Uniform bifurcation of comet-type periodic orbits in the restricted $(n+1)$-body problem with non-Newtonian homogeneous potential

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Abstract

We treat the restricted $(n+1)$-body problem with a non-Newtonian homogeneous potential where the n primaries move on an arbitrary 2π-periodic orbit. We prove that the satellite equation has infinitely many periodic solutions that emerge from the infinity, asymptotically homothetic to the circular solutions of a central force problem. These solutions are obtained as critical solutions of a family of time-dependent perturbed Lagrangian systems, bifurcating uniformly from a compact set of periodic solutions of the unperturbed Lagrangian system.

1 Introduction

We study the planar and spatial non-Newtonian restricted $(n+1)$-body problem. The satellite moves under a non-Newtonian influence with the n primary bodies. The primaries is assumed to move on an arbitrary 2π-periodic orbit. The objective is to find periodic solutions to the equation

$$\ddot{q} = -\sum_{j=1}^{n} m_j \frac{q - q_j(t)}{\|q - q_j(t)\|^{\alpha+1}}. \tag{1}$$

Here, $q \in \mathbb{R}^d$ is the position of the satellite, $q_j(t) \in \mathbb{R}^d$ represents the position of the jth primary body with mass m_j, $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^d$, and $d = 2, 3$. We assume that $\alpha \geq 1$. Notice that $\alpha = 2$ corresponds to the gravitational case and we will omit it. See [2, 3, 7, 12] to find results related with the gravitational case.

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We look for solutions in which the amplitude is very large. These solutions are called *comet solutions*. After rescaling, the strategy will be to write Eq. (1) as a central force problem with a periodic and small perturbation. We want to show that this problem has an infinite number of periodic solutions emerging uniformly from the set of circular orbits of the central force problem for certain values of α.

This problem has been studied before in [2, 3, 7, 12, 15]. The difference between those works and ours is that they obtain a finite number of bifurcations, and therefore a finite (but arbitrarily large) number of periodic solutions. Thanks to the uniformity in the bifurcation, we obtain an infinite number of periodic solutions.

The idea of uniform bifurcation has been employed previously in [19]. In this work, the author uses a Hamiltonian approach and some sophisticated tools to obtain an infinite number of periodic solutions in the more difficult Newtonian situation. The solutions in [19] can have collisions. Our solutions are quasi-circular and thus avoid collisions. We use more elementary techniques, such as a quantitative version of the implicit function theorem, sufficient for our purposes.

In this work, we use a variational approach. We perform a sequence of time rescalings to transform Eq. (1) in a family of perturbed Lagrangian systems with the following structure:

$$L_\varepsilon(t,x,y) = \frac{1}{2}\langle A_\varepsilon(t,x)y, y\rangle + \langle B_\varepsilon(t,x), y\rangle + \mathcal{U}_\varepsilon(t,x). \tag{2}$$

For small ε, the Lagrangian function L_ε can be interpreted as an autonomous part $L_0 = L_\varepsilon|_{\varepsilon=0}$ plus a small and periodic perturbation. In this way we obtain infinite bifurcations from L_0. In principle, the size of each of the branches could depend on the chosen rescaling but we prove that this is not the case. In this sense our bifurcation is uniform, leading to the existence of infinitely many periodic solutions from (1) for small ε.

Then, we look for critical points of the periodic action functional $\mathcal{A}_\varepsilon$ associated with L_ε, given by

$$\mathcal{A}_\varepsilon(x) = \int_0^{2\pi} L_\varepsilon(t, x(t), x'(t))\ \mathrm{d}t. \tag{3}$$

The hypothesis is that the unperturbed action functional $\mathcal{A}_0$ has a compact manifold of critical points. This manifold satisfies a suitable non-degeneracy condition. The idea to obtain critical points of $\mathcal{A}_\varepsilon$ is to reduce the problem to find critical points of a function over a compact manifold. This idea is the theory of nondegenerate critical manifold developed in Section 2 from [5].

Notice that the Lagrangian function given in (2) is a polynomial of degree 2 in its variable y. This is necessary to guarantee that the action functional is sufficiently regular to apply a quantitative version of the implicit function theorem. More precisely, if the action functional $\mathcal{A}_\varepsilon$ given in (3) is twice differentiable and the Lagrangian function L_ε is smooth and satisfies certain quadratic growth conditions, then the associated Lagrangian function is a polynomial of degree at most two in its variable y. This result is proved in Proposition 3.2 from [1].

Going back to the original problem, the manifold of critical points of $\mathcal{A}_0$ is the set of circular orbits of the central force problem with a fixed minimal period. We prove that in the planar case, this manifold has two connected components. Each one is diffeomorphic to $SO(2)$. In the spatial case, there is only one connected component and it is diffeomorphic to $SO(3)$. The Lusternik-Schnirelman category of these manifolds guarantees the minimal number of bifurcations that we can obtain from each connected component. Notice that the Lusternik-Schnirelman category of $SO(2)$ and $SO(3)$ are 2 and 4, respectively.

The manifold of circular periodic solutions satisfies the nondegenerate conditions only for $\alpha \neq 2$. This is because in the gravitational case, the connected component of circular solutions with a fixed minimal period also contains the elliptic solutions. This implies that in this case, the topology of the manifold of periodic solutions is different from the other cases. In particular, it is not a compact manifold unless collisions are regularized. This regularization is treated in [18]. Another way to study the gravitational case is by imposing symmetry conditions in the primaries to exclude elliptic motions, as in [3, 7, 12].

Our results extend to three dimensions those obtained in [7] for the planar case. But even in the planar case, our result can be considered new, since it clarifies and improves the proofs and results in [7]. First of all, we obtain four branches of bifurcation instead of the two obtained in [7]. This is a consequence of the use of both connected components of the manifold of critical points. Secondly, we impose the non-resonance condition $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$ that, although necessary, was not explicitly stated in [7]. Finally, we include the analysis of the uniformity in the bifurcation branches. This was not included in [7] and so the conclusion there should be the existence of an arbitrarily large (but finite) number of periodic solutions.

In the central force problem, we have other manifolds of non-circular periodic solutions. For example, in [8] the authors use a Hamiltonian approach and the Poincaré-Birkhoff theorem to obtain periodic solutions bifurcating from a manifold of non-circular solutions diffeomorphic to a two-dimensional torus. Also, the reader can find more non-circular periodic solutions of the central force problem in Section 2.8 from [6].

The rest of the work proceeds as follows. In Sect. 2, we set Eq. (1) and we describe the set of circular solutions with fixed minimal period of the central force problem. In Theorem 1 we establish the existence of periodic solutions of (1). As in [7], we use a rescaling and we perform an infinite number of changes of variable to write Eq. (1) as a family of perturbed Lagrangian systems of the form (2). In Sect. 3, we discuss in Theorem 2 the existence of critical solutions of Lagrangian systems of the form (2). At the end of this section, we prove Theorem 1 using Theorem 2. Finally, in Sect. 4, we use a functional framework and a quantitative version of the implicit function theorem to prove Theorem 2, using a quantitative result of existence of critical points inspired from Theorem 2.1 from [5].

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2 Time-dependent Restricted $(n+1)$-body Problem

Let $q_j(t) \in \mathbb{R}^d$ $(d = 2, 3)$ be the position of n bodies with masses m_j, called primary bodies, $j = 1, \dots, n$. We assume that $q_j(t)$ is an arbitrary 2π-periodic function of class C^3 for each j satisfying $q_i(t) \neq q_j(t)$ for $t \in \mathbb{R}$ when $i \neq j$. We assume that the center of mass is at the origin,

$$\sum_{j=1}^{n} m_j q_j(t) = 0, \tag{4}$$

and

$$M = \sum_{j=1}^{n} m_j = 1.$$

The position for a satellite with infinitesimal mass $q(t) \in \mathbb{R}^d$ which is influenced by the motion of the primaries under a non-Newtonian homogeneous potential satisfies the equation

$$\ddot{q} = -\sum_{j=1}^{n} m_j \frac{q - q_j(t)}{\|q - q_j(t)\|^{\alpha+1}}, \tag{5}$$

where $\alpha \in [1, \infty[$.

The objective is to find an infinite number of sub-harmonic solutions of (5) in which the satellite is far away from the primaries. These solutions are emerging from the infinity, asymptotically homothetic to the circular solutions of the central force problem, namely

$$\ddot{q} = -\frac{q}{\|q\|^{\alpha+1}}, \qquad q \in \mathbb{R}^d \setminus \{0\}. \tag{6}$$

In fact, if $\alpha \in [1, \infty[$ Eq. (6) has a compact set of periodic solutions with fixed period formed by the circular solutions and they can be given explicitly. Given any $\mathsf{p} \in \mathbb{Z}^+$, we can consider a circular solutions with minimal period $2\pi/\mathsf{p}$ such as

$$\tilde{\gamma}(t; \mathsf{p}) = \mathsf{p}^{-2/(\alpha+1)} \begin{pmatrix} \cos(\mathsf{p}t) \\ \sin(\mathsf{p}t) \end{pmatrix}. \tag{7}$$

Actually, if $\alpha \geq 3$ these circular solutions are the unique periodic solutions with minimal period $2\pi/\mathsf{p}$ of Eq. (6) (see Section 2.b from [4]).

Using (7), we can write every circular solution with minimal period $2\pi/\mathsf{p}$ of (6) such as

$$\gamma(t; \mathsf{p}) = R\tilde{\gamma}(t; \mathsf{p}), \tag{8}$$

where $R \in SO(d)$. In the case $d = 3$, $\tilde{\gamma}$ is assumed to lie in the plane $x_3 = 0$ of $\mathbb{R}^3$. Our main theorem is

Theorem 1. *Let $\mathsf{p} \in \mathbb{Z}^+$, $\alpha \in [1, \infty[$, $d = 2, 3$ and assume that $\mathsf{p}\sqrt{3 - \alpha} \notin \mathbb{Z} \setminus \{0\}$. Then, there is an integer q_0 that only depends on p such that for each integer $\mathsf{q} > \mathsf{q}_0$ co-prime with*

p, *the restricted* $(n+1)$*-body problem* (5) *has at least four different* $2\pi\mathsf{q}$*-periodic solution of the form*

$$Q_l(t) = \mathsf{q}^{2/(\alpha+1)}\gamma_{l_\mathsf{q}}(t/\mathsf{q};\mathsf{p}) + \mathcal{R}_{\mathsf{p},\mathsf{q}}(t), \quad l = 1,2,3,4\ ,$$

where γ_{l_q} *has the form given in* (8), $\mathcal{R}_{\mathsf{p},\mathsf{q}}$ *are* $2\pi\mathsf{q}$*-periodic functions and there is a constant* c_p *that only depends on* p *such that*

$$\|\mathcal{R}_{\mathsf{p},\mathsf{q}}(t)\| \le c_\mathsf{p}\mathsf{q}^{-2/(\alpha+1)}, \quad t\in\mathbb{R}.$$

Since q_0 only depends on p, we can use Theorem 1 to obtain an infinite number of periodic orbits of Eq. (5). This is an improvement with respect of the classical results from [2, 3] where the authors obtain a finite (but arbitrary large) number of periodic orbits.

Notice that $\tilde{\gamma}_l(t/\mathsf{q},\mathsf{p})$ has a minimal period $2\pi\mathsf{q}/\mathsf{p}$. In particular, it is a sub-harmonic function of order q with respect to the period 2π. This means that it is of period $2\pi\mathsf{q}$ but it is not of period $2\pi\mathsf{r}$ for any integer r, $1\le\mathsf{r}<\mathsf{q}$. Given a function with this property, there is a neighborhood in the C^0 topology such that every $2\pi\mathsf{q}$-periodic function in this neighborhood is also a sub-harmonic function of order q with respect to the period 2π and this neighborhood does not depends on q. Applying R^{-1} to Q_l, we have

$$R^{-1}Q_l(t) = \mathsf{q}^{2/(\alpha+1)}\tilde{\gamma}_{l_\mathsf{q}}(t/\mathsf{q};\mathsf{p}) + R^{-1}\mathcal{R}_{\mathsf{p},\mathsf{q}}(t),$$

and since the matrix R represents a rigid rotation, the term $R^{-1}\mathcal{R}_{\mathsf{p},\mathsf{q}}$ has a small amplitude. Therefore, we can ensure that the solution Q_l is a sub-harmonic solution of order q with respect to the period 2π for q large enough.

The parameters p and q are linked with the number of revolutions around the origin of the solutions. Since the solutions Q_l do not pass through the origin, we can write them in polar coordinates,

$$Q_l(t) = |Q_l(t)|\begin{pmatrix}\cos\theta_l(t)\\ \sin\theta_l(t)\end{pmatrix},$$

where the function θ_l is called *argument function* and has the same regularity as Q_l. Using the argument function, we can define the number of revolutions of the solution Q_l in a period as the integer number N_l given by

$$N_l = \frac{\theta_l(2\pi\mathsf{q}) - \theta_l(0)}{2\pi}.$$

The number N_l only depends on the homotopy class of the loop Q_l in $\mathbb{R}^2\setminus\{0\}$. By a direct computation, we can prove that the number of revolutions in a $2\pi\mathsf{q}$-period for $\mathsf{q}^{2/(\alpha+1)}\gamma_l(t/\mathsf{q},\mathsf{p})$ is p. We can construct a continuous homotopy $H_l : \mathbb{R}/2\pi\mathsf{q}\mathbb{Z}\times[0,1]\to\mathbb{R}^2$ given by

$$H_l(t,\lambda) = (1-\lambda)\mathsf{q}^{2/(\alpha+1)}\gamma_{l_\mathsf{q}}(t/\mathsf{q},\mathsf{p}) + \lambda Q_l(t) = \mathsf{q}^{2/(\alpha+1)}\left[\gamma_{l_\mathsf{q}}(t/\mathsf{q},\mathsf{p}) + \lambda\mathcal{R}_{\mathsf{p},\mathsf{q}}(t)\right].$$

Since the amplitude of the periodic remainder $\mathcal{R}_{\mathsf{p},\mathsf{q}}$ is small, this homotopy does not pass through the origin. By the continuity of the number of revolutions, N_l must remain constant along the homotopy in each connected component. Thus, the number of revolutions of the solutions Q_l is also p. Therefore, we are finding solutions where the number of revolutions of the comet in a period $2\pi\mathsf{q}$ is a fixed number p meanwhile the primaries close their orbits q times.

By direct computations, we can also obtain an estimate of the kinetic energy of the remainder $\mathcal{R}_{\mathsf{p},\mathsf{q}}$. That is,

$$\left\| \frac{1}{2\pi\mathsf{q}} \dot{\mathcal{R}}_{\mathsf{p},\mathsf{q}} \right\|_{L^2}^2 \leq \tilde{c}_{\mathsf{p}} \mathsf{q}^{-\frac{\alpha+5}{\alpha+1}},$$

where $\tilde{c}_{\mathsf{p}}$ is a constant that only depend on p. Here, $L^2 = L^2(\mathbb{R}/2\pi\mathsf{q}\mathbb{Z}, \mathbb{R}^d)$ denotes the space of square-integrable periodic paths in $\mathbb{R}^d$ with norm $\|f\|_{L^2} = (\int_0^{2\pi\mathsf{q}} |f(t)|^2 \,\mathrm{d}t)^{1/2}$.

The set of circular periodic solutions of Eq. (6) with fixed minimal period is a manifold with two connected components diffeomorphic to $SO(2)$ if $d = 2$ and a connected manifold diffeomorphic to $SO(3)$ if $d = 3$. As we will see later, the number of solutions found is related with a topological property of these manifolds, called the *Lusternik-Schnirelman category*.

The idea behind the proof of Theorem 1 is to use simultaneously an infinite number of changes of variable. Let $\mathsf{q} \in \mathbb{Z}^+$ and $\varepsilon > 0$ be positive parameters. Let us consider the change of variables

$$q(t) = \varepsilon^{-1} x\left(t/\mathsf{q}\right), \tag{9}$$

and we assume that q and ε are related by

$$\frac{1}{\mathsf{q}^2} = \varepsilon^{\alpha+1}, \tag{10}$$

We will treat ε as a continuous parameter, although the relation (10) restricts the values that ε can take.

If we define a rescaled time variable $\tau = t/\mathsf{q}$ we can transform Eq. (5) in any of the following family of differential equations

$$x'' = -\sum_{j=1}^{n} m_j \frac{x - \varepsilon x_j(\tau; \mathsf{q})}{\|x - \varepsilon x_j(\tau; \mathsf{q})\|^{\alpha+1}}, \tag{11}$$

where the non-autonomous terms are given by

$$x_j(\tau; \mathsf{q}) = q_j(\mathsf{q}\tau), \quad j = 1, \ldots, n. \tag{12}$$

and $'$ denotes the derivative with respect to the variable τ. By a direct computation, we can prove the following

Lemma 1. *If $x(\tau)$ is a 2π-periodic solution of* (11) *and the relation* (10) *is satisfied, then $q(t)$ given by* (9) *is a $2\mathsf{q}\pi$-periodic solution of* (5).

The proof of Theorem 1 will consist in proving that Eq. (11) admits 2π-periodic solutions if ε is small enough independently of q. In this way, for large q it is possible to adjust ε in (10). For this reason we must be careful about the uniformity with respect to the parameter q.

It is easy to see that the family of differential equations (11) is the Euler-Lagrange equation associated with the family of Lagrangian functions

$$L_\varepsilon(\tau, x, y; \mathsf{q}) = \frac{1}{2}\|y\|^2 + \sum_{j=1}^{n} m_j \phi_\alpha(\|x - \varepsilon x_j(\tau; \mathsf{q})\|). \tag{13}$$

where,

$$\phi_\alpha(\lambda) = \begin{cases} \frac{1}{\alpha-1}\lambda^{1-\alpha} & \text{if} \quad \alpha > 1 \\ -\log\lambda & \text{if} \quad \alpha = 1 \end{cases}. \tag{14}$$

The autonomous part L_0 is the Lagrangian function associated with the central force problem,

$$L_0(x, y) = \frac{1}{2}\|y\|^2 + \phi_\alpha(\|x\|), \tag{15}$$

We can see that the autonomous part L_0 is a polynomial of degree 2 in its its variable y. The method used to prove Theorem 1 can be extended to other systems of this type because the theory only depends on this fact. In the next section we will work with Lagrangian systems with this structure, motivated by this discussion.

3 Admissible Families of Lagrangian Functions

Let $\varepsilon_0 > 0$ be a positive number and let $U \subset \mathbb{R}^d$ be an open set. We consider a Lagrangian function $L : \mathcal{D}_U \times [0, \varepsilon_0[\to \mathbb{R}$, $L = L_\varepsilon(\tau, x, y)$ defined on $\mathcal{D}_U = (\mathbb{R}/2\pi\mathbb{Z}) \times U \times \mathbb{R}^d$ with the following form:

$$L_\varepsilon(\tau, x, y) = \frac{1}{2}\langle A_\varepsilon(\tau, x)y, y\rangle + \langle B_\varepsilon(\tau, x), y\rangle + \mathcal{U}_\varepsilon(\tau, x). \tag{16}$$

Here, $\langle\cdot,\cdot\rangle$ is the Euclidean inner product of $\mathbb{R}^d$. We assume that the functions $A : (\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$, $B : (\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0, \varepsilon_0[\to \mathbb{R}^d$ and $\mathcal{U} : (\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0, \varepsilon_0[\to \mathbb{R}$ are in the class $C^{3,2}([(\mathbb{R}/2\pi\mathbb{Z}) \times U] \times [0, \varepsilon_0[)$. Here, $\mathbb{R}^{d\times d}$ denotes the set of matrices of dimension $d \times d$. Let us recall that a function $f : A \times B \to \mathbb{R}$, $f = f(a, b)$, is in the class $C^{p,q}(A \times B)$ if $f(\cdot, b) \in C^p(A)$ for any $b \in B$, $f(a, \cdot) \in C^q(B)$ for any $a \in A$ and the maps

$$(a, b) \to \partial_a^\alpha \partial_b^\beta f(a, b)$$

are continuous for $|\alpha| \leq p$ and $|\beta| \leq q$.

Moreover, we assume that A_0, B_0, and $\mathcal{U}_0$ do not depend on τ and $A_0, B_0, U_0 \in C^4(U)$. Also, we assume that $A_\varepsilon(\tau, x)$ is symmetric for every $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0, \varepsilon_0[$. Under these considerations, the Lagrangian function (16) is in the class $C^{3,2}(\mathcal{D}_U \times [0, \varepsilon_0[)$. In addition, we suppose that there is $\epsilon > 0$ such that

$$\langle A_\varepsilon(\tau, x)y, y\rangle \geq \epsilon\|y\|^2; \qquad (\tau, x, y; \varepsilon) \in \mathcal{D}_U \times [0, \varepsilon_0[. \tag{17}$$

In this section we show the existence of 2π-periodic solutions of the Euler-Lagrange equation associated to a given Lagrangian function L, that is, functions $x_\varepsilon \in C^2$ that satisfy

$$\begin{gathered} \frac{d}{d\tau}\left[\partial_y L_\varepsilon(\tau, x_\varepsilon(\tau), x'_\varepsilon(\tau))\right] = \partial_x L_\varepsilon(\tau, x_\varepsilon(\tau), x'_\varepsilon(\tau)), \\ x_\varepsilon(0) = x_\varepsilon(2\pi), \qquad x'_\varepsilon(0) = x'_\varepsilon(2\pi). \end{gathered} \tag{18}$$

The solutions will emerge from a set of periodic solutions of the autonomous Lagrangian system associated with L_0. This set of periodic solutions is denoted by Γ. By direct computation, we can prove that the variational equation of (18) when $\varepsilon = 0$ around any solution $\gamma \in \Gamma$ becomes

$$\begin{aligned} &\partial^2_{yy} L_0\left(\gamma(\tau), \gamma'(\tau)\right) u'' + \left[\partial^3_{xyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\left[\gamma'(\tau)\right] + \partial^3_{yyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\left[\gamma''(\tau)\right]\right] u' \\ &\quad + \left[\partial^3_{xxy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\left[\gamma'(\tau)\right] + \partial^3_{xyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\left[\gamma''(\tau)\right] - \partial^2_{xx} L_0\left(\gamma, \gamma'\right)\right] u = 0. \end{aligned} \tag{19}$$

We consider only certain sets of periodic solutions of the autonomous Lagrangian system associated with L_0. These sets must satisfy a suitable non-degeneracy condition.

Definition 1. *We say that $\Gamma \subset C^2(\mathbb{R}/2\pi\mathbb{Z}, U)$ is a regular manifold of periodic solutions for the autonomous Lagrangian system associated with $L_0 = L_0(x, y)$ if it satisfies the following properties:*

(i) Every function $\gamma \in \Gamma$ satisfies

$$\frac{d}{d\tau}\left[\partial_y L_0(\gamma(\tau), \gamma'(\tau))\right] = \partial_x L_0(\gamma(\tau), \gamma'(\tau)).$$

(ii) The family Γ is invariant under time translations; that is,

$$\gamma \in \Gamma \Rightarrow T_h\gamma \in \Gamma,$$

where $T_h\gamma(\tau) = \gamma(\tau + h)$.

(iii) The set of initial conditions at $t = 0$, denoted by

$$M_\Gamma = \left\{(\gamma(0), \gamma'(0)) \in U \times \mathbb{R}^d : \gamma \in \Gamma\right\}, \tag{20}$$

is a compact submanifold inside the phase space.

(iv) The dimension of the set of 2π-periodic solutions of the linear equation (19) *is the dimension of M_Γ as a manifold.*

Notice that condition (17) implies that $\det[\partial^2_{yy} L_0(\gamma(\tau), \gamma'(\tau))] > 0$ for every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Then, we can write Eq. (19) in its normal form. This allows us to apply Floquet theory.

Moreover, we also consider only families of Lagrangian functions of the form (16) with the following properties

Definition 2. *We say that a function $L : \mathcal{D}_U \times [0, \varepsilon_0[\to \mathbb{R}$ is an admissible family of Lagrangian functions with respect to the regular manifold Γ if* *(i) We can write L in the form given by Eq.* (16) *and the functions $A = A_\varepsilon(\tau,x)$, $B = B_\varepsilon(\tau,x)$ and $\mathcal{U} = \mathcal{U}_\varepsilon(\tau,x)$ are in the class of class $C^{3,2}([(\mathbb{R}/2\pi\mathbb{Z})\times U]\times[0,\varepsilon_0[)$; $A_\varepsilon(\tau,x)$ satisfies* (17) *form some $\epsilon > 0$; A_0, B_0, and $\mathcal{U}_0$ do not depend on τ and $A_0, B_0, \mathcal{U}_0 \in C^4(U)$.*

(ii) The Lagrangian function L_0 is autonomous and

$$\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau,x,y)\bigg|_{\varepsilon=0} = 0.$$

for every $(\tau,x,y) \in \mathcal{D}_U$.

Using Point *(ii)* of the previous definition, we can write an admissible family of Lagrangian functions as

$$L_\varepsilon(\tau,x,y) = L_0(x,y) + R_\varepsilon(\tau,x,y),$$

where

$$L_0(x,y) = \frac{1}{2}\langle A_0(\tau,x)y,y\rangle + \langle B_0(x),y\rangle + \mathcal{U}_0(x).$$

and $\partial_\varepsilon R_\varepsilon(\tau,x) = 0$ when $\varepsilon = 0$. Therefore, for small ε, the Lagrangian function L_ε can be interpreted as follows: an autonomous part L_0 plus a small and periodic perturbation R_ε.

In the following theorem, $\mathrm{cat}(\Gamma)$ denotes the Lusternik-Schnirelman category of Γ, that is, the smallest number of sets in Γ, open and contractible, needed to cover Γ.

Theorem 2. *Let $\varepsilon_0 > 0$, let $U \subset \mathbb{R}^d$ be an open set and let $L : \mathcal{D}_U \times [0,\varepsilon_0[\to \mathbb{R}$ be an admissible family of Lagrangian functions with respect to the manifold Γ. We suppose that the matrix function A satisfies assumption* (17) *and there is a constant $C > 0$ such that*

$$\begin{aligned} \|A_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} &\le C, \\ \|B_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} &\le C, \\ \|\mathcal{U}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} &\le C, \end{aligned} \tag{21}$$

for every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Then, there are constants $\varepsilon_1 \in]0,\varepsilon_0[$ and $c > 0$ that only depend on ε_0, U, Γ, C and ϵ such that for every $\varepsilon \in [0,\varepsilon_1[$, we have, at least, $n = cat(\Gamma)$ different periodic orbits $x_l(\cdot;\varepsilon)$ $(l = 1,\dots,n)$ of the Lagrangian system (18) *associated to L_ε and*

$$dist_H(x_l,\Gamma) \le c\varepsilon^2, \tag{22}$$

in which $dist_H(x,D)$ denotes the distance between a function $x \in H$ and a closed set $D \subset H$, where $H = H^1(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R}^d)$ is the Sobolev space of 2π-periodic paths on $\mathbb{R}^d$ with one weak derivative in $L^2(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R}^d)$ with norm

$$\|x\|_H = \left(\int_0^{2\pi} \left[\|x(\tau)\|^2 + \|x'(\tau)\|^2\right]\ d\tau\right)^{1/2}.$$

The proof of Theorem 2 is postponed to Section 4.

3.1 Proof of Theorem 1

To apply Theorem 2, we need to prove that the Lagrangian function given in (13) is an admissible family of Lagrangian functions according to Definition 2. Let $\Gamma_{d,\mathsf{p}} \subset C^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ be the set of circular $2\pi/\mathsf{p}$-periodic solutions of (6) with minimal period $2\pi/\mathsf{p}$ ($d = 2, 3$).

Lemma 2. *For any $\mathsf{p} \in \mathbb{Z}^+$, the set $\Gamma_{d,\mathsf{p}}$ is a regular manifold of periodic solutions for the autonomous Lagrangian system associated with the Lagrangian function L_0 given in Eq. (15). If $d = 2$, it has two connected components and each component is diffeomorphic to $SO(2)$. If $d = 3$, it is diffeomorphic to $SO(3)$.*

Proof. We need to prove Points (i)-(iv) from Definition 1. Point (i) is true by definition. Point (ii) is true since L_0 is autonomous. We only need to prove Point (iii) and Point (iv). Using (7) and (8), we can prove that the amplitude A_p and the norm of velocity B_p of every solution in $\Gamma_{d,\mathsf{p}}$ are

$$A_\mathsf{p} = \mathsf{p}^{-2/(\alpha+1)}; \qquad B_\mathsf{p} = \mathsf{p}^{(\alpha-1)/(\alpha+1)}.$$

For $d = 2$, let $\Gamma^+_{2,\mathsf{p}}$ and $\Gamma^-_{2,\mathsf{p}}$ be the sets of solutions of Eq. (6) with positive and negative orientation, respectively. Then, $\Gamma_{2,\mathsf{p}} = \Gamma^+_{2,\mathsf{p}} \cup \Gamma^-_{2,\mathsf{p}}$. In this case, the set of initial conditions at $t = 0$ is given by

$$M_{\Gamma^+_{2,\mathsf{p}}} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x, x\rangle = A^2_\mathsf{p},\ \langle y, y\rangle = B^2_\mathsf{p},\ \langle x, y\rangle = 0,\ \det(x|y) > 0\}.$$

We can construct an explicit diffeomorphism between $SO(2)$ and $M_{\Gamma^+_{2,\mathsf{p}}}$, namely

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto (x, y); \qquad x = A_\mathsf{p} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad y = B_\mathsf{p} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Analogously, we can construct a diffeomorphism between $M_{\Gamma^-_{2,\mathsf{p}}}$ and $SO(2)$, and Point (iii) is followed in this case.

For $d = 3$, the set of initial conditions at $t = 0$ is given by

$$M_{\Gamma_{3,\mathsf{p}}} = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle x, x\rangle = A^2_\mathsf{p},\ \langle x, y\rangle = 0,\ \langle y, y\rangle = B^2_\mathsf{p}\}.$$

After rescaling, we can identify the set $M_{\Gamma_{3,\mathsf{p}}}$ with the unit tangent bundle of S^2, namely,

$$T_1S^2 = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle x, x\rangle = 1,\ \langle x, y\rangle = 0,\ \langle y, y\rangle = 1\}.$$

Moreover, it is possible to construct an explicit diffeomorphism between T_1S^2 and $SO(3)$ (see Section 1.4 of [11]). Therefore, there exists a diffeomorphism between $M_{\Gamma_{3,\mathsf{p}}}$ and $SO(3)$ and Point (iii) is followed in this case.

Finally, to prove Point (iv), we need to compute the dimension of the set of 2π-periodic solutions of Eq. (19). By direct computation and using that $\|\gamma(\tau)\| = A_\mathsf{p}$ and $\sum_{j=1}^n m_j = 1$, we can prove that the variational equation (19) around any $\gamma \in \Gamma_{d,\mathsf{p}}$ becomes

$$u'' + \mathsf{p}^2 \left[I - A^{-2}_\mathsf{p}(\alpha + 1)\gamma(\tau)\gamma(\tau)^T\right] u = 0, \qquad u \in \mathbb{R}^d. \tag{23}$$

Eq. (23) is a linear equation with periodic coefficients. So, the existence of 2π-periodic solutions of Eq. (23) is related to its Floquet exponents. We consider the case $d = 2$ and $d = 3$ separately.

- $d = 2$. Using the diffeomorphism between each connected component of $\Gamma_{2,\mathsf{p}}$ and $SO(2)$ described in Lemma 2, given any $\gamma \in \Gamma^{\pm}_{2,\mathsf{p}}$, we can find $R \in SO(2)$ such that

$$\gamma(\tau) = A_{\mathsf{p}} R \begin{pmatrix} \cos(\mathsf{p}\tau) \\ \pm\sin(\mathsf{p}\tau) \end{pmatrix}.$$

  Now, we can make the following change of variables

$$u = e^{\mp J\mathsf{p}\tau} v, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

  Since $e^{\mp J\mathsf{p}\tau}$ and R conmute,

$$e^{\mp J\mathsf{p}\tau}\gamma(\tau) = A_{\mathsf{p}} R e_1, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

  Then,

$$e^{\mp J\mathsf{p}\tau}\gamma(\tau)\gamma(\tau)^T e^{\pm J\mathsf{p}\tau} = A_{\mathsf{p}}^2 R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R^T.$$

  So, the equation for v becomes a system with constant coefficients, namely

$$v'' \mp 2\mathsf{p}Jv' - \mathsf{p}^2(\alpha+1) R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R^T v = 0.$$

  The new change $w = R^T v$ leads to

$$w'' \mp 2\mathsf{p}Jw' - \mathsf{p}^2(\alpha+1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w = 0. \tag{24}$$

  Since the changes of variables used are 2π-periodic, Floquet exponents of Eq. (23) (when $d = 2$) and Eq. (24) are the same. Thus, we can compute these exponents directly from Eq. (24), obtaining

$$\lambda_{1,2} = 0, \qquad \lambda_{3,4} = \pm\mathsf{p}\sqrt{\alpha - 3}.$$

  If $\alpha > 3$ the solutions associated with $\lambda_{3,4}$ are not periodic and the eigenvalue $\lambda = 0$ has a geometric multiplicity equal to 1. If $1 \le \alpha < 3$ and $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z}$, the solutions associated with $\lambda_{3,4}$ do not have the appropriate period. If $\alpha = 3$, the eigenvalue $\lambda = 0$ has algebraic multiplicity 4, but its geometric multiplicity is 1. Therefore, if $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$, the dimension of the set of 2π-periodic functions is exactly 1.

- $d = 3$. Using the diffeomorphism between $\Gamma_{3,\mathsf{p}}$ and $SO(3)$ described in Lemma 2, given any $\gamma \in \Gamma_{3,\mathsf{p}}$, we can find $R \in SO(3)$ such that

$$\gamma(\tau) = A_{\mathsf{p}} R \begin{pmatrix} \cos(\mathsf{p}\tau) \\ \sin(\mathsf{p}\tau) \\ 0 \end{pmatrix}.$$

Making the change of variables $u = Rz$, and letting $z = (z_1, z_2, z_3)^T$, Eq. (23) can be decomposed in two parts, namely

$$\begin{pmatrix} z_1'' \\ z_2'' \end{pmatrix} + \mathsf{p}^2 \left[I - (\alpha+1) \begin{pmatrix} \cos^2(\mathsf{p}\tau) & \sin(\mathsf{p}\tau)\cos(\mathsf{p}\tau) \\ \sin(\mathsf{p}\tau)\cos(\mathsf{p}\tau) & \sin^2(\mathsf{p}\tau) \end{pmatrix} \right] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0, \qquad (25)$$
$$z_3'' + \mathsf{p}^2 z_3 = 0.$$

We have two linearly independent 2π-periodic solutions of Eq. (25), namely

$$z^{(1)}(\tau) = \begin{pmatrix} 0 \\ 0 \\ \cos(\mathsf{p}\tau) \end{pmatrix}, \quad z^{(2)}(\tau) = \begin{pmatrix} 0 \\ 0 \\ \sin(\mathsf{p}\tau) \end{pmatrix}.$$

The first equation of (25) has the same form as in the case $d = 2$ with $R = I$. Thus, we can compute Floquet exponents of the previous equation in the same way as in the case $d = 2$, obtaining

$$\lambda = 0 \text{ (double)}, \quad \lambda = \pm \mathsf{p}\sqrt{\alpha - 3}.$$

If $\alpha > 3$, the solutions associated with $\lambda = \pm\mathsf{p}\sqrt{\alpha-3}$ are not periodic. Moreover, the eigenvalue $\lambda = 0$ has a geometric multiplicity equal to 1. If $1 \leq \alpha < 3$ and $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z}$ the solutions associated with $\lambda = \pm i\mathsf{p}\sqrt{3-\alpha}$ do not have the appropriate period. Finally, if $\alpha = 3$, $\lambda = 0$ becomes an eigenvalue with algebraic multiplicity equal to 4, but its geometric multiplicity is still 1. In any case, we only have one 2π-periodic solution of (25) with $z_3 = 0$. This solution is linearly independent to $z^{(1)}$ and $z^{(2)}$. So, if $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$ the dimension of the set of 2π-periodic solutions of (23) when $d = 3$ is exactly 3.

In both cases, the dimensions of the set of 2π-periodic solutions of Eq. (23) are the dimensions of $M_{\Gamma_{2,\mathsf{p}}}$ as a manifold when $d = 2$ and $M_{\Gamma_{3,\mathsf{p}}}$ as a manifold when $d = 3$.

□

Lemma 3. *Let* $\mathsf{p} \in \mathbb{Z}^+$, $\alpha \geq 1$ *and assume that* $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$. *Then, the Lagrangian function given in Eq.* (13) *is an admissible family of Lagrangian functions with respect to the manifold* $\Gamma_{d,\mathsf{p}}$.

Proof. First, we need to define the domain U and the constant ε_0 so that the Lagrangian function given in (13) is well-defined. Using (12), we have that $\|x_j(\tau;\mathsf{q})\| \leq \|q_j\|_\infty$ for all $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Since $q_j \in C^3(\mathbb{R}/2\pi\mathbb{Z})$ for each j, there is a constant $c_1 > 0$ (that does not depend on q) such that $\|q_j\|_\infty < c_1$ for all $j = 1, \ldots, n$. Let $\varepsilon_0 > 0$ be a number that satisfies $\varepsilon_0 c_1 < A_\mathsf{p}$ and let $\rho > 0$ be any number in $]0, A_\mathsf{p} - \varepsilon_0 c_1[$. We can define the open set $U \subset \mathbb{R}^n$ such as

$$U = \left\{ x \in \mathbb{R}^d : A_\mathsf{p} - \rho < \|x\| < A_\mathsf{p} + \rho \right\}.$$

Thus, if we take $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0, \varepsilon_0[$ we have

$$\|x - \varepsilon x_j(\tau, \mathsf{q})\| \geq \|x\| - \varepsilon\|q_j\|_\infty > A_\mathsf{p} - \rho - \varepsilon c_1 > 0. \qquad (26)$$

Therefore, the Lagrangian function $L = L_\varepsilon(\tau, x, y)$ given in Eq. (13) is well-defined on $\mathcal{D}_U \times [0, \varepsilon_0[$.

The next step is to prove Points (i) and (ii) from Definition 2. If we set

$$\begin{aligned} A_\varepsilon(\tau, x) &= I, \\ B_\varepsilon(\tau, x) &= 0, \\ \mathcal{U}_\varepsilon(\tau, x) &= \sum_{j=1}^{n} m_j \phi_\alpha(\|x - \varepsilon x_j(\tau; \mathsf{q})\|), \end{aligned} \tag{27}$$

in Eq. (16), we obtain the Lagrangian function (13). Also, it is clear that A, B, and $\mathcal{U}$ are in the class $C^{3,2}([(\mathbb{R}/2\pi\mathbb{Z}) \times U] \times [0, \varepsilon_0[)$ and satisfy the other conditions. Thus Point (i) follows.

The function L becomes the autonomous Lagrangian function L_0 given in (15) when $\varepsilon = 0$. To prove Point (ii), we only need to verify that $\partial_\varepsilon L_\varepsilon(\tau, x; \mathsf{q}) = 0$ when $\varepsilon = 0$. By direct computation,

$$\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau, x; \mathsf{q})\bigg|_{\varepsilon=0} = -\frac{\phi'_\alpha(\|x\|)}{\|x\|} \left\langle x, \sum_{j=1}^{n} m_j x_j(\tau; \mathsf{q}) \right\rangle.$$

On the other hand, using Eq. (4),

$$\sum_{j=1}^{n} m_j x_j(\tau; \mathsf{q}) = \sum_{j=1}^{n} m_j q_j(\mathsf{q}\tau) = 0,$$

and Point (ii) follows. □

Lemma 4. *cat*$(\Gamma^{\pm}_{2,\mathsf{p}}) = 2$ *and cat*$(\Gamma_{3,\mathsf{p}}) = 4$.

Proof. In Lemma 2, we prove that $\Gamma^{\pm}_{2,\mathsf{p}}$ is diffeomorphic to $SO(2)$ and $\Gamma_{3,\mathsf{p}}$ is diffeomorphic to $SO(3)$. Then, it is enough to compute $\mathrm{cat}(SO(d))$ for $d = 2, 3$.

There is a diffeomorphism between $SO(2)$ and S^1. It is possible to cover S^1 with two open and contractible sets. Thus, $\mathrm{cat}(S^1) \leq 2$. Since S^1 is not contractible, $\mathrm{cat}(SO(2)) = \mathrm{cat}(S^1) = 2$. On the other hand, in Corollary 4.2 from [13], the authors prove that $\mathrm{cat}(M) = 4$ if M is a closed 3-manifold and its fundamental group is not free. According to Section 10 from Chapter III from [9], the fundamental group of $SO(3)$ is $\mathbb{Z}_2$, which is not free. Thus, $\mathrm{cat}(\Gamma_{3,\mathsf{p}}) = 4$. □

Proof of Theorem 1. Let $\alpha \geq 1$, $d = 2, 3$ and $\mathsf{p} \in \mathbb{Z}^+$ such that $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z}\backslash\{0\}$. By Lemma 2, the set $\Gamma_{d,\mathsf{p}}$ is a regular manifold for the autonomous Lagrangian system associated with the Lagrangian function L_0 given in (15). Now let $\varepsilon_0 > 0$, $c_1 > 0$, $\rho > 0$, and $U \subset \mathbb{R}^d$ be as in the proof of Lemma 3. Both ε_0, c_1, ρ, and U only depend on p. By Lemma 3, the Lagrangian function (16) (with A, B, and $\mathcal{U}$ given in (27)) is an admissible family of Lagrangian functions with respect to the manifold $\Gamma_{d,\mathsf{p}}$. Also, the matrix function A satisfies assumption (17) with $\epsilon = 1$.

We only need to verify (21). Since A and B are constant functions, the constants $C_1 = 1$ and $C_2 = 0$ are bounds for $A_{(\cdot)}(\tau,\cdot)$ and $B_{(\cdot)}(\tau,\cdot)$ in $C^{3,2}(U \times [0,\varepsilon_0[)$, respectively, for any $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Also, by direct computation, we can prove that there are positive constants δ_1 and δ_2 that only depend on p such that

$$0 < \delta_1 \le \|x - \varepsilon x_j(\tau;\mathsf{q})\| \le \delta_2, \qquad (\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0,\varepsilon_0[, \tag{28}$$

(see Eq. (26)). Since $\mathcal{U}$ and its derivatives only depend on powers of products of $[x - \varepsilon x_j(\tau;\mathsf{q})]$ and $x_j(\tau;\mathsf{q})$, it is possible to find a constant $C_3 > 0$ that only depends on δ_1 and δ_2 (and therefore on p) such that

$$\|\mathcal{U}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U;[0,\varepsilon_0[)} \le C_3, \qquad \tau \in (\mathbb{R}/2\pi\mathbb{Z}) \times [0,\varepsilon_0[.$$

Letting $C = \max\{1, C_3\}$ we verify (21). By construction, the constant C only depends on p. Moreover, from Lemma 4, $\operatorname{cat}(\Gamma_{3,\mathsf{p}}) = 4$.

Therefore, applying Theorem 2 in the case $d = 3$ and using Lemma 4, there are constants $\varepsilon_1 \in [0,\varepsilon_0[$ and $\tilde{c}_{\mathsf{p}} > 0$ that only depend on p (because ε_0, U, $\Gamma_{3,\mathsf{p}}$, and C depend only on p) such that, for any $\varepsilon \in]0,\varepsilon_1[$ we have at least four different periodic orbit x_l ($l = 1,2,3,4$) of (11) that satisfies (22). In particular, since $\Gamma_{3,\mathsf{p}}$ is a compact set on H, there is a function $\tilde{\gamma}_l \in \Gamma_{3,\mathsf{p}}$, $\tilde{\gamma}_l = \tilde{\gamma}_l(\tau;\mathsf{p},\varepsilon)$ such that

$$\operatorname{dist}_H(x_l, \Gamma) = \|x_l(\cdot;\mathsf{p},\varepsilon) - \tilde{\gamma}_l(\cdot;\mathsf{p},\varepsilon)\|_H \le \tilde{c}_{\mathsf{p}}\varepsilon^2.$$

Letting $y_l = x_l - \tilde{\gamma}_l$ we have that

$$x_l(\tau;\mathsf{p},\varepsilon) = \tilde{\gamma}_l(\tau;\mathsf{p},\varepsilon) + y_l(\tau;\mathsf{p},\varepsilon),$$

where

$$\|y_l(\cdot;\mathsf{p},\varepsilon)\|_H \le \tilde{c}_{\mathsf{p}}\varepsilon^2. \tag{29}$$

Since ε_1 does not depends on q, there is an integer $\mathsf{q}_0 \in \mathbb{Z}^+$ such that $1/\mathsf{q}^{2/(\alpha+1)} < \varepsilon_1$ if $\mathsf{q} < \mathsf{q}_0$. Therefore, by Lemma 1 for $\varepsilon = 1/\mathsf{q}^{2/(\alpha+1)}$, the non-Newtonian restricted $(n+1)$-body problem has at least four different comet solutions of the form

$$\begin{aligned} Q_l(t) &= \left(1/\mathsf{q}^{2/(\alpha+1)}\right)^{-1} x_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right) \\ &= \mathsf{q}^{2/(\alpha+1)}\left[\tilde{\gamma}_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right) + y_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right)\right] \\ &= \mathsf{q}^{2/(\alpha+1)}\gamma_{l_{\mathsf{q}}}(t/\mathsf{q};\mathsf{p}) + \mathcal{R}_{\mathsf{p},\mathsf{q}}(t), \end{aligned} \tag{30}$$

where $\gamma_{l_{\mathsf{q}}}(t;\mathsf{p}) = \tilde{\gamma}_l\left(t;\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right)$ and the remainder $\mathcal{R}_{\mathsf{p},\mathsf{q}}$ is a $2\pi\mathsf{q}$ periodic function given by

$$\mathcal{R}_{\mathsf{p},\mathsf{q}}(t) = \mathsf{q}^{2/(\alpha+1)} y_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right).$$

Using the estimate given in Eq. (29) and the embedding $\|\cdot\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})} \le k\|\cdot\|_H$ (see Proposition 1.3 from [16]) we have

$$\|\mathcal{R}_{\mathsf{p},\mathsf{q}}(t)\| \le \mathsf{q}^{2/(\alpha+1)}\|y_l(\cdot;\mathsf{p};1/\mathsf{q}^{2/(\alpha+1)})\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})} \le c_{\mathsf{p}}\mathsf{q}^{-2/(\alpha+1)}$$

where $c_{\mathsf{p}} = k\tilde{c}_{\mathsf{p}}$ only depends on p.

In the case $d = 2$, the set of 2π-periodic solutions has two components. So, we can apply the previous argument in each component. Since cat $(\Gamma^{\pm}_{2,\mathsf{p}}) = \operatorname{cat}(SO(2)) = 2$, we obtain two solutions from each component. That is, we already have four periodic orbits. □

4 Action Functional

In this section, we will use a functional framework and a quantitative version of the implicit function theorem to prove Theorem 2. Let $H = H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ be the Sobolev space of 2π-periodic paths on $\mathbb{R}^d$ with one weak derivative in $L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ and inner product

$$\langle x, y\rangle_H = \int_0^{2\pi} \left[\langle x(\tau), y(\tau)\rangle + \langle x'(\tau), y'(\tau)\rangle\right] \, \mathrm{d}\tau. \tag{31}$$

Here, the product $\langle\cdot,\cdot\rangle$ is the Euclidean inner product of $\mathbb{R}^d$ and x' denotes the (weak) derivative of $x \in H$.

Given a Lagrangian function $L = L_\varepsilon(\tau, x, y)$ of the form (16), its associated action functional $\mathcal{A} = \mathcal{A}_\varepsilon(x)$ is given by

$$\mathcal{A}_\varepsilon(x) = \int_0^{2\pi} L_\varepsilon(\tau, x(\tau), x'(\tau)) \, \mathrm{d}\tau. \tag{32}$$

For $\mathcal{A}$ to be well defined, we need L to be defined on $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$. Since we are studying families of the form given in (16), it will be enough to modify the functions A, B, and $\mathcal{U}$.

Lemma 5. *Let $L = L_\varepsilon(\tau, x, y)$ be a Lagrangian function such as in* (16) *defined on $(\mathbb{R}/2\pi\mathbb{Z}) \times U \times \mathbb{R}^d \times [0, \varepsilon_0[$ and let $\Lambda \subset U$ be a compact set. Then, for any neighborhood V of Λ with compact closure such that $\bar{V} \subset U$, there are functions $\tilde{A} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$, $\tilde{B} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^d$ and $\tilde{\mathcal{U}} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}$ in the class $C^{3,2}(((\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d) \times [0, \varepsilon_0[)$ such that $A = \tilde{A}$, $B = \tilde{B}$ and $\mathcal{U} = \tilde{\mathcal{U}}$ when $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times V \times [0, \varepsilon_0[$; $\tilde{A}_0$, $\tilde{B}_0$, and $\tilde{\mathcal{U}}_0$ do not depend on τ and they are in the class $C^4(\mathbb{R}^d)$; and $\tilde{A}_\varepsilon(\tau, x)$ is a symmetric matrix that satisfies* (17)*. Moreover, there are constants $K > 0$ (that only depends on ε_0, U, and V) and $\tilde{\epsilon} > 0$ (that only depends on ϵ given in* (17)*) such that*

$$\begin{aligned} \|\tilde{A}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(\mathbb{R}^d\times[0,\varepsilon_0[)} &\le K\left[\|A_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} + 1\right], \\ \|\tilde{B}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(\mathbb{R}^d\times[0,\varepsilon_0[)} &\le K\|B_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)}, \\ \|\tilde{\mathcal{U}}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(\mathbb{R}^d\times[0,\varepsilon_0[)} &\le K\|\mathcal{U}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)}, \end{aligned} \tag{33}$$

for any $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, and

$$\langle \tilde{A}_\varepsilon(\tau, x)y, y\rangle \ge \tilde{\epsilon}\|y\|^2; \qquad (\tau, x, y; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d \times [0, \varepsilon_0[.$$

Proof. Let V be an open neighborhood of Λ with compact closure such that $\bar{V} \subset U$. It is well known that there is a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ in the class $C^\infty(\mathbb{R}^d)$ with compact support such that

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad x \in V \\ 0 & \text{if} \quad x \notin U \end{cases}$$

The existence of the function φ is related to the existence of a partition of the unity subordinate to the cover $\{W, \mathbb{R}^d \setminus W\}$, where W is any open set that satisfies $\bar{V} \subset W \subset U$

(see Section 2.2 from [14] for details). Note that φ does not depends on ε. We can extend the function A to zero outside of $(\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0, \varepsilon_0[$ and define the matrix function $\tilde{A} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$ given by

$$\tilde{A}_\varepsilon(\tau, x) = \varphi(x)A_\varepsilon(\tau, x) + (1 - \varphi(x))I.$$

With this, it is clear that $\tilde{A} = A$ when $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times V \times [0, \varepsilon_0[$, $\tilde{A}_0$ does not depends on τ and $\tilde{A}_0 \in C^4(\mathbb{R}^d)$. Since $A_\varepsilon(\tau, x)$ is symmetric and positive definite and $\tilde{A}_\varepsilon(\tau, x)$ is a convex combination of $A_\varepsilon(\tau, x)$ and I, $\tilde{A}$ is symmetric and there is a $\tilde{\epsilon} > 0$ such that

$$\langle \tilde{A}_\varepsilon(\tau, x)y, y\rangle \geq \tilde{\epsilon}\|y\|^2, \qquad (\tau, x, y; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d \times [0, \varepsilon_0[.$$

On the other hand, we can define the function $\tilde{B} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^d$ given by

$$\tilde{B}_\varepsilon(\tau, x) = \begin{cases} \varphi(x)B_\varepsilon(\tau, x) & \text{if} \quad x \in U, \\ 0 & \text{if} \quad x \notin U. \end{cases}$$

and define $\tilde{\mathcal{U}} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}$ using $\mathcal{U}$ in the same way. By construction, $B = \tilde{B}$ and $\mathcal{U} = \tilde{\mathcal{U}}$ when $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times V \times \mathbb{R}^d \times [0, \varepsilon_0[$, $\tilde{B}_0$ and $\mathcal{U}_0$ do not depend on τ and $\tilde{B}_0, \tilde{\mathcal{U}}_0 \in C^4(\mathbb{R}^d)$.

The functions $\tilde{A}$, $\tilde{B}$ and $\tilde{\mathcal{U}}$ are constant when $x \notin U$. Then, the derivatives of $\tilde{A}$, $\tilde{B}$, and $\tilde{\mathcal{U}}$ are bounded by the derivatives of A, B, $\mathcal{U}$, and φ. Moreover, φ has a compact support. Then, it is bounded in $C^3(\mathbb{R}^d)$ by a constant that only depends on U, V, and the choice of φ. Therefore, there is a constant $K > 0$ (related with $\|\varphi\|_{C^3(\mathbb{R}^d)}$) such that the estimates in (33) are valid. □

From here, we assume without loss of generality that the functions A, B and $\mathcal{U}$ are defined on $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[$ and the bounds given in (21) are valid when $U = \mathbb{R}^d$. This implies that the family of Lagrangian function L given in (16) is defined on $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d \times [0, \varepsilon_0[$.

Remark 1. *Assumption* (17) *is only needed to extend the matrix function A. Then, it can be replaced by the more general condition: the function A admits a smooth and symmetric extension $\tilde{A} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$, $\tilde{A} = \tilde{A}_\varepsilon(\tau, x)$, such that $A = \tilde{A}$ when $x \in U$ and there is a constant ϵ such that*

$$\left|\det \tilde{A}_\varepsilon(\tau, x)\right| \geq \epsilon > 0,$$

for any $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[$.

4.1 Regularity

We are going to prove that the action functional and its critical points have the necessary regularity to apply a quantitative version of Theorem 2.1 from [5] for action functionals of the form given in (32). In the following proposition, the set $\mathcal{L}_n = \mathcal{L}(H \times \cdots \times H, \mathbb{R})$ denotes the space of bounded multilinear forms with norm

$$\|M\|_{\mathcal{L}_n} = \sup_{\substack{\|v_i\|_H \leq 1 \\ i=1,\ldots,n}} \left|M[v_1, \ldots, v_n]\right|.$$

For the sake of simplicity, we omit the dependence on ε.

Proposition 1. *Let* $A : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$*,* $B : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $\mathcal{U} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \to \mathbb{R}$ *be functions such that* A*,* B *and* $\mathcal{U}$ *are in the class* $C^{0,3}((\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d)$ *and they satisfy* (33)*. Let* $L : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *be the Lagrangian function given in* (16)*. Then, the action functional* $\mathcal{A} : H \to \mathbb{R}$ *associated with* L *given by* (32) *is three times continuously differentiable.*

Proof. By the regularity of the functions A, B, and $\mathcal{U}$, the Lagrangian function L given in (16) satisfies the hypothesis of Proposition 3.1 from [1]. Also, it is a polynomial of degree 2 in its variable y. Therefore, the action functional $\mathcal{A}$ is in the class $C^2(H)$.

The second-order derivatives of L, $\partial^2_{xx}L$, $\partial^2_{xy}L$, and $\partial^2_{yy}L$ will be interpreted as maps from $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^{d\times d}$. The variations in the direction of $z \in \mathbb{R}^d$ will be matrices denoted by

$$\partial^3_{xxx}L(\tau, x, y)[z], \quad \partial^3_{xxy}L(\tau, x, y)[z], \quad \partial^3_{xyy}L(\tau, x, y)[z]. \quad \partial^3_{yyy}L(\tau, x, y)[z]$$

Now, by standard computations, we can prove that $\delta^2\mathcal{A}$ has, for every $u \in H$, a directional derivative $\delta^3\mathcal{A}(u) \in \mathcal{L}_3$ given by

$$\begin{aligned}\delta^3\mathcal{A}(u)[v_1, v_2, v_3] = \int_0^{2\pi} & \big[\big\langle \partial^3_{xxx}L[v_3(\tau)]v_1(\tau), v_2(\tau)\big\rangle + \big\langle \partial^3_{xxy}L[v_3'(\tau)]v_1(\tau), v_2(\tau)\big\rangle \\ & + \big\langle \partial^3_{xxy}L[v_3(\tau)]v_1'(\tau), v_2(\tau)\big\rangle + \big\langle \partial^3_{xxy}L[v_3(\tau)]v_1(\tau), v_2'(\tau)\big\rangle \\ & + \big\langle \partial^3_{xyy}L[v_3'(\tau)]v_1'(\tau), v_2(\tau)\big\rangle + \big\langle \partial^3_{xyy}L[v_3'(\tau)]v_1(\tau), v_2'(\tau)\big\rangle \\ & + \big\langle \partial^3_{xyy}L[v_3'(\tau)]v_1(\tau), v_2'(\tau)\big\rangle\big] \ \mathrm{d}\tau.\end{aligned}$$

In the previous formula, the derivatives of L are evaluated in $(\tau, u(\tau), u'(\tau))$. Notice that we do not have the term $\partial^3_{yyy}L$ because L is a polynomial of degree 2 in its third variable.

The last step is to prove that the map

$$\delta^3\mathcal{A} : H \to \mathcal{L}_3, \qquad u \to \delta^3\mathcal{A}(u),$$

is continuous. Let $\{u_n\} \subset H$ be a sequence such that $u_n \to u$ in H. Given $v_1, v_2, v_3 \in H$ such that $\|v_i\|_H \leq 1$ $(i = 1, 2, 3)$, we want to prove that the difference

$$\left|\delta^3\mathcal{A}(u_n)[v_1, v_2, v_3] - \delta^3\mathcal{A}(u)[v_1, v_2, v_3]\right|$$

tends to zero uniformly in v_i when $n \to \infty$. Note that $\delta\mathcal{A}(u)$ is formed by a sum of seven terms. Let us analyze one of the terms that involves the derivative $\partial^3_{xyy}L$. From now on, $\partial^3_{xyy}L(\tau)$ will denote $\partial^3_{xyy}L(\tau, u(\tau), u'(\tau))$ and $\partial^3_{xyy}L_n(\tau)$ will denote $\partial^3_{xyy}L(\tau, u_n(\tau), u_n'(\tau))$. This notation will also be used in the other derivatives. The term to analyze is

$$\int_0^{2\pi} \left|\big\langle(\partial^3_{xyy}L_n(\tau) - \partial^3_{xyy}L(\tau))[v_3'(\tau)]v_1'(\tau), v_2(\tau)\big\rangle\right| \ \mathrm{d}\tau.$$

Taking the derivatives in (16), we have that $\partial^3_{xyy}L(\tau, x, y) = D_xA(\tau, x)$. Since the matrix $A(\tau, \cdot)$ is bounded in $C^3(\mathbb{R}^d)$ (see (33)), we have that $\{\partial^3_{xyy}L_n\}$ converges uniformly to $\partial^3_{xyy}L$. That is

$$\|\partial^3_{xyy}L_n - \partial^3_{xyy}L\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})} \xrightarrow[n\to\infty]{} 0. \tag{34}$$

By Proposition 1.3 from [16], there is a constant $k > 0$ such that $\|v_2\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})} \leq k\|v_2\|_H$. Using this fact, $\|v_i\|_H \leq 1$ and Hölder inequality, we have

$$\int_0^{2\pi} \left|\left\langle (\partial^3_{xyy}L_n - \partial^3_{xyy}L)[v_3']v_1', v_2\right\rangle\right| \leq k\|\partial^3_{xyy}L_n - \partial^3_{xyy}L\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})}\|v_1\|_H\|v_2\|_H\|v_3\|_H$$
$$\leq k\|\partial^3_{xyy}L_n - \partial^3_{xyy}L\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})}.$$

Using (34), we have

$$\sup_{\substack{\|v_i\|_H\leq 1\\ i=1,2,3}} \int_0^{2\pi} \left|\left\langle (\partial_{xyy}L^3_n(\tau) - \partial^3_{xyy}L(\tau))[v_3'(\tau)]v_1'(\tau), v_2(\tau)\right\rangle\right| \, \mathrm{d}\tau \xrightarrow[n\to\infty]{} 0.$$

For the other terms, we can proceed in a similar way: using the bounds given in (33), we can prove that

$$\|\partial^3_{xxx}L_n - \partial^3_{xxx}L\|_{L^1(\mathbb{R}/2\pi\mathbb{Z})} \xrightarrow[n\to\infty]{} 0,$$
$$\|\partial^3_{xxy}L_n - \partial^3_{xxy}L\|_{L^2(\mathbb{R}/2\pi\mathbb{Z})} \xrightarrow[n\to\infty]{} 0.$$

With a similar analysis, we have

$$\|\delta^3\mathcal{A}(u_n) - \delta^3\mathcal{A}(u)\|_{\mathcal{L}_3} = \sup_{\|v_i\|_H\leq 1} \left|\delta^3\mathcal{A}(u_n)[v_1,v_2,v_3] - \delta^3\mathcal{A}(u)[v_1,v_2,v_3]\right| \xrightarrow[n\to\infty]{} 0,$$

and the result follows. □

Under the same hypothesis of Proposition 1, if $\gamma \in H$ is a critical point of the action functional $\mathcal{A}$, then $\gamma \in H^2 = H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ (that is, $\gamma, \gamma' \in H$). Moreover, the derivative of γ', denoted by γ'', satisfies

$$A(\tau,\gamma(\tau))\gamma'' + \partial^2_{xy}L(\tau,\gamma(\tau),\gamma'(\tau))\gamma'(\tau) + \partial^2_{\tau y}L(\tau,\gamma(\tau),\gamma'(\tau)) = \partial_x L(\tau,\gamma(\tau),\gamma'(\tau)), \quad (35)$$

for almost every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ (see the proof of Proposition 3.1 from [1]. This implies that $\gamma \in C^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$, and hence it is a classical solution of Eq. (18).

4.2 Gradient and Hessian maps

Since H is a Hilbert space, we can introduce the gradient map of a functional $\mathcal{A}$ as the function that associates any $x \in H$ with the unique vector $\nabla\mathcal{A}(x) \in H$ that satisfies

$$\langle \nabla\mathcal{A}(x), v\rangle_H = \delta\mathcal{A}(x)v, \quad (36)$$

for any $v \in H$. Also, we can define the Hessian map $D^2\mathcal{A}(x) : H \to H$ using the second variation $\delta^2\mathcal{A}(x)$ as follows: given $u \in H$, $D^2\mathcal{A}(x)u \in H$ is the unique vector that satisfies

$$\left\langle D^2\mathcal{A}(x)u, v\right\rangle_H = \delta^2\mathcal{A}(x)[u,v], \quad (37)$$

for any $v \in H$. Also, since $\mathcal{A}$ is in the class $C^2(H)$, the map $D^2\mathcal{A}(x)$ is a symmetric operator with respect to the inner product given in (31). Let us recall that $K \in \mathcal{L}(H,H)$ is a compact linear operator if $K(\mathcal{U})$ has a compact closure in H for every bounded subset $\mathcal{U} \subset H$.

Lemma 6. *If $x \in H^2$, the operator $D^2\mathcal{A}(x)$ can be written as*

$$D^2\mathcal{A}(x) = \Phi + K. \tag{38}$$

where $\Phi : H \to H$ is an isomorphism and $K : H \to H$ is a compact operator.

Proof. From now on, $\mathfrak{A}(\tau)$ will denote $A(\tau, x(\tau))$ and $\partial^2_{xy}L(\tau)$ will denote $\partial^2_{xy}L(\tau, x(\tau), x'(\tau))$. This notation will also be used in the other derivatives. Let us consider the linear map $u \mapsto p$, where

$$p(\tau) = f_1(\tau)u(\tau) + f_2(\tau)u'(\tau)$$

and the functions f_1 and f_2 are given by

$$\begin{aligned} f_1(\tau) &= -\mathfrak{A}(\tau) + \mathfrak{A}''(\tau) - \left(\partial^2_{xy}L(\tau)\right)' + \partial^2_{yx}L(\tau) + \partial^2_{xx}L(\tau), \\ f_2(\tau) &= \mathfrak{A}'(\tau) - \partial^2_{xy}L(\tau). \end{aligned}$$

Using that $x \in H^2$ and the bounds given in (33), we can prove that the operator $F : H \to L^2$ given by $Fu = p$ is a bounded linear operator. On the other hand, we can define the linear operator $\tilde{K} : L^2 \to H$ such that $z = \tilde{K}p$ is the unique 2π-periodic solution of

$$-z'' + z = p(\tau).$$

The operator $\tilde{K}$ maps bounded sets of L^2 in bounded sets of H^2 and H^2 has a compact embedding in H. Then, $\tilde{K}$ is a compact operator.

Let us define the linear operators $\Phi, K : H \to H$ given by $\Phi u = \mathfrak{A}u$ and $K = \tilde{K} \circ F$. The map Φ is an isomorphism. In fact, since the matrix A satisfies (33) and (17), we have that $\Phi u \in H$ and the inverse map is given by $\Phi^{-1}v = \mathfrak{A}^{-1}v$. On the other hand, K is compact since it is the composition of a bounded linear operator and a compact operator. Finally, using (37) and letting $w = Ku$ we have

$$\begin{aligned} \delta^2\mathcal{A}(x)[u, v] &= \int_0^{2\pi} \left[\langle \mathfrak{A}u', v' \rangle + \langle \partial^2_{xy}Lu, v' \rangle + \langle \partial^2_{yx}Lu', v \rangle + \langle \partial^2_{xx}Lu, v \rangle\right] \\ &= \int_0^{2\pi} \left[\langle \mathfrak{A}u, v \rangle + \langle (\mathfrak{A}u)', v' \rangle + \langle w, v \rangle + \langle w', v' \rangle\right] \\ &= \langle \mathfrak{A}u, v \rangle_H + \langle w, v \rangle_H = \langle D^2\mathcal{A}(x)u, v \rangle_H. \end{aligned}$$

From here, we deduce that $D^2\mathcal{A}(x) = \Phi + K$. □

4.3 Regular Critical Manifold

Let us recall that L_0 is an autonomous Lagrangian function and $\Gamma \subset C^2$ is a manifold of periodic solutions. The action functional associated to L_0 will be denote by $\mathcal{A}_0$.

Since $\Gamma \subset C^2$, we can consider Γ as a differential C^k-submanifold of (the Hilbert space) H as in Definition 10.1 from [16]. Thus, we can interpret Γ as a critical manifold of $\mathcal{A}_0$, that is, a manifold filled with critical points.

Definition 3. *We say that a C^k-submanifold $\Gamma \subset H$ is a regular critical manifold of $\mathcal{A}_0$ if*

(i) all points of Γ are critical points of $\mathcal{A}_0$,

(ii) the nullity of $D^2\mathcal{A}_0(\gamma)$ for each $\gamma \in \Gamma$ is equal to the dimension of Γ,

(iii) $D^2\mathcal{A}_0(\gamma)$ is a Fredholm operator of index 0 for each $\gamma \in \Gamma$.

Recall that a Fredholm operator is an operator with finite-dimensional kernel, its range is closed and the co-dimension of its range coincides with the dimension of its kernel.

There is a connection between the concept of regular critical manifold of $\mathcal{A}_0$ (as in Definition 3) and regular manifold of periodic solutions for the Lagrangian system associated with L_0 (as in Definition 1). In the following lemma, we denote by $T_\gamma\Gamma$ the tangent space of Γ at γ. From now on, the set $R(T)$ denotes the range of a linear operator on a Hilbert space T and T^* denotes its adjoint. The orthogonal complements and the adjoint are taken with respect to the inner product of H given in (31)

Lemma 7. *If Γ is a regular manifold of periodic solutions for the autonomous Lagrangian system associated to L_0, then Γ is a regular critical C^3-manifold of the action functional $\mathcal{A}_0$.*

Proof. First, we need to prove that Γ is a compact C^3-submanifold of the Hilbert space H, according to Definition 10.1 from [16]. The main tool will be the Theorem 10.1 of [16]. Since A_0 satisfies (17), we can write Eq. (35) as

$$x'' = F(x, x'). \tag{39}$$

The function F is related to the derivatives of A_0, B_0 and $\mathcal{U}_0$. Since $A_0, B_0, \mathcal{U}_0 \in C^4(\mathbb{R}^d)$ we have that $F \in C^3(\mathbb{R}^d \times \mathbb{R}^d)$. Let M_Γ be the set of initial conditions given in (20). Given $(x_0, v_0) \in M_\Gamma$, we denote $x(\tau; x_0, y_0)$ as the 2π-periodic solution of Eq. (39) with initial conditions $x(0) = x_0$ and $x'(0) = v_0$. By the differentiable dependence of solutions on the initial condition, the map

$$\mathbb{R} \times M_\Gamma \to \mathbb{R}^d \times \mathbb{R}^d, \quad (\tau; x_0, v_0) \to (x(\tau; x_0, y_0), x'(\tau; x_0, y_0))$$

is in the class C^3. As a consequence, the map $\varphi : M_\Gamma \to C^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ given by

$$\varphi(x_0, v_0) = x(\,\cdot\,; x_0, v_0)$$

is in the class C^3.

We define the map $f : M_\Gamma \to H$ by composing the function φ with the inclusion $C^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \hookrightarrow H$. Clearly, $f \in C^3$. Thus, we need to verify points (a), (b), and (c) from Theorem 10.1 of [16]. To prove Point (a), we need to verify that Γ is a compact manifold. This is true because M_Γ is a compact set by hypothesis and they are homeomorphic. To prove Point (b), we need to verify that f is an injective map. This is valid by the uniqueness of the initial value problem in Eq. (39). Finally, for Point (c) we need to verify that $\delta f_{(x_0,v_0)}$ is an injective map for every $(x_0, v_0) \in M_\Gamma$. By chain rule, it is enough to verify that $\delta\varphi_{(x_0,v_0)}$

is injective, for every $(x_0, v_0) \in M_\Gamma$. Let $(x_0, v_0) \in M_\Gamma$ and $(u_1, u_2) \in T_{(x_0,v_0)}(M_\Gamma)$. By direct computation, we can prove that $u = (\delta\varphi)_{(x_0,v_0)}(u_1, u_2)$ is the 2π-periodic solution of the variational equation (19) with initial conditions $u(0) = u_1$ and $u'(0) = u_2$. The uniqueness of the initial value problem for Eq. (19) implies that $(\delta\varphi)_{(x_0,v_0)}$ is an injective map (notice that we can write Eq. (19) in its normal form due the assumption (17)). Therefore, $\Gamma = f(M_\Gamma)$ is a compact C^3-submanifold of H.

Now, we need to prove Points (*i*)-(*iii*) from Definition 3. Point (*i*) is true since every function $\gamma \in \Gamma$ is a 2π-periodic solution of the Lagrangian system associated to L_0.

For the second point, let $\gamma \in \Gamma$ and $u \in \ker D^2\mathcal{A}_0(\gamma)$. Then

$$\left\langle D^2\mathcal{A}_0(\gamma)u, v\right\rangle_H = 0, \quad \text{for all } v \in H.$$

Using (37), the function u must satisfy

$$\begin{aligned} A\left(\tau,\gamma(\tau)\right)u'' =& \big[-\partial^3_{xyy}L_0\left(\gamma(\tau),\gamma'(\tau)\right)\gamma'(\tau) - \partial^3_{yyy}L_0\left(\gamma(\tau),\gamma'(\tau)\right)\gamma''(\tau) \\ &+ \partial^2_{yx}L_0\left(\gamma(\tau),\gamma'(\tau)\right) - \partial^2_{xy}L_0\left(\gamma(\tau),\gamma'(\tau)\right)\big]u' \\ &+ \big[-\partial^3_{xxy}L_0\left(\gamma(\tau),\gamma'(\tau)\right)\gamma'(\tau) - \partial^3_{yxy}L_0\left(\gamma(\tau),\gamma'(\tau)\right)\gamma''(\tau) \\ &+ \partial^2_{xx}L_0\left(\gamma(\tau),\gamma'(\tau)\right)\big]u. \end{aligned} \tag{40}$$

Thus, the function u satisfies the variational equation (19). This implies that $\ker D^2\mathcal{A}_0(\gamma) = T_\gamma\Gamma$. Since Γ is a regular set of periodic solutions, $\dim\ker D^2\mathcal{A}_0(\gamma) = \dim T_\gamma\Gamma = \dim\Gamma$, and Point (*ii*) follows.

For the third point, we need to prove that $D^2\mathcal{A}_0(\gamma)$ is a Fredholm operator of index 0, that is, $R(D^2\mathcal{A}_0(\gamma))$ is closed and $\mathrm{codim} R(D^2\mathcal{A}_0(\gamma)) = \dim\Gamma$. Let us recall that γ is a critical point of $\mathcal{A}_0$. This implies that $\gamma \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$. Then, we can use Lemma 6 to write $D^2\mathcal{A}_0(\gamma) = \Phi + K$, where Φ is an isomorphism and K is compact. We have

$$R\big(D^2\mathcal{A}_0(\gamma)\big) = R\big((I + K\Phi^{-1}) \circ \Phi\big) = R\big(I + K\Phi^{-1}\big).$$

Since K is a compact operator, $K\Phi^{-1}$ is also a compact operator. Thus, we can apply the Fredholm alternative (Theorem 6.6 from [10]) to the operator $I + K\Phi^{-1}$. In particular, this implies that $R\big(D^2\mathcal{A}_0(\gamma)\big)$ is closed. Also, $D^2\mathcal{A}_0(\gamma)$ is symmetric. Then, $D^2\mathcal{A}_0(\gamma)$ is self-adjoint. Applying Corollary 2.18 from [10] we have

$$R\big(D^2\mathcal{A}_0(\gamma)\big) = \left[\ker\left(D^2\mathcal{A}_0(\gamma)\right)\right]^\perp. \tag{41}$$

The previous equation implies that $\mathrm{codim} R(D^2\mathcal{A}_0(\gamma)) = \dim\Gamma < \infty$ and the proof is complete. □

Remark 2. *In [7], the authors verify the non-degeneracy condition by writing the hessian of the action functional in its Fourier components.*

4.4 Proof of Theorem 2

In this section, we obtain a quantitative result about the existence of critical points of a family of action functionals associated with an admissible family of Lagrangian functions.

This result is based on Theorem 2.1 from [5] and will be used to prove Theorem 2. We need to introduce the quantitative version of the implicit function theorem that we will use. In the following lemma, we denote by $B_E(x, r)$ the open ball in the Banach space E with center at $x \in E$ and radius $r > 0$.

Lemma 8. *Let E, F and G be Banach spaces, let $U \subset E \times F$ be an open set and let $\mathcal{F} : U \to G$, $\mathcal{F} = \mathcal{F}(x, y)$ be a function of class C^2 such that $\mathcal{F}(x_0, y_0) = 0$ and the map $\delta_y \mathcal{F}(x_0, y_0)$ is invertible, for some $(x_0, y_0) \in U$. Assume that there exist a uniform bound $C > 0$ such that $\|\delta_x \mathcal{F}\| \le C$, $\|\delta_{xy} \mathcal{F}\| \le C$, $\|\delta_{yy} \mathcal{F}\| \le C$, and $\|\delta_y \mathcal{F}(x_0, y_0)^{-1}\| \le C$. Then, there are constants $R, r > 0$ that only depend on C such that $B_E(x_0, R) \times B_F(y_0, r) \subset U$ and there is a function $\varphi : B_E(x_0, R) \to B_F(y_0, r)$ in the class C^2 that satisfies $\varphi(x_0) = y_0$ and φ is the unique solution of the equation*

$$\mathcal{F}(x, \varphi(x)) = 0, \quad x \in B_E(x_0, R).$$

The proof Lemma 8 is an adaptation of the ideas presented in Lemma 4.2 from [17].

Remark 3. *In Lemma 8, we need a bound for the inverse of $\delta_y \mathcal{F}$ only at (x_0, y_0). As we will see later, we also need a uniform bound for the inverse of the derivative $\delta_y \mathcal{F}$ in a neighborhood of (x_0, y_0) (see Proof of Theorem 2 below). We can obtain uniform bounds for $\delta_y \mathcal{F}$ using the bounds in $\delta_{xy} \mathcal{F}$ and $\delta_{yy} \mathcal{F}$. That is, we can find constants $\tilde{R} < R$ and $\tilde{r} < r$ such that*

$$\|\delta_y \mathcal{F}(x, y)^{-1}\| \le C/2, \qquad (x, y) \in B_E(x_0, \tilde{R}) \times B_F(y_0, \tilde{r}).$$

We denote by $N_\gamma \Gamma$ the orthogonal complement to the tangent space $T_\gamma \Gamma$, called normal space. Let P_γ and Q_γ be the orthogonal projectors onto $N_\gamma \Gamma$ and $T_\gamma \Gamma$, respectively. These projectors define two functions $\mathcal{P}, \mathcal{Q} : \Gamma \to \mathcal{L}(H, H)$ given by

$$\mathcal{P}(\gamma) = P_\gamma; \qquad \mathcal{Q}(\gamma) = Q_\gamma.$$

Since Γ is a C^3-manifold, the functions $\mathcal{P}$ and $\mathcal{Q}$ are in the class $C^2(\Gamma)$ (according to [16]) and their differentials will be denoted by

$$T_\gamma \Gamma \ni v \xrightarrow{D_\gamma \mathcal{P}} P'_\gamma v \in \mathcal{L}(H, H),$$
$$T_\gamma \Gamma \ni v \xrightarrow{D_\gamma \mathcal{Q}} Q'_\gamma v \in \mathcal{L}(H, H).$$

Lemma 9. *Let $\mathcal{A} = \mathcal{A}_\varepsilon(x)$ be a family of action functionals associated with an admissible family of Lagrangian functions $L = L_\varepsilon(\tau, x, y)$ with respect to the regular manifold Γ. Then there exist a constant $\varepsilon_1 > 0$ and a neighborhood $\mathcal{G}$ of Γ (that only depend on the bounds given in (21), ε_0, Γ and ϵ) such that for any $0 < \varepsilon < \varepsilon_1$ the action functional $\mathcal{A}_\varepsilon$ has, at least, cat(Γ) critical points $x_l \in \mathcal{G}$.*

Proof. Let $\mathcal{F} : \hat{\Gamma} \to H$ be a function defined on $\hat{\Gamma} = \Gamma \times H \times [0, \varepsilon_0[$ given by

$$\mathcal{F}_\varepsilon(\gamma, y) = P_\gamma \nabla \mathcal{A}_\varepsilon(\gamma + y) + Q_\gamma y. \tag{42}$$

By Lemma 7, Γ is a regular critical C^3-manifold of the action functional $\mathcal{A}_0$. Since the projections are twice differentiable, $\mathcal{F}$ is in the class $C^2(\hat{\Gamma})$. Also, it is clear that $\mathcal{F}_0(\gamma,0)=0$. The derivative $\delta_y\mathcal{F}:\hat{\Gamma}\to\mathcal{L}(H,H)$ at $(\gamma,0;0)\in\hat{\Gamma}$ is given by

$$\delta_y\mathcal{F}_0(\gamma,0)=P_\gamma\circ D^2\mathcal{A}_0(\gamma)+Q_\gamma.$$

We want to prove that the map $v\mapsto\delta_y\mathcal{F}_0(\gamma,0)v$ is an isomorphism of H. First, we will see that $\delta_y\mathcal{F}_0(\gamma,0)$ is injective. Let $v\in\ker\delta_y\mathcal{F}_0(\gamma,0)$. This implies that

$$\begin{aligned} D^2\mathcal{A}_0(\gamma)v &\in \ker P_\gamma,\\ v&\in N_\gamma\Gamma. \end{aligned}$$

Using Eq. (41) and the fact that Γ is a regular manifold of periodic solutions, we have

$$D^2\mathcal{A}_0(\gamma)v\in R\big(D^2\mathcal{A}_0(\gamma)\big)=\big[\ker\big(D^2\mathcal{A}_0(\gamma)\big)\big]^\perp=\big(T_\gamma\Gamma\big)^\perp=\ker Q_\gamma.$$

Since $\ker P_\gamma\cap\ker Q_\gamma=\{0\}$, we have that $v\in\ker\big(D^2\mathcal{A}_0(\gamma)\big)=T_\gamma\Gamma$. But $v\in N_\gamma\Gamma$, so $v=0$ and $\delta_y\mathcal{F}_0(\gamma,0)$ is an injective map.

To prove that $\delta_y\mathcal{F}_0(\gamma,0)$ is a surjective map, let $w\in H$ be an arbitrary vector. Then, there exist two vectors $w_1\in N_\gamma\Gamma$ and $w_2\in T_\gamma\Gamma$ such that $w=w_1+w_2$. Since $N_\gamma\Gamma=R(D^2\mathcal{A}_0(\gamma))$, there is a vector $v_1\in N_\gamma\Gamma$ such that $D^2\mathcal{A}_0(\gamma)v_1=w_1$. Letting $v=v_1+w_2$, we have that $\delta_y\mathcal{F}_0(\gamma,0)v=w$. Thus, $\delta_y\mathcal{F}_0(\gamma,0)$ is surjective, and hence, an isomorphism on H.

From now on, $\mathrm{Inv}(H)\subset\mathcal{L}(H,H)$ denotes the set of invertible bounded linear maps. Let us recall that the map $\gamma\in\Gamma\mapsto D^2\mathcal{A}_0(\gamma)\in\mathcal{L}(H,H)$ is continuous since $\mathcal{A}_0$ is in the class $C^2(\Gamma)$. Therefore, $\gamma\in\Gamma\mapsto\delta_y\mathcal{F}_0(\gamma,0)\in\mathrm{Inv}(H)$ is continuous by the continuity of the projections and the continuity of the composition of continuous maps. The map $\mathcal{I}:\mathrm{Inv}(H)\to\mathrm{Inv}(H)$ given by $\mathcal{I}(L)=L^{-1}$ is also a continuous map. Therefore, the composition

$$\gamma\in\Gamma\mapsto\big[\delta_y\mathcal{F}_0(\gamma,0)\big]^{-1}\in\mathrm{Inv}(H),$$

is a continuous map defined over a compact set Γ. This implies that there is a constant $C_1>0$ that only depends on Γ, A_0, B_0, and $\mathcal{U}_0$ such that

$$\Big\|\big[\delta_y\mathcal{F}_0(\gamma,0)\big]^{-1}\Big\|_{\mathcal{L}(H,H)}\le C_1,\quad \gamma\in\Gamma.$$

Now, we need to find uniform bounds of the derivatives

$$\begin{aligned} &\delta_\varepsilon\mathcal{F}:\hat{\Gamma}\to H,\quad \delta_\gamma\mathcal{F}:\hat{\Gamma}\to\mathcal{L}(H,H),\quad \delta^2_{\varepsilon y}\mathcal{F}:\hat{\Gamma}\to\mathcal{L}(H,H)\\ &\delta^2_{\gamma y}\mathcal{F}:\hat{\Gamma}\to\mathcal{L}(H,\mathcal{L}(H,H)),\quad \delta^2_{yy}\mathcal{F}:\hat{\Gamma}\to\mathcal{L}(H,\mathcal{L}(H,H)). \end{aligned}\tag{43}$$

In the following, the term $D^3\mathcal{A}_\varepsilon(x)[w]\in\mathcal{L}(H,H)$ denotes the variation of $D^2_\gamma\mathcal{A}_\varepsilon(x)$ in the direction of $w\in H$. The derivatives in (43) becomes

$$\begin{aligned} \delta_\varepsilon\mathcal{F}_\varepsilon(\gamma,y)&=P_\gamma\big(\nabla(\partial_\varepsilon A_\varepsilon)(\gamma+y)\big)\\ \delta_\gamma\mathcal{F}_\varepsilon(\gamma,y)&=P'_\gamma\big(\nabla\mathcal{A}_\varepsilon(\gamma+y)\big)+P_\gamma\circ D^2\mathcal{A}_\varepsilon(\gamma+y)+Q'_\gamma y,\\ \delta^2_{\varepsilon y}\mathcal{F}_\varepsilon(\gamma,y)&=P_\gamma\circ D^2(\partial_\varepsilon\mathcal{A}_\varepsilon)(\gamma+y)\\ \delta^2_{\gamma y}\mathcal{F}_\varepsilon(\gamma,y)[w]&=P'_\gamma\big(D^2\mathcal{A}_\varepsilon(\gamma+y)w\big)+P_\gamma\circ D^3\mathcal{A}_\varepsilon(\gamma+y)[w]+Q'_\gamma w,\\ \delta^2_{yy}\mathcal{F}_\varepsilon(\gamma,y)[w]&=P_\gamma\circ D^3_\gamma\mathcal{A}_\varepsilon(\gamma+y)[w]. \end{aligned}\tag{44}$$

The previous derivatives are related with the derivatives of L and hence, with the derivatives of A, B and $\mathcal{U}$. Thus, they are uniformly bounded by a constant $C_2 > 0$ that only depends on the bound C given in (21), ε_0, Γ, and ϵ.

Applying Lemma 8, for each $\gamma_0 \in \Gamma$, there exist three positive numbers R_{γ_0}, r_{γ_0} and ε_{γ_0} that only depend on C_1, C_2 and there exists a function $Y_{\gamma_0} : B_H(\gamma_0, R_{\gamma_0}) \times [0, \varepsilon_{\gamma_0}[\to B_H(0, r_{\gamma_0})$ in the class $C^2(B_H(\gamma_0, R_{\gamma_0}) \times [0, \varepsilon_{\gamma_0}[)$ satisfying $Y_{\gamma_0}(\gamma_0, 0) = 0$ and Y_{γ_0} is the unique solution of

$$P_\gamma \nabla \mathcal{A}_\varepsilon(\gamma + Y_{\gamma_0}(\gamma; \varepsilon)) + Q_\gamma Y_{\gamma_0}(\gamma; \varepsilon) = 0, \qquad (\gamma; \varepsilon) \in B_H(\gamma_0, R_{\gamma_0}) \times [0, \varepsilon_{\gamma_0}[.$$

Using the uniqueness of Y_{γ_0} for each $\gamma_0 \in \Gamma$ and the compactness of Γ, we can construct a neighborhood $\mathcal{G}$ of Γ and a function $Y : \Gamma \times [0, \varepsilon_1[\to H$, where ε_1 is a positive constant, $\gamma + Y(\gamma; \varepsilon) \in \mathcal{G}$ for every $(\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[$ and Y is the unique function in the class $C^2(\Gamma \times [0, \varepsilon_1[)$ that satisfies $Y(\gamma; 0) = 0$, and

$$P_\gamma \nabla \mathcal{A}_\varepsilon(\gamma + Y(\gamma; \varepsilon)) + Q_\gamma Y(\gamma; \varepsilon) = 0, \qquad (\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[. \tag{45}$$

Both $\mathcal{G}$ and ε_1 only depend on the bounds given in (21), ε_0 and Γ.

Now, we can define the function $\mathcal{B} : \Gamma \times [0, \varepsilon_1[\to \mathbb{R}$ in the class $C^1(\Gamma \times [0, \varepsilon_1[)$ given by

$$\mathcal{B}_\varepsilon(\gamma) = \mathcal{A}_\varepsilon(\gamma + Y(\gamma; \varepsilon)).$$

According to Lemma 10.13 from [16], if $\nabla \mathcal{B}_\varepsilon(\gamma) = 0$, then $\nabla \mathcal{A}_\varepsilon(\gamma + Y(\gamma; \varepsilon)) = 0$. So, we only need to find critical points of $\mathcal{B}_\varepsilon$.

Since $\mathcal{B}_\varepsilon$ is a function in the class $C^1(\Gamma)$ and Γ is a compact set, the function $\mathcal{B}_\varepsilon$ has at least $n = \mathrm{cat}(\Gamma)$ critical points, denoted by $\chi_l \in \Gamma$, $\chi_l = \chi_l(\tau; \varepsilon)$, $(l = 1, \ldots, n)$. Using the equivalence between critical points of $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$, the action functional $\mathcal{A}_\varepsilon$ has $\mathrm{cat}(\Gamma)$ critical points $x_l \in \mathcal{G}$ of the form

$$x_l(\tau; \varepsilon) = \chi_l(\tau; \varepsilon) + Y(\chi_l; \varepsilon)(\tau), \quad l = 1, \ldots, n\ , \tag{46}$$

and the proof is complete. □

Remark 4. *In the previous lemma, we need the action functional to be in the class $C^3(H)$. According to [1], this is valid if and only if the associated Lagrangian system is of the form* (16)*.*

Remark 5. *The fact that $D^2\mathcal{A}_0(\gamma)$ is a Fredholm operator of index 0 is only needed to be able to use Fredholm alternative. Here, the action functional $\mathcal{A}_0$ is in the class $C^2(\Gamma)$. This implies that $D^2\mathcal{A}_0(\gamma)$ is a symmetric linear operator. Therefore, Point (iii) of Definition 3 can be replaced by the more general condition: The map $D^2\mathcal{A}_0(\gamma)$ is a closed linear operator and its kernel is finite-dimensional.*

Proof of Theorem 2. Let $\mathcal{A} : H \times [0, \varepsilon_0[\to \mathbb{R}$ be the action functional given by (32). Applying Lemma 9, there exists $\varepsilon_1' < \varepsilon_0$ that only depends on the bounds given in (21), U, ε_0, Γ and ϵ such that, if $\varepsilon \in\ ]0, \varepsilon_1'[$, there are $n = \mathrm{cat}(\Gamma)$ critical point $x_l \in H$, $(l = 1, \ldots, n)$ of $\mathcal{A}_\varepsilon$. In addition, $x_l' \in H$ (see Eq. (35)) and x_l are a critical solution of L_ε. This critical points are of the form given in (46). Let us recall that the function Y satisfies Eq. (45).

On the other hand, let us recall that the map $\mathcal{I} : \mathrm{Inv}(H) \to \mathrm{Inv}(H)$ given by $\mathcal{I}(L) = L^{-1}$ is continuous. We know that the compact set $K = \{\delta_y \mathcal{F}_0(\gamma, 0) : \gamma \in \Gamma\}$ satisfies $K \subset \mathrm{Inv}(H)$ (see the proof of Lemma 9). By the continuous dependence of the spectrum of an operator with respect to parameters, we can find a bounded closed neighborhood $\tilde{K} \subset \mathrm{Inv}(H)$ of K that only depends on ε_0, U, Γ, C and ϵ such that $\mathcal{I}(\tilde{K}) \subset \mathrm{Inv}(H)$. Moreover, since $\mathcal{I}$ maps bounded closed sets into bounded sets, there is a constant C_2 that only depends on $\tilde{K}$ such that $\|L^{-1}\|_{\mathcal{L}(H,H)} \leq C_2$ for every $L \in \tilde{K}$. Since $\mathcal{F}$ is in the class C^2, the map $\varepsilon \mapsto \delta_y \mathcal{F}_\varepsilon(\gamma, Y(\gamma; \varepsilon))$ is continuous for every $\gamma \in \Gamma$. Therefore, there is a constant $\varepsilon_1 < \varepsilon_1'$ such that $\delta_y \mathcal{F}_\varepsilon(\gamma, Y(\gamma; \varepsilon)) \in \tilde{K}$ if $(\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[$ and

$$\left\| \left[\delta_y \mathcal{F}_\varepsilon(\gamma, Y(\gamma; \varepsilon))\right]^{-1} \right\|_{\mathcal{L}(H,H)} \leq C_2, \qquad (\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[.$$

Using implicit derivation in (45), the first derivative $\partial_\varepsilon Y$ becomes

$$\partial_\varepsilon Y(\gamma; \varepsilon) = -\left[\delta_y \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma; \varepsilon)\big)\right]^{-1} \delta_\varepsilon \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma; \varepsilon)\big), \qquad (\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[.$$

The term $\partial_\varepsilon \mathcal{F}_\varepsilon(\gamma, y)$ is uniformly bounded (see (44)). The term $[\delta_y \mathcal{F}_\varepsilon(\gamma, y)]^{-1}$ is uniformly bounded, by Remark 3. Thus, the first derivative $\partial_\varepsilon Y(\gamma; \varepsilon)$ is uniformly bounded. Also, using the second equation of (44) we have

$$\delta_\varepsilon \mathcal{F}_0(\gamma, 0) = P_\gamma \left(\nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)(\gamma)\Big|_{\varepsilon=0} \right).$$

We can compute $\nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)$ from (36), obtaining

$$\begin{aligned}\langle \nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)(\gamma), v \rangle_H = \int_0^{2\pi} \Bigg[&\left\langle \frac{\partial}{\partial x} \left(\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau, \gamma(\tau), \gamma'(\tau)) \right), v(\tau) \right\rangle + \\ &\left\langle \frac{\partial}{\partial y} \left(\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau, \gamma(\tau), \gamma'(\tau)) \right), v'(\tau) \right\rangle \Bigg] \, \mathrm{d}\tau.\end{aligned}$$

Using Point (ii) of Definition 2, we have

$$\left\langle \nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)(\gamma)\Big|_{\varepsilon=0}, v \right\rangle_H = 0,$$

for every $v \in H$. Therefore, $\delta_\varepsilon \mathcal{F}_0(\gamma, 0) = 0$ and

$$\partial_\varepsilon Y(\gamma; 0) = 0, \qquad \gamma \in \Gamma.$$

Proceeding in a similar way, we can prove that the second derivative is given by

$$\begin{aligned}\partial^2_{\varepsilon\varepsilon} Y(\gamma; \varepsilon) = -\left[\delta_y \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma; \varepsilon)\big)\right]^{-1} \Big\{ &\delta^2_{\varepsilon\varepsilon} \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma; \varepsilon)\big) + 2\delta^2_{\varepsilon y} \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma; \varepsilon)\big) \partial_\varepsilon Y(\gamma; \varepsilon) + \\ &\delta^2_{yy} \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma; \varepsilon)\big) \left[\partial_\varepsilon Y(\gamma; \varepsilon), \partial_\varepsilon Y(\gamma; \varepsilon)\right] \Big\}, \qquad (\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[.\end{aligned}$$

The term $\delta^2_{\varepsilon\varepsilon}\mathcal{F}_\varepsilon(\gamma, y)$ is given by

$$\delta^2_{\varepsilon\varepsilon}\mathcal{F}_\varepsilon\big(\gamma, y\big) = P_\gamma\big(\nabla(\partial^2_\varepsilon\mathcal{A}_\varepsilon)(\gamma + y)\big)$$

and it is uniformly bounded by the bounds given in (33) and ϵ (the dependence on ϵ is because we use (17) to write the differential equation that satisfies $\nabla(\partial^2_\varepsilon\mathcal{A}_\varepsilon)(\gamma + y)$ in its normal form). The other terms in the previous equation are also uniformly bounded (see (44) and Remark 3). Therefore, the second derivative $\partial^2_{\varepsilon\varepsilon}Y$ is uniformly bounded. Using the Taylor expansion for Y, there is a constant c that only depends on ε_0, U, Γ and ϵ such that

$$\|Y(\gamma;\varepsilon)\|_H \leq c\varepsilon^2, \qquad (\gamma;\varepsilon) \in \Gamma \times [0, \varepsilon_1[.$$

Finally, the distance between the solutions x_l and Γ satisfies

$$\mathrm{dist}_H(x_l, \Gamma) \leq \|x_l(\cdot;\varepsilon) - \chi_l(\cdot;\varepsilon)\|_H = \|Y(x_l;\varepsilon)\|_H \leq c\varepsilon^2, \qquad l = 1, \ldots, n$$

and the proof is complete. □

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