

Periodic solutions of a perturbed Kepler problem in the plane: from existence to stability

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Abstract

The existence of elliptic periodic solutions of a perturbed Kepler problem is proved. The equations are in the plane and the perturbation depends periodically on time. The proof is based on a local description of the symplectic group in two degrees of freedom.

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1 Introduction

Perturbations of the Kepler problem appear naturally in Celestial Mechanics. These equations are relevant for applications but they also have an intrinsic mathematical interest. In particular the existence and stability of periodic solutions have been discussed by a large number of authors. After Poincaré these questions are usually treated via the averaging method. We refer to the papers [7, 11] for results on the autonomous case and for useful lists of references. We will be interested in periodic time dependent perturbations, a class of equations already considered by Fatou in [5]. More recently Ambrosetti and Coti Zelati treated in [1] a class of periodic perturbations with symmetries and presented the averaging method in a variational framework.

We are going to consider the perturbed Kepler problem in the plane

$$\ddot{x} = -\frac{x}{|x|^3} + \varepsilon \nabla_x U(t, x), \quad x \in \mathbb{R}^2 \setminus \{0\}, \quad (1)$$

where ε is a small parameter and U is a smooth function with period 2π in the variable t . In principle U could also depend on ε but we have eliminated this dependence for simplicity. We are interested in the stability properties of 2π -periodic solutions obtained as a continuation from the integrable case $\varepsilon = 0$. These solutions are understood without collisions and so the restriction to an interval of length 2π , say $x : [0, 2\pi] \rightarrow \mathbb{R}^2 \setminus \{0\}$, defines a loop in $\mathbb{R}^2 \setminus \{0\}$. In particular, each 2π -periodic solution has a winding number $N \in \mathbb{Z}$. For $\varepsilon = 0$ the system has 2π -periodic solutions with any winding number $N \neq 0$. They are produced by the elliptic (or

circular) orbits with major half-axis $a_N = |N|^{-2/3}$. This quantity appears as a consequence of Kepler's third law when we look for solutions with minimal period $\frac{2\pi}{|N|}$. The sign of N corresponds to the orientation of the orbit. Let Σ_N denote the set of initial conditions (x_0, y_0) in $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ producing 2π -periodic solution with winding number $N \neq 0$. In view of the relationship between the energy and the major axis of a Keplerian ellipse we can describe Σ_N by the equations

$$\frac{1}{2}|y_0|^2 - \frac{1}{|x_0|} = -\frac{1}{2}N^{2/3}, \quad N(x_{01}y_{02} - x_{02}y_{01}) > 0.$$

Later we shall see that Σ_N is a three dimensional manifold diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}$, where \mathbb{D} is the unit open disk.

Let $\phi_t(x_0, y_0) = (x(t; x_0, y_0), y(t; x_0, y_0))$ be the flow associated to the the Kepler problem

$$\dot{x} = y, \quad \dot{y} = -\frac{x}{|x|^3}.$$

Then Σ_N is invariant under ϕ_t and we can average $U(t, x)$ with respect to the flow over the manifold Σ_N to obtain the function

$$\Gamma_N : \Sigma_N \rightarrow \mathbb{R}, \quad \Gamma_N(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} U(t, x(t; x_0, y_0)) dt.$$

Any continuation from $\varepsilon = 0$ of 2π -periodic solutions with winding number N must emanate from the set of critical points of Γ_N . Conversely, if the critical point satisfies some non-degeneracy condition, such a continuation always exists. This type of result can be obtained using an abstract variational framework as in [2] or by a more traditional averaging method. We will follow the second alternative and then it is convenient to employ a system of coordinates which is natural to the equation for $\varepsilon = 0$. Since the Kepler problem is integrable the action-angle variables seem a natural choice (see [6]). These are the well-known Delaunay variables and they work well for the continuation from positive eccentricity. However these coordinates present a blow-up at eccentricity $e = 0$ and they do not seem suitable to deal with the continuation from circular solutions. Poincaré proposed a variant of the Delaunay variables which solves this difficulty (see [9] and [4]). They are of the form $(\lambda, \Lambda, \eta, \xi)$ with $\lambda \in \mathbb{S}^1$, $\Lambda > 0$ and $\eta^2 + \xi^2 < 2\Lambda$. The sub-manifold Σ_N has the simple equation $\Lambda = |N|^{-2/3}$ and (λ, η, ξ) are a one-to-one parameterization of Σ_N . Poincaré coordinates are probably the most natural coordinates to deal with the continuation problem (1) and they have been already employed in [11]. In this paper we will show that they are also very useful in the study of the properties of stability. After expressing the function Γ_N in terms of these coordinates, $\gamma_N = \gamma_N(\lambda, \eta, \xi)$, we will prove the following

Theorem 1.1. *Assume that $(\lambda^*, \eta^*, \xi^*)$ is a non-degenerate critical point of γ_N , i.e., $\nabla \gamma_N = 0$, $\det(D^2 \gamma_N) \neq 0$ at $(\lambda^*, \eta^*, \xi^*)$. Then, for any small $\varepsilon > 0$ the bifurcating solution $x_\varepsilon(t)$ is elliptic if*

$$\partial_{\lambda\lambda}^2 \gamma_N(\lambda^*, \eta^*, \xi^*) > 0 \quad \text{and} \quad \det D^2 \gamma_N(\lambda^*, \eta^*, \xi^*) > 0.$$

Conversely, if $\partial_{\lambda\lambda}^2\gamma_N(\lambda^*, \eta^*, \xi^*) < 0$ or $\det D^2\gamma_N(\lambda^*, \eta^*, \xi^*) < 0$ then $x_\varepsilon(t)$ is linearly unstable.

We say that a periodic solution $x_\varepsilon(t)$ is elliptic if all Floquet multipliers μ_i satisfy $|\mu_i| = 1$, $\mu_i \neq \pm 1$. The solution is linearly unstable if some μ_i satisfies $|\mu_i| \neq 1$. These multipliers are the eigenvalues of the monodromy matrix of the linearized periodic system

$$\ddot{y} + D_x^2W(t, x_\varepsilon(t); \varepsilon)y = 0, \quad W(t, x; \varepsilon) = -\frac{1}{|x|} - \varepsilon U(t, x). \quad (2)$$

The condition of ellipticity is not sufficient to guarantee the stability in the Lyapunov sense of the periodic solution or even the stability of the linearized system. It can be interpreted as a type of spectral stability and our result can be seen as a first step towards the understanding of this delicate stability problem. Further progress could lead to proofs of KAM stability and stability in measure. See [3, 8] for more information on the different notions of stability.

Typically the known results on linearized stability via the averaging method assume that some of the multipliers of the unperturbed system are different from ± 1 , see for instance [2] and [11]. However it is well known that the only multiplier of the periodic solutions of Kepler problem is 1. Indeed the monodromy matrix P_* is given by

$$P_* = \begin{pmatrix} I_2 & T \\ 0 & I_2 \end{pmatrix}, \quad T = \begin{pmatrix} \tau & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau \neq 0. \quad (3)$$

To overcome this difficulty with the multipliers we will observe that the matrix P_* enjoys the following symplectic property. Let $\text{Sp}(\mathbb{R}^4)$ be the symplectic group of 4×4 matrices, then there exists a neighborhood $\mathcal{U} \subset \text{Sp}(\mathbb{R}^4)$ of P_* such that the spectrum of every matrix $S \in \mathcal{U}$ satisfies

$$\sigma(S) \subset (\mathbb{R} \setminus \{0\}) \cup \mathbb{S}^1.$$

In other words, the eigenvalues of S can only lie on the real line or on the unit circle. This is an important fact because the monodromy matrix of the system (2) lies in \mathcal{U} for small ε and so the Floquet multipliers can only get out of the unit circle through the real line. This fact will play a role in the proof of Theorem 1.1. Incidentally we note that the previous discussion explains why we work on the plane and not in 3d.

In practice the conditions of the theorem are not easy to check because the solutions of the Kepler problem are implicitly defined by the Kepler equation $u - e \sin u = \ell$. In the circular case $u = \ell$ and these solutions are just trigonometric functions and this will allow us to compute the second derivatives of the Poincaré coordinates at $\eta = \xi = 0$. In this way we will be led to fully explicit results in the circular case. For instance, given $a \in \mathbb{C}$, the forced Kepler problem

$$\ddot{z} = -\frac{z}{|z|^3} - \varepsilon (e^{it} + ae^{-it}), \quad z \in \mathbb{C} \setminus \{0\},$$

has two families of 2π -periodic solutions emanating respectively from e^{it} and $-e^{it}$ if $|a| \neq 4$. Moreover both families are unstable if $|a| > 4$ while the first family is elliptic and the second is unstable if $|a| < 4$.

The rest of the paper is divided in six sections. Section 2 deals with certain properties of 4×4 symplectic matrices. The Poincaré coordinates are reviewed in Section 3. The next Section 4 deals with a class of degenerate Hamiltonian systems inspired by the Kepler problem, the typical computations in the averaging method are presented. The proof of Theorem 1.1 is presented in Section 5 and Section 6 contains the application of the Theorem to the circular case. Finally we add an appendix with the computations of the derivatives of the Poincaré coordinates.

2 A property of symplectic matrices

The symplectic group $\text{Sp}(\mathbb{R}^{2d})$ is composed by the real matrices S of dimension $(2d) \times (2d)$ satisfying

$$S^t J S = J,$$

where S^t is the transpose of S and $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$.

The spectrum of S will be denoted by $\sigma(S)$. It is well known that for symplectic matrices the spectrum is invariant under inversion. This means that if μ is an eigenvalue of S then $\mu \neq 0$ and also $\frac{1}{\mu}$ is an eigenvalue.

A matrix $S \in \text{Sp}(\mathbb{R}^{2d})$ is called *elliptic* if every eigenvalue $\mu \in \sigma(S)$ satisfies $|\mu| = 1$, $\mu \neq \pm 1$. In one degree of freedom ($d = 1$) elliptic matrices are easily characterized in terms of the trace. More precisely, $S \in \text{Sp}(\mathbb{R}^2)$ is elliptic whenever $|\text{tr } S| < 2$. This property is very useful when dealing with stability problems in the plane. Next we are going to present an analogous criterion in two degrees of freedom. In this case the characterization of elliptic matrices will have a local nature since it will hold in a neighborhood of a matrix P with the following properties

$$P \in \text{Sp}(\mathbb{R}^4), \quad P \neq I, \quad \dim \ker(P - I) \neq 2, \quad \sigma(P) = \{1\}. \quad (4)$$

As an example consider the matrix P_* previously defined by (3).

Proposition 2.1. *Assume that P is a matrix satisfying (4). Then there exists a neighborhood $\mathcal{U} \subset \text{Sp}(\mathbb{R}^4)$ of P such that a matrix $S \in \mathcal{U}$ is elliptic if and only if the following conditions hold*

$$\det(S - I) > 0 \quad \text{and} \quad \text{tr } S < 4. \quad (5)$$

Remark 2.2. The condition $\dim \ker(P - I) \neq 2$ is essential. To show this we prove that the above Proposition does not hold if

$$P = \begin{pmatrix} \beta^t & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \quad \text{with} \quad \beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The matrix P satisfies all the conditions of assumption (4) excepting that $\dim \ker (P - I) = 2$.

For each $\varepsilon \neq 0$ consider the matrix $S_\varepsilon = Q_\varepsilon B_\varepsilon Q_\varepsilon^{-1}$ with

$$Q_\varepsilon = \begin{pmatrix} A_\varepsilon & M \\ 0 & A_\varepsilon^{-1} \end{pmatrix}, \quad A_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$B_\varepsilon = \begin{pmatrix} C_\varepsilon^t & 0 \\ 0 & C_\varepsilon^{-1} \end{pmatrix}, \quad C_\varepsilon = \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix}.$$

The matrices Q_ε and B_ε belong to $\text{Sp}(\mathbb{R}^4)$ and the inverse of Q_ε is given by $Q_\varepsilon^{-1} = \begin{pmatrix} A_\varepsilon^{-1} & -M \\ 0 & A_\varepsilon \end{pmatrix}$. Note that M and A_ε commute. The spectrum of S_ε is

$$\sigma(S_\varepsilon) = \sigma(B_\varepsilon) = \{1 \pm i\varepsilon, (1 \pm i\varepsilon)^{-1}\}$$

and so S_ε is not elliptic. Moreover $\det(S_\varepsilon - I) = \det(B_\varepsilon - I) > 0$ and $\text{tr } S_\varepsilon = \text{tr } B_\varepsilon < 4$. Therefore the equivalence between ellipticity and (5) is broken for S_ε . Finally we observe that

$$S_\varepsilon = \begin{pmatrix} A_\varepsilon C_\varepsilon^t A_\varepsilon^{-1} & -A_\varepsilon C_\varepsilon^t M + M C_\varepsilon^{-1} A_\varepsilon \\ 0 & A_\varepsilon^{-1} C_\varepsilon^{-1} A_\varepsilon \end{pmatrix} \rightarrow P \quad \text{as } \varepsilon \rightarrow 0.$$

Remark 2.3. A matrix $S \in \text{Sp}(\mathbb{R}^{2d})$ is called stable if the sequence $\{\|S^n\|\}_{n \in \mathbb{Z}}$ is bounded. Here $\|\cdot\|$ denotes any matrix norm. Most elliptic matrices are stable but there are exceptions if $d \geq 2$. Actually an elliptic matrix is stable if and only if it can be diagonalized. Unstable elliptic matrices can appear in the neighborhood \mathcal{U} given by Proposition 2.1. We illustrate this fact with an example concerning the matrix P_* given by (3). The notations introduced in the previous remark remain valid. Consider

$$E_\varepsilon = \begin{pmatrix} R_\varepsilon & \mathcal{T} \\ 0 & R_\varepsilon \end{pmatrix}$$

where

$$R_\varepsilon = \begin{pmatrix} \cos(\varepsilon^2) & -\sin(\varepsilon^2) \\ \sin(\varepsilon^2) & \cos(\varepsilon^2) \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix}$$

with $\varepsilon > 0$ and $\tau \neq 0$. The matrix E_ε belongs to $\text{Sp}(\mathbb{R}^4)$. Moreover it is elliptic if $\varepsilon^2 < \pi$ and it cannot be diagonalized. Hence the same properties hold for $\mathcal{E}_\varepsilon = Q_\varepsilon E_\varepsilon Q_\varepsilon^{-1}$. The matrix \mathcal{E}_ε converges to P_* as $\varepsilon \rightarrow 0$.

To prepare the proof of Proposition 2.1 we need to introduce some terminology coming from Symplectic Geometry. Given two vectors $\xi, \eta \in \mathbb{R}^4$ we consider the symplectic form acting on them

$$\omega(\xi, \eta) = \xi^t J \eta.$$

Sometimes the same form will act on vectors lying in \mathbb{C}^4 . A Lagrangian plane is a linear manifold $V \subset \mathbb{R}^4$ of dimension two such that the symplectic form vanishes on V , that is

$$\omega(\xi, \eta) = 0 \quad \text{for each } \xi, \eta \in V.$$

The planes $V = \{\xi \in \mathbb{R}^4 : \xi_3 = \xi_4 = 0\}$ and $W = \{\xi \in \mathbb{R}^4 : \xi_1 = \xi_2 = 0\}$ are Lagrangian. The couple (V, W) is an example of a Lagrangian splitting of the space since $\mathbb{R}^4 = V \oplus W$. Next we present an algebraic result where this notion plays a role.

Lemma 2.4. *Assume that 1 is an eigenvalue of the matrix $S \in \text{Sp}(\mathbb{R}^4)$ and there is a splitting by invariant Lagrangian planes, that is, there exist two Lagrangian planes V and W satisfying*

$$V \oplus W = \mathbb{R}^4, \quad S(V) = V, \quad S(W) = W.$$

Then either $S = I$ or $\dim \ker(S - I) = 2$.

Proof. It follows from Lemma 3.2.4 in the book [6] that S is conjugate in $\text{Sp}(\mathbb{R}^4)$ to a matrix of the type

$$\begin{pmatrix} \beta^t & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$

where β is an invertible 2×2 matrix. From the assumptions on S we deduce that 1 is also an eigenvalue of β . Then either β is the identity or $\ker(\beta - I)$ has dimension one. In the first case $S = I$ while in the second the kernel of $S - I$ has dimension two. \square

The next result shows that the class of matrices having a splitting by invariant Lagrangian planes is closed in the symplectic group.

Lemma 2.5. *Let $\{S_n\}$ be a sequence of matrices in $\text{Sp}(\mathbb{R}^4)$ converging to S . In addition assume that for each $n \geq 0$ there exists a splitting of \mathbb{R}^4 by Lagrangian planes that are invariant under S_n . Then there exists another splitting by Lagrangian planes that are invariant under S .*

Proof. Assume that (V_n, W_n) is a Lagrangian splitting of \mathbb{R}^4 with $S_n(V_n) = V_n$, $S_n(W_n) = W_n$. We can apply Lemma 3.2.3. in the book [6]. After selecting an orthonormal basis of V_n , say $\{\alpha_n, \beta_n\}$, we extend it to a symplectic basis of \mathbb{R}^4 , say $\{\alpha_n, \beta_n, \gamma_n, \delta_n\}$ with $\gamma_n, \delta_n \in W_n$. In particular

$$\omega(\alpha_n, \delta_n) = \omega(\beta_n, \gamma_n) = 0, \quad \omega(\alpha_n, \gamma_n) = \omega(\beta_n, \delta_n) = 1.$$

Moreover, from the proof of the Lemma mentioned above,

$$\omega(\alpha_n, \eta) = \langle \gamma_n, \eta \rangle, \quad \omega(\beta_n, \eta) = \langle \delta_n, \eta \rangle$$

for each $\eta \in W_n$. Thus, $\{\gamma_n, \delta_n\}$ is an orthonormal basis of W_n . After extracting subsequences we can assume that the vectors $\alpha_n, \beta_n, \gamma_n, \delta_n$ converge as $n \rightarrow +\infty$, say

$$\alpha_n \rightarrow \alpha, \quad \beta_n \rightarrow \beta, \quad \gamma_n \rightarrow \gamma, \quad \delta_n \rightarrow \delta.$$

The plane V spanned by α and β is Lagrangian and the same can be said of the plane W spanned by γ and δ . Note that $\omega(\alpha_n, \beta_n) \rightarrow \omega(\alpha, \beta)$. From $S_n \alpha_n \rightarrow S\alpha$, $S_n \beta_n \rightarrow S\beta$ we deduce that $S(V) \subset V$. Since S is an isomorphism, $S(V) = V$. Similarly, $S(W) = W$. In this way we have found that S admits a splitting by invariant Lagrangian planes. \square

For matrices in $\text{Sp}(\mathbb{R}^2)$ the spectrum is contained in $\mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$, where $\mathbb{S}^1 = \{\mu \in \mathbb{C} : |\mu| = 1\}$. This is not always the case in $\text{Sp}(\mathbb{R}^4)$ but we will show that it still holds in the neighborhood of certain matrices.

Lemma 2.6. *Assume that P satisfies (4). Then there exists a neighborhood $\mathcal{U}_1 \subset \text{Sp}(\mathbb{R}^4)$ such that*

$$\sigma(S) \subset \mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$$

for each $S \in \mathcal{U}_1$.

Proof. By a contradiction argument assume the existence of a sequence of matrices S_n in $\text{Sp}(\mathbb{R}^4)$ converging to P and such that $\sigma(S_n)$ is not contained in $\mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$. Hence S_n has four different eigenvalues denoted by $\mu_n, \bar{\mu}_n, \mu_n^{-1}, \bar{\mu}_n^{-1}$ with $|\mu_n| > 1$ and $\Im \mu_n \neq 0$. In addition we know that $\mu_n \rightarrow 1$ as $n \rightarrow +\infty$.

Let v_n and w_n be vectors in $\mathbb{C}^4 \setminus \{0\}$ satisfying

$$S_n v_n = \mu_n v_n, \quad S_n w_n = \mu_n^{-1} w_n.$$

The plane $V_n \subset \mathbb{R}^4$ spanned by $\Re v_n$ and $\Im v_n$ is invariant under S_n . Similarly the plane $W_n \subset \mathbb{R}^4$ spanned by $\Re w_n$ and $\Im w_n$ satisfies $S_n(W_n) = W_n$. Since V_n and W_n correspond to different eigenvalues we deduce that they are a splitting of \mathbb{R}^4 . We claim that V_n and W_n are Lagrangian. To prove it we observe that the form ω is preserved by symplectic matrices and so

$$\omega(v_n, \bar{v}_n) = \omega(S_n v_n, S_n \bar{v}_n) = \mu_n \bar{\mu}_n \omega(v_n, \bar{v}_n).$$

From $|\mu_n| \neq 1$ we deduce that $\omega(v_n, \bar{v}_n) = 0$. This implies that ω vanishes on $V_n \times V_n$ and so V_n is Lagrangian. The same can be said about W_n .

Once we have a splitting by Lagrangian planes that are invariant under S_n we can apply Lemma 2.5 to find another Lagrangian splitting invariant under P . Then we can apply Lemma 2.4 to find a contraction with (4). \square

Proof of Proposition 2.1. Given $S \in \text{Sp}(\mathbb{R}^4)$ we label its eigenvalues by $\mu_1, \mu_2, \mu_3, \mu_4$ with $\mu_1 \mu_3 = 1$ and $\mu_2 \mu_4 = 1$. Note that eigenvalues are counted according to their multiplicity. We define the numbers

$$\Delta_1 = \mu_1 + \mu_3, \quad \Delta_2 = \mu_2 + \mu_4$$

and observe that $\Delta_1 = \Delta_2 = 2$ when $S = P$. Here we are using the assumption $\sigma(P) = \{1\}$. In consequence we can find a neighborhood $\mathcal{U}_2 \subset \text{Sp}(\mathbb{R}^4)$ of the matrix P such that $\Delta_1 > 0$ and $\Delta_2 > 0$ if $S \in \mathcal{U}_2$. Define $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$, where \mathcal{U}_1 is given by Lemma 2.6. From the definition of Δ_1 and Δ_2 ,

$$\text{tr } S = \Delta_1 + \Delta_2.$$

Also

$$\det(S - I) = (2 - \Delta_1)(2 - \Delta_2).$$

To prove this identity we observe that the characteristic polynomial of S can be factorized as

$$\det(S - \lambda I) = (\lambda^2 - \Delta_1\lambda + 1)(\lambda^2 - \Delta_2\lambda + 1).$$

Then it is sufficient to let $\lambda = 1$.

We are ready to prove that if S is elliptic then (5) holds. The ellipticity of S implies that $\mu_3 = \bar{\mu}_1$, $\mu_4 = \bar{\mu}_2$ with $\mu_i \neq 1$ and $|\mu_i| = 1$. Then $\Delta_i = 2\Re\mu_i < 2$ and the condition (5) becomes a consequence of the previous formulas for $\text{tr } S$ and $\det(S - I)$.

Assume now that $S \in \mathcal{U}$ satisfies (5). It follows from Lemma 2.6 that either $\mu_i \in \mathbb{R}$ or $|\mu_i| = 1$ with $\bar{\mu}_{i+2} = \mu_i$. In any of the two cases the numbers Δ_i are real and the conditions in (5) are equivalent to

$$(2 - \Delta_1)(2 - \Delta_2) > 0 \quad \text{and} \quad \Delta_1 + \Delta_2 < 4.$$

Once we know that $\Delta_i > 0$, $i = 1, 2$, they become equivalent to $|\Delta_i| < 2$ and so $\mu_i \in \mathbb{S}^1$, $\mu_i \neq \pm 1$. □

Remark 2.7. It is not hard to check that in \mathcal{U} the conditions

$$\det(S - I) > 0, \quad \text{tr } S > 4$$

correspond to the hyperbolic case $\mu_i \in \mathbb{R}$, $\mu_i \neq \pm 1$, $i = 1, 2$. Finally,

$$\det(S - I) < 0$$

appears in the mixed elliptic-hyperbolic case, say $\mu_1 \in \mathbb{R}$, $|\mu_2| = 1$, $\mu_i \neq \pm 1$, $i = 1, 2$.

3 Poincaré coordinates

In connection with Kepler problem

$$\dot{x} = y, \quad \dot{y} = -\frac{x}{|x|^3}, \quad (x, y) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2, \quad (6)$$

we consider the functions

$$E(x, y) = \frac{1}{2}|y|^2 - \frac{1}{|x|} \quad \text{and} \quad M(x, y) = x_1y_2 - x_2y_1$$

and the set

$$\mathcal{E}_+ = \{(x, y) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 : E(x, y) < 0, M(x, y) > 0\}.$$

Elliptic and circular motions with positive orientation have states lying in \mathcal{E}_+ . In particular the circular motions have states lying on the surface

$$\mathcal{C} = \{(x, y) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 : 2E(x, y)M(x, y)^2 = -1\}.$$

Incidentally we observe that the inequality $2EM^2 \geq -1$ holds everywhere.

The astronomical coordinates (a, e, l, g) are related to the Cartesian coordinates by the formulas

$$x = R[g] \begin{pmatrix} a(\cos u - e) \\ a\sqrt{1-e^2} \sin u \end{pmatrix}, \quad y = R[g] \begin{pmatrix} -\frac{\sin u}{a^{1/2}(1-e \cos u)} \\ \frac{\sqrt{1-e^2} \cos u}{a^{1/2}(1-e \cos u)} \end{pmatrix},$$

where

$$R[g] = \begin{pmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{pmatrix} \quad \text{and} \quad u - e \sin u = l.$$

The elliptic and circular orbits of (6), $x = x(t)$, $y = y(t)$, are obtained by letting $l = \frac{t}{a^{3/2}}$.

The change of coordinates

$$\mathcal{A} : (a, e, l, g) \in]0, \infty[\times]0, 1[\times \mathbb{T}^2 \mapsto (x, y) \in \mathcal{E}_+ \setminus \mathcal{C}$$

is a real analytic diffeomorphism. From now on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Indeed the map \mathcal{A} is also analytic on $\{e = 0\}$ but the extension of \mathcal{A} to $\{e \geq 0\}$ is no longer a diffeomorphism. The map \mathcal{A} collapses the three-dimensional manifold $]0, \infty[\times \{0\} \times \mathbb{T}^2$ into the surface \mathcal{C} . For further reference we write explicitly the formula of \mathcal{A} on $\{e = 0\}$,

$$x = a \begin{pmatrix} \cos \lambda \\ \sin \lambda \end{pmatrix}, \quad y = a^{-1/2} \begin{pmatrix} -\sin \lambda \\ \cos \lambda \end{pmatrix}, \quad \lambda = l + g. \quad (7)$$

Next we introduce the Delaunay variables $(l, L; g, G)$ by the formulas

$$L = \sqrt{a}, \quad G = \sqrt{a(1-e^2)}.$$

If we define

$$\Sigma = \{(L, G) \in \mathbb{R}^2 : 0 < G < L\}$$

then the map

$$\mathcal{D} : (L, G, l, g) \in \Sigma \times \mathbb{T}^2 \mapsto (x, y) \in \mathcal{E}_+ \setminus \mathcal{C}$$

is also an analytic diffeomorphism. The advantage with respect to the previous coordinates is that \mathcal{D} defines a symplectic transformation; that is,

$$dl \wedge dL + dg \wedge dG = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

The problem with the circular motions still remains in these new coordinates, now $\{e = 0\}$ is replaced by $\{L = G\}$. To overcome this difficulty, Poincaré coordinates $(\lambda, \Lambda; \eta, \xi)$ are introduced by the formulas

$$\lambda = l + g, \quad \Lambda = L, \quad \eta = \sqrt{2H} \sin h, \quad \xi = \sqrt{2H} \cos h,$$

where $h = -g$ and $H = L - G$.

A straightforward computation shows that

$$d\lambda \wedge d\Lambda + d\eta \wedge d\xi = dl \wedge dL + dg \wedge dG. \quad (8)$$

The above discussion follows along the lines of the original exposition by Poincaré in Sections 56-57 of [9]. The notation is taken from an unpublished report by A. Chenciner. In the book by Poincaré the coordinates H and h are called ρ_1 and ω_1 . In principle the formula (8) is valid on the domain

$$\Omega^* = \{(\lambda, \Lambda; \eta, \xi) \in \mathbb{T} \times]0, \infty[\times \mathbb{R}^2 : 0 < \eta^2 + \xi^2 < 2\Lambda\}$$

and so the map

$$\mathcal{P} : (\lambda, \Lambda, \eta, \xi) \in \Omega^* \mapsto (x, y) \in \mathcal{E}_+ \setminus \mathcal{C}$$

is a symplectic diffeomorphism.

This map has a continuous extension defined on

$$\Omega = \{(\lambda, \Lambda; \eta, \xi) \in \mathbb{T} \times]0, \infty[\times \mathbb{R}^2 : \eta^2 + \xi^2 < 2\Lambda\}$$

and, in contrast to the previous cases, the manifold $\Omega \setminus \Omega^*$ has the same dimension of \mathcal{C} . The map \mathcal{P} on $\Omega \setminus \Omega^*$ is given by the formulas in (7) with $a = \Lambda^2$. It defines a diffeomorphism between $\Omega \setminus \Omega^*$ and \mathcal{C} . From the formulas it is not obvious at all that the extended map is analytic in the variables $(\lambda, \Lambda, \eta, \xi)$. Poincaré proved this in Sections 65-69 of his book [9]. For further reference we state the main conclusion of this Section as a proposition.

Proposition 3.1. *The map $\mathcal{P} : \Omega \rightarrow \mathcal{E}_+$ is a real analytic symplectic diffeomorphism.*

For completeness a proof of this result will be included in the Appendix. Also in the Appendix we will explain how to compute the successive derivatives of $\mathcal{P} = (x, y)$ at $\eta = \xi = 0$. To obtain explicit formulas in the circular case we need to know the derivatives of x of first and second order with respect to λ, η and ξ . These derivatives are

$$\partial_{(\lambda, \eta, \xi)} x(\lambda, \Lambda, 0, 0) = \left(\Lambda^2 i e^{i\lambda}, \frac{\Lambda^{3/2}}{2} (3i + i e^{2i\lambda}), \frac{\Lambda^{3/2}}{2} (-3 + e^{2i\lambda}) \right)$$

and

$$\partial_{(\lambda, \eta, \xi)}^2 x(\lambda, \Lambda, 0, 0) = \begin{pmatrix} -\Lambda^2 e^{i\lambda} & -\Lambda^{3/2} e^{2i\lambda} & \Lambda^{3/2} i e^{2i\lambda} \\ * & -\Lambda e^{i\lambda} - \frac{\Lambda}{4} (e^{-i\lambda} + 3e^{3i\lambda}) & \frac{\Lambda}{4} (-i e^{-i\lambda} + 3i e^{3i\lambda}) \\ * & * & -\Lambda e^{i\lambda} + \frac{\Lambda}{4} (e^{-i\lambda} + 3e^{3i\lambda}) \end{pmatrix}.$$

To simplify the formulas we use complex notation and identify $x = (x_1, x_2)$ to $x = x_1 + i x_2$.

4 Local continuation of periodic solutions

The system (1) has a Hamiltonian structure with

$$\dot{x} = K_y, \quad \dot{y} = -K_x,$$

and

$$K(t, x, y; \varepsilon) = \frac{1}{2}|y|^2 - \frac{1}{|x|} - \varepsilon U(t, x).$$

From now on we assume that $U : \mathbb{T} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function having partial derivatives with respect to x up to the third order. Moreover the functions $(t, x) \mapsto \partial_x^\alpha U(t, x)$ with $|\alpha| \leq 3$ are continuous. These assumptions will be summarized by $U \in C^{0,3}(\mathbb{T} \times \mathbb{R}^2)$.

In the elliptic region \mathcal{E}_+ this system can be transformed via the Poincaré coordinates into a new system with Hamiltonian function

$$\mathcal{K}_\varepsilon(t, \lambda, \Lambda, \eta, \xi) = -\frac{1}{2\Lambda^2} - \varepsilon \mathcal{U}(t, \lambda, \Lambda, \eta, \xi)$$

where $(\lambda, \Lambda, \eta, \xi)$ lies in Ω and $\mathcal{U}(t, \lambda, \Lambda, \eta, \xi) = U(t, x(\lambda, \Lambda, \eta, \xi))$. Note that $x = x(\lambda, \Lambda, \eta, \xi)$ is defined by \mathcal{P} . In agreement with (8) the system becomes

$$\dot{\lambda} = \frac{1}{\Lambda^3} - \varepsilon \partial_\Lambda \mathcal{U}, \quad \dot{\Lambda} = \varepsilon \partial_\lambda \mathcal{U}, \quad \dot{\eta} = -\varepsilon \partial_\xi \mathcal{U}, \quad \dot{\xi} = \varepsilon \partial_\eta \mathcal{U}, \quad (9)$$

and we observe that for $\varepsilon = 0$ and $N \geq 1$ this system has a continuum of 2π -periodic solutions defined by

$$\lambda(t) = Nt + \lambda_0, \quad \Lambda(t) = \frac{1}{N^{1/3}}, \quad \eta(t) = \eta_0, \quad \xi(t) = \xi_0.$$

Note that λ is an angular variable and the minimal period of these solutions is $\frac{2\pi}{N}$.

The perturbed system (9) is in the framework of averaging theory for periodic systems. We consider the function

$$\gamma_N(\lambda_0, \eta_0, \xi_0) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{U} \left(t, Nt + \lambda_0, \frac{1}{N^{1/3}}, \eta_0, \xi_0 \right) dt$$

defined on the solid torus $\lambda_0 \in \mathbb{T}$, $\eta_0^2 + \xi_0^2 < \frac{2}{N^{1/3}}$. The next result establishes a link between this function and the continuation of 2π -periodic solutions.

Proposition 4.1. *Assume that $(\lambda_0, \eta_0, \xi_0)$ is a non-degenerate critical point of γ_N . Then there exists a C^1 family of 2π -periodic solutions of (9), $\lambda = \lambda(t, \varepsilon)$, $\Lambda = \Lambda(t, \varepsilon)$, $\eta = \eta(t, \varepsilon)$, $\xi = \xi(t, \varepsilon)$, defined for ε small and satisfying*

$$\lambda(t, 0) = Nt + \lambda_0, \quad \Lambda(t, 0) = \frac{1}{N^{1/3}}, \quad \eta(t, 0) = \eta_0, \quad \xi(t, 0) = \xi_0.$$

Remark 4.2. 1. The point $(\lambda_0, \eta_0, \xi_0)$ satisfies $\nabla \gamma_N(\lambda_0, \eta_0, \xi_0) = 0$ and also $\det(D^2 \gamma_N(\lambda_0, \eta_0, \xi_0)) \neq 0$. If we pass to Cartesian coordinates the corresponding periodic solution $x(t, \varepsilon)$ has winding number N . This is clear for $\varepsilon = 0$ and then we observe that the winding number is independent of ε .

2. To connect γ_N with the function Γ_N defined in the introduction we observe that the map \mathcal{P} defines a diffeomorphism between $\Omega \cap \left\{ \Lambda = \frac{1}{N^{1/3}} \right\}$ (the domain of γ_N) and Σ_N as defined in the introduction. Then $\Gamma_N \circ \mathcal{P} = \gamma_N$ and so the critical points of both functions are in a one-to-one correspondence. In addition the Morse index coincides.

3. The result is also valid for negative N . Indeed it is sufficient to change the direction of time in (9).

The above result is a consequence of the general theory of averaging but we will present a direct proof. This will allow us to perform some computations that will be useful later.

The Kepler problem can be seen as a model for a large class of degenerate Hamiltonian systems and we find convenient to work in a general setting. From now on we consider a system of symplectic coordinates $(\theta, q; r, p)$ associated to the 2-form

$$d\theta \wedge dr + \sum_{i=1}^d dq_i \wedge dp_i.$$

Here θ is an angular variable, the points $q = (q_1, \dots, q_d)$, $p = (p_1, \dots, p_d)$ lie in \mathbb{R}^d and the phase space is of the type

$$\Omega = \left\{ z = (\bar{\theta}, q, r, p) : \bar{\theta} \in \mathbb{T}, r \in I, (q, p) \in D_r \right\}$$

where I is an open interval in \mathbb{R} and D_r is an open subset of $\mathbb{R}^d \times \mathbb{R}^d$ which may change continuously with r .

Consider also the Hamiltonian function

$$\mathcal{K}_\varepsilon(t, \theta, q, r, p) = h(r) - \varepsilon \mathcal{U}(t, \theta, q, r, p)$$

with $h \in C^3(I)$ and $\mathcal{U} \in C^{0,3}(\mathbb{T} \times \Omega)$.

The associated system is

$$\dot{z} = J \nabla \mathcal{K}_\varepsilon(t, z)$$

or, in coordinates,

$$\dot{\theta} = h'(r) - \varepsilon \partial_r \mathcal{U}, \quad \dot{q} = -\varepsilon \partial_p \mathcal{U}, \quad \dot{r} = \varepsilon \partial_\theta \mathcal{U}, \quad \dot{p} = \varepsilon \partial_q \mathcal{U}. \quad (10)$$

The perturbed Kepler problem (9) is recovered with $\theta = \lambda$, $r = \Lambda$, $q = \eta$, $p = \xi$ and $h(r) = -\frac{1}{2}r^{-2}$.

Given an integer N we select $r_N \in I$ such that

$$h'(r_N) = N \quad (11)$$

The number r_N (when it exists) produces a family of 2π -periodic solutions of the system (10) for $\varepsilon = 0$, namely

$$\theta(t) = \theta_0 + Nt, \quad q(t) = q_0, \quad r(t) = r_N, \quad p(t) = p_0 \quad (12)$$

with parameters $\bar{\theta}_0 \in \mathbb{T}$ and $(q_0, p_0) \in D_{r_N}$.

From now on the initial conditions will be simply denoted by (θ, q, r, p) . The Poincaré map of (10) is denoted by

$$\Pi_\varepsilon : (\theta, q, r, p) \mapsto (\theta', q', r', p')$$

and sends an initial condition at time $t = 0$ to the value of the corresponding solution at time $t = 2\pi$. For $\varepsilon = 0$ this map is well defined on the whole domain Ω and can be computed explicitly,

$$\Pi_0(\theta, q, r, p) = (\theta + 2\pi h'(r), q, r, p). \quad (13)$$

Once a point $(\theta_*, q_*, r_*, p_*)$ lying in Ω has been selected, the map Π is well defined and of class C^2 in a neighborhood of $(\theta_*, q_*, r_*, p_*, \varepsilon = 0)$.

We want to find periodic solutions emanating from the family (12) at a point satisfying $r_* = r_N$. From now on we work on a small neighborhood of this point and try to solve the system

$$\theta' = \theta + 2\pi N, \quad q' = q, \quad r' = r, \quad p' = p. \quad (14)$$

Let us assume that

$$h''(r_N) \neq 0. \quad (15)$$

Then $\frac{\partial \theta'}{\partial r}$ does not vanish at $\varepsilon = 0$. The implicit function theorem can be applied to the equation $\theta' = \theta + 2\pi N$ in order to express the unknown r as a C^2 function of the remaining variables,

$$r = \varphi(\theta, q, p; \varepsilon) \quad \text{with} \quad \varphi(\theta, q, p; 0) = r_N.$$

Then the system (14) is reduced to

$$q' = q, \quad r' = r, \quad p' = p$$

where r is now a dependent variable. In other words, we are looking for zeros of the map $\sigma = (\sigma_1, \sigma_2, \sigma_3)$,

$$\sigma(\theta, q, p; \varepsilon) = (q' - q, r' - r, p' - p).$$

From the explicit formula for Π_0 given by (13) and the identity $r = r_N$ for $\varepsilon = 0$ we deduce that the restriction of σ to $\varepsilon = 0$ vanishes everywhere. To obtain a bifurcation for $\varepsilon \neq 0$ we must look for zeros of the map

$$\Phi(\theta, q, p) = \partial_\varepsilon \sigma(\theta, q, p; 0).$$

By chain rule,

$$\begin{aligned}\partial_\varepsilon \sigma_1 &= \frac{\partial q'}{\partial \varepsilon} + \frac{\partial q'}{\partial r} \frac{\partial r}{\partial \varepsilon} \\ \partial_\varepsilon \sigma_2 &= \frac{\partial r'}{\partial \varepsilon} + \frac{\partial r'}{\partial r} \frac{\partial r}{\partial \varepsilon} - \frac{\partial r}{\partial \varepsilon} \\ \partial_\varepsilon \sigma_3 &= \frac{\partial p'}{\partial \varepsilon} + \frac{\partial p'}{\partial r} \frac{\partial r}{\partial \varepsilon}.\end{aligned}$$

Again from (13) it follows that $\frac{\partial r'}{\partial r} = 1$ and $\frac{\partial q'}{\partial r} = \frac{\partial p'}{\partial r} = 0$ at $\varepsilon = 0$. Hence,

$$\Phi = \left(\frac{\partial q'}{\partial \varepsilon}, \frac{\partial r'}{\partial \varepsilon}, \frac{\partial p'}{\partial \varepsilon} \right) \Big|_{\varepsilon=0} = \pi \circ \partial_\varepsilon \Pi_\varepsilon \Big|_{\varepsilon=0},$$

where π is the projection defined by

$$\pi(\theta, q, r, p) = (q, r, p).$$

To compute $\partial_\varepsilon \Pi_\varepsilon \Big|_{\varepsilon=0}$ we first consider the variational equation (with respect to ε) for the system (10) around a solution of the family (12) with $\varepsilon = 0$,

$$\dot{\delta}_1 = h''(r_N) \delta_3 - \partial_r \mathcal{U}, \quad \dot{\delta}_2 = -\partial_p \mathcal{U}, \quad \dot{\delta}_3 = \partial_\theta \mathcal{U}, \quad \dot{\delta}_4 = \partial_q \mathcal{U}.$$

The last three coordinates are easily computed by direct integration and, after imposing the initial condition $\delta_i(0) = 0$, $1 \leq i \leq 4$, we are led to the identity

$$\Phi = 2\pi \mathcal{R} \nabla \mathcal{U}_\# \tag{16}$$

where

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & -I_d \\ 1 & 0 & 0 \\ 0 & I_d & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{U}_\#(\theta, q, p) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{U}(t, \theta + Nt, q, r_N, p) dt.$$

A sufficient condition for the bifurcation of $\sigma = 0$ is $\Phi = 0$ and $\det \Phi' \neq 0$. This is equivalent to finding non-degenerate critical points of $\mathcal{U}_\#$. In many aspects the previous analysis is reminiscent of Section 46 in [10].

The conditions (11) and (15) are satisfied by the Kepler problem with $r_N = \frac{1}{N^{1/3}}$. Since $\gamma_N = \mathcal{U}_\#$ in this case, the proof of Proposition 4.1 is complete.

It is possible to obtain other bifurcation results with more relaxed conditions on the critical point. For instance, a bifurcation exists when the function γ_N has an isolated critical point and $\nabla \gamma_N$ has non-zero Brouwer index at this point. This bifurcation does not necessarily leads to a smooth family of periodic solutions.

5 Periodic solutions of elliptic type

In this Section we prove the main result of the paper, Theorem 1.1. First we work with the general class of systems introduced in Section 4. This will allow us to obtain an instability result valid in arbitrary dimension.

Let $z(t, \varepsilon) = (\theta(t, \varepsilon), q(t, \varepsilon), r(t, \varepsilon), p(t, \varepsilon))$ be a C^1 family of 2π -periodic solutions of the system (10). We assume that the conditions (11) and (15) hold and

$$z(t, 0) = (\theta_* + Nt, q_*, r_N, p_*).$$

Proposition 5.1. *In the previous setting assume that one of the conditions below hold,*

$$h''(r_N) \partial_{\theta\theta}^2 \mathcal{U}_{\#}(\theta_*, q_*, p_*) > 0, \quad (17)$$

$$h''(r_N) \det [D^2 \mathcal{U}_{\#}(\theta_*, q_*, p_*)] > 0, \quad (18)$$

then, for small $\varepsilon > 0$, the solution $z(t, \varepsilon)$ is linearly unstable.

We will prove that some Floquet multiplier of the variational equation

$$\dot{\delta} = JD_z^2 \mathcal{K}_{\varepsilon}(t, z(t, \varepsilon)) \delta. \quad (19)$$

lies outside the unit circle. Incidentally, this in particular implies that $z(t, \varepsilon)$ is Lyapunov unstable.

Proof. The Poincaré map associated to (10) is denoted by $\Pi_{\varepsilon}(z) = \Pi(z, \varepsilon)$. The monodromy matrix associated to (19) is given by

$$S(\varepsilon) = \partial_z \Pi(z(0, \varepsilon), \varepsilon).$$

This matrix defines a path in the symplectic group $\text{Sp}(\mathbb{R}^{2(d+1)})$. We will compute the Taylor expansion of $S(\varepsilon)$ at $\varepsilon = 0$ but first we go back to the formula (13) in order to compute the successive derivatives of $\Pi(z, 0)$. The Jacobian matrix is

$$\partial_z \Pi(z, 0) = \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_d \end{pmatrix} \quad \text{with } \tau = 2\pi h''(r), \quad (20)$$

and all second derivatives vanish excepting $\frac{\partial^2 \theta'}{\partial r^2} = 2\pi h'''(r)$.

Differentiating $S(\varepsilon)$ with respect to ε and evaluating at $\varepsilon = 0$,

$$S'(0) = H = \partial_{\varepsilon z}^2 \Pi(z_*, 0) + \mathcal{N}$$

where $z_* = (\theta_*, q_*, r_N, p_*)$ and $\mathcal{N} = (n_{ij})$ satisfies $n_{ij} = 0$ if $(i, j) \neq (1, d+2)$ and

$$n_{1,d+2} = \frac{\partial^2 \theta'}{\partial r^2}(z_*, 0) \frac{\partial r}{\partial \varepsilon}(0, 0).$$

Note that this coefficient occupies the same position at \mathcal{N} as the coefficient τ in the above matrix.

Thus

$$S(\varepsilon) = P_* + \varepsilon H + o(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0, \quad (21)$$

with $P_* = \partial_z \Pi(z_*, 0)$. From the previous expansion it is not difficult to see that

$$\det(S(\varepsilon) - I) = (-1)^{d+1} \tau_N \varepsilon^{2d+1} \det M + o(\varepsilon^{2d+1})$$

where $\tau_N = 2\pi h''(r_N)$ and M is the sub-matrix of $\partial_{\varepsilon z}^2 \Pi(z_*, 0)$ obtained after eliminating the first row and the column $d+2$.

To determine M we first define the map

$$\tilde{\Pi}(\theta, q, p, \varepsilon) = (q', r', p')$$

where $(\theta', q', r', p') = \Pi_\varepsilon(\theta, q, r_N, p)$. Some computations of the previous Section leading to (16) show that

$$\partial_\varepsilon \tilde{\Pi}(\theta, q, p, 0) = 2\pi \mathcal{R} \nabla \mathcal{U}_\#(\theta, q, p).$$

Differentiating this identity with respect to θ, q and p and evaluating at (θ_*, q_*, p_*) we obtain

$$M = 2\pi \mathcal{R} D^2 \mathcal{U}_\#(\theta_*, q_*, p_*).$$

Since $\det \mathcal{R} = (-1)^d$ we deduce that

$$\det(S(\varepsilon) - I) = -\tau_N (2\pi)^{2d+1} \varepsilon^{2d+1} \det [D^2 \mathcal{U}_\#(\theta_*, q_*, p_*)] + o(\varepsilon^{2d+1}). \quad (22)$$

If we assume that (18) holds then the determinant of $S(\varepsilon) - I$ is negative if $\varepsilon > 0$ is small enough. Let $\mu_1(\varepsilon), \dots, \mu_{2d+2}(\varepsilon)$ be the eigenvalues of $S(\varepsilon)$. They satisfy $\mu_k(\varepsilon) \rightarrow 1$ and $\mu_k(\varepsilon) \neq 1$ if $\varepsilon \neq 0$ is small. We prove by a contradiction argument that some $\mu_k(\varepsilon)$ must lie outside \mathbb{S}^1 , for otherwise the eigenvalues could be labeled in such a way that $\overline{\mu_i(\varepsilon)} = \mu_{i+d+1}(\varepsilon)$ and

$$\det(S(\varepsilon) - I) = \prod_{i=1}^{d+1} |1 - \mu_i(\varepsilon)|^2 > 0.$$

This proves the Proposition when (18) holds. To deal with the other assumption (17) we go back to the expansion (21) and compute the trace

$$\text{tr } S(\varepsilon) = 2d + 2 + \varepsilon \text{tr } H + o(\varepsilon) = 2d + 2 + \varepsilon \text{tr } \partial_{\varepsilon z}^2 \Pi(z_*, 0) + o(\varepsilon).$$

The Poincaré map is canonical and so

$$d\theta' \wedge dr' + \sum_{i=1}^d dq'_i \wedge dp'_i = d\theta \wedge dr + \sum_{i=1}^d dq_i \wedge dp_i,$$

implying

$$\frac{\partial \theta'}{\partial \theta} \frac{\partial r'}{\partial r} - \frac{\partial \theta'}{\partial r} \frac{\partial r}{\partial \theta} + \sum_{j=1}^d \left(\frac{\partial q'_j}{\partial \theta} \frac{\partial p'_j}{\partial r} - \frac{\partial q'_j}{\partial r} \frac{\partial p'_j}{\partial \theta} \right) = 1$$

and

$$\frac{\partial \theta'}{\partial q_i} \frac{\partial r'}{\partial p_i} - \frac{\partial \theta'}{\partial p_i} \frac{\partial r}{\partial q_i} + \sum_{j=1}^d \left(\frac{\partial q'_j}{\partial q_i} \frac{\partial p'_j}{\partial p_i} - \frac{\partial q'_j}{\partial p_i} \frac{\partial p'_j}{\partial q_i} \right) = 1.$$

Differentiating these identities with respect to ε and evaluating at $\varepsilon = 0$,

$$\partial_{\varepsilon \theta}^2 \theta' + \partial_{\varepsilon r}^2 r' - \frac{\partial \theta'}{\partial r} \partial_{\varepsilon \theta}^2 r' = 0$$

and

$$\partial_{\varepsilon q_i}^2 q'_i + \partial_{\varepsilon p_i}^2 p'_i = 0.$$

Here we are using the formula (20). In particular,

$$\frac{\partial \theta'}{\partial r} \Big|_{\varepsilon=0} = 2\pi h''(r).$$

In consequence, if $\varepsilon = 0$

$$\partial_{\varepsilon \theta}^2 \theta' + \partial_{\varepsilon r}^2 r' + \sum_{i=1}^d (\partial_{\varepsilon q_i}^2 q'_i + \partial_{\varepsilon p_i}^2 p'_i) = 2\pi h''(r) \partial_{\varepsilon \theta}^2 r'.$$

We know from the previous computations on the matrix M that

$$\partial_{\varepsilon \theta}^2 r' = 2\pi \partial_{\theta \theta}^2 \mathcal{U}_{\#} \quad \text{if } \varepsilon = 0, r = r_N.$$

From the previous expansion for the trace we deduce now that

$$\text{tr } S(\varepsilon) = 2d + 2 + \varepsilon 4\pi^2 h''(r_N) \partial_{\theta \theta}^2 \mathcal{U}_{\#}(\theta_*, q_*, p_*) + o(\varepsilon). \quad (23)$$

Therefore $\text{tr } S(\varepsilon) > 2d + 2$ if (15) holds and $\varepsilon > 0$ is small enough. We claim that some eigenvalues must lie outside \mathbb{S}^1 . Otherwise we could label the eigenvalues as before and deduce that

$$\text{tr } S(\varepsilon) = 2 \sum_{i=1}^d \Re(\mu_i(\varepsilon)) \leq 2(d+1).$$

Note that now $\mu_i(\varepsilon) = 1$ is admissible. □

The previous Proposition can be applied to the Kepler problem. It leads to the conclusions of the main Theorem 1.1 on instability. To obtain the conclusion on ellipticity the dimension will play a role. Since the monodromy matrix $S(\varepsilon)$ now belongs to $\text{Sp}(\mathbb{R}^4)$, the results of Section 2 can be employed. From the formula (20) we deduce that $S(0) = P_*$ where P_* is given by (3) with $\tau = \tau_N = -6\pi N^{4/3}$. Then Proposition 2.1 implies that, for $\varepsilon > 0$ small, the matrix $S(\varepsilon)$ will be elliptic if

$$\det(S(\varepsilon) - I) > 0 \quad \text{and} \quad \text{tr } S(\varepsilon) < 4.$$

The expansions (22) and (23) imply that these inequalities are equivalent to

$$\det[D^2 \mathcal{U}_{\#}] > 0 \quad \text{and} \quad \partial_{\theta \theta}^2 \mathcal{U}_{\#} > 0$$

with $\mathcal{U}_{\#} = \gamma_N$.

6 Explicit computations for the circular case

In this section we will discuss the continuation from circular orbits for the forced Kepler problem

$$\ddot{x} = -\frac{x}{|x|^3} - \varepsilon p(t) \quad (24)$$

where $p : \mathbb{R} \rightarrow \mathbb{R}^2$, $p = (p_1, p_2)^t$, is a continuous and 2π -periodic function.

The associated potential is

$$U(t, x) = -\langle p(t), x \rangle$$

and, after fixing an integer $N > 0$, the function γ_N introduced in Section 4 can be expressed as

$$\gamma_N(\lambda, \eta, \xi) = -\frac{1}{2\pi} \int_0^{2\pi} \langle p(t), x(\lambda + Nt, \Lambda_N, \eta, \xi) \rangle dt$$

where $\Lambda_N = \frac{1}{N^{1/3}}$ and $x = x(\lambda, \Lambda, \eta, \xi)$ is defined by the Poincaré coordinates. The domain of γ_N is the solid torus $\mathbb{T} \times \mathbb{B}_N$ with

$$\mathbb{B}_N = \{(\eta, \xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 < 2\Lambda_N\}.$$

In order to apply the results of Sections 4 and 5 at the circular orbits we must look for critical points of γ_N lying on the equator $\mathbb{T} \times \{0\}$. It will be convenient to employ the complex notation that is typical in Fourier analysis. To this end we identify $p = p_1 + ip_2$, $x = x_1 + ix_2$ and rewrite γ_N by the expression

$$\gamma_N(\lambda, \eta, \xi) = -\frac{1}{2\pi} \int_0^{2\pi} \Re \left(p(t) \overline{x(\lambda + Nt, \Lambda_N, \eta, \xi)} \right) dt.$$

The Fourier coefficients of $p(t)$ are given by the formulas

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} p(t) e^{-int} dt, \quad n \in \mathbb{Z}.$$

Using the identities presented at the end of Section 3 we can express the gradient of γ_N along $\mathbb{T} \times \{0\}$ in terms of c_0, c_N and c_{2N} . In particular

$$\nabla \gamma_N(\lambda, 0, 0) = 0$$

is equivalent to

$$\Im(e^{-i\lambda} c_N) = 0, \quad e^{-2i\lambda} c_{2N} = 3\overline{c_0}. \quad (25)$$

This system in λ has a solution if and only if

$$3|c_0| = |c_{2N}| \quad \text{and} \quad c_0 c_{2N} \overline{c_N}^2 \in [0, \infty[. \quad (26)$$

When some of these Fourier coefficients do not vanish this condition implies that γ_N has exactly two critical points on $\mathbb{T} \times \{0\}$ and they are antipodal, that is, $(\lambda^*, 0, 0)$

and $(\lambda^* + \pi, 0, 0)$ for some λ^* . In the degenerate case $c_0 = c_N = c_{2N} = 0$ the function γ_N has a continuum of critical points along the equator.

In the real Banach space $C(\mathbb{T}, \mathbb{R}^2)$ the subset described by (26) contains a manifold of codimension two ($c_0 \neq 0, c_N \neq 0$) but it also contains a linear manifold of codimension 4 described by

$$c_0 = c_{2N} = 0. \quad (27)$$

For simplicity in the computations we will restrict to this linear manifold. From now on we assume that (27) holds. We also assume that

$$c_N \neq 0,$$

for otherwise critical points on $\mathbb{T} \times \{0\}$ would not be isolated.

Let us choose λ^* such that

$$e^{i\lambda^*} = \frac{c_N}{|c_N|}.$$

In agreement with (25) we deduce that $(\lambda^*, 0, 0)$ and $(\lambda^* + \pi, 0, 0)$ are critical points. From the identities for the second derivatives of x (Section 3), we deduce that

$$D^2\gamma_N(\lambda^*, 0, 0) = \left(\begin{array}{c|cc} \Lambda_N^2 |c_N| & 0 & 0 \\ \hline 0 & & \\ 0 & \Lambda_N \mathcal{M}(p) & \end{array} \right)$$

with

$$\mathcal{M}(p) = \begin{pmatrix} |c_N| + \frac{1}{4} \Re \left(\frac{c_N c_{-N}}{|c_N|} + 3 \frac{\overline{c_N}^3}{|c_N|^3} c_{3N} \right) & \frac{1}{4} \Im \left(\frac{c_N c_{-N}}{|c_N|} - 3 \frac{\overline{c_N}^3}{|c_N|^3} c_{3N} \right) \\ * & |c_N| - \frac{1}{4} \Re \left(\frac{c_N c_{-N}}{|c_N|} + 3 \frac{\overline{c_N}^3}{|c_N|^3} c_{3N} \right) \end{pmatrix}.$$

In addition

$$D^2\gamma_N(\lambda^* + \pi, 0, 0) = -D^2\gamma_N(\lambda^*, 0, 0).$$

If the determinant of $\mathcal{M}(p)$ does not vanish then Proposition 4.1 and Theorem 1.1 apply. In this way we are led to a very explicit result.

Proposition 6.1. *Assume that $p(t)$ is such that $c_0 = c_{2N} = 0$ and $c_N \neq 0$. In addition*

$$\det \mathcal{M}(p) \neq 0.$$

Then, for small $\varepsilon > 0$, the equation (24) has two smooth families of 2π -periodic solutions satisfying

$$x(t, \varepsilon) = e^{i(\lambda^* + Nt)} + O(\varepsilon), \quad \tilde{x}(t, \varepsilon) = -e^{i(\lambda^* + Nt)} + O(\varepsilon)$$

for λ^ satisfying $e^{i\lambda^*} = \frac{c_N}{|c_N|}$. If $\det \mathcal{M}(p) > 0$ the first family is elliptic and the second is unstable in the Lyapunov sense; if $\det \mathcal{M}(p) < 0$ both families are unstable.*

The example in the Introduction corresponds to $N = 1$, $\lambda^* = 0$ and

$$\mathcal{M}(p) = \begin{pmatrix} 1 + \frac{1}{4}\Re(a) & \frac{1}{4}\Im(a) \\ * & 1 - \frac{1}{4}\Re(a) \end{pmatrix}.$$

It is also possible to obtain some more or less explicit conditions in the non-circular case. The computations are more involved and we have been unable to find simple conditions like in Proposition 6.1. It seems that the classical Fourier expansion solving the Kepler equation $u - e \sin u = t$ plays a role when $p(t)$ is a trigonometric polynomial.

7 Appendix

Proof of Proposition 3.1. The main point in the proof is to show that the function $\mathcal{P} = \mathcal{P}(\lambda, \Lambda; \eta, \xi)$ is real analytic at each point $(\lambda, \Lambda; 0, 0) \in \Omega \setminus \Omega^*$. If this is given for granted the proof follows easily. We already know that \mathcal{P} is a symplectic diffeomorphism between Ω^* and $\mathcal{E}_+ \setminus \mathcal{C}$ and a homeomorphism from Ω onto \mathcal{E}_+ . Therefore it is enough to prove that \mathcal{P} is a local symplectic diffeomorphism around each point on $\Omega \setminus \Omega^*$. To this end we invoke the identity (8). In principle it is valid on Ω^* but the continuity of the derivatives of \mathcal{P} implies that it also holds on Ω . In consequence $\det \mathcal{P}' = 1$ on Ω and the inverse function theorem can be applied at each point $(\lambda, \Lambda; 0, 0)$.

Let us now concentrate on the analyticity of the coordinates of \mathcal{P} . We start with some auxiliary functions. By direct computations the eccentricity can be expressed in terms of Poincaré coordinates,

$$e = \sqrt{\eta^2 + \xi^2} \sqrt{\frac{1}{\Lambda} - \frac{\eta^2 + \xi^2}{4\Lambda^2}}.$$

This implies that e^2 is analytic in (Λ, η, ξ) . The same can be said for $\gamma := \frac{e}{\sqrt{\eta^2 + \xi^2}}$. Define $A := e \cos h = \gamma\xi$, $B := e \sin h = \gamma\eta$. They are also analytic and $A(\Lambda, 0, 0) = B(\Lambda, 0, 0) = 0$. Define now $\alpha := e \cos l$, $\beta := e \sin l$. Using $\lambda = l + g$ we can express them in terms of $A, B, \cos \lambda, \sin \lambda$. This implies that α and β are analytic functions in the variables $(\lambda, \Lambda; \eta, \xi)$ and $\alpha(\lambda, \Lambda; 0, 0) = \beta(\lambda, \Lambda; 0, 0) = 0$.

Kepler's equation $u - e \sin u = l$ can be rewritten as

$$u - l = \alpha \sin(u - l) + \beta \cos(u - l)$$

and the implicit function theorem can be applied to deduce that $u - l = e \sin u$ is analytic in $(\lambda, \Lambda; \eta, \xi)$. Also $e \cos u = \alpha \cos(u - l) - \beta \sin(u - l)$ is analytic. From $u + g = u - l + \lambda$ we deduce that $\cos(u + g)$ and $\sin(u + g)$ are analytic. Finally we observe that $e^2 \cos(u - g)$ and $e^2 \sin(u - g)$ are analytic. This follows from $u - g = u + l - \lambda$ and

$$e^2 \cos(u + l) = (\alpha^2 - \beta^2) \cos(u - l) - 2\alpha\beta \sin(u - l),$$

$$e^2 \sin(u + l) = (\alpha^2 - \beta^2) \sin(u - l) + 2\alpha\beta \cos(u - l).$$

To complete the proof we express the coordinates of \mathcal{P} in terms of the previous functions. In particular, for x_1 and y_1 ,

$$\frac{x_1}{\Lambda^2} = \frac{1 + \sqrt{1 - e^2}}{2} \cos(u + g) + \frac{1}{2(1 + \sqrt{1 - e^2})} e^2 \cos(u - g) - A$$

and

$$\Lambda y_1 = \frac{1}{1 - e \cos u} \left[-\frac{1}{2(1 + \sqrt{1 - e^2})} e^2 \sin(u - g) - \frac{1 + \sqrt{1 - e^2}}{2} \sin(u + g) \right].$$

□

We shall now explain how to obtain the formulas presented at the end of Section 3. Given $\mathcal{P} = (x, y)^t$ and $x = x(\lambda, \Lambda, \eta, \xi)$, $x = (x_1, x_2)^t$, we deduce from the definition of the Poincaré coordinates that

$$x(\lambda, \Lambda, 0, 0) = \Lambda^2 \begin{pmatrix} \cos \lambda \\ \sin \lambda \end{pmatrix}.$$

The derivatives $\frac{\partial x}{\partial \lambda}$ and $\frac{\partial^2 x}{\partial \lambda^2}$ can be computed from here. To compute the remaining derivatives we consider the change of variables $r^2 = 2H$, $\eta = r \sin h$, $\xi = r \cos h$ and observe that the functions $e = e(\Lambda, r)$ and $u = u(\lambda, \Lambda, r, h)$ are analytic at $r = 0$. This is a consequence of the formulas

$$e = r \sqrt{\frac{1}{\Lambda} - \frac{r^2}{4\Lambda^2}}, \quad u - e \sin u = \lambda + h.$$

In consequence the function

$$\tilde{x}(\lambda, \Lambda, r, h) = \Lambda^2 R[-h] \begin{pmatrix} \cos u - e \\ \sqrt{1 - e^2} \sin u \end{pmatrix} \quad (28)$$

is analytic at $r = 0$, $\Lambda > 0$, $\lambda, h \in \mathbb{T}$.

Since $x(\lambda, \Lambda, \eta, \xi) = \tilde{x}(\lambda, \Lambda, r, h)$ we can differentiate at each point with $r > 0$ to obtain

$$\frac{\partial \tilde{x}}{\partial r} = \frac{\partial x}{\partial \eta} \sin h + \frac{\partial x}{\partial \xi} \cos h$$

and

$$\frac{\partial^2 \tilde{x}}{\partial r^2} = \frac{\partial^2 x}{\partial \eta^2} \sin^2 h + 2 \frac{\partial^2 x}{\partial \eta \partial \xi} \sin h \cos h + \frac{\partial^2 x}{\partial \xi^2} \cos^2 h.$$

Letting $r \rightarrow 0^+$ and selecting appropriate values of h we express the derivatives of x with respect to η and ξ in terms of $\frac{\partial \tilde{x}}{\partial r}$ and $\frac{\partial^2 \tilde{x}}{\partial r^2}$. For instance, the choice $h = \frac{\pi}{2}$ leads to the identities

$$\frac{\partial x}{\partial \eta}(\lambda, \Lambda, 0, 0) = \frac{\partial \tilde{x}}{\partial r} \left(\lambda, \Lambda, 0, \frac{\pi}{2} \right) \quad (29)$$

and

$$\frac{\partial^2 x}{\partial \eta^2}(\lambda, \Lambda, 0, 0) = \frac{\partial^2 \tilde{x}}{\partial r^2} \left(\lambda, \Lambda, 0, \frac{\pi}{2} \right). \quad (30)$$

Similar identities for the derivatives with respect to ξ are obtained with the choice $h = 0$. Finally the identity

$$\frac{\partial^2 x}{\partial \eta \partial \xi}(\lambda, \Lambda, 0, 0) = \frac{\partial^2 \tilde{x}}{\partial r^2} \left(\lambda, \Lambda, 0, \frac{\pi}{4} \right) - \frac{1}{2} \left[\frac{\partial^2 x}{\partial \eta^2}(\lambda, \Lambda, 0, 0) + \frac{\partial^2 x}{\partial \xi^2}(\lambda, \Lambda, 0, 0) \right] \quad (31)$$

is obtained for $h = \frac{\pi}{4}$.

Our next task is to compute $\frac{\partial \tilde{x}}{\partial r}$ and $\frac{\partial^2 \tilde{x}}{\partial r^2}$ at $r = 0$. Differentiating (28) with respect to r and evaluating at $r = 0$ we obtain

$$\frac{\partial \tilde{x}}{\partial r}(\lambda, \Lambda, 0, h) = \frac{1}{2} \Lambda^{3/2} R[-h] \begin{pmatrix} -3 + \cos 2(\lambda + h) \\ \sin 2(\lambda + h) \end{pmatrix}, \quad (32)$$

where we have employed

$$e(\Lambda, 0) = 0, \quad \frac{\partial e}{\partial r}(\Lambda, 0) = \sqrt{\frac{1}{\Lambda}},$$

$$u(\lambda, \Lambda, 0, h) = \lambda + h, \quad \frac{\partial u}{\partial r}(\lambda, \Lambda, 0, h) = \sqrt{\frac{1}{\Lambda}} \sin(\lambda + h)$$

together with the trigonometric identity $1 + \sin^2 \theta = \frac{3 - \cos 2\theta}{2}$.

Combining (32) with (29) we compute $\frac{\partial x}{\partial \eta}$ and $\frac{\partial x}{\partial \xi}$ is computed in a similar way. Differentiating with respect to λ the identities

$$\frac{\partial x}{\partial \eta}(\lambda, \Lambda, 0, 0) = \frac{\Lambda^{3/2}}{2} (3i + ie^{2i\lambda}), \quad \frac{\partial x}{\partial \xi}(\lambda, \Lambda, 0, 0) = \frac{\Lambda^{3/2}}{2} (-3 + e^{2i\lambda})$$

we compute $\frac{\partial^2 x}{\partial \lambda \partial \eta}$ and $\frac{\partial^2 x}{\partial \lambda \partial \xi}$. By now the only derivatives that have not been computed are $\frac{\partial^2 x}{\partial \eta^2}$, $\frac{\partial^2 x}{\partial \xi^2}$ and $\frac{\partial^2 x}{\partial \eta \partial \xi}$. They will follow from (30), (31) and the computation of $\frac{\partial^2 \tilde{x}}{\partial r^2}$. To this end we differentiate twice the identity (28) with respect to r and evaluate at $r = 0$ to obtain

$$\frac{\partial^2 \tilde{x}}{\partial r^2}(\lambda, \Lambda, 0, h) = \Lambda R[-h] \begin{pmatrix} -3 \sin^2(\lambda + h) \cos(\lambda + h) \\ -\sin(\lambda + h) - \sin^3(\lambda + h) + 2 \cos^2(\lambda + h) \sin(\lambda + h) \end{pmatrix},$$

where we are using

$$\frac{\partial^2 e}{\partial r^2}(\Lambda, 0) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial r^2}(\lambda, \Lambda, 0, h) = \frac{2 \sin(\lambda + h) \cos(\lambda + h)}{\Lambda}.$$

From the trigonometric identities

$$\sin^2 \theta \cos \theta = \frac{1}{4}(\cos \theta - \cos 3\theta), \quad -\sin \theta - \sin^3 \theta + 2 \cos^2 \theta \sin \theta = \frac{-5 \sin \theta + 3 \sin 3\theta}{4}$$

we express $\frac{\partial^2 \tilde{x}}{\partial r^2}$ in terms of $e^{i(\lambda+h)}$ and $e^{3i(\lambda+h)}$, precisely

$$\frac{\partial^2 \tilde{x}}{\partial r^2}(\lambda, \Lambda, 0, h) = \Lambda e^{-ih} \left(-e^{i(\lambda+h)} + \frac{1}{4} \left(e^{-i(\lambda+h)} + 3e^{3i(\lambda+h)} \right) \right).$$

The derivatives $\frac{\partial^2 x}{\partial \eta^2}$ and $\frac{\partial^2 x}{\partial \xi^2}$ are computed evaluating $\frac{\partial^2 \tilde{x}}{\partial r^2}$ for $h = \frac{\pi}{2}$ and $h = 0$. Finally we compute $\frac{\partial^2 x}{\partial \eta \partial \xi}$ by an evaluation of $\frac{\partial^2 \tilde{x}}{\partial r^2}$ at $h = \frac{\pi}{4}$ combined with (31).

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