Persistence of equilibria as periodic solutions of forced systems

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1 Introduction

Let $X : U \subset \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz continuous vector field defined on an open and bounded neighborhood of the origin. Assume that the origin is the only zero of X, that is

$$X(0) = 0 \text{ and } X(x) \neq 0 \text{ if } x \in U \setminus \{0\}.$$

Associated to this vector field is the autonomous system

$$\dot{x} = X(x), \tag{1}$$

having $x = 0$ as the unique equilibrium. Next we consider the perturbed system

$$\dot{x} = X(x) + p(t, x, \varepsilon) \tag{2}$$

where $p : \mathbb{R} \times U \times [0, 1] \to \mathbb{R}^d$ is continuous, T-periodic in its first variable and such that

$$p(t, x, 0) \equiv 0.$$

We say that the equilibrium $x = 0$ *persists as a T-periodic solution* if, given any p in the above conditions, there exists a T-periodic solution $\varphi_\varepsilon(t)$ of (2) for small $\varepsilon > 0$ and

$$\lim_{\varepsilon \searrow 0} ||\varphi_\varepsilon(t)|| = 0 \text{ uniformly in } t \in \mathbb{R}. \tag{3}$$

There are classical results on persistence based on the use of the Implicit Function Theorem or in Degree Theory (see [5, 2, 6]). In this paper we are interested in finding sharp

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conditions on X for persistence. In the search of these conditions we will find connections with several topological notions.

Let us discuss first the link with topological degree. Very simple examples show that there are non-persistent equilibria such the degree $\deg(X, U, 0)$ takes any integer value. The notation deg will be employed for *Brouwer degree* in $\mathbb{R}^d$. One of these examples is the vector field $X(z, w) = (z^n, iw)$ where $(z, w) \in \mathbb{C} \times \mathbb{C} \equiv \mathbb{R}^4$ and $n = 1, 2, \ldots$ In this case $\deg(X, U, 0) = n$ and the system (2) has no T-periodic solutions if $T = 2\pi$ and $p(t, z, w, \varepsilon) = (0, \varepsilon e^{it})$. To obtain examples with negative degree it is sufficient to replace z^n by $\overline{z}^n$. However the condition

$$\deg(X, U, 0) \neq 0$$

implies persistence as soon as one imposes a separation condition to the origin. This means that $x = 0$ is not immersed in a continuum of T-periodic solutions of (1).

A diffeotopy is a path of diffeomorphisms. This notion is usually employed in Differential Topology but it is also very natural in the context of non-autonomous differential equations. Actually the general solution of one of these equations can be interpreted as a diffeotopy. We will use this concept to obtain a negative result. It will be shown that $x = 0$ is not persistent (period T) if the time T-map associated to (1) can be deformed (via a diffeotopy) to a map without fixed points. In particular this result can be applied to T-isochronous systems. The system (1) is *T-isochronous around the origin* if there exists a neighborhood V of $x = 0$ such that every solution of (1) passing through V is T-periodic.

Hopf's classification theorem in the sphere $\mathbb{S}^d$ says that two continuous maps $f_1, f_2 : \mathbb{S}^d \to \mathbb{S}^d$ are homotopic if and only if $\deg_{\mathbb{S}^d}(f_1) = \deg_{\mathbb{S}^d}(f_2)$. Here $\deg_{\mathbb{S}^d}$ denotes the degree on the compact manifold $\mathbb{S}^d$. We will employ this classical result to prove that $x = 0$ does not persist if $\deg(X, U, 0) = 0$. Collecting all these results it seems that we are close to a characterization of persistence. This is indeed the case in dimension $d = 2$, as shown by the following result.

Theorem 1. *Assume $d = 2$. In the previous setting the following statements are equivalent.*

(i) $x = 0$ persists as a T-periodic solution.

(ii) The system (1) *is not T-isochronous around the origin and* $\deg(X, U, 0) \neq 0$.

The above characterization is no longer valid for $d \geq 3$. This is shown by the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1, \quad \dot{x}_3 = -x_3.$$

In this case (ii) holds with $\deg(X, U, 0) = -1$ but $x = 0$ does not persist, as it is easily shown using the perturbation $p(t, x, \varepsilon) = (0, \varepsilon \sin t, 0)$. Notice that in this example the plane $x_3 = 0$ is T-isochronous. Later we will construct another example of non-persistence where the "isochronous set" is a cone instead of a plane. It seems that an appropriate notion of partial isochronicity would be required in order to extend the above theorem to higher dimensions. We refer to the end of the paper for more comments in this direction.

2 A sufficient condition for persistence

In this section we work in arbitrary dimension. We will introduce the notion of *separated equilibrium (period T)* but first we need to present some notation. It will be assumed that the vector field is Lipschitz-continuous and so there is uniqueness for the initial value problem associated to (1). The solution of (1) satisfying $x(0) = p$ will be denoted by $\phi_t(p)$. By continuous dependence we know that $\phi_t(p)$ is well defined on $[0, T]$ when p is sufficiently close to the origin. Let us consider the set of T-periodic points

$$\mathrm{Per}_T = \{p \in U \ / \ \phi_T(p) = p\}.$$

Definition 2. *The equilibrium $x = 0$ is separated (period T) if $\{0\}$ is the connected component of Per_T containing the origin.*

To illustrate this notion let us consider the 2-dimensional center

$$\dot{x} = \omega \left(||x||^2 \right) Jx$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\omega : [0, \delta[\to \mathbb{R}$ is a smooth function with $\omega > 0$ everywhere. The quantity $||x||^2$ is a first integral and the set Per_T is composed by the union of the origin and all circumferences having radius $\rho < \sqrt{\delta}$ and such that

$$\omega(\rho^2) \in \frac{2\pi}{T}\mathbb{Z}.$$

Thus, $x = 0$ is separated unless ω is constant in a neighborhood of the origin with $\omega \equiv \frac{2\pi}{T}N, \quad N = 1, 2, \ldots$.

Notice that $x = 0$ can be separated even if it is not isolated as a T-periodic solution. This is the case if ω is a C^∞-function having sequences $\rho_n \searrow 0$ and $r_n \searrow 0$ such that

$$\omega(\rho_n^2) = \frac{2\pi}{T} \ \text{ and } \ \omega(r_n^2) \neq \frac{2\pi}{T} \ \text{ for each } \ n.$$

Next result shows that this notion is useful for persistence.

Proposition 3. *Assume that $\deg(X, U, 0) \neq 0$ and $x = 0$ is separated (period T). Then $x = 0$ persists as a T-periodic solution.*

For the proof we need a preliminary result.

Lemma 4. *Assume that A is a compact subset of $\mathbb{R}^d$ containing the origin and satisfying the following property. There exists U neighborhood of the origin such that for any $V \subset U$ an open set with $0 \in V$, we have $A \cap \partial V \neq \emptyset$. Then A contains a non-trivial continuum K with $0 \in K$.*

Proof. Assume by contradiction that the connected component of A containing the origin is a singleton. Then there exist closed and non-void sets F_1 and F_2 with

$$F_1 \cap F_2 = \emptyset, \quad F_1 \cup F_2 = A, \quad 0 \in F_1 \subset U.$$

This can be justified using Corollary 1, page 83 of [7]. The two sets F_1, F_2 are compact and so there exists $\delta > 0$ such that

$$||x - y|| \geq \delta \text{ for each } x \in F_1 \text{ and } y \in F_2.$$

Let $F_1^\varepsilon = \left\{x \in \mathbb{R}^d \mid \operatorname{dist}(x, F_1) < \varepsilon\right\}$ be the ε-neighborhood of F_1. We select $\varepsilon < \delta$ so that $F_1^\varepsilon \subset U$ and

$$F_1^\varepsilon \cap F_2 = \emptyset.$$

Since F_1^ε is an open neighborhood of the origin contained in U, we know by assumption that

$$A \cap \partial F_1^\varepsilon \neq \emptyset.$$

Points in this intersection satisfy $\operatorname{dist}(x, F_1) = \varepsilon$ and so they cannot lie in F_1 or F_2. This is a contradiction since their union covers A. □

Proof of Proposition 3. We claim that the following property holds

(D) *there exists a sequence* $U_n \subset \mathbb{R}^d$ *of open neighborhoods of the origin with* $\operatorname{diam} U_n \to 0$ *as* $n \to \infty$ *and such that* $\operatorname{Per}_T \cap \partial U_n = \emptyset$ *for each* n.

To prove this claim we are going to employ Lemma 4. Let W be a compact neighborhood of $x = 0$ with $W \subset U$ and such that $\phi_t(p)$ is well-defined on $[0, T]$ if $p \in W$. Define $A = \operatorname{Per}_T \cap W$ and assume, by a contradiction argument, that (D) does not hold. Then Lemma 4 implies that Per_T contains a non-trivial continuum emanating from the origin. This is not possible since we are assuming that $x = 0$ is separated.

In order to prove that $x = 0$ persists as T-periodic solution we employ Corollary 7 from [1] for each bounded and open set U_n given by property (D). Since $U_n \subset U$ and 0 is the only zero of X in U, the hypothesis $\deg(X, U, 0) \neq 0$ assures that $\deg(X, U_n, 0) \neq 0$. It is easy to see, based on (D) and $\deg(X, U_n, 0) \neq 0$, that the hypotheses of Corollary 7 from [1] are fulfilled. Then there is $\varepsilon_n > 0$ such that for $0 < \varepsilon < \varepsilon_n$ system (2) has at least one T-periodic solution $\varphi_\varepsilon(t)$ such that $\varphi_\varepsilon(t) \in \overline{U}_n$ for all $t \in [0, T]$. Since $\operatorname{diam} U_n \to 0$ as $n \to \infty$, there exists $\varphi_\varepsilon(t)$ that satisfies (3). □

3 Diffeotopies and non-persistence

A diffeotopy of class C^k is a C^k-map

$$h : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d, \quad (x, \lambda) \mapsto h(x, \lambda)$$

such that, for each λ, $h_\lambda = h(\cdot, \lambda)$ is a diffeomorphism of $\mathbb{R}^d$.

Theorem 5. *Assume that there exists a neighborhood* W *of* $0 \in \mathbb{R}^d$ *and a* C^2*-diffeotopy* h *such that* $h_0 = id$ *and* $h_\lambda \circ \phi_T$ *has no fixed points on* W *when* $\lambda \in]0, 1]$. *Then* $x = 0$ *does not persist as a* T*-periodic solution.*

Before the proof we present two examples where the theorem applies.

Example 1. Isochronous systems.

When the system (1) is isochronous the diffeotopy

$$h(x,\lambda) = x + \lambda v,$$

can be used. Here $v \in \mathbb{R}^d \setminus \{0\}$ is a fixed vector. Since ϕ_T is the identity on a neighborhood of the origin,

$$h_\lambda(\phi_T(x)) = h_\lambda(x) = x + \lambda v \neq x \quad \text{if } \lambda \in]0,1].$$

From the theorem we conclude that isochronicity implies non-persistence. We stress that in the previous discussion the vector field was only Lipschitz-continuous. For vector fields of class C^1 we can give a more direct proof of non-persistence. Fix $v \in \mathbb{R}^d \setminus \{0\}$ and define $y(t,q,\epsilon) = \phi(t, q + \epsilon vt)$. Here we are using the notation $\phi(t,p) = \phi_t(p)$ for the flow. The functions $y(t,q,\epsilon)$ are the solutions of

$$\dot{y} = X(y) + \epsilon \frac{\partial \phi}{\partial p}\,(t, \phi_{-t}(y))\, v.$$

The perturbation $p(t,y,\epsilon) = \epsilon \frac{\partial \phi}{\partial p}\,(t,\phi_{-t}(y))\, v$ is T-periodic in the region where $\phi_T(y) = y$ and the solutions are not T-periodic. This shows that $y = 0$ is not persistent.

Example 2. An "isochronous cone".

Consider the polynomial

$$q(x_1,x_2,x_3) = (x_3^2 - x_1^2 - x_2^2)^2$$

and the vector field

$$X(x) = (x_2 - q(x)x_1, -x_1 - q(x)x_2, -q(x))^t.$$

Let C be the double cone defined by the equation $q = 0$. This set is invariant under the flow ϕ_t. The 2π-periodic solutions $x_1(t) = \alpha \sin t$, $x_2(t) = \alpha \cos t$, $x_3(t) = \alpha$, $(\alpha \in \mathbb{R})$ produce closed orbits foliating C. Outside C the system satisfies

$$\frac{d}{dt}(x_3(t)) < 0 \tag{4}$$

and so the remaining orbits are not closed. The diffeotopy $h(x,\lambda) = x + \lambda e_1$, $e_1 = (1,0,0)^t$ is such that $h_\lambda \circ \phi_{2\pi}$ has no fixed points if $\lambda > 0$. To prove this we notice that $\pi_3 \circ h_\lambda = \pi_3$, where π_3 is the projection defined by $\pi_3(x) = x_3$. If $p \in \mathbb{R}^3 \setminus C$ we use (4) and obtain

$$\pi_3(h_\lambda \circ \phi_{2\pi}(p)) = \pi_3(\phi_{2\pi}(p)) < \pi_3(p).$$

This implies that $h_\lambda \circ \phi_{2\pi}(p) \neq p$ if $p \notin C$. If $p \in C$ we notice that $\phi_{2\pi}(p) = p$ and so $h_\lambda(\phi_{2\pi}(p)) = p + \lambda e_1 \neq p$. It is easy to prove that $x = 0$ is the only critical point of X.

Remark. In the previous examples the diffeotopy was just a translation. The use of a general h gives more flexibility to the previous result. As a simple instance assume that $y = \varphi(x)$ is a diffeomorphism of $\mathbb{R}^3$ with $\varphi(0) = 0$. If we change variables in the above example the diffeotopy $h(y,\lambda) = \varphi(\varphi^{-1}(y) + \lambda e_1)$ would be admissible.

Proof. Step 1. Construction of a Carathéodory perturbation.
Define $\Phi(t,x,\varepsilon) = h\left(x, \frac{\varepsilon t}{T}\right)$, $t \in [0,T]$. Then $\Phi : [0,T] \times \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ is of class C^2 and satisfies

$$\begin{aligned}
&\Phi(0,x,\varepsilon) = x, \quad \Phi(t,x,0) = x \\
&\Phi(T, \phi_T(x), \varepsilon) \neq x \text{ if } x \in W, \ \varepsilon \in]0,1] \\
&\Phi(t,\cdot,\varepsilon) \text{ is a diffeomorphism of } \mathbb{R}^d \text{ with inverse } \Psi(t,\cdot,\varepsilon) = h^{-1}_{\varepsilon t/T}\,.
\end{aligned}$$

From the Implicit Function Theorem we deduce that also Ψ is C^2.

Next we transform $\dot{x} = X(x)$ by means of the change of variables $y = \Phi(t,x,\varepsilon)$. Notice that it is not T-periodic in t and we only work on $[0,T]$. The new system is

$$\begin{aligned}
\dot{y} &= \frac{\partial \Phi}{\partial t}(t, \Psi(t,y,\varepsilon), \varepsilon) + \frac{\partial \Phi}{\partial x}(t, \Psi(t,y,\varepsilon), \varepsilon)\, X\left(\Psi(t,y,\varepsilon)\right) \\
&=: F_*(t,y,\varepsilon)
\end{aligned}$$

Since $\Psi(t,y,0) = y$ for all $(t,y) \in [0,T] \times \mathbb{R}^d$ we can find $\varepsilon_* > 0$ and V an open neighborhood of the origin, such that if $(t,y,\varepsilon) \in [0,T] \times V \times [0,\varepsilon_*]$ then $\Psi(t,y,\varepsilon) \in U$. Therefore $F_* : [0,T] \times V \times [0,\varepsilon_*] \to \mathbb{R}^d$ is continuous and Lipschitz-continuous with respect to y. We notice that $F_*(0,y,\varepsilon)$ and $F_*(T,y,\varepsilon)$ probably do not coincide but nevertheless consider the T-periodic extension

$$F : \mathbb{R} \times V \times [0,\varepsilon_*] \to \mathbb{R}^d, \ F(t+T,y,\varepsilon) = F(t,y,\varepsilon), \ F(t,y,\varepsilon) = F_*(t,y,\varepsilon) \text{ if } t \in [0,T[.$$

This function is not continuous but satisfies Carathéodory conditions. The solution of the initial value problem (for each $q \in V$)

$$\dot{y} = F(t,y,\varepsilon), \quad y(0) = q$$

on the interval $[0,T]$ is

$$y(t,q,\varepsilon) = \Phi\left(t, \phi_t(q), \varepsilon\right) \tag{5}$$

where ϕ_t is the flow associated to $\dot{x} = X(x)$.

Let $K_0 \subset K \subset V \cap W$ be compact neighborhoods of the origin such that if $q \in K_0$ then $y(t,q,\varepsilon) \in \mathrm{int}(K)$ (the interior of K) if $t \in [0,T]$. Again in this step it may be necessary to decrease ε. We notice that

$$\dot{y} = F(t,y,\varepsilon) \tag{6}$$

has no T-periodic solutions starting at K_0 if $\varepsilon > 0$. Finally we observe that

$$p(t,y,\varepsilon) := F(t,y,\varepsilon) - X(y)$$

satisfies

$$\lim_{\varepsilon \to 0} p(t,y,\varepsilon) = 0 \text{ uniformly in } t \in [0,T], \ y \in K. \tag{7}$$

To justify this, notice that $\Psi(t,y,0) = y$, $\dfrac{\partial \Phi}{\partial t}(t,x,0) = 0$, $\dfrac{\partial \Phi}{\partial x}(t,x,0) = I_d$. Once we know that the system (6) can be seen as a perturbation of $\dot{x} = X(x)$ it could seem that the result is already proven. This is not the case since the perturbation p is not continuous, but only satisfies Carathéodory conditions.

Step 2. Regularization

Given $\delta > 0$ define

$$F_\delta(t,y,\varepsilon) = \frac{1}{2\delta}\int\limits_{t-\delta}^{t+\delta} F(\tau,y,\varepsilon)d\tau.$$

Then $F_\delta : \mathbb{R}\times V\times[0,\varepsilon_*] \to \mathbb{R}^d$ is T-periodic and continuous.

Assume now that $\Delta : [0,\varepsilon_*] \to \mathbb{R}$ is a given continuous function with $\Delta(0) = 0$ and $\Delta(\varepsilon) > 0$ if $\varepsilon > 0$. The function

$$G(t,y,\varepsilon) = \begin{cases} F_{\Delta(\varepsilon)}(t,y,\varepsilon), & \varepsilon \in]0,\varepsilon_*] \\ X(y) & , \ \varepsilon = 0 \end{cases}$$

is T-periodic in t and continuous on $\mathbb{R}\times K\times[0,\varepsilon_*]$. The continuity can be proved using (7).

We will prove that, for an appropriate choice of Δ, the equation

$$\dot{y} = G(t,y,\varepsilon) =: X(y) + \hat{p}(t,y,\varepsilon) \tag{8}$$

has no T-periodic solutions starting at K_0 when ε is positive and small. This will prove that the origin does not persist as T-periodic solution.

To construct Δ we define

$$\rho(\varepsilon) = \min_{q\in K_0} ||h\left(\phi_T(q),\varepsilon\right) - q||$$

and

$$\omega(\delta,\varepsilon) = \sup_{y\in K}\int_0^T ||F(t,y,\varepsilon) - F_\delta(t,y,\varepsilon)||dt.$$

Notice that ρ is continuous with $\rho(0) = 0$, $\rho(\varepsilon) > 0$ if $\varepsilon > 0$ and

$$\lim_{\delta\searrow 0}\omega(\delta,\varepsilon) = 0 \text{ uniformly in } \varepsilon\in[0,\varepsilon_*]. \tag{9}$$

Finally, let $L > 0$ be a Lipschitz constant for F in K with respect to y. It is easily checked that the same Lipschitz constant can be employed for F_δ, that is

$$||F_\delta(t,y_1,\varepsilon) - F_\delta(t,y_2,\varepsilon)|| \le L||y_1 - y_2|| \text{ if } y_1, y_2 \in K.$$

In particular this property will also hold for G. Using (9) we can find a continuous Δ : $[0,\varepsilon_*] \to \mathbb{R}$ with $\Delta(0) = 0$, $\Delta(\varepsilon) > 0$ if $\varepsilon > 0$ and

$$e^{LT}\omega\left(\Delta(\varepsilon),\varepsilon\right) \le \frac{\rho(\varepsilon)}{2} \text{ for each } \varepsilon.$$

Next we compare the initial value problems associated to (6) and (8).

The corresponding solutions are denoted by $y(t,q,\varepsilon)$ and $\hat{y}(t,q,\varepsilon)$. The size of ε is restricted so that if $q \in K_0$ then $\hat{y}(t,q,\varepsilon) \in \operatorname{int}(K)$ on $[0,T]$. The associated Volterra integral equations lead to the inequalities for $t \in [0,T]$,

$$\begin{aligned} ||y(t,q,\varepsilon)-\hat{y}(t,q,\varepsilon)|| &\leq \int_0^t ||F\left(s,y(s,q,\varepsilon),\varepsilon\right)-G\left(s,\hat{y}(s,q,\varepsilon),\varepsilon\right)||ds \\ &\leq \int_0^t ||F\left(s,y(s,q,\varepsilon),\varepsilon\right)-F_{\Delta(\varepsilon)}\left(s,y(s,q,\varepsilon),\varepsilon\right)||ds \\ &\quad + \int_0^t ||G\left(s,y(s,q,\varepsilon),\varepsilon\right)-G\left(s,\hat{y}(s,q,\varepsilon),\varepsilon\right)||ds \\ &\leq \omega\left(\Delta(\varepsilon),\varepsilon\right)+L\int_0^t ||y(s,q,\varepsilon)-\hat{y}(s,q,\varepsilon)|\,|ds. \end{aligned}$$

From Gronwall's Lemma,

$$||\hat{y}(T,q,\varepsilon)-q|| \geq ||y(T,q,\varepsilon)-q||-||y(T,q,\varepsilon)-\hat{y}(T,q,\varepsilon)|| \geq \rho(\varepsilon)-e^{LT}\omega\left(\Delta(\varepsilon),\varepsilon\right)>0,$$

where we have used (5). □

4 Vector fields with zero degree

This section deals with general continuous vector fields $X : U \subset \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$X(0)=0 \text{ and } X(x)\neq 0 \text{ if } x\in U\setminus\{0\}.$$

The assumption of Lipschitz-continuity will play no role. The main result of this section is the following.

Theorem 6. *In the previous setting assume that*

$$\deg(X,U,0)=0.$$

Then $x=0$ does not persists as a T-periodic solution.

The proof of this theorem will require two preliminary results which can be of independent interest. We employ the notations $B_r=\left\{x\in\mathbb{R}^d \,:\, ||x||<r\right\}$ and $\mathbb{S}^{d-1}=\left\{x\in\mathbb{R}^d \,:\, ||x||=1\right\}$.

Proposition 7. *Let $Y:\overline{B}_r\subset\mathbb{R}^d\to\mathbb{R}^d$ be a continuous vector field defined on the closure of the ball B_r and assume that it satisfies*

$$Y(0)=0 \;\; \textit{and} \;\; Y(x)\neq 0 \;\; \textit{if} \;\; x\in\overline{B}_r\setminus\{0\}, \quad \deg(Y,B_r,0)=0.$$

Then for each $\varepsilon\in]0,r[$ there exists a continuous vector field $Y_\varepsilon:\overline{B}_r\subset\mathbb{R}^d\to\mathbb{R}^d$ such that

$$\begin{aligned} &Y_\varepsilon(x)=Y(x) \quad \textit{if} \quad \varepsilon\leq||x||\leq r, \\ &Y_\varepsilon(x)\neq 0 \quad \textit{for all} \quad x\in\overline{B}_r \quad \textit{and} \\ &\lim_{\varepsilon\downarrow 0}Y_\varepsilon(x)=Y(x) \quad \textit{uniformly in} \quad x\in\overline{B}_r. \end{aligned}$$

Proof. It is inspired by the proof of Theorem 5.2 of Chapter 1 in the book [5]. First we consider the map on the sphere

$$\varphi_\varepsilon : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}, \quad x \mapsto \frac{Y(\varepsilon x)}{||Y(\varepsilon x)||}.$$

There is a well known connection between the degree in balls of $\mathbb{R}^d$ and the degree in $\mathbb{S}^{d-1}$. See in particular [3]. It implies that $\deg_{\mathbb{S}^{d-1}}(\varphi) = \deg(Y, B_r, 0) = 0$. Hopf's theorem says that homotopy classes in the sphere are classified by degree (see [5] or [4]). This implies that φ_ε must be homotopic to constant maps. Let us fix a point y^* in $\mathbb{S}^{d-1}$ and consider a continuous map

$$\Phi^\varepsilon : \mathbb{S}^{d-1} \times [0,1] \to \mathbb{S}^{d-1}, \quad (x,\lambda) \mapsto \Phi^\varepsilon(x,\lambda) = \Phi^\varepsilon_\lambda(x)$$

with $\Phi^\varepsilon_1 = \varphi_\varepsilon$ and $\Phi^\varepsilon_0(x) = y^*$ for each $x \in \mathbb{S}^{d-1}$. Once the homotopy Φ^ε has been constructed, we define the modified vector field

$$Y_\varepsilon(x) = \begin{cases} \varepsilon\, y^* & \text{if} \quad x = 0 \\ \left(\frac{||x||}{\varepsilon}||Y(x)|| + \varepsilon - ||x||\right) \Phi^\varepsilon\left(\frac{x}{||x||}, \frac{||x||}{\varepsilon}\right) & \text{if} \quad 0 < ||x|| \leq \varepsilon \\ Y(x) & \text{if} \quad \varepsilon < x \leq r. \end{cases}$$

The claimed properties of Y_ε are easily checked. Perhaps the most delicate point is the uniform convergence of Y_ε to Y. This follows from the estimate

$$||Y(x) - Y_\varepsilon(x)|| \leq \varepsilon + 2 \max_{||x|| \leq \varepsilon} ||Y(x)||, \quad x \in \overline{B}_r.$$

$\square$

Proposition 8. *Let us fix $T > 0$, $\rho < r$ and a vector field $Z : \overline{B}_r \subset \mathbb{R}^d \to \mathbb{R}^d$ of class C^1 and such that*

$$Z(x) \neq 0 \quad \textit{for all} \quad x \in \overline{B}_r.$$

Then there exists a sequence of vector fields (Z_n) of class $C^1\left(\overline{B}_r, \mathbb{R}^d\right)$ such that $Z_n \to Z$ uniformly in $\overline{B}_r$ and, for each n, the system

$$\dot{x} = Z_n(x)$$

has no T-periodic solutions in B_ρ.

Proof. Let us first recall some results on vector fields defined on compact manifolds which are obtained in the proof of Kupka-Smale Theorem. We follow [8] for the proof of this theorem and the notations.

Let M be a compact manifold and let $\mathfrak{X}^1(M)$ be the Banach space of C^1 vector fields endowed with the C^1 norm. It is known that the class of vector fields having a finite number of periodic orbits with period $\leq T$ is dense in $\mathfrak{X}^1(M)$. We shall employ this fact for the case $M = \mathbb{S}^d$. Notice that it was also employed in the paper [1].

Let us place the sphere $\mathbb{S}^d$ so that it is tangent to $\mathbb{R}^d$ at the south pole S and consider the stereographic projection from the north pole N, $\pi : \mathbb{S}^{d-1} \setminus \{N\} \to \mathbb{R}^d$. We can transport locally the vector field Z to the sphere. This means that we consider a vector field $\hat{Z} \in \mathfrak{X}^1\left(\mathbb{S}^d\right)$ such that on the ball of radius $\rho < r$,

$$Z(x) = d\pi_y\left(\hat{Z}(y)\right) \quad \text{if} \quad x \in \overline{B}_\rho \quad \text{and} \quad \pi(y) = x.$$

Notice that, by construction, the flows induced by Z and $\hat{Z}$ are equivalent in small neighborhoods of $x = 0$ and $y = S$. Let $\mu > 0$ be such that

$$||Z(x)|| \geq \mu \quad \text{if} \quad x \in \overline{B}_\rho.$$

This is possible because we know that Z does not vanish on $\overline{B}_r$.

The vector field $\hat{Z}$ can be approximated by a sequence $\left(\hat{Z}_n\right)$ in $\mathfrak{X}^1\left(\mathbb{S}^d\right)$ such that the associated flows have only a finite number of closed orbits with period $\leq T$. We transport these vector fields to $\overline{B}_\rho$ by means of the formula

$$\tilde{Z}_n(x) := d\pi_y\left(\hat{Z}_n(y)\right) \quad \text{if} \quad x \in \overline{B}_\rho, \quad \pi(y) = x.$$

Then $\tilde{Z}_n$ converges to Z in $C^1\left(\overline{B}_\rho, \mathbb{R}^d\right)$. In particular, for large n,

$$||\tilde{Z}_n(x)|| \geq \frac{\mu}{2} \quad \text{if} \quad x \in \overline{B}_\rho$$

and so $\tilde{Z}_n$ has no equilibria in $\overline{B}_\rho$.

Let us fix n and denote by $\tau_1, \ldots, \tau_k$ the minimal periods of the closed orbits of the system

$$\dot{x} = \tilde{Z}_n(x), \quad x \in \overline{B}_\rho.$$

Notice that these periods as well as the number k may depend upon n. Let us fix a number γ_n lying in $]1, 1+1/n]$ and such that

$$\tau_i \neq \gamma_n \frac{T}{N} \quad \text{for each} \quad 1 \leq i \leq k \quad \text{and} \quad N = 1, 2, \ldots\ .$$

This is always possible because the set $\{N\tau_i/T \,:\, i = 1, \ldots, k,\ N = 1, 2, \ldots\}$ is countable and so cannot cover the whole interval $]1, 1+1/n]$. Finally we consider the modified vector fields

$$Z_n = \gamma_n \tilde{Z}_n.$$

It is clear that Z_n converges to Z in $C^1\left(\overline{B}_\rho, \mathbb{R}^d\right)$. Moreover Z_n has no equilibria on $\overline{B}_\rho$ and the admissible periods for a closed orbit of $\dot{x} = Z_n(x)$ are the numbers $N\tau_i/\gamma_n$. Neither of them can coincide with T and so this system has no T-periodic solutions.

Proof of Theorem 6. Let us fix positive numbers $\rho < r$ such that $\overline{B}_r \subset U$. We claim that there exists a sequence of vector fields $W_n \in C^1\left(\overline{B}_\rho, \mathbb{R}^d\right)$ such that

$$\begin{array}{l} ||W_n - X||_{L^\infty(B_\rho)} \to 0 \\ \dot{x} = W_n(x) \quad \text{has no } T\text{-periodic solutions lying in} \quad \overline{B}_\rho. \end{array}$$

Assuming for the moment that the claim holds it is easy to prove that $x = 0$ cannot persist. Let us consider the perturbation

$$p(t, x, \varepsilon) = \sum_n \eta_n(\varepsilon) \left(W_n(x) - X(x)\right)$$

where $\eta_n : [0, \rho] \to \mathbb{R}$ are continuous functions satisfying

$$0 \leq \eta_n \leq 1, \quad \eta_n(1/n) = 1 \quad \text{and} \quad \eta_n \cdot \eta_m = 0 \quad \text{if} \quad n \neq m.$$

Notice that p is well-defined because the sum can only contain one non-zero term. The convergence of W_n to X implies the continuity of p with $p(t, x, 0) \equiv 0$. For $\varepsilon = 1/n$ the system (2) is precisely $\dot{x} = W_n(x)$ and so the perturbed system does not have T-periodic solutions close to $x = 0$.

To prove the claim we consider the restriction of X to $\overline{B}_r$ and find a sequence $Y_n \in C\left(\overline{B}_r, \mathbb{R}^d\right)$ such that

$$||X - Y_n||_{L^\infty(B_r)} \leq \frac{1}{n} \quad \text{and} \quad Y_n(x) \neq 0 \quad \text{if} \quad x \in \overline{B}_r.$$

Here we have used Proposition 7.

Next we approximate Y_n by smooth vector fields $Z_n \in C^1\left(\overline{B}_r, \mathbb{R}^d\right)$. They can be chosen so that

$$||Y_n - Z_n||_{L^\infty(B_r)} \leq \frac{1}{n} \quad \text{and} \quad Z_n(x) \neq 0 \quad \text{if} \quad x \in \overline{B}_r.$$

Finally we apply Proposition 8 and find, for each Z_n, a vector field $W_n \in C^1\left(\overline{B}_r, \mathbb{R}^d\right)$ such that $||Z_n - W_n||_{L^\infty(B_r)} \leq \frac{1}{n}$ and $\dot{x} = W_n(x)$ has no T-periodic solutions on $\overline{B}_\rho$. □

5 Dimension two

The theorem stated in the introduction for $d = 2$ can be derived from the previous results. To prove $(i) \Rightarrow (ii)$ we use Theorems 5 and 6. To prove $(ii) \Rightarrow (i)$ we can use Proposition 3 together with the following result.

Proposition 9. *Assume that $d = 2$. Then the autonomous system* (1) *is T-isochronous around the origin if and only if $x = 0$ is not separated (period T).*

Proof. If the system is isochronous then Per_T contains a neighborhood of the origin so $\{0\}$ cannot be a component. This argument is valid in any dimension.

To prove the converse we will use some tools from planar topology, in particular Jordan Curve Theorem and the index of a point with respect to a circuit. Let us start by finding two closed disks D and Δ, centered at the origin, and such that

$$\phi_t(D) \subset \Delta \subset U, \quad \text{for each } t \in [0, T].$$

This is possible because $\phi_t(0) = 0$ for each t. Given $p \in D \cap Per_T$, $p \neq 0$, the orbit of (1) passing through p is denoted by γ_p. This orbit is closed and the bounded connected component of $\mathbb{R}^2 \setminus \gamma_p$ is denoted by $R_i(\gamma_p)$. The unbounded component is denoted as $R_e(\gamma_p)$. We notice that $\gamma_p \cup R_i(\gamma_p)$ is contained in Δ, and hence in U. Since the origin is the only equilibrium of X, the region $R_i(\gamma_p)$ is a neighborhood of $x = 0$. After these preliminaries we are ready to prove the converse. We proceed by contradiction and assume that $x = 0$ is neither separated nor isochronous. Let C be the connected component of Per_T containing the origin. We know that it is not a singleton and so we can find a point $p_* \neq 0$ lying in $C \cap D$. Since $x = 0$ is not isochronous, the region $R_i(\gamma_{p_*})$ must contain some point q outside Per_T. Let us define the function

$$\eta : C \to \mathbb{R}, \quad \eta(p) = \frac{1}{2\pi i} \int_{\gamma_p} \frac{dz}{z - q} \quad \text{if } p \neq 0, \quad \eta(0) = 0.$$

This is just the index of q with respect to the loop γ_p and so η can only take the values 0 and 1. Notice that it really takes both values since $\eta(0) = 0$ and $\eta(p_*) = 1$. Next we prove that η is continuous. Given a sequence (p_n) in C converging to $p \in C$, we observe that

$$\phi_t(p_n) \to \phi_t(p), \quad \text{uniformly in } t \in [0, T].$$

Moreover, from the equation (1),

$$\dot{\phi}_t(p_n) \to \dot{\phi}_t(p), \quad \text{uniformly in } t \in [0, T].$$

If $p \neq 0$ then, for large n,

$$\eta(p_n) = \frac{1}{2\pi i} \int_0^T \frac{\dot{\phi}_t(p_n) dt}{\phi_t(p_n) - q} \to \frac{1}{2\pi i} \int_0^T \frac{\dot{\phi}_t(p) dt}{\phi_t(p) - q} = \eta(p).$$

If $p = 0$ the orbits γ_{p_n} will collapse at the origin and so $q_* \in R_e(\gamma_{p_n})$ for large n. Thus $\eta(p_n) = 0$. Summing up the previous discussions, we have constructed a continuous function $\eta : C \to \mathbb{R}$ taking exactly two values. This is impossible if C is connected and therefore we have reached a contradiction. □

We do not know if the converse of Proposition 3 is valid for $d \geq 3$. Such a result would be a satisfactory extension of Theorem 1 to higher dimensions.

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