35B40, 35K57, 35K40

1 Introduction and main results

The aim of this paper is to analyze the existence of travelling waves associated to a heterogeneous nonlinear diffusion partial differential equation coupled to a reaction term of Fisher–Kolmogorov–Petrovskii–Piskunov type. The nonlinear diffusion term has been motivated in different contexts and from

*This work was partially supported by MCI (Spain) and FEDER, Projects MTM2008–05271 and MTM2008-02502 and Junta de Andalucía Projects FQM–4267 and FQM2216.

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different points of view (see the pioneering work [29]). Also, it has been deduced in the Monge-Kantorovich's optimal mass transport framework where it is usually called the relativistic heat equation [15] or in astrophysics [26]. The existence and uniqueness of entropy solutions for the nonlinear parabolic flux diffusion was proved in [4], while in [5] the finite speed of propagation was analyzed. The resulting reaction–flux–limited–diffusion system exhibits new properties compared to the classical reaction coupled to the linear diffusion equation, such as the existence of singular travelling waves which opens new perspectives of application to biology or traffic flow frameworks.

Reaction–diffusion systems consist in mathematical models describing the dynamics of the concentration of one or more populations distributed in space under the influence of two processes: local reactions in which the populations interact with each other, and diffusion which provokes the populations to spread out in space. In the context of reaction-diffusion the notion of population can be understood in a wide sense such as particles or concentrations in chemical processes, but also examples can be found in biology (cells, morphogens), geology, combustion, physics and ecology or more recently in computer science or complex systems, see for instance [16, 18, 21, 24, 25, 27, 30, 32, 34]. This fact has motivated the attention by both formal and rigorous work on a variety of applications starting from linear diffusion of type

$$\frac{\partial u}{\partial t} = \nu \, \Delta u + f(u), \qquad u(t=0,x) = u_0(x), \tag{1}$$

where ν is the so called diffusion coefficient and f represents the reaction term. Cooperative behavior often stems from diffusive coupling of nonlinear elements and reaction-diffusion equations provide the prototypical description of such systems.

In many applications and in particular in complex systems reaction-diffusion equations often provide a natural mathematical description of these dynamical networks since the elements of the networks are coupled through diffusion in many instances. The correct description of reaction-diffusion phenomena requires a detailed knowledge of the interactions between individuals and groups of individuals. This line of research motivates the study of nonlinear cooperative behavior in complex systems [9], which is a closed subject interconnected with reaction–diffusion systems. There is a wide literature raising the universality of application of reaction–diffusion systems. Nevertheless, there are limitations to the reaction-diffusion description. In biochemical networks constituted by small cellular geometries a macroscopic reaction-diffusion model may be inappropriate. In some circumstances the coupling among elements is not diffusive or the diffusive processes are nonlinear, which will strongly influence the dynamical behavior of the network. In [31] it is proposed a nonlinear degenerate density-dependent diffusion motivated by the fact that there are biological (mating, attracting and repelling substances, overcrowding, spatial distribution of food, social behavior, etc.) and physical (light, temperature, humidity, etc.) factors which imply that the probability is no longer a space-symmetric function, i.e., it looses the homogeneity, and so linear diffusion is not a good approach. This heterogeneity property of the diffusion operator comes from the heterogeneous character of the equation and/or from the underlying domain, we refer also to [12, 13, 10, 11]. The same problems with the universality in the applicability occurs when we have not a mean-field interaction between particles or when the particles are dilute or large with respect to the vessel or the media where they are moving [6, 30]. In these cases the linear diffusion approximation might not be the most appropriate. The above processes probably require to incorporate one or various phenomena not included in linear diffusion such as the finite speed of propagation of matter or the existence of nonsmooth densities (singular travelling waves), for example. The mathematical argument justifying that even if the solution has not compact support the size (mass or concentration, depending on the case dealt with) is very small out of some ball with large radius could be unrealistic because in several applications in biology (morphogenesis) [1, 11, 34, 33], social sciences [9] or traffic flow [14] this kind of situations (solutions with large queues) could activate other processes which is the case, for example, of the biochemical processes inside the cells whose activation depends on the time of exposure as well as on the received concentration of morphogen, see [1]. Then, exploring or modeling new nonlinear transport/diffusion phenomena is an interesting subject not only from the viewpoint of applications but also from a mathematical perspective.

Reaction–diffusion systems have also attracted the attention as prototype models for pattern formation which is, in particular, connected with the study of travelling waves, i.e. solutions of the type $u(t,x) = u(x - \sigma t)$ playing an important role in concrete applications. The problem when considering travelling waves for (1) is that the evolution of the support could have an infinite speed of propagation, which would contradict the fact that the speed should not exceed the propagation rate of the real transport process.

Motivated by the above considerations the objective of this paper consists in analyzing the existence of travelling waves for the one–dimensional, nonlinear, flux limited reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \nu \partial_x \left(\frac{u \partial_x u}{\sqrt{|u|^2 + \frac{\nu^2}{c^2} |\partial_x u|^2}} \right) + f(u), \qquad u(t=0,x) = u_0(x), \tag{2}$$

where ν is the viscosity and c is a constant velocity related to the inner properties of the particles. Why this election for the nonlinear diffusion term? First of all, the solutions to this system have finite speed of propagation as opposite to the linear heat equation, i.e. for an initial data with compact support the velocity of growth of the support of the solution is bounded by c (see [2]). Furthermore, this is an extension of the heat equation in the following sense: rewrite the heat equation as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial}{\partial x} \left[u \frac{\partial}{\partial x} \ln u \right] = \nu \frac{\partial}{\partial x} [u\, v], \tag{3}$$

where v is a microscopic velocity. In this form the heat equation can be seen as a transport kinetic equation. The velocity v is determined by the entropy of the system, $S(u) = u \ln u$, and by the concentration u, via the following formula

$$v = \frac{\partial}{\partial x} \left(\frac{S(u)}{u} \right). \tag{4}$$

Note that $\frac{S(u)}{u} = \ln u$ is known as the chemical potential. We propose to modify the form of the flux in (3) by considering a new microscopic velocity averaged with respect to the line element associated with the motion of the particle, so that the new velocity is given by $\frac{\partial}{\partial v} \sqrt{1 + |v|^2} = \frac{v}{\sqrt{1+|v|^2}}$ with

$$v = \frac{\partial_x (S(u)/u)}{\sqrt{1 + [\partial_x (S(u)/u)]^2}}, \tag{5}$$

arriving at the flux limited equation (2). This implies that the chemical potential is now finite, which is not the case for the linear heat equation. Thus, the velocity for which the concentration or density u is transported depends on the entropy of the system (determining the disorder) as well as on its density under an appropriate measure. This is the situation in which one can think in a traffic flow or in a biological context, for example.

For the reaction term, we will consider one canonical model of Fisher [19] or KPP [23] (for Kolmogorov, Petrovsky and Piskunov) type to analyze travelling waves, called FKPP from now on. For the linear diffusion case, the properties associated with this system are well understood in the homogeneous framework, see for example [7, 8, 19, 23]. The above equation (2) with $f = 0$ is known as the relativistic heat equation and is one among the various flux limited diffusion equations used in the theory of radiation hydrodynamics [26].

The term $f(u)$ is written as $uK(u)$, where K is known in biology as the growth rate of the population. The main hypotheses on the FKPP reaction term $K \in C^1([0,1])$ are typically written as

$$\text{(i) } K(1) = 0\,, \qquad \text{(ii) } K'(s) < 0\,,\; s \in (0,1]\,. \tag{6}$$

These hypotheses on $K(u)$ have some consequences on $f(u)$ such us $f(0) = f(1) = 0$, $f'(1) < 0$, $f'(0) > 0$, $f > 0$ in $(0,1)$. Hypothesis (i) in (6) is a normalization property of the carrying capacity and (ii) represents a saturation of the media when the concentration is increasing. Typical examples of such nonlinearities are $K(s) = k(1-s)$ or $K(s) = k(1-s^2)$, where $k = K(0) = f'(0)$ is a constant related to the growth rate of the (biological) particles, usually called intrinsic growth rate. In [19, 23] it was proved that, under the above assumptions, there is a threshold value $\sigma^* = 2\sqrt{\nu k}$ for the speed σ associated with the linear diffusion system (1). Namely, no fronts exist for $\sigma < \sigma^*$, and there is a unique front (up to space or time shifts) for all $\sigma \geq \sigma^*$.

The study of existence and uniqueness of solutions to the flux limited reaction-diffusion equation (2) has been done in [2], see also the references therein for a complete study of the "relativistic" heat equation. The natural concept of solution for this problem implies the use of Kruzhkoz's entropy solutions. In fact, in [2] it is proved that for any initial datum $0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, there exists a unique entropy solution u of (2) in the N-dimensional case $[0,T) \times \mathbb{R}^N$, for every $T > 0$, such that $u(t=0) = u_0$. In addition, solutions live in a subspace of Bounded Variation functions. Moreover, if $u(t)$, $\bar{u}(t)$ are the entropy solutions corresponding to initial data u_0, $\bar{u}_0 \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$, respectively, then

$$\|u(t) - \bar{u}(t)\|_{L^1(\mathbb{R}^N)} \leq e^{t\|f\|_{Lip}} \|u_0 - \bar{u}_0\|_{L^1(\mathbb{R}^N)}\,,\ \forall t \geq 0\,,$$

where $\|f\|_{Lip}$ denotes the Lipschitz constant for f in $[0,1]$. The the existence of entropy solutions to initial data only in in L^∞ was extended in Proposition 3.14 of [2].

One of the most important differences between the linear (1) and the nonlinear (2) diffusion models emerges, besides the existence theory reported above, in the study of travelling waves. A travelling wave is a solution having a constant profile which moves with constant speed, i.e. a solution of the equation of the form $u(t,x) = u(\xi)$ with $\xi = x - \sigma t$ for some constant σ. The function u is usually called the wave profile and the constant σ is the wave speed. Let us give a simple example that may illustrate the results obtained in this paper for (2) by means of a simplified reaction-flux–limited-diffusion equation,

$$\partial_t u = \partial_x \left(u \frac{\partial_x u}{|\partial_x u|} \right) + u(1-u)\,, \tag{7}$$

which allows us to compute explicit travelling waves. Given (7), the equation satisfied by a decreasing wave front profile $u(\xi) = u(x - \sigma t)$ is

$$-\sigma \dot{u} = -\dot{u} + u(1-u)\,.$$

Then, it can be easily proved the existence of a unique, global classical solution given by

$$u_\sigma(\xi) = \frac{1}{e^{-\frac{1}{\sigma-1}\xi} + 1}\,, \quad \xi \in \mathbb{R}\,,$$

only if $\sigma > 1$ up to space or time shifts. Furthermore, the step function $u(\xi) = 1$ if $\xi < 0$ and null otherwise, gives the travelling wave profile of an entropy solution to (7) with $\sigma = 1$. Let us observe how regular and discontinuous solutions coexist in this simplified model. To complete the above results see [2].

As in the previous case, we find singular profiles for the travelling waves of (2) which to a certain extent constitute the equivalent notion of shock waves in hyperbolic models for traffic flow. On the other hand, there is a wide variety and significant differences for the possible choices of the velocity σ for the travelling wave solutions to the nonlinear reaction–diffusion equations (2) with respect to those associated with (1).

In this paper, we look for a particular kind of travelling waves called *wave front*, determined by a decreasing wave profile $u \in (0,1)$ such that $\lim_{\xi\to-\infty} u(\xi) = 1$, $\lim_{\xi\to\infty} u(\xi) = 0$, verifying (2) in a sense specified later. By the degenerate character of the flux limiter if $u \equiv 0$, we split the analysis of the wave front in two steps. For the positive part $u(\xi) > 0$ $\forall \xi \in (-\infty, \xi_0)$, we impose that $u \in C^2$ solves the equation in a classical sense. Thus, if $\xi_0 = \infty$ we will have a *classical solution* verifying the equation everywhere in the domain of definition. If $\xi_0 < \infty$, we will see that the null extension of the positive part can be an entropy solution under certain conditions, these solutions being *discontinuous*. The entropy criterium is necessary in this problem since it selects travelling waves of discontinuous type.

Our main result is the following.

Theorem 1.1. *In terms of a value $\sigma^* \leq c$, depending on ν, c, and k, there exists a wave front which is*

(i) a classical solution to (2), with wave speed $\sigma > \sigma^*$ or $\sigma = \sigma^* < c$;

(ii) a discontinuous entropy solution to (2), with wave speed $\sigma = \sigma^* = c$.

Remark 1. *The existence of travelling wave solutions in the case $\sigma < \sigma^*$ is an open problem. Also, the existence of other kind of travelling waves such as those with pulses or soliton-type shape could be explored, see for example [28] or [17] in another context.*

In Section 2 we will analyze the necessary and sufficient condition for the parameters ν, c, and k in order to determine σ^*. The analytical theory dealing with the existence of a solution-set-structure follows from the associated asymptotic initial value problem satisfied by the travelling wave profile. This problem is framed in the analysis of a planar dynamical system where the wave speed σ is a parameter.

Another fundamental property of equation (1) concerns the asymptotic speed of spreading and was established in [8]: If $u_0 \geq 0$ is a continuous function in $\mathbb{R}^N$ with compact support and $u_0 \not\equiv 0$, then the solution $u(t,x)$ with initial data $u(t=0,x) = u_0(x)$ spreads out with speed σ^* in all directions as $t \to +\infty$, i.e. $\max_{|x| \leq \sigma t} |u(t,x) - 1| \to 0$ for each $\sigma \in [0, \sigma^*)$, and $\max_{|x| \geq \sigma t} u(t,x) \to 0$ for each $\sigma > \sigma^*$. A similar result may fit our context by the control of the bound of the entropy solution in the set $\{x > \sigma\, t\}$ by means of an exponential function with negative exponent (see Proposition 3.4 below).

The paper is organized as follows. In Section 2 we pose the asymptotic initial value problem associated with travelling wave solutions and deal with the existence and uniqueness of regular travelling waves. Finally, in Section 3 we analyze the singular wave profiles that can be identified as entropy solutions.

2 An equivalent problem for classical travelling waves

As we mentioned before, the aim of this section is to analyze the classical wave front solutions to (2).

2.1 Travelling wave equations

The existence of a regular travelling wave $u(x-\sigma t)$ of the equation (2) leads to the problem of finding a solution of the following equation

$$\nu \left(\frac{u\, u'}{\sqrt{|u|^2 + \frac{\nu^2}{c^2}|u'|^2}} \right)' + \sigma u' + f(u) = 0\,, \tag{8}$$

which is defined on $(-\infty, \xi_0)$ and satisfies

$$\lim_{\xi \to -\infty} u(\xi) = 1 \tag{9}$$

and

$$u'(\xi) < 0 \text{ for any } \xi \in (-\infty, \xi_0)\,. \tag{10}$$

The constant σ is a further unknown of the problem. Let us analyze this asymptotic initial value problem where $f(u) = uK(u)$ and K fulfills (6). The following result contributes to deduce the asymptotic value of the derivative of u.

Lemma 2.1. *Let $u : (-\infty, \xi_0) \to (0,1)$ be a solution of (8) that satisfies (9)-(10). Then,*

$$\lim_{\xi \to -\infty} u'(\xi) = 0\,. \tag{11}$$

Proof. Take $\xi_n \to -\infty$ with $\xi_n < \xi_0$. For any fixed $n \in \mathbb{N}$ we use the mean value theorem in the interval $[\xi_n - 1, \xi_n]$ to obtain the existence of a sequence $s_n \in [\xi_n - 1, \xi_n]$ satisfying

$$u'(s_n) = u(\xi_n) - u(\xi_n - 1) \to 0.$$

Then, we integrate (8) over $[s_n, \xi_n]$ and analyze the terms of the following equality

$$\int_{s_n}^{\xi_n} \nu \left(\frac{u(\delta)u'(\delta)}{\sqrt{|u(\delta)|^2 + \frac{\nu^2}{c^2}|u'(\delta)|^2}} \right)' d\delta + \int_{s_n}^{\xi_n} \sigma u'(\delta) d\delta + \int_{s_n}^{\xi_n} f(u(\delta)) d\delta = 0\,.$$

The third term

$$\int_{s_n}^{\xi_n} f(u(\delta)) d\delta \to 0\,,$$

since the interval is bounded and the integrand converges uniformly to zero. The second term, using Barrow's rule, is

$$\sigma(u(\xi_n) - u(s_n))$$

that tends to zero because of (9). The first term, again from Barrow's rule, takes the form

$$\nu \frac{u(\xi_n)u'(\xi_n)}{\sqrt{|u(\xi_n)|^2 + \frac{\nu^2}{c^2}|u'(\xi_n)|^2}} - \nu \frac{u(s_n)u'(s_n)}{\sqrt{|u(s_n)|^2 + \frac{\nu^2}{c^2}|u'(s_n)|^2}}\,,$$

which tends to zero since $u'(s_n) \to 0$ and

$$\nu \frac{u(\xi_n)u'(\xi_n)}{\sqrt{|u(\xi_n)|^2 + \frac{\nu^2}{c^2}|u'(\xi_n)|^2}} \to 0.$$

Using (10) one gets

$$\frac{u(\xi_n)u'(\xi_n)}{\sqrt{|u(\xi_n)|^2 + \frac{\nu^2}{c^2}|u'(\xi_n)|^2}} = \frac{-1}{\sqrt{\frac{1}{|u'(\xi_n)|^2} + \frac{\nu^2}{c^2}\frac{1}{|u(\xi_n)|^2}}}\,,$$

therefore

$$\frac{1}{|u'(\xi_n)|^2} + \frac{\nu^2}{c^2}\frac{1}{|u(\xi_n)|^2} \to \infty\,.$$

As the second term tends to $\frac{\nu^2}{c^2}$, then $\frac{1}{|u'(\xi_n)|^2} \to \infty$ and finally $u'(\xi_n) \to 0$. We have then shown that for any $\xi_n \to -\infty$, $u'(\xi_n) \to 0$. This proves (11). □

In a classical framework, looking for travelling wave solutions is equivalent to finding heteroclinic trajectories of a planar system of ODE's which arises from transforming the original problem into travelling wave coordinates (see

[19, 23, 31]). The same ideas in the search of travelling waves of (2) leads to a system which is not uniquely derived from heteroclinic trajectories. Hence, a more detailed analysis of the phase diagram for the planar system of ODE's is required. Define

$$r(\xi) = -\frac{\nu}{c}\frac{u'(\xi)}{\sqrt{|u(\xi)|^2 + \frac{\nu^2}{c^2}|u'(\xi)|^2}}\,, \tag{12}$$

where u is any positive solution of (8)–(9)–(10). Then (u, r) satisfies the first order differential system

$$\left.\begin{aligned} u' &= -\frac{c}{\nu}\frac{|u|\,r}{\sqrt{1-r^2}}\,, \\ r' &= \frac{c}{\nu}\frac{r(r-\frac{\sigma}{c})}{\sqrt{1-r^2}} + \frac{1}{c}K(u)\,. \end{aligned}\right\} \tag{13}$$

By using that $u' < 0$, (12) yields $r \in (0, 1)$. Also, Lemma 2.1 implies $\lim_{\xi\to-\infty} r(\xi) = 0$. As a consequence, the problem of finding a maximal solution of (8)–(10) is equivalent to look for a solution $(u, r) : (-\infty, \xi_0) \to (0, 1)^2$ of (13), maximal in $(0, 1)^2$, that satisfies

$$\lim_{\xi\to-\infty} u(\xi) = 1, \quad \lim_{\xi\to-\infty} r(\xi) = 0. \tag{14}$$

We now analyze the equilibrium points of the system (13) which are $(1, 0)$ and $(0, r^*)$, where $r^* \in (0, 1)$ is a possible root of

$$\frac{c}{\nu}\frac{r(r-\frac{\sigma}{c})}{\sqrt{1-r^2}} + \frac{1}{c}k = 0\,, \tag{15}$$

with $k = K(0) = f'(0)$. The existence of equilibrium points $(u, r) = (0, r^*)$ will determine the behavior of the solution to (13)-(14) and consequently of the solution to (8)–(10). More precisely, we obtain the following result.

Proposition 2.1. *There always exists a solution u of (8) that satisfies (9) and (10). This solution is unique up to a time translation and verifies:*

(i) If there exist no roots $r^* \in (0, 1)$ of (15), then the existence interval for u can be extended to $(-\infty, \xi_0)$, with $\xi_0 < \infty$, and

$$\lim_{\xi\to\xi_0} u(\xi) > 0, \quad \lim_{\xi\to\xi_0} u'(\xi) = -\infty. \tag{16}$$

(ii) If there exist roots of (15), then $\xi_0 = \infty$ and u satisfies

$$\lim_{\xi\to\infty} u(\xi) = 0\,. \tag{17}$$

As a consequence, this solution is maximal in $\mathbb{R}\times(-1,1)$ and is located in $(0,1)^2$.

To prove Proposition 2.1 we will need two preliminary results describing some properties of r and u.

Lemma 2.2. *Let $-\infty < \xi_0 \le \infty$ and $(u,r) : (-\infty,\xi_0) \to (0,1)^2$ be a solution of (13) that satisfies (14). Then, $r'(\xi) > 0$. The same holds true for any extension of (u,r). In particular, the maximal solution (u_M, r_M) associated with (u,r) remains in $(0,1)^2$ and verifies $r'_M(\xi) > 0$.*

We will give the proof of this result at the end of this Section by analyzing in detail the zeros of r' in (13) and describing the phase diagram associated with (13)-(14).

The following result deals with the strict positivity of u.

Lemma 2.3. *Let $(u,r) : (\xi_1,\xi_0) \to (0,1)^2$ be a solution of (13), where $-\infty \le \xi_1 < \xi_0 \le \infty$ are such that*

$$\lim_{\xi\to\xi_0} r(\xi) = 1\,, \quad r'(\xi) > 0\,.$$

Then

$$\lim_{\xi\to\xi_0} u(\xi) > 0\,.$$

Proof. Denote $(\bar u, \bar r)$ this particular solution. A contradiction argument allows to define $\tilde u(r) := \bar u(\bar r^{-1}(r))$ in an interval $(1-\varepsilon, 1)$ that satisfies

$$z' = \frac{-zr}{r(r-\frac{\sigma}{c}) + \frac{\nu}{c^2}K(\tilde u(r))\sqrt{1-r^2}}\,, \quad z(1) = 0\,.$$

If $\frac{\sigma}{c} \neq 1$, this equation is locally Lischitsz-continuous in z and the point $(1,0)$ is regular. Then, by using the uniqueness of the initial value problem z must vanish identically, which is a contradiction. If $\frac{\sigma}{c} = 1$, then the differential equation is singular. However, $\tilde u$ is a solution of the differential equation

$$z' = -z\frac{h(r)}{\sqrt{1-r}}$$

With

$$h(r) = \frac{r}{-r\sqrt{1-r} + \frac{\nu}{c^2}K(\tilde{u}(r))\sqrt{1+r}}\,.$$

The term $\frac{h(r)}{\sqrt{1-r}}$ is singular but improperly integrable and the associated differential equation has uniqueness again by arguing via the separated variables theory. $\square$

We are now in a position to prove Proposition 2.1.

2.2 Proof of Proposition 2.1

A local analysis of (13) gives the following Jacobian matrix in (u, r)

$$J[u,r] = \begin{pmatrix} -\dfrac{c}{\nu}\dfrac{r}{\sqrt{1-r^2}} & -\dfrac{c}{\nu}\dfrac{u}{(1-r^2)^{3/2}} \\ \\ \dfrac{K'(u)}{c} & -\dfrac{c}{\nu}\dfrac{\frac{\sigma}{c} - 2r + r^3}{(1-r^2)^{3/2}} \end{pmatrix}.$$

Clearly,

$$J[1,0] = \begin{pmatrix} 0 & -\dfrac{c}{\nu} \\ \\ \dfrac{K'(1)}{c} & -\dfrac{\sigma}{\nu} \end{pmatrix}$$

has two eigenvalues $\lambda_- < 0 < \lambda_+$ (because $K'(1) < 0$) which are given by $\lambda_\pm = -\frac{\sigma}{2\nu} \pm \sqrt{\left(\frac{\sigma}{2\nu}\right)^2 - \frac{K'(1)}{\nu}}$. The local unstable manifold theorem (see [20, 22]) guarantees the existence of a curve with initial condition γ for which the corresponding solution satisfies (14). As the slope of the eigenvector corresponding to λ_+ is negative (see Remark 2 for an explicit calculus of the eigenvector) only one branch of $\gamma - \{(1,0)\}$ is locally contained in $(0,1)^2$. Let us take γ maximal in $(0,1)^2$. Then, there exist solutions of (13) satisfying (14). Uniqueness up to a time translation comes up from the local uniqueness of the branch γ. Now, Lemmata 2.2 and 2.3 can be applied.

From the fact that u' has opposite sign to r we can deduce that u satisfies (9) and (10). According to the existence of roots of equation (15) we will prove the statements, (1) or (2), of Proposition 2.1. Let us choose (u, r) :

$(-\infty, \xi_0) \to (0,1)^2$ to be a particular solution of (13) satisfying (14). Then, Lemma 2.2 implies that the following limit exists

$$\lim_{\xi \to \xi_0} r(\xi) \,=\, r_L \,.$$

Let us prove that r_L is a lower bound for any possible root r^* of (15), i.e. $r_L \leq r^*$. In fact, if $r(\bar{\xi}) = r^*$ for $\bar{\xi} \in (-\infty, \xi_0)$, then (6) leads to

$$r'(\bar{\xi}) = \frac{c}{\nu} \frac{r(\bar{\xi})(r(\bar{\xi}) - \frac{\sigma}{c})}{\sqrt{1 - r^2(\bar{\xi})}} + \frac{1}{c} K(u(\bar{\xi})) < \frac{c}{\nu} \frac{r(\bar{\xi})(r(\bar{\xi}) - \frac{\sigma}{c})}{\sqrt{1 - r^2(\bar{\xi})}} + \frac{1}{c} k = 0 \,,$$

which contradicts Lemma 2.2. We focus now on the case in which there exists r^* a root of (15). Assume $u < 1$ and $r(\xi) < r^*$ for any $\xi \in (-\infty, \bar{\xi})$. Thus, $0 < r(\xi) < r_L < 1$ and the pair $(u(\xi), r(\xi))$ lives in a compact set for ξ near ξ_0, away from $r = 0$, $r = 1$, and maximal also in $\mathbb{R} \times (-1, 1)$. Global continuation theorems imply $\xi_0 = \infty$.

To prove (17) we observe that

$$\lim_{\xi \to \infty} \frac{u'(\xi)}{u(\xi)} = -\frac{c}{\nu} \lim_{\xi \to \infty} \frac{r(\xi)}{\sqrt{1 - (r(\xi))^2}} = -\frac{c}{\nu} \frac{r_L}{\sqrt{1 - r_L^2}} < 0 \,. \tag{18}$$

Hence, we can use a Gronwall-type estimate in an interval $(\xi^*, +\infty)$ with ξ^* large enough so that $u'(\xi) \leq -\alpha u(\xi)$ holds, where α is a positive constant and $\xi > \xi^*$.

In the case that (15) has no roots, let us first prove that $r_L = 1$. Arguing by contradiction (by assuming $r_L < 1$), we can use a similar argument as in the previous case by using r_L instead of r^*. In this way, we will obtain that $\xi_0 = +\infty$, and also (17). On the other hand, since r has a limit as ξ goes to $+\infty$, then $r'(\xi_n) \to 0$ up to a subsequence. Using this fact in the second equation of (13) we obtain that r_L is a root of (15), which contradicts our assumption. Hence, $r_L = 1$ holds and the first equation of (13) leads to

$$\lim_{\xi \to \xi_0} \frac{u'(\xi)}{u(\xi)} = -\infty \,. \tag{19}$$

Now, we use Lemma 2.3 to show the first part of (16). There only remains to prove that $\xi_0 < \infty$. This statement can be achieved by a contradiction argument again. Actually, if $\xi_0 = +\infty$ we get a sequence ξ_n for which $u'(\xi_n) \to 0$, which contradicts (19). $\square$

Remark 2. *It is possible to follow very precisely the track of the solution of (13) starting from the point $(u,r)=(0,1)$. Denote $r=\tilde{r}(u)$ the smallest root of*

$$\frac{1}{K(u)}\frac{c^2}{\nu}\left(\frac{\sigma}{c}-\tilde{r}(u)\right)=\frac{\sqrt{1-(\tilde{r}(u))^2}}{\tilde{r}(u)}\,,\qquad u\in(0,1)\,.$$

The eigenfunction associated with the eigenvalue $\lambda_+=-\frac{\sigma}{2\nu}+\sqrt{\left(\frac{\sigma}{2\nu}\right)^2-\frac{K'(1)}{\nu}}$, defined at the beginning of the proof of Proposition 2.1, determines the local unstable manifold and is defined by $\left(c\frac{\sigma+\sqrt{-4K'(1)\nu+\sigma^2}}{2K'(1)\nu},\,1\right)$. On the other hand, it is easy to check that the following identity

$$\lim_{u\to 1}\ \tilde{r}(u)\ =\ \frac{\nu}{c\,\sigma}K'(1)$$

holds. Then, $\left(1,\frac{\nu}{c\,\sigma}K'(1)\right)$ is the tangent vector to the solution curve $r=\tilde{r}(u)$. Comparing the slopes of the above vectors leads to the following unrestricted inequality

$$\frac{2K'(1)\nu}{c(\sigma+\sqrt{-4K'(1)\nu+\sigma^2})}>\frac{\nu}{c\,\sigma}K'(1)\,.$$

Therefore, the curve $r=\tilde{r}(u)$ starting at $u=1$ verifies that $r'|_{u=1}<0$.

2.3 Existence of roots for (15)

To conclude the section we describe the existence of roots in (15) depending on σ, c, ν and $k=K(0)$. This problem is equivalent to find zeros of the equation

$$\frac{c^2}{\nu k}\left(\frac{\sigma}{c}-r\right)=g(r)\,,\quad r\in(0,1)\,,\tag{20}$$

where g is defined as

$$g(r)=\frac{\sqrt{1-r^2}}{r}\,,$$

which is a decreasing function with a pole at $r=0$. The left–hand side is a decreasing linear function that touches the r–axis at $\frac{\sigma}{c}$ with slope $-\frac{1}{k}\frac{c^2}{\nu}$. So, when

$$\frac{\sigma}{c}>1\tag{21}$$

there exists at least one root of (20), see Figure 1 (first two cases). Define $\tilde{r}$ as the smallest root of (20) in $(0,1)$.

Let us now focus our attention on the case

$$\frac{\sigma}{c} \leq 1. \tag{22}$$

Now, the existence of roots of (20) depends on $\frac{\sigma}{c}$ as well as on the slope $-\frac{c^2}{\nu}\frac{1}{k}$ of the straight line in the left–hand side of (20). Let us prove that for a range of values $m = \frac{c^2}{\nu}\frac{1}{k}$, there exists $\sigma^* = \sigma^*(m)$ such that for every $\frac{\sigma}{c} \in (\frac{\sigma^*}{c}, 1)$ there exists a root of (20). Note that $g'(r)$ has a unique maximum in $(0,1)$, around it the function is inceasing and then decreasing, verifying $g'(r) \leq -\frac{3\sqrt{3}}{2} = g'(\sqrt{2/3})$ and $\lim_{r\to 0} g'(r) = \lim_{r\to 1} g'(r) = -\infty$. Then, if $-m \leq -\frac{3\sqrt{3}}{2}$, we can claim that there exist roots in $(0,1)$ of the equation

$$g'(r) = -m\,. \tag{23}$$

In fact, when the inequality is strict, i.e. $-m < -\frac{3\sqrt{3}}{2}$, there are two roots in $(0,1)$ while there is only one if the equality is fulfilled, see Figure 1. Let us denote $\tilde{r}$ the smallest real root of (20), $\tilde{r} \in (0, \sqrt{2/3})$. Consider the intersection $\tilde{\delta}$ of the tangent to g at $\tilde{r}$ with the abscissa, which has the expression

$$\tilde{\delta} = \tilde{\delta}\,(m) = \tilde{r} - \frac{g(\tilde{r})}{g'(\tilde{r})} = 2\tilde{r} - \tilde{r}^3. \tag{24}$$

Clearly, we have that for any $\frac{\sigma}{c} \geq \tilde{\delta}\,(m)$ the equation (20), with $m = \frac{c^2}{\nu}\frac{1}{k}$, has at least one root in $(0,1)$. To analyze the case $\frac{\sigma}{c} < 1$ we will check the range of values m for which $\tilde{\delta}\,(m) \leq 1$. By using (24) we deduce that $\tilde{\delta}\,(m) \leq 1$ if and only if $\tilde{r} \leq \frac{\sqrt{5}-1}{2}$ or, according to (23),

$$m \geq \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{5}{2}}. \tag{25}$$

In conclusion, under condition (25) there exists a root of (20) in $(0,1)$, for every $\frac{\sigma}{c} \geq \tilde{\delta}\,(m)$.

Define $\sigma^*(m)$ as follows

$$\frac{\sigma^*(m)}{c} = \begin{cases} \tilde{\delta}(m), & \text{when } m \geq \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{5}{2}}, \\ 1, & \text{otherwise.} \end{cases} \tag{26}$$

Then, we have proved the following result

Proposition 2.2. *There exists a solution of (15) in $r \in (0,1)$ if and only if $\sigma > \sigma^*$ or $\sigma = \sigma^* < c$, where σ^* is defined by (26).*

As a consequence, combining Propositions 2.2 and 2.1 allows to deduce the existence of a classical solution in Theorem 1.1.

2.4 Proof of Lemma 2.2

In order to prove Lemma 2.2, let us provide a description of the positive invariant set associated with the flux defined by the planar system (13). The values (u, r) for which $r' = 0$ are defined by the equation

$$K(u) = -\frac{c^2}{\nu} \frac{r(r - \frac{\sigma}{c})}{\sqrt{1 - r^2}} \,. \tag{27}$$

The roots of this equation can be equivalently obtained as the intersections between $g(r) = \frac{\sqrt{1-r^2}}{r}$ and the straight line $-\frac{c^2}{K(u)\nu}\left(r - \frac{\sigma}{c}\right)$. The straight line is determined by the point $\left(\frac{\sigma}{c}, 0\right)$ and the slope $-\frac{c^2}{K(u)\nu}$, where only the last one depends on u. Using (6), we have that the slope is a decreasing function of u verifying

$$-\infty < -\frac{c^2}{K(u)\nu} \leq -\frac{c^2}{K(0)\nu} = -\frac{c^2}{k\nu}, \qquad u \in [0, 1).$$

Our purpose now is to describe the function $\tilde{r}(u)$, which is defined by the smallest root of (27) for σ, c and ν fixed. We will prove that he number of these roots as well as their existence depend on the value $\frac{\sigma}{c}$. Simple calculus gives that the tangent to g passing by $(\frac{\sigma}{c}, 0)$ satisfies

$$r\left(2 - r^2\right) = -\frac{g(r)}{g'(r)} + r = \frac{\sigma}{c}.$$

The maximum value of the function $r\,(2 - r^2)$, reached at $\sqrt{2/3}$, is $8/(3\sqrt{6})$. The value of $\frac{\sigma}{c}$ in relation to 1 and $8/(3\sqrt{6})$ will determine the different cases. In Figure 1 the curved lines describe the function $g(r)$ while the straight lines represent the function $\frac{1}{K(u)}\frac{c^2}{\nu}\left(\frac{\sigma}{c} - r\right)$.

In the first case (left–hand side in Figure 1), $\frac{\sigma}{c} \geq 8/(3\sqrt{6})$, the straight lines have an unique intersection with the curve $g(r)$ and consequently $\tilde{r}(u)$ is

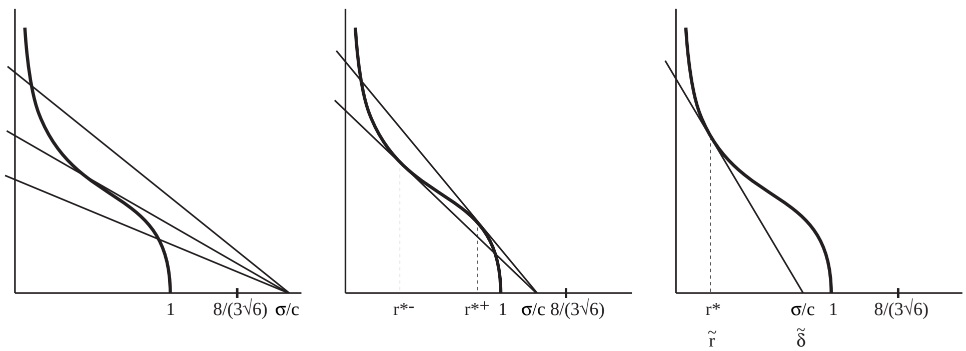

Figure 1: The curved lines represent the function $g(r)$ and the straight lines the functions $\frac{c^2}{\nu K(u)}\left(\frac{\sigma}{c} - r\right)$ for different values u.

uniquely determined and is a decreasing function. The second case (central picture in Figure 1) corresponds to $1 < \frac{\sigma}{c} < 8/(3\sqrt{6})$. It is easy to check that again $\tilde{r}(u)$ is uniquely determined and is a decreasing function which has the shape given in Figure 2 in terms of the two critical values r^{*+} and r^{*-}. Finally, the third case $0 \leq \frac{\sigma}{c} \leq 1$ is represented by the picture in the right–hand side of Figure 1. The function $\tilde{r}(u)$ has the same monotonicity and well–definition properties that in the previous cases, but now the critical value r^* determines the range of definition. The analysis represented in Figure 1 leads to the complete definition of $\tilde{r}(u)$.

Let us now prove that the region

$$S = \left((u,r) \in (0,1)^2, \left\{ \begin{array}{ll} 0 < r < \tilde{r}(u), & \text{if } \tilde{r}(u) \text{ is defined,} \\ 0 < r < 1, & \text{otherwise} \end{array} \right. \right) \tag{28}$$

is positively invariant. In order to prove the positive invariance of S we will describe the flux at the boundary. First, we observe that the segment $\{(u,r),\, 0 \leq r < 1,\, u = 0\}$ at the left–hand side of the square $(0,1)^2$ is invariant, which prevents the solutions to escape through it. Every point of the segment $\{(u,r),\, 0 < u < 1,\, r = 0\}$ at the bottom of the square $(0,1)^2$ has an strict incoming flux because the vector field is vertical through this segment. The arrow coming from the corner $(u,r) = (1,0)$ corresponds to the discussion about the eigenvector for the local unstable manifold theorem in Remark 2. The solid lines in Figure 2 correspond to the curves $\tilde{r}(u)$ and satisfy that the vertical components of the flux are zero because $r' = 0$ while $u' < 0$. The dashed lines corresponding to the slopes in the curves $\tilde{r}(u)$ are

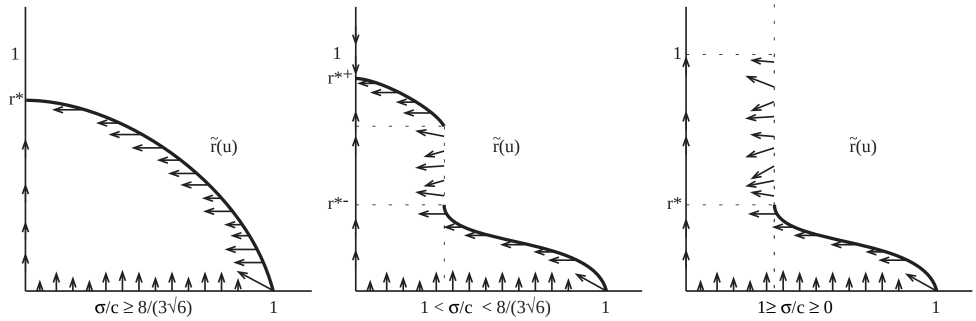

Figure 2: Description of the positive invariant regions S in terms of the curves $\tilde{r}(u)$.

also incoming points since $u' < 0$ there. Then, in Figure 2 we have plotted the phase diagram (slope field) of the planar system (13), $(u, r) : (-\infty, \xi_0) \to (0,1)^2$ with boundary conditions (14) and (17). Therefore, we have proved that if there exists $\bar{\xi}$ such that $(u(\bar{\xi}), r(\bar{\xi})) \in S$, then $(u(\xi), r(\xi)) \in S$, for any $\xi \geq \bar{\xi}$.

We shall be done with the proof once we prove the existence of a sequence of values $\bar{\xi}_n$ such that $\bar{\xi}_n \to -\infty$ and $(u(\bar{\xi}_n), r(\bar{\xi}_n)) \in S$. Using (14), we can deduce the existence of a sequence $\bar{\xi}_n \to -\infty$ for which $r'(\bar{\xi}_n) > 0$. Now, we observe that the graphic of $\tilde{r}(u)$ splits $(0,1) \times (0, r^*)$ into two components characterized by $r' > 0$ or $r' < 0$. Since $(u(\bar{\xi}_n), r(\bar{\xi}_n)) \to (1, 0)$, then $(u(\bar{\xi}_n), r(\bar{\xi}_n)) \in S \cap (0,1) \times (0, r^*)$ for n large enough. $\square$

3 Entropy solutions and consequences

In this section we deal with the study of discontinuous traveling waves. So far ad authors know, there is no previous literature reporting on the existence of singular traveling waves. In this case it is necessary to use the notion of entropy solution for this equation, which has been introduced in [2].

The main result of this section is the following Theorem about existence of singular travelling wave solutions.

Theorem 3.2. *Assume $\sigma = \sigma^* = c$. Then, there exists a discontinuous entropy travelling wave solution of (2).*

The existence of entropy, travelling wave solutions if $\sigma < \sigma^*$ is an open problem.

Define

$$v(t,x) = \begin{cases} u(x-\sigma t), & x-\sigma t < \xi_0, \\ 0, & \text{otherwise}, \end{cases} \tag{29}$$

where $\sigma \le \sigma^*$ and $u : (-\infty, \xi_0) \to (0,1)$, $\xi_0 < \infty$, is a solution of (8) given by Proposition 2.1. (16) implies that v is discontinuous.

It is not trivial to prove that some of these functions v are entropy solutions. This follows from the next two results.

Lemma 3.4. *Any solution of (8) satisfying (9)–(10) is log-concave in* $(-\infty, \xi_0)$.

Proof. To see that $\log(v(\xi))$ is concave, it is enough to prove that $\frac{v'(\xi)}{v(\xi)}$ is decreasing. Using the system (13) we have

$$\frac{v'(\xi)}{v(\xi)} = -\frac{c}{\nu}\frac{r(\xi)}{\sqrt{1-r(\xi)^2}}\,.$$

The result follows from Lemma 2.2, since the function $r \to \frac{r}{\sqrt{1-r^2}}$, $r \in (0,1)$, is strictly increasing. □

The following Proposition characterizes the entropy solutions. The proof follows the same lines of Proposition 6.6 in [3], where a similar result was obtained in the case of compact support solutions for the equation without the reaction FKPP term. Thus, combining Theorem 3.4 and Proposition 6.6 in [3] together with the null flux at infinity for non–compact support solutions and Proposition 3.15 in [2] we have

Proposition 3.3. *Let* $v : [0,T) \times \mathbb{R} \to [0,1)$ *and* $\Omega = supp(v(0,\cdot))$ *be such that for any* $t \in [0,T)$*:*

(i) *supp*$(v(t,\cdot)) = \overline{\Omega_t}$*, where* $\Omega_t = \Omega + B(0,\, c\,t)$.

(ii) $v \in C^2(\Omega_t)$ *and satisfies the differential equation (2).*

(iii) $v(t,x)$ *has a vertical contact angle at the boundary of* Ω_t*, for any* $t \in (0,T)$.

(iv) $v(t,x)$ *is log-concave in* Ω_t.

Then, v *is an entropy solution.*

This result allows to select an entropy solution v from those defined by (29). Properties (ii) and (iv) of Proposition 3.3 are satisfied by any v, but only when $\sigma = \sigma^* = c$ the statement (i) holds, i.e. $supp(v) = \Omega(t)$. Moreover, we conclude the proof of Theorem 3.2 by proving that, in this case, v has a vertical contact angle at the boundary of $\Omega(t)$, and therefore (iii) is also satisfied.

The following result can be deduced directly from Proposition 2.1. We give here a more explicit behavior of the vertical angle near ξ_0.

Lemma 3.5. *Let u be a discontinuous travelling wave for $\sigma = \sigma^* = c$. Then, the vertical angle near ξ_0 is of order $(\xi_0 - \xi)^{-\frac{1}{2}}$.*

Proof. Our starting point is system (13). By using Lemma 2.3 we can assure, when $\sigma \leq \sigma^*$, that there exists a constant $\alpha_\sigma > 0$ and ξ_0 such that $u(\xi_0) = \alpha_\sigma$ and $r(\xi_0) = 1$. In the case $\sigma = \sigma^* = c$, (13) leads to

$$r' = \frac{1}{c} K(u) - \frac{c}{\nu} r \frac{\sqrt{1-r}}{\sqrt{1+r}} .$$

Clearly $r'(\xi_0) = \frac{1}{c} K(\alpha_\sigma) < \infty$. An expansion of $r(\xi)$ in Taylor series around ξ_0 allows to find $r(\xi) = 1 + \frac{1}{c} K(\alpha_\sigma)(\xi - \xi_0) + O((\xi - \xi_0)^2)$. Now, combining this expression with the equation for u and integrating between ξ_0 y ξ, $0 < \xi_0 - \xi \ll 1$, we obtain

$$-\log(u(\xi_0)) + \log(u(\xi)) = \frac{c}{\nu} \frac{2}{\left(2\frac{1}{c} K(\alpha_\sigma)\right)^{\frac{1}{2}}} (\xi_0 - \xi)^{\frac{1}{2}} - \frac{c}{\nu} \left(\frac{K(\alpha_\sigma)}{2c} \right) (\xi_0 - \xi)^{\frac{3}{2}} .$$

Neglecting higher–order terms we find $u(\xi) = \alpha_\sigma e^{\frac{2}{\left(2\frac{1}{c} K(\alpha_\sigma)\right)^{\frac{1}{2}}} (\xi_0 - \xi)^{\frac{1}{2}}}$ or

$$u(\xi) = \alpha_\sigma + \alpha_\sigma \frac{2}{\left(2\frac{1}{c} K(\alpha_\sigma)\right)^{\frac{1}{2}}} (\xi_0 - \xi)^{\frac{1}{2}} , \quad \text{for } 0 < \xi_0 - \xi \ll 1 ,$$

after Taylor expansion. □

Remark 3. *Since classical solutions are in particular entropy solutions, the existence of travelling waves for $\sigma \geq \sigma^*$ is completed. The existence of an entropy solution for $\sigma < \sigma^*$ is an open question, we can only assure that the corresponding function v, defined by (29), is not an entropy solution. This follows from the fact established in Theorem 3.9 of [2], that the support of any log-concave solution moves with speed c while the support of $v(t, \cdot)$ moves with $\sigma < c$.*

Remark 4. *The existence of travelling waves having different profiles from wave fronts is also an open question. It can be proved that no more classical (C^2) wave fronts exist. The authors conjecture that no more entropy travelling wave solution will exist, but it is likely to be a much harder problem.*

To conclude this section we propose an application of the travelling wave solutions with $\sigma^* < c$ that allows to bound entropy solutions.

Proposition 3.4. *Let $u_0 : \mathbb{R} \to [0,1)$ be a measurable function with compact support and* $\text{ess sup}(u_0) < 1$. *Let $u(t,x)$ be an entropy solution of (2) with initial data u_0. Then,*

$$\text{ess sup}_{x\in\mathbb{R}}(u(t,x)) < 1$$

and for any $c > \sigma > \sigma^*$ there exist positive constants α and β not depending on σ such that

$$\text{ess sup}_{|x|>\sigma t} u(t,x) \le \alpha e^{-\beta(\sigma-\sigma^*)t}.$$

In addition, if $\sigma > c$ we have

$$\text{ess sup}_{|x|>\sigma t} u(t,x) = 0$$

for large t.

Proof. Let $v^*(t,x) = u^*(x-\sigma^* t)$ be a C^2 travelling wave solution of (2) defined by Theorem 1.1. Then, we can take a translation of u^*, still denoted u^* for simplicity, such that $u^*(\xi) \ge u_0(\xi)$. A comparison principle for entropy solutions, see Theorem 3.8 in [2], leads to

$$u(t,x) \le u^*(x-\sigma^* t), \quad \text{a.e. } (t,x) \in \mathbb{R}^2.$$

On the other hand, for a classical travelling wave there exist positive constants α and β such that

$$u(\xi) \le \alpha e^{-\beta\xi}\,, \quad \xi \in \mathbb{R}\,.$$

This upper estimate is a consequence of the fact that u^* is uniformly bounded and $\lim_{\xi\to\infty} \frac{(u^*(\xi))'}{u(\xi)}$ is strictly negative as pointed out in (18). Hence, we find

$$u(t,x) \le u^*(x-\sigma^* t) \le \alpha e^{-\beta(x-\sigma^* t)}, \quad \text{a.e. } (t,x) \in \mathbb{R}^2. \tag{30}$$

Assuming now that $x > \sigma\, t$, we deduce from (30) the inequality

$$u(t,x) \le \alpha e^{-\beta(\sigma-\sigma^*)t}, \quad \text{a.e. } (t,x) \in \mathbb{R}^2,\ x > \sigma\, t. \tag{31}$$

In the case $x < -\sigma t$ we can argue in a similar way by using a classical travelling wave $\tilde{u}^*(\sigma^* t - x)$ such that $u_0(\xi) < \tilde{u}^*(-\xi)$.

The second assertion follows by a comparison argument with the singular travelling wave defined by (29). $\square$

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