

Regularized variational principles for the perturbed Kepler problem

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Abstract

The goal of the paper is to develop a method that will combine the use of variational techniques with regularization methods in order to study existence and multiplicity results for the periodic and the Dirichlet problem associated to the perturbed Kepler system

$$\ddot{x} = -\frac{x}{|x|^3} + p(t), \quad x \in \mathbb{R}^d,$$

where $d \geq 1$, and $p : \mathbb{R} \rightarrow \mathbb{R}^d$ is smooth and T -periodic, $T > 0$.

The existence of critical points for the action functional associated to the problem is proved via a non-local change of variables inspired by Levi-Civita and Kustaanheimo-Stiefel techniques. As an application we will prove that the perturbed Kepler problem has infinitely many generalized T -periodic solutions for $d = 2$ and $d = 3$, without any symmetry assumptions on p .

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1 Introduction

Consider the perturbed Kepler problem

$$\ddot{x} = -\frac{x}{|x|^3} + p(t), \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $d \geq 1$, and $p : \mathbb{R} \rightarrow \mathbb{R}^d$ is smooth and T -periodic, $T > 0$.

Recently several papers have examined the existence of generalized T -periodic solutions ([14, 17, 23] for $d = 1$, [7, 6] for $d = 2, 3$), according to the following definition.

Definition 1.1. A *generalized solution* of (1.1) on the interval $J \subset \mathbb{R}$ is a continuous function $x : J \rightarrow \mathbb{R}^d$ satisfying the following conditions:

1. $Z \cup \partial J$ is discrete, where $Z = \{t \in J : x(t) = 0\}$ is the set of *collisions* and ∂J is the boundary of J ,
2. for any open interval $I \subset J \setminus Z$, the function x is $C^2(I)$ and satisfies (1.1) on I ,
3. for any $t_* \in Z$, the limits

$$\lim_{t \rightarrow t_*} \frac{x(t)}{|x(t)|} \quad \text{and} \quad \lim_{t \rightarrow t_*} \left(\frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{|x(t)|} \right) \tag{1.2}$$

of collision direction and collision energy exist and are finite.

The existence of the right and left limits in 3. of Definition 1.1 is a consequence of [18] whenever t_* is isolated in the collision set Z ; therefore, in case of isolated collisions, the only thing to check is that left/right limits agree. In particular if $t_* \in Z \cap \partial J$ (e.g. in the Dirichlet problem) we just have to check that they are isolated collision instants.

We refer to [7] for a discussion on the significance of these solutions. Using the Poincaré-Birkhoff theorem it was proven in [14, 17] that the equation (1.1) in one dimension has infinitely many generalized T -periodic solutions. For $d \geq 2$ the knowledge of the periodic problem for (1.1) is more fragmentary. Authors in [6] proved the existence of at least one periodic solution if $d = 2$, while in [7] the existence of any number of periodic solutions has been proved when $d = 2, 3$ and p is small enough. The approach employed in [7] and [6] are quite different. The basic idea in [7] is to regularize the system (1.1) by a change of variables from $x = x(t)$ to $y = y(\tau)$, where

$$\tau = \int_0^t \frac{d\xi}{|x(\xi)|}, \quad x = \Phi(y). \quad (1.3)$$

Here Φ is an appropriate homogeneous function of degree 2 (different choices are available, depending on the spatial dimension d). The definition of τ comes from the well known Sundman integral [20]. In the variables (τ, y) the system has no singularities, and it is possible to apply some classical bifurcation results due to Weinstein. The approach in [6] is variational, searching for critical points of the action functional

$$\mathcal{A}(x) = \int_0^T \left[\frac{1}{2} |\dot{x}(t)|^2 + \frac{1}{|x(t)|} + \langle p(t), x(t) \rangle \right] dt. \quad (1.4)$$

They prove the existence of critical points by minimization among loops around the origin having non-trivial winding number, but they also prove that these critical points do produce generalized solutions. Incidentally, in the literature the variational framework have been exploited also in connection with different notions of periodic generalized solutions, see e.g. [4, 2, 21, 15, 10]; anyhow, most of these papers deal with autonomous singular Hamiltonian systems, and their (weaker) notion of generalized solutions requires only conservation of energy across collisions (i.e. only the second condition in (1.2)).

Our goal in this paper is to develop a method that will combine the use of variational techniques with the regularization method. In Section 2, after some heuristic computations, we will show that the change of variables (1.3) transforms the functional \mathcal{A} into a new functional $\mathcal{B} = \mathcal{B}(y)$. Although local changes of variables are traditional in Calculus of Variation (see for instance [13]), the change (1.3) is not local and the new functional will not be in a standard class. More precisely, \mathcal{B} takes the form

$$\mathcal{B}(y) = \int_0^1 \beta \left(\tau, y(\tau), y'(\tau), \int_0^\tau |y(\xi)|^2 d\xi, \|y\|_2 \right) d\tau,$$

where $\|\cdot\|_2$ is the norm in $L^2((0, 1); \mathbb{R}^d)$. Furthermore, the transformation (1.3) does not induce a diffeomorphism between the natural classes of functions, and the consistency between (1.1), \mathcal{A} and \mathcal{B} becomes subtle. We will discuss such consistency in Section 3, establishing some regularized variational principles. As a rule of thumb, critical points of \mathcal{B} correspond to generalized solutions of (1.1) only imposing further conditions. In Section 3.4 we show that such conditions are sharp, providing several counterexamples.

Once the variational principles are established, the rest of the paper is devoted to applications, for either Dirichlet or periodic boundary conditions. First, in Section 4, we consider the minimization \mathcal{B} in the Sobolev space $H_0^1(0, 1; \mathbb{R}^d)$. This will lead to the existence of a generalized solution of (1.1) satisfying the Dirichlet boundary conditions $x(0) = x(T) = 0$. This can be done for any dimension $d \geq 1$.

In our second application, illustrated in Section 5, we assume $d = 2$ and adapt the definition of Φ in (1.3) to the classical Levi-Civita change of variables. The consequence is the discovery of a hidden

symmetry: the functional $\mathcal{B} = \mathcal{B}_{LC}$ is even. When we look for T -periodic solutions of (1.1), there are two possible choices for the domain of \mathcal{B}_{LC} . We can consider the Hilbert spaces W_1 and W_{-1} , where

$$W_{\pm 1} := \{z \in H^1(0, 1; \mathbb{C}) : z(1) = \pm z(0)\}.$$

The most convenient choice is W_{-1} because in this case \mathcal{B}_{LC} satisfies the Palais-Smale condition. Classical minimax theory for even functionals implies that (1.1) has infinitely many generalized T -periodic solutions in dimension $d = 2$. It is worthwhile noticing that anti-periodic functions z produce periodic solutions x , without imposing any symmetry condition on the forcing p .

As we show in Section 6, an analogous conclusion is also valid in dimension $d = 3$ but the proof is more delicate. The definition of Φ is now inspired by the Kustaanheimo-Stiefel change of variables. The consistency of the periodic problem with $\mathcal{B} = \mathcal{B}_{KS}$ requires a more sophisticated domain, a symmetric Hilbert manifold \mathcal{M} that will be defined later. Again \mathcal{B}_{KS} is even but the Palais-Smale condition does not hold. Nonetheless, the more flexible Cerami condition is satisfied, and the existence of infinitely many critical points of \mathcal{B}_{KS} over \mathcal{M} can be proved.

We already mentioned the different notions of generalized solution that have been introduced in the literature on variational methods. The classical theory of holomorphic differential equations and the modern theory of dynamical systems have also led to other notions of generalized solutions (see [12] for more information). The notion employed in the present paper is in perfect correspondence with the regularization theories by Levi-Civita and Kustaanheimo-Stiefel, but it has the advantage of having an intrinsic formulation. The possible relevance of this notion of solution for non-Newtonian potentials is a question to be analyzed.

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Notation

- $|\cdot|$ is the euclidean norm in \mathbb{R}^d and, sometimes, the Lebesgue measure of a set.

- $\mathbb{S}^d := \{x \in \mathbb{R}^d : |x| = 1\}$.
- $\langle \cdot, \cdot \rangle$ is the euclidean scalar product in \mathbb{R}^d .
- $\|\cdot\|_p = \|\cdot\|_{L^p(0,1;\mathbb{R}^k)}$, with $k = 1$ or d .
- $a_n = O(b_n)$ (as $n \rightarrow +\infty$) if, for some N, C , $n \geq N \implies |a_n| \leq C|b_n|$.

2 Heuristic derivation of a class of regularized functionals

Let us consider the *action functional*, associated to the perturbed Kepler problem (1.1),

$$\mathcal{A}(x) = \int_0^T \left[\frac{1}{2} |\dot{x}(t)|^2 + \frac{1}{|x(t)|} + \langle p(t), x(t) \rangle \right] dt,$$

which is (well defined and) finite in

$$\mathcal{X} := \left\{ x \in H^1(0, T; \mathbb{R}^d) : \int_0^T \frac{dt}{|x(t)|} < \infty \right\}.$$

We want to exploit some changes of variables (both for t and x) in such a way that the action functional in the new variables is *regularized*, i.e. it does not contain a singular potential term. The different choices for such changes of variables are inspired by classical regularization techniques for the Kepler problem (Sundman, Levi-Civita, Kustaanheimo-Stiefel). Accordingly, we will obtain different functionals.

As far as the time scale is concerned, the common change of variables we use is that introduced by Sundman [20]. For any $x \in \mathcal{X}$ let L_x be the corresponding (strictly positive and finite) quantity

$$L_x := \int_0^T \frac{dt}{|x(t)|},$$

and let us define

$$\tau = \tau(t) := \frac{1}{L_x} \int_0^t \frac{d\xi}{|x(\xi)|}, \quad t \in [0, T]. \quad (2.1)$$

In order to enlighten the properties of τ , we use the following result.

Lemma 2.1. *Let $a : [0, T] \rightarrow \mathbb{R}$ be a continuous and non-negative function such that*

$$\frac{1}{a} \in L^1(0, T).$$

Define

$$A(t) = \int_0^t \frac{d\xi}{a(\xi)}, \quad t \in [0, T].$$

Then A is a homeomorphism between $[0, T]$ and $[0, \Xi]$, with

$$\Xi = \int_0^T \frac{d\xi}{a(\xi)}.$$

Moreover, the inverse function $B = B(\tau)$ belongs to $C^1([0, \Xi])$ and satisfies

$$B'(\tau) = a(B(\tau)) \quad \text{for each } \tau \in [0, \Xi].$$

We postpone the proof of the above lemma to Appendix A.1. The application of such lemma to $a(t) = L_x|x(t)|$ implies that (2.1) is invertible, and that the inverse function $t = t(\tau)$ is of class $C^1([0, 1])$, with

$$t'(\tau) = L_x|x(t(\tau))|, \quad \text{for any } \tau \in [0, 1].$$

Let now $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a map such that:

$$(\Phi 1) \quad \Phi|_{\mathbb{S}^{d-1}} \text{ is smooth and } \Phi(\mathbb{S}^{d-1}) \subseteq \mathbb{S}^{d-1},$$

$$(\Phi 2) \quad \Phi(\lambda y) = \lambda^2 \Phi(y) \quad \forall y \in \mathbb{R}^d \text{ and } \lambda > 0.$$

Notice that, for every $y \neq 0$,

$$|\Phi(y)| = |y|^2 \left| \Phi\left(\frac{y}{|y|}\right) \right| = |y|^2. \quad (2.2)$$

Let $x \in \mathcal{X}$, τ defined as above, and let us assume at this point that we can find $y : [0, 1] \rightarrow \mathbb{R}^d$ such that

$$x(t) = \Phi(y(\tau(t))). \quad (2.3)$$

Under such assumption we want to write $\mathcal{A}(x)$ in terms of y . To start with we remark that, by (2.2),

$$\dot{t}(t) = \frac{1}{L_x |y(\tau(t))|^2}, \quad \frac{1}{L_x} = |y(\tau(t))|^2 \dot{t}(t) \quad \text{hence} \quad \frac{T}{L_x} = \int_0^1 |y(\tau)|^2 d\tau.$$

Then we can write

$$L_x = \frac{T}{\|y\|_{L^2(0,1)}^2} =: \mathcal{L}(y) \quad \text{and} \quad t_y = t_y(\tau) = \mathcal{L}(y) \int_0^\tau |y(\xi)|^2 d\xi.$$

Since, at least formally,

$$\dot{x}(t) = D\Phi(y(\tau(t)))y'(\tau(t))\dot{t}(t),$$

the kinetic part transforms into

$$\int_0^T \frac{1}{2} |\dot{x}(t)|^2 dt = \frac{1}{\mathcal{L}(y)} \int_0^1 \frac{1}{2} \frac{\langle [D\Phi(y(\tau))]^T D\Phi(y(\tau))y'(\tau), y'(\tau) \rangle}{|y(\tau)|^2} d\tau =: \frac{1}{\mathcal{L}(y)} \mathcal{Q}(y).$$

On the other hand,

$$\int_0^T \frac{dt}{|x(t)|} = \mathcal{L}(y)$$

and

$$\int_0^T \langle p(t), x(t) \rangle dt = \mathcal{L}(y) \int_0^1 |y(\tau)|^2 \langle p(t_y(\tau)), \Phi(y(\tau)) \rangle d\tau =: \mathcal{L}(y) \mathcal{R}(y).$$

Resuming we have that, at least formally, (2.3) implies $\mathcal{A}(x) = \mathcal{B}(y)$, where

$$\mathcal{B}(y) := \frac{1}{\mathcal{L}(y)} \mathcal{Q}(y) + \mathcal{L}(y) [1 + \mathcal{R}(y)]. \quad (2.4)$$

We notice that, in the functional above, only \mathcal{Q} and \mathcal{R} depend on the actual choice of Φ .

As we mentioned, we will deal with three different choices of Φ .

Case 1 (the functional \mathcal{B} in any d). The more direct choice for Φ , which works in any dimension, consists in taking $\Phi|_{\mathbb{S}^{d-1}}$ to be the identity on the sphere. Then $(\Phi 1)$ is obvious, and $(\Phi 2)$ forces

$$\Phi(y) = |y|y.$$

Then $D\Phi(y)$ is symmetric and

$$[D\Phi(y)]^T D\Phi(y) = \left(\frac{yy^T}{|y|} + |y|\text{Id} \right)^2 = 3yy^T + |y|^2\text{Id}$$

(indeed $(yy^T)^2 = |y|^2 yy^T$). We obtain that \mathcal{B} is as in (2.4), with

$$\begin{aligned} Q(y) &= \frac{1}{2} \int_0^1 \left(3 \frac{\langle y, y' \rangle^2}{|y|^2} + |y'|^2 \right) d\tau, \\ \mathcal{R}(y) &= \int_0^1 |y|^3 \langle p \circ t_y, y \rangle d\tau. \end{aligned}$$

We recall that, given any $y \in H^1(0, 1; \mathbb{R}^d)$, also $|y| \in H^1(0, 1; \mathbb{R})$ and its (weak) derivative writes

$$|y|' = \begin{cases} \frac{\langle y, y' \rangle}{|y|}, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases} \quad (2.5)$$

Then we can write

$$Q(y) := \frac{1}{2} \int_0^1 [3(|y|')^2 + |y'|^2] d\tau,$$

so that \mathcal{B} is well defined in $H^1(0, 1; \mathbb{R}^d) \setminus \{0\}$ (otherwise $\mathcal{L}(y)$ is not defined). Actually, it is possible to read \mathcal{B} as an extended valued functional on $H^1(0, 1; \mathbb{R}^d)$ by choosing $\mathcal{B}(0) = +\infty$.

Case 2 (The Levi-Civita regularization in $d = 2$ and the functional \mathcal{B}_{LC}). In dimension $d = 2$ we can exploit the complex structure of the plane and define

$$\Phi(z) = \Phi_{\text{LC}}(z) = z^2 \sim \begin{pmatrix} z_1^2 - z_2^2 \\ 2z_1 z_2 \end{pmatrix}, \quad z \in \mathbb{C} \cong \mathbb{R}^2.$$

It is immediate to check that Φ_{LC} verifies $(\Phi 1)$ and $(\Phi 2)$, and that

$$[D\Phi_{\text{LC}}(z)]^T D\Phi_{\text{LC}}(z) = 4 \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix} \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix} = 4|z|^2 \text{Id}.$$

Writing $z(\tau)$ instead of $y(\tau)$ we obtain that in this case the functional \mathcal{B} writes

$$\mathcal{B}_{\text{LC}}(z) = \frac{2}{T} \int_0^1 |z|^2 d\tau \int_0^1 |z'|^2 d\tau + \frac{T}{\int_0^1 |z|^2 d\tau} \left[1 + \int_0^1 |z|^2 \langle p \circ t_z, z^2 \rangle d\tau \right].$$

Also \mathcal{B}_{LC} is well defined in $H^1(0, 1; \mathbb{C}) \setminus \{0\}$, and it can be extended as $\mathcal{B}_{\text{LC}}(0) = +\infty$.

Case 3 (The Kustaanheimo-Stiefel regularization in $d = 3$ and the functional \mathcal{B}_{KS}). As it is well known, the regularization in the three-dimensional case is more involved, as it requires to consider $\Phi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, induced by the quaternionic structure. To this aim, following [22], we denote with \mathbb{H} the skew-field of quaternions, and with \mathbb{IH} the subset of purely imaginary quaternions:

$$\mathbb{H} := \{z = z_0 + z_1i + z_2j + z_3k : (z_0, z_1, z_2, z_3) \in \mathbb{R}^4\}, \quad \mathbb{IH} := \{z \in \mathbb{H} : \Re(z) = 0\},$$

where the real part of a quaternion is defined as $\Re(z_0 + z_1i + z_2j + z_3k) = z_0$. Notice that, in a trivial way, both $\mathbb{H} \cong \mathbb{R}^4$ and $\mathbb{IH} \cong \mathbb{R}^3$, in the sense of (real) vector spaces. Defining the conjugate of a quaternion as $\bar{z} := 2\Re(z) - z$, direct computations show that, for any $z \in \mathbb{H}$,

$$\bar{z}iz = (z_0^2 + z_1^2 - z_2^2 - z_3^2)i + 2(z_1z_2 - z_0z_3)j + 2(z_1z_3 + z_0z_2)k \in \mathbb{IH}.$$

Then we define $\Phi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ as

$$\Phi(z) = \Phi_{\text{KS}}(z) = \bar{z}iz \sim \begin{pmatrix} z_0^2 + z_1^2 - z_2^2 - z_3^2 \\ 2(z_1z_2 - z_0z_3) \\ 2(z_1z_3 + z_0z_2) \end{pmatrix}, \quad z \in \mathbb{H} \cong \mathbb{R}^4.$$

We observe that Φ_{KS} is invariant under the following action of \mathbb{S}^1 :

$$\Phi_{\text{KS}}(e^{i\vartheta}z) = \Phi_{\text{KS}}(z), \quad \text{for every } \vartheta \in \mathbb{R}, z \in \mathbb{H}.$$

Since $|\bar{z}iz| = |z|^2$, this implies that $\Phi_{\text{KS}} : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ induces the Hopf fibration (for more details see [11]). Although Φ_{KS} is not exactly in the previous framework (\mathbb{H} and \mathbb{IH} have different dimension), Φ_{KS} is again homogeneous of degree 2 and (Φ2) and (2.2) make sense. Now, assume that

$$x(t) = \Phi_{\text{KS}}(z(\tau(t))). \quad (2.6)$$

Then

$$\langle [D\Phi_{\text{KS}}(z)]^T D\Phi_{\text{KS}}(z)z', z' \rangle = |\bar{z}'iz + \bar{z}iz'|^2,$$

and

$$\int_0^T \frac{1}{2} |\dot{x}(t)|^2 dt = \frac{1}{\mathcal{L}(z)} \int_0^1 \frac{1}{2} \frac{|\bar{z}'iz + \bar{z}iz'|^2}{|z|^2} d\tau.$$

Motivated by the role of the bilinear form $(z, w) \mapsto \Re(\bar{w}iz)$ in Kustaanheimo-Stiefel regularization, we assume that $z(\tau)$ satisfies the further condition

$$\Re(\bar{z}'iz) = -\Re(\bar{z}iz') = \langle z', iz \rangle = -\langle iz', z \rangle = 0 \quad \text{for every } \tau, \quad (2.7)$$

then $|\bar{z}'iz + \bar{z}iz'| = 2|\bar{z}'iz| = 2|z'| |z|$. As a consequence, with similar calculations as in the previous cases, we obtain that (2.6) and (2.7) imply $\mathcal{A}(x) = \mathcal{B}_{\text{KS}}(z)$, where

$$\mathcal{B}_{\text{KS}}(z) = \frac{2}{T} \int_0^1 |z|^2 d\tau \int_0^1 |z'|^2 d\tau + \frac{T}{\int_0^1 |z|^2 d\tau} \left[1 + \int_0^1 |z|^2 \langle p \circ t_z, \bar{z}iz \rangle d\tau \right].$$

Notice that, since $\bar{z}iz \in \mathbb{IH}$, in the expression of \mathcal{B}_{KS} also $\mathbb{R}^3 \ni p = (p_1, p_2, p_3)$ can be interpreted as an element of \mathbb{IH} (even though any other choice of p_0 has no effect on the functional).

Remark 2.2 (On the connection between \mathcal{B}_{LC} and \mathcal{B}_{KS} when p takes values in \mathbb{R}^2). Let $\Pi \subset \mathbb{H}$ be a Levi-Civita plane. This means that $\Re(\bar{w}iz) = 0$ for every $w, z \in \Pi$ (Π is indeed a Lagrangian plane with respect to the corresponding symplectic structure). Let $\{r_1, r_2\}$ be an orthonormal basis of Π . Define

$$\hat{r}_1 = \Phi_{\text{KS}}(r_1) = \bar{r}_1 i r_1, \quad \hat{r}_2 = \bar{r}_1 i r_2.$$

Then $\hat{r}_1, \hat{r}_2 \in \mathbb{IH}$ are linearly independent unit vectors (see [22, Lemma 3.4]).

Given $c_1, c_2 \in \mathbb{R}$ and $z = c_1 r_1 + c_2 r_2 \in \Pi$, it turns out that $\Phi_{\text{KS}}(z) = (c_1^2 - c_2^2)\hat{r}_1 + 2c_1 c_2 \hat{r}_2$. Let $\hat{\Pi}$ be the plane spanned by \hat{r}_1, \hat{r}_2 . Then $\Phi_{\text{KS}}(\Pi) = \hat{\Pi}$. Define the isomorphisms of real vector spaces

$$\begin{aligned} \varphi : \mathbb{C} &\rightarrow \Pi, & \varphi(1) &= r_1, \quad \varphi(i) = r_2, \\ \psi : \hat{\Pi} &\rightarrow \mathbb{C}, & \psi(\hat{r}_1) &= 1, \quad \psi(\hat{r}_2) = i. \end{aligned}$$

Then $\psi \circ \Phi_{\text{KS}} \circ \varphi(z) = z^2 = \Phi_{\text{LC}}(z)$.

Let $\psi^*: \mathbb{C} \rightarrow \hat{\Pi}$ be the adjoint of ψ :

$$\langle \psi(p), z \rangle_{\mathbb{C}} = \langle p, \psi^*(z) \rangle_{\mathbb{H}} \quad \text{if } z \in \mathbb{C}, p \in \hat{\Pi}.$$

Let us now assume that $p: [0, T] \rightarrow \mathbb{C}$, $p = p(t)$, and let us consider $p^*: [0, T] \rightarrow \hat{\Pi} \subset \mathbb{H}$, $p^* = \psi^* \circ p$. Consider the functionals \mathcal{B}_{KS} , associated to $p^*(t)$, and \mathcal{B}_{LC} , associated to $p(t)$. Then, given $z \in H^1(0, 1; \mathbb{C})$, $\varphi \circ z \in H^1(0, 1; \Pi)$, for some time interval I , we have that

$$\mathcal{B}_{\text{KS}, p^*}(\varphi \circ z) = \mathcal{B}_{\text{LC}, p}(z).$$

Hence \mathcal{B}_{LC} can be seen as \mathcal{B}_{KS} restricted to $H^1(0, 1; \Pi)$.

3 Critical points of the regularized functionals and generalized solutions of (1.1)

In this section we deal with critical points of the functionals defined in Section 2, and with their relations with generalized solutions of the perturbed Kepler problem (1.1).

Throughout the section we assume that p is a function of class C^1 . As we will see, this will imply that the term \mathcal{R} is of class C^1 in each regularized functional. On the contrary, this is not true for the term \mathcal{Q} : as a matter of fact, the main difference between \mathcal{B} and \mathcal{B}_{LC} , \mathcal{B}_{KS} , is that, while the latter are differentiable at any non identically zero function, the former needs not to be Gateaux-differentiable at $y \in H^1(0, 1; \mathbb{R}^d)$, whenever y vanishes at some point.

Along this section, we do not take into account boundary conditions. For this reason, we consider points y in H^1 , which are critical with respect to smooth variations φ , compactly supported in $(0, 1)$ (or outside the collision set of y , for the functional \mathcal{B}). Of course, to impose boundary conditions, one has to choose critical points y in a suitable subspace of H^1 , and/or variations φ in a suitable space containing $\mathcal{D} = C_0^\infty$. This will be done, case by case, in the subsequent sections.

For each functional, once the Euler-Lagrange equations are derived, we analyze when their solutions actually correspond to solutions of the perturbed Kepler problem. As we will see, both for \mathcal{B} and for \mathcal{B}_{KS} this will require further conditions. In the last subsection we are going to show that such conditions are sharp, providing some counterexamples.

3.1 Euler-Lagrange equations for \mathcal{B} outside collisions

First we choose $\Phi(y) = |y|y$, see Case 1 in the previous section, and we deal with the functional

$$\mathcal{B}: H^1(0, 1; \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$\mathcal{B}(y) := \frac{1}{\mathcal{L}(y)} \mathcal{Q}(y) + \mathcal{L}(y) \left[1 + \mathcal{R}(y) \right], \quad \mathcal{B}(0) := +\infty, \quad (3.1)$$

where

$$\begin{aligned}
\mathcal{L}(y) &:= \frac{T}{\int_0^1 |y|^2 d\tau}, \\
\mathcal{Q}(y) &:= \frac{1}{2} \int_0^1 [3(|y|')^2 + |y'|^2] d\tau, \\
\mathcal{R}(y) &:= \int_0^1 |y|^3 \langle p \circ t_y, y \rangle d\tau, \\
t_y(\tau) &:= \mathcal{L}(y) \int_0^\tau |y(\xi)|^2 d\xi.
\end{aligned} \tag{3.2}$$

As we will see, in case $y(\bar{\tau}) = 0$ and $\varphi \in \mathcal{D}(0, 1; \mathbb{R}^d)$, $\varphi(\bar{\tau}) \neq 0$, it is not clear whether the function

$$\varepsilon \mapsto \mathcal{Q}(y + \varepsilon\varphi)$$

is differentiable at $\varepsilon = 0$ (see Remark 3.3 ahead). For this reason, when searching for the Euler-Lagrange equation associated to \mathcal{B} , it is natural to work on intervals where its argument y is collision free.

We will show the following.

Proposition 3.1. *Let $y \in H^1(0, 1; \mathbb{R}^d)$ be such that*

$$|y(\tau)| > 0 \quad \text{for } \tau \in (\tau_1, \tau_2) \subset [0, 1]$$

and

$$\frac{d}{d\varepsilon} [\mathcal{B}(y + \varepsilon\varphi)]_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in \mathcal{D}(\tau_1, \tau_2; \mathbb{R}^d). \tag{3.3}$$

Then the map $\tau \mapsto t_y(\tau)$ is $C^3(\tau_1, \tau_2) \cap C^1([\tau_1, \tau_2])$, with inverse $t \mapsto \tau_y(t)$ which is $C^3(t_1, t_2) \cap C([t_1, t_2])$, where $t_i = t_y(\tau_i)$. Moreover, writing

$$x(t) = x_y(t) = |y(\tau_y(t))|y(\tau_y(t)),$$

we have that there exists a constant $\mu \in \mathbb{R}$ such that

$$\ddot{x} = -\mu \frac{x}{|x|^3} + p(t), \quad t \in (t_1, t_2). \tag{3.4}$$

Finally, in case $(\tau_1, \tau_2) = (0, 1)$, (3.4) holds true in $(0, T)$ with $\mu = 1$.

Notice that in Section 2 we obtained \mathcal{B} starting from the action related to the Kepler problem (1.1). Here, to go back to (1.1), we have two main problems: the first one is that we have to restrict to collisionless intervals; the second one is that “critical points” y of \mathcal{B} (in the sense of the above proposition) solve (3.4), which agrees with (1.1) only when $\mu = 1$. As we mentioned, we will show that the absence of internal collisions implies also $\mu = 1$. On the other hand, in case an internal collision occurs, one can only expect that (3.4) holds true on each collisionless subinterval, with $\mu \geq 0$ possibly depending on the interval. Furthermore, even though x satisfies (3.4) with $\mu = 1$ on any collisionless subinterval, it may fail to be a generalized solution at collisions. Examples in these directions are provided in Section 3.4.

We will prove Proposition 3.1 through a sequence of lemmas. As a first step, we determine the Euler-Lagrange equation associated to \mathcal{B} .

Lemma 3.2. Let $y = y(\tau)$ satisfy the assumptions of Proposition 3.1. Then, in distributional sense in (τ_1, τ_2) ,

$$\frac{d}{d\tau} \left(y' + 3 \frac{\langle y, y' \rangle}{|y|^2} y \right) = \alpha_y + \frac{2}{T} \mathcal{L} \left(\mathcal{Q} - \mathcal{L}^2(1 + \mathcal{R}) \right) y + \beta_y, \quad (3.5)$$

where \mathcal{L} , \mathcal{Q} and \mathcal{R} are evaluated at y and

$$\begin{aligned} \alpha_y &:= 3 \left(\frac{\langle y, y' \rangle}{|y|^2} y' - \frac{\langle y, y' \rangle^2}{|y|^4} y \right) \\ \beta_y &:= \mathcal{L}^2 \left[3|y| \langle p \circ t_y, y \rangle y + |y|^3 p \circ t_y \right] + \frac{2\mathcal{L}^4}{T} y \Gamma_y, \end{aligned}$$

where

$$\Gamma_y(\tau) = \int_0^1 |y(\xi)|^2 \left(\int_\tau^\xi |y(s)|^3 \langle \dot{p}(t_y(s)), y(s) \rangle ds \right) d\xi.$$

Proof. Notice that, by assumption, $\mathcal{B}(y) < +\infty$. Let $\varphi \in \mathcal{D}(\tau_1, \tau_2)$. In particular, the support of φ is contained in some $[\hat{\tau}_1, \hat{\tau}_2] \subset (\tau_1, \tau_2)$, where $y(\tau)$ does not vanish, and the functions $\varepsilon \mapsto \mathcal{L}(y + \varepsilon\varphi)$, $\varepsilon \mapsto \mathcal{Q}(y + \varepsilon\varphi)$ and $\varepsilon \mapsto \mathcal{R}(y + \varepsilon\varphi)$ are differentiable at $\varepsilon = 0$. Recalling that $\mathcal{L}(y) = T\|y\|_2^{-2}$, we have

$$\frac{d}{d\varepsilon} \left[\frac{1}{\mathcal{L}(y + \varepsilon\varphi)} \right]_{\varepsilon=0} = \frac{2}{T} \int_{\tau_1}^{\tau_2} \langle y, \varphi \rangle, \quad \frac{d}{d\varepsilon} [\mathcal{L}(y + \varepsilon\varphi)]_{\varepsilon=0} = -\frac{2}{T} \mathcal{L}^2(y) \int_{\tau_1}^{\tau_2} \langle y, \varphi \rangle;$$

and

$$\frac{d}{d\varepsilon} [t_{y+\varepsilon\varphi}]_{\varepsilon=0} = -\frac{2}{T} \mathcal{L}^2(y) \left(\int_{\tau_1}^{\tau_2} \langle y, \varphi \rangle \right) \int_0^\tau |y|^2 + 2\mathcal{L}(y) \int_{\tau_1}^\tau \langle y, \varphi \rangle.$$

Furthermore

$$\frac{d}{d\varepsilon} [\mathcal{Q}(y + \varepsilon\varphi)]_{\varepsilon=0} = \int_{\tau_1}^{\tau_2} \left(\langle \alpha_y, \varphi \rangle + \left\langle 3 \frac{\langle y, y' \rangle}{|y|^2} y + y', \varphi' \right\rangle \right). \quad (3.6)$$

On the other hand,

$$\begin{aligned} \frac{d}{d\varepsilon} [\mathcal{R}(y + \varepsilon\varphi)]_{\varepsilon=0} &= \int_{\tau_1}^{\tau_2} \left[3|y| \langle y, \varphi \rangle \langle p \circ t_y, y \rangle + |y|^3 \langle p \circ t_y, \varphi \rangle \right] \\ &\quad + \int_0^1 |y|^3 \langle \dot{p} \circ t_y, y \rangle \left(-\frac{2}{T} \mathcal{L}^2(y) \int_{\tau_1}^{\tau_2} \langle y, \varphi \rangle \int_0^\tau |y|^2 + 2\mathcal{L}(y) \int_{\tau_1}^\tau \langle y, \varphi \rangle \right) d\tau \\ &= \int_{\tau_1}^{\tau_2} \langle 3|y| \langle p \circ t_y, y \rangle y + |y|^3 p \circ t_y, \varphi \rangle d\tau \\ &\quad + \frac{2\mathcal{L}^2(y)}{T} \int_0^1 |y|^3 \langle \dot{p} \circ t_y, y \rangle \left(-\int_{\tau_1}^{\tau_2} \langle y, \varphi \rangle \int_0^\tau |y|^2 + \int_{\tau_1}^\tau \langle y, \varphi \rangle \int_0^1 |y|^2 \right) d\tau. \end{aligned}$$

Noticing that

$$\begin{aligned} \int_0^1 \left(\int_{\tau_1}^{\tau_2} \left(\int_0^\tau a(\tau) b(\xi) c(s) ds \right) d\xi \right) d\tau &= \int_{\tau_1}^{\tau_2} b(\xi) \left(\int_0^1 c(s) \left(\int_s^1 a(\tau) d\tau \right) ds \right) d\xi \\ \int_0^1 \left(\int_{\tau_1}^\tau \left(\int_0^1 a(\tau) b(\xi) c(s) ds \right) d\xi \right) d\tau &= \int_{\tau_1}^1 b(\xi) \left(\int_0^1 c(s) \left(\int_\xi^1 a(\tau) d\tau \right) ds \right) d\xi \\ &= \int_{\tau_1}^{\tau_2} b(\xi) \left(\int_0^1 c(s) \left(\int_\xi^1 a(\tau) d\tau \right) ds \right) d\xi \end{aligned}$$

provided $b(\tau) \equiv 0$ on $[\tau_2, 1]$, we obtain that the last line in the previous identity can be rewritten as

$$\frac{2\mathcal{L}^2(y)}{T} \int_{\tau_1}^{\tau_2} \left[\langle y(\tau), \varphi(\tau) \rangle \int_0^1 |y(\xi)|^2 \left(\int_{\tau}^{\xi} |y|^3 \langle \dot{p} \circ t_y, y \rangle ds \right) d\xi \right] d\tau.$$

from which the lemma follows. \square

Remark 3.3. Notice that, in the previous lemma, the assumption $|y(\tau)| > 0$ on (τ_1, τ_2) is essential, because it is not clear whether Q may be differentiable or not in case of collisions. More precisely, the terms in α_y , equation (3.6), i.e.

$$\frac{\langle y, y' \rangle}{|y|^2} y' \quad \text{and} \quad \frac{\langle y, y' \rangle^2}{|y|^4} y,$$

need not to be L^1 if y vanishes somewhere in (τ_1, τ_2) . On the other hand, cancellations may occur, so that α_y may be L^1 also when collisions occur.

To show regularity of y we need the following lemma.

Lemma 3.4. *Let I be an open interval, $y \in H^1(I; \mathbb{R}^d)$ be such that*

$$y(\tau) \neq 0 \quad \text{if } \tau \in \bar{I}$$

and, for some $\lambda \neq -1$,

$$y' + \lambda \frac{\langle y, y' \rangle}{|y|^2} y \in W^{1,1}(I)$$

(i.e. it is absolutely continuous on \bar{I}). Then also $y' \in W^{1,1}(I)$.

Remark 3.5. The condition $\lambda \neq -1$ is essential. Consider $y(\tau) = |\tau|U$, with $U \in \mathbb{S}^{d-1}$ constant and $I = (-1, 1)$. Then

$$y' - \frac{\langle y, y' \rangle}{|y|^2} y = 0,$$

but y' is not continuous.

Proof. Define

$$r(\tau) = |y(\tau)|, \quad U(\tau) = \frac{1}{r(\tau)} y(\tau).$$

Then $r \in H^1(I)$ with $r' = \langle y, y' \rangle / |y|$, and $U \in H^1(I; \mathbb{R}^d)$ with $\langle U, U' \rangle = 0$ a.e. in \bar{I} . By assumption

$$w := y' + \lambda \frac{\langle y, y' \rangle}{|y|^2} y = (1 + \lambda)r'U + rU' \in W^{1,1}(I).$$

Since the space of absolutely continuous functions is a Banach algebra, we infer that

$$(1 + \lambda)r' = \langle w, U \rangle \in W^{1,1}(I).$$

Using the assumption $1 + \lambda \neq 0$, we deduce that $\lambda r'U \in W^{1,1}(I)$ too, and finally

$$y' = w - \lambda r'U \in W^{1,1}(I). \quad \square$$

Corollary 3.6. *Let $y = y(\tau)$ satisfy the assumptions of Proposition 3.1, and let $[\hat{\tau}_1, \hat{\tau}_2] \subset (\tau_1, \tau_2)$. Then $y \in W^{2,1}(\hat{\tau}_1, \hat{\tau}_2)$.*

Proof. It follows from Lemma 3.2 and Lemma 3.4 (with $\lambda = 3$), after noticing that $\alpha_y \in L^1(\hat{\tau}_1, \hat{\tau}_2)$ and $\beta_y \in C([\hat{\tau}_1, \hat{\tau}_2])$ (recall that p is C^1). \square

Lemma 3.7. *Let $y = y(\tau)$ satisfy the assumptions of Proposition 3.1. Then $y \in C^2(\tau_1, \tau_2)$ satisfies*

$$y'' = \gamma_y + \frac{\mathcal{L}}{2T} \left(\mathcal{Q} - \mathcal{L}^2(1 + \mathcal{R}) \right) y + \delta_y, \quad (3.7)$$

where \mathcal{L} , \mathcal{Q} and \mathcal{R} are evaluated at y and

$$\begin{aligned} \gamma_y &:= \frac{3}{4} \frac{\langle y, y' \rangle^2 - |y'|^2 |y|^2}{|y|^4} y \\ \delta_y &:= \mathcal{L}^2 |y|^3 p \circ t_y + \frac{\mathcal{L}^4}{2T} y \Gamma_y, \end{aligned}$$

where Γ_y has been introduced in Lemma 3.2.

Proof. By Corollary 3.6 we have that, both weakly and a.e. in $[\hat{\tau}_1, \hat{\tau}_2] \subset (\tau_1, \tau_2)$,

$$\begin{aligned} & \frac{1}{\mathcal{L}} \left(-y'' + 3 \frac{\langle y, y' \rangle^2 - |y'|^2 |y|^2 - \langle y, y'' \rangle |y|^2}{|y|^4} y \right) + \frac{2}{T} [\mathcal{Q} - \mathcal{L}^2(1 + \mathcal{R})] y + \\ & + \mathcal{L} [3|y| \langle p \circ t_y, y \rangle y + |y|^3 p \circ t_y] + \frac{2\mathcal{L}^3(y)}{T} y \int_0^1 |y(\xi)|^2 \left(\int_\tau^\xi |y|^3 \langle \dot{p} \circ t_y, y \rangle \right) d\xi = 0. \end{aligned} \quad (3.8)$$

Then we multiply (3.8) with y in order to solve for $\langle y, y'' \rangle$ and substitute in (3.8) itself. After some cancellations, we deduce that (3.7) holds, weakly and a.e. in $[\hat{\tau}_1, \hat{\tau}_2]$. Since $y \in W^{2,1}$ and the functions γ_y and δ_y are continuous, we obtain that $y \in C^2([\hat{\tau}_1, \hat{\tau}_2])$. Since in $[\hat{\tau}_1, \hat{\tau}_2]$ is arbitrary, the lemma follows. \square

Lemma 3.8. *Let y, t_y and x be as in Proposition 3.1. Then:*

- $t_y \in C^3(\tau_1, \tau_2) \cap C^1([\tau_1, \tau_2])$, with inverse $\tau_y \in C^3(t_1, t_2) \cap C([t_1, t_2])$;
- $x \in C^2(t_1, t_2)$.

Proof. Once the regularity of y is proved as in Lemma 3.7, the claims follow by the chain rule and the elementary inverse function theorem. \square

Lemma 3.9. *Let x be as in Proposition 3.1. Then there exists $\mu \in \mathbb{R}$ such that (3.4) holds true.*

Proof. Notice that, in (τ_1, τ_2) and (t_1, t_2) respectively,

$$x(t) = |y(\tau_y(t))| y(\tau_y(t)) \quad \Longleftrightarrow \quad y(\tau) = |x(t_y(\tau))|^{-1/2} x(t_y(\tau))$$

where y satisfy (3.7). Our aim is to substitute the second relation above into (3.7); this can be done by the regularity properties obtained in Lemma 3.8. At the end, no explicit dependence on τ will appear, and substituting $t = t_y(\tau)$ we will obtain the differential equation for $x = x(t)$.

We have that $|y(\tau)| = |x(t(\tau))|^{1/2}$, while

$$\begin{aligned} y'(\tau) &= \mathcal{L} \left(-\frac{1}{2} |x|^{-3/2} \langle x/|x|, \dot{x} \rangle x + |x|^{-1/2} \dot{x} \right) |x| = \mathcal{L} \left(-\frac{1}{2} |x|^{-3/2} \langle x, \dot{x} \rangle x + |x|^{1/2} \dot{x} \right) \\ y''(\tau) &= \mathcal{L}^2 \left(\frac{3}{4} |x|^{-5/2} \langle x, \dot{x} \rangle^2 x - \frac{1}{2} |x|^{-1/2} |\dot{x}|^2 x - \frac{1}{2} |x|^{-1/2} \langle x, \ddot{x} \rangle x + |x|^{3/2} \ddot{x} \right), \end{aligned}$$

In order to substitute in the first line of (3.7), we compute separately

$$\begin{aligned}\langle y, y' \rangle^2 &= \frac{\mathcal{L}^2}{4} \langle x, \dot{x} \rangle^2 \\ |y|^2 |y'|^2 &= |x| \mathcal{L}^2 \left(|x| |\dot{x}|^2 - \frac{3}{4} |x|^{-1} \langle x, \dot{x} \rangle^2 \right) = \mathcal{L}^2 \left(|x|^2 |\dot{x}|^2 - \frac{3}{4} \langle x, \dot{x} \rangle^2 \right).\end{aligned}$$

Hence

$$y'' - \gamma_y = \mathcal{L}^2 |x|^{3/2} \left(\ddot{x} - \frac{1}{2} |x|^{-2} \langle x, \ddot{x} \rangle x + \frac{1}{4} |x|^{-2} |\dot{x}|^2 x \right).$$

On the other hand, by a change of variables in the integrals,

$$\mathcal{L} = \int_0^T \frac{dt}{|x|}, \quad \mathcal{Q} = \frac{\mathcal{L}}{2} \int_0^T |\dot{x}|^2 dt, \quad \mathcal{R} = \frac{1}{\mathcal{L}} \int_0^T \langle p, x \rangle dt,$$

while

$$\mathcal{L}^2 |y|^3 (p \circ t_y) = \left(\mathcal{L}^2 |x|^{3/2} p \right) \circ t_y$$

and

$$\begin{aligned}\mathcal{L}^2 \int_0^1 |y(\xi)|^2 \left(\int_\tau^\xi |y|^3 \langle \dot{p} \circ t_y, y \rangle \right) d\xi &= \int_0^T \left(\int_t^s \langle \dot{p}, x \rangle \right) ds \\ &= \int_0^T \left(\int_t^T \langle \dot{p}, x \rangle \right) ds - \int_0^T \left(\int_s^T \langle \dot{p}, x \rangle \right) ds \\ &= T \int_t^T \langle \dot{p}, x \rangle - \int_0^T t \langle \dot{p}, x \rangle dt\end{aligned}$$

We conclude that equation (3.7) transforms into the following equation for $x = x(t)$:

$$\ddot{x} - \frac{1}{2} |x|^{-2} \langle x, \ddot{x} \rangle x + \frac{1}{4} |x|^{-2} |\dot{x}|^2 x = \frac{1}{2} \left[\frac{1}{T} \int_0^T \left(\frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|} - \langle p, x \rangle - t \langle \dot{p}, x \rangle \right) dt + \int_t^T \langle \dot{p}, x \rangle \right] |x|^{-2} x + p,$$

that is

$$\ddot{x} = \frac{1}{2} \left[\langle x, \ddot{x} \rangle - \frac{1}{2} |\dot{x}|^2 + C + \int_t^T \langle \dot{p}, x \rangle \right] |x|^{-2} x + p \quad (3.9)$$

where

$$C := \frac{1}{T} \int_0^T \left(\frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|} - \langle p, x \rangle - t \langle \dot{p}, x \rangle \right) dt. \quad (3.10)$$

In particular, multiplying (3.9) by x we obtain

$$\langle x, \ddot{x} \rangle = -\frac{1}{2} |\dot{x}|^2 + C + \int_t^T \langle \dot{p}, x \rangle + 2 \langle p, x \rangle$$

and, substituting into (3.9),

$$\ddot{x} = \left[C - \frac{1}{2} |\dot{x}|^2 + \langle p, x \rangle + \int_t^T \langle \dot{p}, x \rangle \right] |x|^{-2} x + p, \quad t \in (t_1, t_2). \quad (3.11)$$

Let $\beta = \beta(t)$ be such that $\ddot{x} - p = \beta x$. Then

$$\beta|x|^2 = C - \frac{1}{2}|\dot{x}|^2 + \langle p, x \rangle + \int_t^T \langle \dot{p}, x \rangle, \quad t \in (t_1, t_2). \quad (3.12)$$

Since x is C^2 , we have that β is of class C^1 in (t_1, t_2) . Hence we can differentiate to obtain

$$\dot{\beta}|x|^2 + 2\beta\langle x, \dot{x} \rangle = -\langle \ddot{x} - p, \dot{x} \rangle = -\langle \beta x, \dot{x} \rangle, \quad t \in (t_1, t_2).$$

Hence $\dot{\beta}|x|^2 + 3\beta\langle x, \dot{x} \rangle = 0$, which implies $\beta = -\mu/|x|^3$ for some constant $\mu \in \mathbb{R}$. \square

Remark 3.10. By the proof of Proposition 3.1 we have that the constant μ appearing in (3.4) is related to the constant C defined in (3.10). More precisely, substituting (3.4) into (3.11) we infer

$$C = \frac{1}{2}|\dot{x}|^2 - \frac{\mu}{|x|} - \langle p, x \rangle - \int_t^T \langle \dot{p}, x \rangle, \quad t \in (t_1, t_2). \quad (3.13)$$

Lemma 3.11. *Under the assumptions of Proposition 3.1 we have that*

- if either $y(\tau_1^+) = 0$ or $y(\tau_2^-) = 0$ then, in (3.4), $\mu \geq 0$;
- if $(\tau_1, \tau_2) = (0, 1)$ (with either collisions at the extrema or not) then, in (3.4), $\mu = 1$.

Proof. First of all, for concreteness, let us assume $y(\tau_1^+) = 0$. Then, by (3.13) we obtain

$$-\frac{1}{|x|}\mu \leq C + \langle p, x \rangle + \int_t^T \langle \dot{p}, x \rangle.$$

Taking the limit as $t \rightarrow t_1^+$ we deduce that the constant μ can not be strictly negative.

On the other hand, let $(\tau_1, \tau_2) = (0, 1)$. Then (3.13) holds true with $(t_1, t_2) = (0, T)$. Integrating on $(0, T)$ and recalling the definition of C in (3.10) we obtain that $\mu = 1$. \square

End of the proof of Proposition 3.1. The proposition follows by Lemmas 3.8, 3.9 and 3.11. \square

3.2 Analysis of \mathcal{B}_{LC}

Now, for $z = z_1 + iz_2 \in \mathbb{C} \cong \mathbb{R}^2$, let $\Phi(z) = \Phi_{\text{LC}}(z) = z^2$, see Case 2 in the Section 2. The corresponding functional is

$$\mathcal{B}_{\text{LC}} : H^1(0, 1; \mathbb{C}) \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$\mathcal{B}_{\text{LC}}(z) := \frac{1}{\mathcal{L}(z)} \mathcal{Q}(z) + \mathcal{L}(z) \left[1 + \mathcal{R}(z) \right], \quad \mathcal{B}(0) := +\infty, \quad (3.14)$$

where

$$\begin{aligned} \mathcal{L}(z) &:= \frac{T}{\int_0^1 |z|^2 d\tau}, \\ \mathcal{Q}(z) &:= 2 \int_0^1 |z'|^2 d\tau, \\ \mathcal{R}(z) &:= \int_0^1 |z|^2 \langle p \circ t_z, z^2 \rangle d\tau, \\ t_z(\tau) &:= \mathcal{L}(z) \int_0^\tau |z(\xi)|^2 d\xi. \end{aligned} \quad (3.15)$$

The main difference with respect to the previous section consists in the fact that \mathcal{B}_{LC} is now of class C^1 in the whole $H^1(0, 1; \mathbb{C}) \setminus \{0\}$, regardless of possible collisions.

Proposition 3.12. Let $z \in H^1(0, 1; \mathbb{C}) \setminus \{0\}$ satisfy

$$\frac{d}{d\varepsilon} [\mathcal{B}_{LC}(z + \varepsilon\varphi)]_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in \mathcal{D}(0, 1; \mathbb{C}).$$

Then $z \in C^4([0, 1])$, the map $\tau \mapsto t_z(\tau)$ is invertible on $[0, 1]$ with inverse $t \mapsto \tau_z(t)$, and

$$x(t) = z^2(\tau_z(t))$$

is a generalized solution of equation (1.1).

To prove Proposition 3.12, as a first step, we determine the Euler-Lagrange equation associated to \mathcal{B}_{LC} .

Lemma 3.13. Let $z \in H^1(0, 1; \mathbb{C}) \setminus \{0\}$ be as in Proposition 3.12, then

$$z'' = \frac{\mathcal{L}}{2T} \left(\mathcal{Q} - \mathcal{L}^2(1 + \mathcal{R}) \right) z + \delta_z, \quad (3.16)$$

where \mathcal{L} , \mathcal{Q} and \mathcal{R} are evaluated at z and

$$\delta_z := \frac{\mathcal{L}^2}{2} \left[\langle p \circ t_z, z^2 \rangle + \bar{z}^2(p \circ t_z) + \frac{\mathcal{L}^2}{T} \Delta_z \right] z,$$

where

$$\Delta_z := \int_0^1 |z(\xi)|^2 \left(\int_\tau^\xi |z(s)|^2 \langle \dot{p}(t_z(s)), z^2(s) \rangle d\xi \right) d\xi.$$

In particular $z \in C^3([0, 1])$.

Proof. The proof mainly retraces the one of Lemmas 3.2 and 3.7. The main difference consists in the term arising from the differentiation of $\mathcal{Q}(z)$, indeed in this case the analogous of (3.6) simplifies into

$$\frac{d}{d\varepsilon} [\mathcal{Q}(y + \varepsilon\varphi)]_{\varepsilon=0} = 4 \int_0^1 \langle z', \varphi' \rangle,$$

for every $\varphi \in \mathcal{D}(0, 1; \mathbb{C})$. Regularity of z and equation (3.16) follow at once (recall that we are assuming $p \in C^1$). \square

Remark 3.14. For future purposes we notice that, for every $z \neq 0$, $v \in H^1(0, 1; \mathbb{C})$,

$$\mathcal{B}_{LC}'(z)[v] = \frac{4}{\mathcal{L}} \int_0^1 \left[\langle z', v' \rangle + \left\langle \frac{\mathcal{L}}{2T} \left(\mathcal{Q} - \mathcal{L}^2(1 + \mathcal{R}) \right) z + \delta_z, v \right\rangle \right],$$

where δ_z has been introduced in the previous lemma.

Lemma 3.15. Let z be a critical point of \mathcal{B}_{LC} and $\tau^* \in [0, 1]$ be such that $z(\tau^*) = 0$. Then $z'(\tau^*) \neq 0$ and the set $Z := \{\tau \in [0, 1] : z(\tau) = 0\}$ is finite.

Proof. Notice that equation (3.16) can be written as

$$z'' = g(\tau)z \quad (3.17)$$

for some continuous, complex valued, function g . Let us assume that $z'(\tau^*) = 0$; then, by uniqueness of the Cauchy problem associated to the previous equation, $z \equiv 0$, a contradiction. Finally, if Z is not finite, then it must have some accumulation point which can not be a simple zero of z . \square

Corollary 3.16. *The function $t_z \in C^4([0, 1])$ is invertible, with inverse $\tau_z \in C([0, T])$ which is C^4 outside the finite set $t_z(Z)$.*

Proof. The result follows by the definition of t_z and by the elementary inverse function theorem. \square

Motivated by the previous corollary we define, for a suitable $N \geq 1$, the points $0 = t_0 < t_1 < \dots < t_N = 1$ in such a way that

$$(0, T) \setminus t_z(Z) = \bigcup_{i=1}^N (t_{i-1}, t_i).$$

Notice that $\tau_i := \tau_z(t_i)$ is such that $z(\tau_i) = 0$ at least for $i = 1, \dots, N-1$ ($\tau_0 = 0$ and $\tau_N = 1$ may or may not be collision instants).

Lemma 3.17. *Let x as in Proposition 3.12 and let $\{t_0, \dots, t_N\}$ as above. Then $x \in C([0, 1]; \mathbb{R}^2)$ is C^2 outside collisions. Moreover, the function*

$$t \mapsto |\dot{x}(t)|^2 |x(t)| = \frac{4}{\mathcal{L}^2} |z'(\tau_z(t))|^2 \quad (3.18)$$

is continuous in $[0, 1]$ and, for every $i = 1, \dots, N$ there exists $\mu_i > 0$ such that

$$\ddot{x} = -\mu_i \frac{x}{|x|^3} + p(t), \quad t \in (t_{i-1}, t_i). \quad (3.19)$$

Proof. By Corollary 3.16, for any $t \in [0, T]$ and $\tau \in [0, 1]$ we have that $x(t) = z^2(\tau_z(t))$. Then,

$$\zeta(t) := z(\tau_z(t)) \quad \implies \quad x(t) = \zeta^2(t).$$

In particular x is continuous in $[0, T]$. Restricting to (τ_{i-1}, τ_i) and (t_{i-1}, t_i) respectively, we compute

$$z'(\tau_z(t)) = \frac{\mathcal{L}}{2} |x(t)| \zeta^{-1}(t) \dot{x}(t)$$

so that equation (3.18) follows on each (t_{i-1}, t_i) . By Corollary 3.16, $|\dot{x}|^2 |x|$ can be extended to a continuous function in the whole $[0, T]$, still satisfying (3.18). Differentiating once more and recalling that $2 \langle a, b \rangle = a\bar{b} + \bar{a}b$ we obtain

$$\begin{aligned} z'' \circ \tau_z &= \frac{\mathcal{L}^2}{2} |x|^2 \zeta^{-1} \left(\ddot{x} - \frac{1}{2} x^{-1} \dot{x}^2 + \frac{\langle x, \dot{x} \rangle}{|x|^2} \dot{x} \right) \\ &= \frac{\mathcal{L}^2}{2} |x|^2 \zeta^{-1} \left(\ddot{x} - \frac{\bar{x} \dot{x}^2}{2|x|^2} + \frac{x \dot{x} + \bar{x} \dot{x}}{2|x|^2} \dot{x} \right) = \frac{\mathcal{L}^2}{2} |x|^2 \zeta^{-1} \left(\ddot{x} + \frac{1}{2} \frac{|\dot{x}|^2}{|x|^2} x \right). \end{aligned}$$

On the other hand, reasoning as in the proof of Lemma 3.9 we obtain

$$\frac{\mathcal{L}^3}{2T} \left[\frac{Q}{\mathcal{L}^2} - 1 - \mathcal{R} + \mathcal{L} \int_0^1 |z(\xi)|^2 \left(\int_\tau^\xi |z|^2 \langle \dot{p} \circ t_z, z^2 \rangle \right) d\xi \right] z = \frac{\mathcal{L}^2}{2} \left[C + \int_t^T \langle \dot{p}, x \rangle \right] \zeta$$

where

$$C := \frac{1}{T} \int_0^T \left(\frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|} - \langle p, x \rangle - t \langle \dot{p}, x \rangle \right) dt. \quad (3.20)$$

Finally

$$\frac{\mathcal{L}^2}{2} [\langle p \circ t_z, z^2 \rangle + \bar{z}^2 (p \circ t_z)] z = \frac{\mathcal{L}^2}{2} [\langle p, x \rangle \zeta + |x|^2 \zeta^{-1} p].$$

Substituting in (3.16) we obtain

$$\ddot{x} = \left[C - \frac{1}{2}|\dot{x}|^2 + \langle p, x \rangle + \int_t^T \langle \dot{p}, x \rangle \right] |x|^{-2}x + p, \quad t \in (t_{i-1}, t_i), \quad (3.21)$$

which is the same equation obtained in (3.11). Arguing as in Lemma 3.9 we obtain the existence of a constant $\mu_i \in \mathbb{R}$ such that (3.19) holds; furthermore, by an analogue of Lemma 3.11 for \mathcal{B}_{LC} , $\mu_i \geq 0$ and, if $(\tau_0, \tau_1) = (0, 1)$ (with or without collisions at the extrema), then $\mu_1 = 1$. Hence we are left to prove that $\mu_i > 0$ in case, say, $x(t_i) = 0$. Assume by contradiction that $\mu_i = 0$. Then $\ddot{x} = p$ in (t_{i-1}, t_i) and we deduce that \dot{x} is continuous up to t_i^- . Equation (3.18) implies that

$$z(\tau_i) = z'(\tau_i) = 0,$$

in contradiction with Lemma 3.15. \square

Lemma 3.18. *In the same assumptions of Lemma 3.17 we have, for every $i = 1, \dots, N$,*

$$\mu_i = 1 \quad \text{and} \quad |z'(\tau_i)|^2 = \frac{\mathcal{L}^2}{2}. \quad (3.22)$$

Proof. To start with, we prove that $\mu_i = \mu_{i+1}$, for every i . From (3.19) and (3.21), for every i ,

$$-\mu_i = C|x| - \frac{1}{2}|\dot{x}|^2|x| + |x| \langle p, x \rangle + |x| \int_t^T \langle \dot{p}, x \rangle, \quad t \in (t_{i-1}, t_i).$$

Letting $t \rightarrow t_i^\pm$ and using (3.18),

$$\mu_i = \frac{2}{\mathcal{L}^2} |z'(\tau_i)|^2 = \mu_{i+1}. \quad (3.23)$$

This shows that $\mu_1 = \dots = \mu_N =: \mu$. Using (3.19), (3.20) and (3.21) we obtain

$$\frac{1}{2}|\dot{x}|^2 - \frac{\mu}{|x|} - \langle p, x \rangle - \int_t^T \langle \dot{p}, x \rangle = \frac{1}{T} \int_0^T \left(\frac{1}{2}|\dot{x}|^2 - \frac{1}{|x|} - \langle p, x \rangle - t \langle \dot{p}, x \rangle \right) dt, \quad t \notin \{t_1, \dots, t_N\}. \quad (3.24)$$

Integrating on $[0, T]$ we obtain $\mu = 1$. Then the lemma follows by (3.23). \square

End of the proof of Proposition 3.12. We are left to prove the third point of Definition 1.1, that is, continuity of energy and direction across collisions. Let $t_* \in (0, T)$ be such that $x(t_*) = z(\tau_*) = 0$. Recall that, by Lemma 3.13, $z \in C^3([0, 1], \mathbb{C})$. Moreover, by (3.22) and (3.17), $|z'(\tau_*)| = \mathcal{L}/\sqrt{2}$ and $z''(\tau_*) = g(\tau_*)z(\tau_*) = 0$. In particular, for τ approaching τ_* ,

$$\begin{aligned} z(\tau) &= (\tau - \tau_*)\alpha(\tau) \quad \text{with } |\alpha(\tau_*)| = \frac{\mathcal{L}}{\sqrt{2}} \neq 0 \text{ and } \alpha \text{ is continuous,} \\ z'(\tau) &= z'(\tau_*) + (\tau - \tau_*)^2\beta(\tau) \quad \text{with } |z'(\tau_*)| = \frac{\mathcal{L}}{\sqrt{2}} \text{ and } \beta \text{ is continuous.} \end{aligned} \quad (3.25)$$

As far as the energy continuity is concerned, we have that, by (3.18),

$$h(t) := \frac{1}{2}|\dot{x}|^2 - \frac{1}{|x|} = \frac{2|z'|^2 - \mathcal{L}^2}{\mathcal{L}^2|z|^2}.$$

Using (3.25) we obtain

$$\lim_{t \rightarrow t_*^\pm} h(t) = \lim_{\tau \rightarrow \tau_*^\pm} \frac{4 \langle \beta(\tau), z'(\tau_*) \rangle (\tau - \tau_*)^2 + o((\tau - \tau_*)^2)}{\mathcal{L}^2 |\alpha(\tau)|^2 (\tau - \tau_*)^2} = \frac{8}{\mathcal{L}^4} \langle \beta(\tau_*), z'(\tau_*) \rangle.$$

Analogously,

$$\lim_{t \rightarrow t_*^\pm} \frac{x}{|x|} = \lim_{\tau \rightarrow \tau_*^\pm} \frac{z^2}{|z|^2} = \lim_{\tau \rightarrow \tau_*^\pm} \frac{\alpha^2(\tau)(\tau - \tau_*)^2}{|\alpha^2(\tau)|(\tau - \tau_*)^2} = \frac{2}{\mathcal{L}^2} z'(\tau_*)^2. \quad \square$$

3.3 Analysis of \mathcal{B}_{KS}

Finally, for $z = z_0 + z_1i + z_2j + z_3k \in \mathbb{H} \cong \mathbb{R}^4$, let $\Phi: \mathbb{H} \rightarrow \mathbb{IH} \cong \mathbb{R}^3$, $\Phi(z) = \Phi_{\text{KS}}(z) = \bar{z}iz$, see Case 3 in Section 2. In this case, it is natural to choose $p \in \mathbb{IH}$. Now the corresponding functional is

$$\mathcal{B}_{\text{KS}} : H^1(0, 1; \mathbb{H}) \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$\mathcal{B}_{\text{KS}}(z) := \frac{1}{\mathcal{L}(z)} \mathcal{Q}(z) + \mathcal{L}(z) \left[1 + \mathcal{R}(z) \right], \quad \mathcal{B}(0) := +\infty, \quad (3.26)$$

where

$$\begin{aligned} \mathcal{L}(z) &:= \frac{T}{\int_0^1 |z|^2 d\tau}, \\ \mathcal{Q}(z) &:= 2 \int_0^1 |z'|^2 d\tau, \\ \mathcal{R}(z) &:= \int_0^1 |z|^2 \langle p \circ t_z, \bar{z}iz \rangle d\tau, \\ t_z(\tau) &:= \mathcal{L}(z) \int_0^\tau |z(\xi)|^2 d\xi. \end{aligned} \quad (3.27)$$

Recall that, in Section 2, we established the correspondence between \mathcal{B}_{KS} under the validity of condition (2.7). Actually, as we are going to show, this condition can be appreciably weakened.

Proposition 3.19. *Let $p \in C^1([0, T]; \mathbb{IH})$, and let $z \in H^1(0, 1; \mathbb{H}) \setminus \{0\}$ satisfy*

$$\frac{d}{d\varepsilon} [\mathcal{B}_{\text{KS}}(z + \varepsilon\varphi)]_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in \mathcal{D}(0, 1; \mathbb{H}). \quad (3.28)$$

Then $z \in C^3([0, 1])$, and the map $\tau \mapsto t_z(\tau)$ is invertible on $[0, 1]$, with inverse $t \mapsto \tau_z(t)$.

Furthermore, if z satisfies also

$$\langle z'(\tau^*), iz(\tau^*) \rangle = 0, \quad \text{for some } \tau^* \in [0, 1], \quad (3.29)$$

then

$$x(t) = \bar{z}(\tau_z(t))iz(\tau_z(t))$$

is a generalized solution of equation (1.1).

As usual, we start deducing the Euler-Lagrange equations and proving some regularity results.

Lemma 3.20. *Let $z \in H^1(0, 1; \mathbb{H}) \setminus \{0\}$ satisfy (3.28). Then*

$$z'' = \frac{\mathcal{L}}{2T} \left(\mathcal{Q} - \mathcal{L}^2(1 + \mathcal{R}) \right) z + \delta_z, \quad (3.30)$$

where \mathcal{L} , \mathcal{Q} and \mathcal{R} are evaluated at z and

$$\delta_z := \frac{\mathcal{L}^2}{2} \left[\langle p \circ t_z, \bar{z}iz \rangle z - iz|z|^2(p \circ t_z) + \frac{\mathcal{L}^2}{T} \Delta_z z \right]$$

where

$$\Delta_z := \int_0^1 |z(\xi)|^2 \left(\int_\tau^\xi |z(s)|^2 \langle \dot{p}(t_z(s)), \bar{z}(s)iz(s) \rangle d\xi \right) d\xi.$$

In particular $z \in C^3([0, 1])$. Moreover, if $\tau^ \in [0, 1]$ is such that $z(\tau^*) = 0$ then $z'(\tau^*) \neq 0$ and the set $Z := \{\tau \in [0, 1] : z(\tau) = 0\}$ is finite. Furthermore, the function $t_z \in C^3([0, 1])$ is invertible, with inverse $\tau_z \in C([0, T])$ which is C^3 outside the finite set $t_z(Z)$.*

Proof. The lemma follows reasoning as in Lemmas 3.2, 3.13 and taking into account that, since $\langle v, w \rangle = \Re(\bar{w}v) = \langle \bar{v}, \bar{w} \rangle$,

$$\begin{aligned} \frac{d}{d\varepsilon} \left[\int_0^1 |z|^2 \langle p \circ t_z, (\bar{z} + \varepsilon \bar{\varphi}) i(z + \varepsilon \varphi) \rangle d\tau \right]_{\varepsilon=0} &= \int_0^1 |z|^2 \langle p \circ t_z, \bar{\varphi} i z + \bar{z} i \varphi \rangle d\tau \\ &= \int_0^1 |z|^2 \langle \bar{p} \circ t_z, -\bar{z} i \varphi \rangle d\tau + \int_0^1 |z|^2 \langle p \circ t_z, \bar{z} i \varphi \rangle d\tau = -2 \int_0^1 |z|^2 \langle i z p \circ t_z, \varphi \rangle d\tau \end{aligned}$$

where we used the fact that $\bar{p} = -p$. We proceed in the proof writing equation (3.30) as a linear second order equation in \mathbb{R}^4 (with coefficients depending on τ via z), and we conclude arguing as in the proofs of Lemma 3.15 and Corollary 3.16. \square

The equation satisfied by z allows to relate conditions (3.29) and (2.7).

Lemma 3.21. *Let $z \in H^1(0, 1; \mathbb{H}) \setminus \{0\}$ satisfy (3.28). Then*

$$\langle z', i z \rangle \quad \text{is constant on } [0, 1].$$

In particular, if z satisfies also (3.29), then

$$\langle z', i z \rangle \equiv 0 \quad \text{on } [0, 1].$$

Proof. By Lemma 3.20 z is regular up to the boundary, and it solves equation (3.30). Such equation can be written as

$$z'' = a(\tau)z + b(\tau)izp$$

for some real-valued functions a and b . Then

$$\frac{d}{dt} \langle z', i z \rangle = b(\tau) \langle i z p, i z \rangle = b(\tau) |z|^2 \langle p, 1 \rangle = 0. \quad \square$$

To proceed, we use Lemma 3.20 in order to write the interval $(0, T)$ as the union of disjoint interval (t_{i-1}, t_i) , $i = 1, \dots, N$, for some $N \geq 1$, in such a way that $z(\tau_i) = 0$, $\tau_i := \tau_z(t_i)$, at least for $i = 1, \dots, N - 1$.

Lemma 3.22. *Let (3.28) and (3.29) hold true. Let x be defined as in Proposition 3.19 and $\{t_0, \dots, t_N\}$ as above. Then $x \in C([0, 1])$ is C^2 outside collisions. Moreover, the function*

$$t \mapsto -\dot{x}^2(t)|x(t)| = |\dot{x}(t)|^2|x(t)| = \frac{4}{\mathcal{L}^2} |z'(\tau_z(t))|^2 \quad (3.31)$$

is continuous in $[0, 1]$ and, for every $i = 1, \dots, N$ there exists $\mu_i > 0$ such that

$$\ddot{x} = -\mu_i \frac{x}{|x|^3} + p(t), \quad t \in (t_{i-1}, t_i). \quad (3.32)$$

Proof. Let us define $\zeta(t) := z(\tau_z(t))$ so that

$$x(t) = \bar{\zeta}(t) i \zeta(t), \quad |x(t)| = |\zeta(t)|^2$$

are continuous in $[0, T]$. Notice that, by Lemma 3.21,

$$\langle \dot{\zeta}(t), i \zeta(t) \rangle = \dot{\tau}_z(t) \langle z'(\tau_z(t)), i z(\tau_z(t)) \rangle = 0$$

for every t . Restricting to (τ_{i-1}, τ_i) and (t_{i-1}, t_i) respectively, we can compute

$$\dot{x}(t) = \dot{\zeta}(t)i\zeta(t) + \bar{\zeta}(t)i\dot{\zeta}(t) = 2\bar{\zeta}(t)i\dot{\zeta}(t).$$

Since $\dot{\tau}_z(t) = 1/(\mathcal{L}|x(t)|)$, we have

$$\dot{\zeta}(t) = -\frac{1}{2|x(t)}i\zeta(t)\dot{x}(t) \quad \text{and} \quad z'(\tau_z(t)) = \mathcal{L}|x(t)|\dot{\zeta}(t) = -\frac{\mathcal{L}}{2}i\zeta(t)\dot{x}(t)$$

so that equation (3.31) follows on each (t_{i-1}, t_i) . By Lemma 3.20, $|\dot{x}|^2|x|$ can be extended to a continuous function in the whole $[0, T]$, still satisfying (3.31). Differentiating once more we obtain

$$\begin{aligned} z''(\tau_z(t)) &= \mathcal{L}|x(t)|\frac{d}{dt}z'(\tau_z(t)) = -\frac{\mathcal{L}^2}{2}\left(\frac{1}{2}\zeta(t)\dot{x}^2(t) + |x(t)|i\zeta(t)\ddot{x}(t)\right) \\ &= \frac{\mathcal{L}^2}{2}\left(\frac{1}{2}|\dot{x}(t)|^2\zeta(t) - |x(t)|i\zeta(t)\ddot{x}(t)\right) \end{aligned}$$

(recall that, since $x \in \mathbb{H}$, then $\dot{x}^2 = -|\dot{x}|^2$). On the other hand, reasoning as in the proof of Lemmas 3.9, 3.17 we obtain

$$\frac{\mathcal{L}^3}{2T}\left[\frac{Q}{\mathcal{L}^2} - 1 - \mathcal{R} + \mathcal{L}\int_0^1|z(\xi)|^2\left(\int_\tau^\xi|z|^2\langle\dot{p}\circ t_z, \bar{z}iz\rangle\right)d\xi\right]z = \frac{\mathcal{L}^2}{2}\left[C + \int_t^T\langle\dot{p}, x\rangle\right]\zeta$$

where, as usual,

$$C := \frac{1}{T}\int_0^T\left(\frac{1}{2}|\dot{x}|^2 - \frac{1}{|x|} - \langle p, x\rangle - t\langle\dot{p}, x\rangle\right)dt. \quad (3.33)$$

Finally

$$\frac{\mathcal{L}^2}{2}\left[\langle p\circ t_z, \bar{z}iz\rangle z - iz|z|^2(p\circ t_z)\right] = \frac{\mathcal{L}^2}{2}\left[\langle p, x\rangle\zeta - i\zeta|x|p\right]\circ t_z.$$

Substituting in (3.30) we obtain

$$-|x|i\zeta\ddot{x} = \left[C - \frac{1}{2}|\dot{x}|^2 + \langle p, x\rangle + \int_t^T\langle\dot{p}, x\rangle\right]\zeta - i\zeta|x|p.$$

and finally, multiplying on the left by $\bar{\zeta}i$,

$$\ddot{x} = \left[C - \frac{1}{2}|\dot{x}|^2 + \langle p, x\rangle + \int_t^T\langle\dot{p}, x\rangle\right]|x|^{-2}x + p, \quad t \in (t_{i-1}, t_i), \quad (3.34)$$

which is the same equation obtained in (3.11) and (3.21). At this point, we can conclude by reasoning as at the end of the proof of Lemma 3.17. \square

Remark 3.23. We stress for future reference that the calculations above yield

$$\dot{x}(t) = \frac{2}{|z(\tau_z(t))|^2\mathcal{L}(z)}\bar{z}(\tau_z(t))iz'(\tau_z(t)).$$

whenever z satisfies (3.28), (3.29) and $x(t) \neq 0$.

End of the proof of Proposition 3.19. Starting from Lemmas 3.20, 3.21, 3.22, and in particular from equations (3.31), (3.33), (3.34), we can first show an analogue of Lemma 3.18, and then conclude as in the end of the proof of Proposition 3.12. \square

3.4 Critical points may not correspond to generalized solutions

In this section we provide examples of solutions of the Euler-Lagrange equations associated to \mathcal{B} and \mathcal{B}_{KS} which do not correspond to generalized solutions of (1.1). Of course, we will construct such examples by violating the additional assumptions of Propositions 3.1 and 3.19.

Example 3.24. Assume that $y \in H^1(0, 1; \mathbb{R}^d)$ satisfies (3.3) both on $(0, \tau_*)$ and $(\tau_*, 1)$, with $y(\tau_*) = 0$. Assume that y is differentiable in τ_* , with $y'(\tau_*) = a \neq 0$. Then it can not correspond to a generalized solution of (1.1).

Indeed, let x be defined on $(0, t^*)$ and (t^*, T) , according to Proposition 3.1. Since $y(\tau) = a(\tau - \tau_*) + o(\tau - \tau_*)$ as $\tau \rightarrow \tau_*$ we obtain

$$\lim_{t \rightarrow t_*^\pm} \frac{x(t)}{|x(t)|} = \lim_{\tau \rightarrow \tau_*^\pm} \frac{y(\tau)|y(\tau)|}{|y(\tau)|^2} = \pm \frac{a}{|a|},$$

so that x does not satisfy Definition 1.1 at t_* .

Example 3.25. In dimension $d = 1$ the functional \mathcal{B} is of class C^1 on $H^1(0, 1; \mathbb{R})$. Nonetheless, its critical points do not necessarily correspond to generalized solutions of (1.1).

We consider the functional \mathcal{B} in dimension $d = 1$, with $p \equiv 0$:

$$\mathcal{B}(y) = \frac{2}{T} \int_0^1 y^2 \int_0^1 (y')^2 + \frac{T}{\int_0^1 y^2}.$$

For concreteness, we work with Dirichlet boundary conditions, i.e. we consider the restriction of \mathcal{B} to $H_0^1(0, 1; \mathbb{R}) \setminus \{0\}$. Assume that $y \in H_0^1(0, 1; \mathbb{R}) \setminus \{0\}$ satisfies $\mathcal{B}'(y)[\varphi] = 0$ for every $\varphi \in \mathcal{D}(0, 1; \mathbb{R})$. By direct computations we obtain that y satisfies

$$\begin{cases} \frac{2}{T} Ay - \frac{2}{T} By'' - \frac{T}{B^2} y = 0 \\ y(0) = y(1) = 0, \end{cases} \quad (3.35)$$

where

$$A := \int_0^1 (y')^2, \quad B := \int_0^1 y^2.$$

We obtain that problem (3.35) admits nontrivial solutions if and only if

$$\frac{T^2}{2B^3} - \frac{A}{B} = n^2 \pi^2, \quad n = 1, 2, \dots$$

with solutions

$$y_n(\tau) = k_n \sin(n\pi\tau), \quad \text{where } k_n = \pm \left(\frac{T\sqrt{2}}{\pi n} \right)^{1/3}.$$

Now, if $n = 1$ then $y_1(\tau) > 0$ in $(0, 1)$ and all the assumptions of Proposition 3.1 are satisfied on such interval; as a consequence, the corresponding x is a generalized solution to the unperturbed Kepler problem (indeed, it corresponds to a ejection-collision motion, with collisions in the first and last time instants). On the other hand, in case $n \geq 2$ y_n vanishes at some interior point, but it is C^1 and its zeroes are simple, therefore it can not correspond to a generalized solution by the previous example.

Example 3.26. Assume that $y \in H^1(0, 1; \mathbb{R}^d)$ satisfies (3.3) on any collisionless subinterval. Then (3.4) may hold true with different values of μ , depending on the corresponding interval.

In general, if some internal collision occurs, one can not expect that $\mu = 1$, even though the assumptions of Proposition 3.1 hold in any collision-free subinterval. More precisely, let us assume that $y \in H^1(0, 1; \mathbb{R}^d)$ is such that

$$(0, 1) \setminus \{\tau : y(\tau) = 0\} =: \bigcup_{j \in \mathcal{J}} I_j,$$

where each I_j is an open interval and the index set \mathcal{J} is at most countable. If the assumptions of Proposition 3.1 hold true in each I_j , then there exist coefficients $\mu_j \geq 0$ such that

$$\ddot{x} = -\mu_j \frac{x}{|x|^3} + p(t), \quad t \in I_j.$$

Then, reasoning as in the proof of Lemma 3.11, we can only show that

$$\sum_j \int_{I_j} \frac{\mu_i}{|x|} = \int_0^T \frac{1}{|x|}. \quad (3.36)$$

On the other hand, we can construct an example in which the above facts actually occur, with different μ_i . Let us consider the unperturbed Kepler problem $p \equiv 0$, and let us assume that $y \in H_0^1(0, 1; \mathbb{R}^d)$ satisfies the assumptions of Proposition 3.1, with $(\tau_1, \tau_2) = (0, 1)$. The existence of such a y , which corresponds to a degenerate Keplerian ellipse, can be obtained for instance as a corollary of Theorem 4.1 ahead. For every $h_1, h_2 > 0$ such that

$$h_1^2 + h_2^2 = 2^{1/3},$$

we define the function $w \in H_0^1(0, 1; \mathbb{R}^d)$ as

$$w(\tau) := \begin{cases} h_1 y(2\tau) & 0 < \tau \leq 1/2 \\ h_2 y(2 - 2\tau) & 1/2 \leq \tau < 1. \end{cases}$$

We claim that

$$\frac{d}{d\varepsilon} [\mathcal{B}(w + \varepsilon\varphi)]_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in \mathcal{D}(0, 1/2) \cup \mathcal{D}(1/2, 1). \quad (3.37)$$

Then Proposition 3.1 applies on both subintervals, so that we can define as usual the function $x(t) = x_w(t)$. Finally, by direct calculations, one can show that

$$\begin{aligned} \ddot{x} &= -2^{2/3} h_1^2 \frac{x}{|x|^3} && \text{in } (0, 2^{-1/3} h_1^2 T), \\ \ddot{x} &= -2^{2/3} h_2^2 \frac{x}{|x|^3} && \text{in } (2^{-1/3} h_1^2 T, T). \end{aligned} \quad (3.38)$$

We postpone the proofs of (3.37) and (3.38) in the appendix.

Example 3.27. *Periodic critical points of \mathcal{B}_{KS} may not correspond to generalized solutions of (1.1), in case (3.29) fails.*

In Section 3.3 we proved that if z is stationary for \mathcal{B}_{KS} with respect to compactly supported variations, and furthermore (3.29) holds true, i.e.

$$\langle z'(\tau^*), iz(\tau^*) \rangle = 0, \quad \text{for some } \tau^* \in [0, 1],$$

then $x(t) = \bar{z}(\tau_z(t))iz(\tau_z(t))$ is a solution of the perturbed Kepler problem (under the identification $\mathbb{H} \cong \mathbb{R}^3$). Of course, if z satisfies either Dirichlet or Neumann homogeneous boundary conditions on

$\{0, 1\}$, then the above condition follows. Then a natural question is whether periodic critical points of \mathcal{B}_{KS} do satisfy such condition, too. In the following we provide an example showing that in general this is not the case, and furthermore the corresponding x is not a generalized solution of the perturbed Kepler problem.

For concreteness, let $p \equiv 0$ and $T = 1$, so that the functional \mathcal{B}_{KS} writes as

$$\mathcal{B}_{\text{KS}}(z) = \frac{1}{\mathcal{L}(z)}\mathcal{Q}(z) + \mathcal{L}(z) = 2 \int_0^1 |z|^2 \int_0^1 |z'|^2 + \frac{1}{\int_0^1 |z|^2}.$$

Then, as before, z is a critical point of \mathcal{B}_{KS} in $H^1(\mathbb{R}/\mathbb{Z}; \mathbb{H})$ if and only if

$$\begin{cases} 2Az - 2Bz'' - \frac{1}{B^2}z = 0 \\ z(0) = z(1), \quad z'(0) = z'(1), \end{cases} \quad \text{where } A := \int_0^1 |z'|^2, \quad B := \int_0^1 |z|^2. \quad (3.39)$$

Now, let

$$z(\tau) := \Lambda (\sin 2\pi\tau + i \cos 2\pi\tau + j \sin 2\pi\tau + k \cos 2\pi\tau)$$

Then $A = 8\pi^2\Lambda^2$, $B = 2\Lambda^2$, and z satisfies (3.39) provided $\Lambda^{-6} = 128\pi^2$. On the other hand,

$$\langle z', iz \rangle = 2\pi\Lambda^2 \left(-\cos^2 2\pi\tau - \sin^2 2\pi\tau - \cos^2 2\pi\tau - \sin^2 2\pi\tau \right) = -4\pi\Lambda^2.$$

Hence (3.29) does not hold. Actually, if we try to perform the usual change of variable we obtain $\tau = \tau_z(t) = t$ and

$$x(t) = \bar{z}(\tau_z(t))iz(\tau_z(t)) = 2\Lambda^2k,$$

which is not a solution of the Kepler problem.

On the contrary, let us consider

$$w(\tau) := \Lambda [\sin 2\pi\tau + i \cos 2\pi\tau + j \cos 2\pi\tau + k \sin 2\pi\tau].$$

Also w satisfies (3.39), with the same value of Λ . Then $\langle w', iw \rangle \equiv 0$, and actually

$$\begin{aligned} x(t) &= \bar{w}(\tau_w(t))iw(\tau_w(t)) = 2\Lambda^2 [j(\cos^2 2\pi t - \sin^2 2\pi t) + k(2 \sin 2\pi t \cos 2\pi t)] \\ &= 2\Lambda^2 [j \cos 4\pi t + k \sin 4\pi t], \end{aligned}$$

which is a Keplerian circular motion in the jk -plane:

$$\ddot{x} = -16\pi^2x = -\frac{x}{8\Lambda^6} = -\frac{x}{|x|^3}.$$

4 The Dirichlet problem for \mathcal{B}

This section deals with the functional \mathcal{B} introduced in (3.1) (corresponding to Case 1 discussed in Section 2). Actually the same arguments can be applied with minor simplifications also to \mathcal{B}_{LC} and \mathcal{B}_{KS} .

We will prove the following result.

Theorem 4.1. *Let $p \in C([0, T]; \mathbb{R}^d)$. Then*

$$\min_{H_0^1(0,1; \mathbb{R}^d)} \mathcal{B}$$

is achieved and any minimizer y is such that $y(\tau) \neq 0$, for any $\tau \in (0, 1)$. In particular if $p \in C^1(0, T)$ then any minimizer y corresponds to a generalized solution x of (1.1), with collisions in $t = 0$ and $t = T$.

Corollary 4.2. *Assume that $p \in C^1(\mathbb{R}; \mathbb{R}^d)$ is $2T$ -periodic and even. Then the minimizer y in the above theorem can be extended as an even, 2 -periodic loop such that the corresponding x is an even, $2T$ -periodic generalized solution of (1.1), with collision at $t = kT$, $k \in \mathbb{Z}$.*

A result similar to the above corollary was obtained by Rabinowitz in [15], for a different class of problems. More precisely, he deals with autonomous Hamiltonian systems, although treating more general singularities. His construction of periodic generalized solutions is based on that of brake-collision orbits, which can be extended to periodic ones. It is worth mentioning that, even though he deals with a weaker notion of generalized solution, by construction he finds solutions which fulfill also Definition 1.1.

Lemma 4.3. *For every $y \in H_0^1(0, 1; \mathbb{R}^d)$, $y \neq 0$, it holds*

$$\mathcal{B}(y) \geq \frac{1}{4} \frac{\|y'\|_2^2}{\mathcal{L}(y)} + \mathcal{L}(y) - 4\|p\|_\infty^2 T^3 \geq \|y'\|_2 - 4\|p\|_\infty^2 T^3.$$

Proof. To start with, we infer that

$$\mathcal{B}(y) \geq \frac{\|y'\|_2^2}{2\mathcal{L}(y)} + \mathcal{L}(y) - \mathcal{L}(y)\|p\|_\infty \int_0^1 |y|^4.$$

Since

$$|y(\tau)|^2 = \int_0^\tau 2\langle y, y' \rangle \leq 2\|y\|_2 \|y'\|_2$$

we deduce the Gagliardo-Nirenberg inequality

$$\int_0^1 |y|^4 \leq \|y\|_\infty^2 \|y\|_2^2 \leq 2\|y\|_2^3 \|y'\|_2.$$

Recalling the definition of $\mathcal{L}(y)$ we obtain

$$\mathcal{B}(y) \geq \frac{\|y'\|_2^2}{2\mathcal{L}(y)} + \mathcal{L}(y) - 2\|p\|_\infty T^{3/2} \frac{\|y'\|_2}{\sqrt{\mathcal{L}(y)}}.$$

Recalling the elementary inequalities

$$2\|p\|_\infty T^{3/2} \frac{\|y'\|_2}{\sqrt{\mathcal{L}(y)}} \leq \frac{1}{4} \frac{\|y'\|_2^2}{\mathcal{L}(y)} + 4\|p\|_\infty^2 T^3,$$

and

$$\|y'\|_2 \leq \frac{1}{4} \frac{\|y'\|_2^2}{\mathcal{L}(y)} + \mathcal{L}(y),$$

we easily conclude. □

Corollary 4.4. *Let $(y_n)_n \subset H_0^1(0, 1; \mathbb{R}^d)$. Then*

$$\mathcal{B}(y_n) \leq M \quad \implies \quad \|y_n'\|_2 \leq C_1(M), \quad 0 < C_2(M) \leq \mathcal{L}(y_n) \leq C_3(M),$$

for some constants $C_1(M) = C_3(M) = M + 4\|p\|_\infty^2 T^3$ and $C_2(M) = \pi^2 T [C_1(M)]^{-2}$ (by Poincaré inequality).

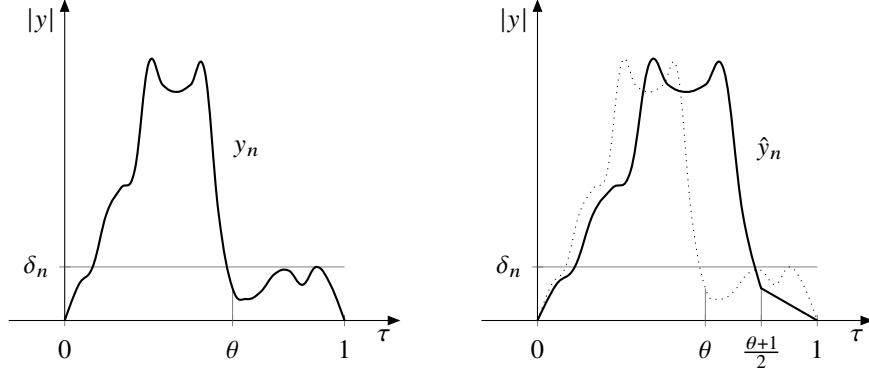


Figure 1: test function for Lemma 4.5, as defined in equation (4.1).

Lemma 4.5. Let $(y_n)_n \subset H_0^1(0, 1; \mathbb{R}^d)$ be a minimizing sequence for the functional \mathcal{B} on $H_0^1(0, 1; \mathbb{R}^d)$. Then for any $\theta \in (0, 1)$ there exist $\bar{\delta} > 0$ and $N \in \mathbb{N}$ such that

$$\max_{[\theta, 1]} |y_n| \geq \bar{\delta}, \quad \forall n \geq N.$$

An analogous result holds in $[0, \theta]$.

Proof. By contradiction let us assume that, for some $\theta \in (0, 1)$, there exists a sequence $(\delta_n)_n$ such that

$$\max_{[\theta, 1]} |y_n| = \delta_n \quad \text{and} \quad \delta_n \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

For $k = \frac{2\theta}{\theta+1} < 1$ and $\ell_n(\tau) = \frac{2y_n(\theta)}{\theta-1}(\tau-1)$, we define (see Fig. 1)

$$\hat{y}_n(\tau) := \begin{cases} y_n(k\tau), & \tau \in [0, \frac{\theta+1}{2}], \\ \ell_n(\tau), & \tau \in (\frac{\theta+1}{2}, 1]. \end{cases} \quad (4.1)$$

We claim that there exists $C > 0$, independent of n , and a sequence γ_n , with $\gamma_n \rightarrow 0$ as $n \rightarrow +\infty$, such that

$$\mathcal{B}(\hat{y}_n) \leq \mathcal{B}(y_n) - C + \gamma_n, \text{ as } n \rightarrow +\infty. \quad (4.2)$$

In order to do that, let us first estimate the terms involving first derivatives:

$$\begin{aligned} \int_0^1 |\hat{y}_n'|^2 &= k^2 \int_0^{\frac{\theta+1}{2}} |y_n'(k\tau)|^2 d\tau + \frac{4}{(\theta-1)^2} |y_n(\theta)|^2 \frac{1-\theta}{2} = \\ &= k \int_0^\theta |y_n'|^2 + \frac{2}{\theta-1} |y_n(\theta)|^2 \leq k \int_0^1 |y_n'|^2 + \frac{2}{1-\theta} \delta_n^2, \end{aligned}$$

and, with similar computations,

$$\int_0^1 \frac{\langle \hat{y}_n, \hat{y}_n' \rangle^2}{|\hat{y}_n|^2} \leq k \int_0^1 \frac{\langle y_n, y_n' \rangle^2}{|y_n|^2} + \frac{2}{1-\theta} \delta_n^2.$$

These estimates imply that

$$\mathcal{Q}(\hat{y}_n) = k\mathcal{Q}(y_n) + \frac{4}{1-\theta}\delta_n^2. \quad (4.3)$$

Next, we obtain asymptotic expansions for the remaining terms. Since $|\hat{y}_n| \leq \delta_n$ on $[\frac{\theta+1}{2}, 1]$ and $|y_n| \leq \delta_n$ on $[\theta, 1]$, a direct computation shows that

$$\|\hat{y}_n\|_2^2 = \frac{1}{k}\|y_n\|_2^2 + O(\delta_n^2).$$

This is equivalent to

$$\frac{1}{\mathcal{L}(\hat{y}_n)} = \frac{1}{k} \frac{1}{\mathcal{L}(y_n)} + O(\delta_n^2). \quad (4.4)$$

We know from Corollary 4.4 that $\mathcal{L}(y_n)$ lies between two positive constants. Hence the previous expansion leads automatically to

$$\mathcal{L}(\hat{y}_n) = k\mathcal{L}(y_n) + O(\delta_n^2). \quad (4.5)$$

The last term to estimate is $\mathcal{R}(\hat{y}_n)$. First we define

$$\hat{t}_n(\tau) = \mathcal{L}(\hat{y}_n) \int_0^\tau |\hat{y}_n(\xi)|^2 d\xi, \quad t_n(\tau) = \mathcal{L}(y_n) \int_0^\tau |y_n(\xi)|^2 d\xi.$$

These functions are related by the identity

$$\hat{t}_n(\tau/k) = \frac{\mathcal{L}(\hat{y}_n)}{k\mathcal{L}(y_n)} t_n(\tau) \quad \text{if } \tau \in [0, \theta]. \quad (4.6)$$

From the definition of \mathcal{R} and Corollary 4.4,

$$\begin{aligned} \mathcal{R}(\hat{y}_n) &= \int_0^1 |\hat{y}_n|^3 \langle \hat{y}_n, p \circ \hat{t}_n \rangle = \\ &= \int_0^{\frac{\theta+1}{2}} |y_n(k\tau)|^3 \langle y_n(k\tau), p(\hat{t}_n(\tau)) \rangle d\tau + O(\delta_n^4) = \\ &= \frac{1}{k} \int_0^\theta |y_n(\tau)|^3 \langle y_n(\tau), p(t_n(\tau)) \rangle d\tau + \alpha_n + O(\delta_n^4), \end{aligned}$$

where

$$\alpha_n = \frac{1}{k} \int_0^\theta |y_n(\tau)|^3 \langle y_n(\tau), p(\hat{t}_n(\tau/k)) - p(t_n(\tau)) \rangle d\tau.$$

In view of (4.5), (4.6) and the uniform continuity of p , we conclude that $\alpha_n \rightarrow 0$. Note that the rate of convergence will be of order of $\omega(\delta_n^2)$, where ω is a modulus of continuity of the function p . Using once again that $|y_n| \leq \delta_n$ on $[\theta, 1]$, we conclude that

$$\mathcal{R}(\hat{y}_n) = \frac{1}{k}\mathcal{R}(y_n) + o(1), \quad \text{as } n \rightarrow +\infty. \quad (4.7)$$

From (4.4), (4.3), (4.5) and (4.7) we deduce that

$$\mathcal{B}(\hat{y}_n) \leq \mathcal{B}(y_n) - (1-k)\mathcal{L}(y_n) + \gamma_n,$$

with $\gamma_n \rightarrow 0$. Since $k \in (0, 1)$ and $\mathcal{L}(y_n)$ is bounded away from 0, we deduce the existence of a strictly positive constant C such that (4.2) holds, which contradicts the nature of the sequence $(y_n)_n$. \square

Proposition 4.6. *Let $(y_n)_n \subset H_0^1(0, 1; \mathbb{R}^d)$ be a minimizing sequence for the functional \mathcal{B} on $H_0^1(0, 1; \mathbb{R}^d)$, such that $y_n \rightharpoonup y_\infty$ in $H_0^1(0, 1; \mathbb{R}^d)$. Then*

$$|y_\infty(\tau)| > 0, \quad \forall \tau \in (0, 1).$$

Proof. Recall that any minimizing sequence has a limit point, by Corollary 4.4. By contradiction, let us assume that the limit function y_∞ vanishes at some $\tau_0 \in (0, 1)$. We define the interval $[\tau^*, \tau^{**}]$ as the connected component of $\{\tau : y_\infty(\tau) = 0\}$ containing τ_0 , in such a way that

$$y_\infty|_{[\tau^*, \tau^{**}]} \equiv 0, \quad y_\infty|_{[a, b]} \not\equiv 0 \text{ for any } [a, b] \supsetneq [\tau^*, \tau^{**}] \quad (4.8)$$

(notice that it may happen that $\tau^* = \tau^{**} = \tau_0$). By Lemma 4.5 and by the uniform convergence of the minimizing sequence, we deduce the existence of $\bar{\delta} > 0$ such that

$$\max_{[0, \tau^*]} |y_\infty(\tau)| \geq \bar{\delta} \quad \text{and} \quad \max_{[\tau^{**}, 1]} |y_\infty(\tau)| \geq \bar{\delta},$$

so that $0 < \tau^* \leq \tau^{**} < 1$. Let us now fix $\delta \in (0, \bar{\delta})$ which will be specified in the following. The uniform convergence of y_n to y_∞ guarantees that, for n sufficiently large,

$$\max_{[0, \tau^*]} |y_n(\tau)| > \delta, \quad \max_{[\tau^{**}, 1]} |y_n(\tau)| > \delta, \quad |y_n(\tau^*)| < \frac{\delta}{10}, \quad \text{and} \quad |y_n(\tau^{**})| < \frac{\delta}{10}. \quad (4.9)$$

Consequently, the following sequences are well defined

$$\begin{aligned} a_n &= a_n(\delta) := \max\{\tau < \tau^* : |y_n(\tau)| = \delta\}, \\ b_n &= b_n(\delta) := \min\{\tau > \tau^{**} : |y_n(\tau)| = \delta\}; \end{aligned}$$

furthermore

$$|y_n(\tau)| < \delta \text{ on } (a_n, b_n) \quad \text{and} \quad |y_n(a_n)| = |y_n(b_n)| = \delta.$$

Note that, up to subsequences, we have

$$a_n = a_n(\delta) \rightarrow a_\infty(\delta) \quad \text{and} \quad b_n = b_n(\delta) \rightarrow b_\infty(\delta),$$

where, by uniform convergence, $|y_\infty(a_\infty)| = |y_\infty(b_\infty)| = \delta$, $|y_\infty| \leq \delta$ on (a_∞, b_∞) . Then (4.8) implies

$$a_\infty(\delta) \rightarrow \tau^* \quad \text{and} \quad b_\infty(\delta) \rightarrow \tau^{**}, \quad \text{as } \delta \rightarrow 0.$$

Recalling that $(y_n)_n$ converges uniformly to y_∞ we conclude that for any $\varepsilon > 0$

$$\tau^* - a_n < \varepsilon \quad \text{and} \quad b_n - \tau^{**} < \varepsilon, \quad (4.10)$$

for δ sufficiently small and $n > N(\delta)$.

In order to obtain a contradiction we consider, for n large, the sequence (see Fig. 2)

$$\hat{y}_n(\tau) := \begin{cases} \delta \hat{U}_n(\tau), & \text{on } (a_n, b_n), \\ y_n(\tau), & \text{on } [0, 1] \setminus (a_n, b_n), \end{cases} \quad (4.11)$$

where the path $\hat{U}_n(\tau)$, $\tau \in (a_n, b_n)$, traces the (shortest) arc of geodesic on the unitary sphere in \mathbb{R}^d joining $y_n(a_n)/\delta$ with $y_n(b_n)/\delta$ and $|\hat{U}_n'(\tau)|$ is constant.

As in (4.2), we claim to prove that $\mathcal{B}(\hat{y}_n)$ strictly lowers $\mathcal{B}(y_n)$ uniformly, at least for n sufficiently large. We argue similarly to the previous lemma estimating $\mathcal{L}(\hat{y}_n)$, $\mathcal{Q}(\hat{y}_n)$, $\mathcal{R}(\hat{y}_n)$ in terms of the same

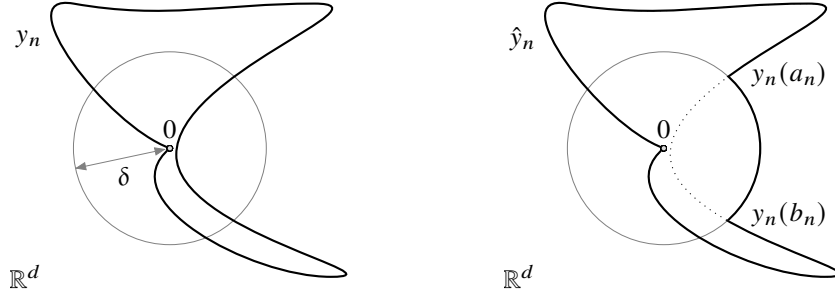


Figure 2: test function for Proposition 4.5, as defined in equation (4.11).

functionals evaluated at y_n . In the present case the variation \hat{y}_n differs from y_n on the interval (a_n, b_n) hence we have

$$\|y_n\|_2^2 \leq \|\hat{y}_n\|_2^2 \leq \|y_n\|_2^2 + \delta^2(b_n - a_n),$$

so that, being $(a_n, b_n) \subset (0, 1)$

$$0 \leq \frac{1}{\mathcal{L}(\hat{y}_n)} - \frac{1}{\mathcal{L}(y_n)} \leq \frac{\delta^2}{T}(b_n - a_n) \leq \frac{\delta^2}{T} \quad (4.12)$$

and

$$\mathcal{L}(y_n)(1 - \beta_n) \leq \mathcal{L}(\hat{y}_n) \leq \mathcal{L}(y_n), \quad \beta_n = \delta^2 \frac{(b_n - a_n)}{\|y_n\|_2^2 + \delta^2(b_n - a_n)}. \quad (4.13)$$

Furthermore, using the continuity of p and equations (4.12) and (4.13) we obtain

$$\mathcal{R}(\hat{y}_n) = \mathcal{R}(y_n) + o(1) \text{ as } n \rightarrow +\infty. \quad (4.14)$$

In order to compute the difference between the kinetic terms we introduce the radial and angular variables in \mathbb{R}^d ; for any $y \in \mathbb{R}^d$ we write

$$y = rU, \text{ with } r \geq 0, \text{ and } U \in \mathbb{S}^{d-1}.$$

Given $y \in H_0^1(0, 1; \mathbb{R}^d)$, we have that $\langle U(\tau), U'(\tau) \rangle = 0$ and $|y(\tau)|' = r'(\tau)$, hence

$$\mathcal{Q}(y) = \frac{1}{2} \int_0^1 [4(r')^2 + r^2|U'|^2].$$

Since the radial part of \hat{y}_n is constant and equal to δ on (a_n, b_n) , we obtain

$$\begin{aligned} \mathcal{Q}(\hat{y}_n) - \mathcal{Q}(y_n) &= \frac{1}{2} \int_{a_n}^{b_n} [\delta^2|\hat{U}'_n|^2 - 4(r'_n)^2 - r_n^2|U'_n|^2] \leq \\ &\leq \frac{1}{2} \delta^2 \frac{|U_n(b_n) - U_n(a_n)|^2}{b_n - a_n} - 2 \int_{a_n}^{b_n} (r'_n)^2 \leq 2\delta^2 \underbrace{\frac{1}{b_n - a_n}}_{(A)} - 2 \underbrace{\int_{a_n}^{b_n} (r'_n)^2}_{(B)}. \end{aligned}$$

Now, on the one hand,

$$4(A) = \min_{\tau \in (a_n, b_n)} \left(\frac{1}{\tau - a_n} + \frac{1}{b_n - \tau} \right) \leq \frac{1}{\tau^* - a_n} + \frac{1}{b_n - \tau^{**}};$$

on the other hand, using Cauchy-Schwarz inequality and Eq. (4.9),

$$\begin{aligned} (B) &\geq \int_{a_n}^{\tau^*} (r'_n)^2 + \int_{\tau^{**}}^{b_n} (r'_n)^2 \geq \frac{|r_n(\tau^*) - r_n(a_n)|^2}{\tau^* - a_n} + \frac{|r_n(b_n) - r_n(\tau^{**})|^2}{b_n - \tau^{**}} \\ &\geq \left(\frac{9}{10} \right)^2 \delta^2 \left(\frac{1}{\tau^* - a_n} + \frac{1}{b_n - \tau^{**}} \right). \end{aligned}$$

We can then deduce

$$\mathcal{Q}(\hat{y}_n) - \mathcal{Q}(y_n) \leq \delta^2 \left[\frac{1}{2} - 2 \left(\frac{9}{10} \right)^2 \right] \left(\frac{1}{\tau^* - a_n} + \frac{1}{b_n - \tau^{**}} \right) < -\delta^2 \left(\frac{1}{\tau^* - a_n} + \frac{1}{b_n - \tau^{**}} \right). \quad (4.15)$$

Using Corollary 4.4 and Eqs. (4.12)-(4.13)-(4.14)-(4.15), we can estimate the difference

$$\begin{aligned} \mathcal{B}(\hat{y}_n) - \mathcal{B}(y_n) &= [\mathcal{Q}(\hat{y}_n) - \mathcal{L}(y_n) \mathcal{L}(\hat{y}_n) \mathcal{R}(\hat{y}_n)] \left(\frac{1}{\mathcal{L}(\hat{y}_n)} - \frac{1}{\mathcal{L}(y_n)} \right) + \\ &\quad \frac{1}{\mathcal{L}(y_n)} [\mathcal{Q}(\hat{y}_n) - \mathcal{Q}(y_n)] + \mathcal{L}(y_n) [\mathcal{R}(\hat{y}_n) - \mathcal{R}(y_n)] \leq \\ &\leq \delta^2 \left[C_1 - C_2 \left(\frac{1}{\tau^* - a_n} + \frac{1}{b_n - \tau^{**}} \right) \right], \end{aligned}$$

where C_1 and C_2 are positive constants not depending on δ and n . By virtue of Eq. (4.10), choosing $\delta > 0$ sufficiently small, we contradict the minimality of the sequence $(y_n)_n$. \square

In order to conclude the proof of Theorem 4.1 we need the following quite general result.

Lemma 4.7. *Let $y_\infty \in H_0^1(0, 1; \mathbb{R}^d)$ and $(y_n)_n \subset H_0^1(0, 1; \mathbb{R}^d)$ be such that*

- (i) $y_n \rightharpoonup y_\infty$ in $H_0^1(0, 1; \mathbb{R}^d)$;
- (ii) $|Z| = 0$, where $Z := \{\tau \in (0, 1) : y_\infty(\tau) = 0\}$.

Then $|y_n| \rightharpoonup |y_\infty|$ in $H_0^1(0, 1; \mathbb{R})$.

Proof. We have already observed that $|y_\infty|, |y_n| \in H_0^1(0, 1; \mathbb{R})$, for any n ; we denote $|y_\infty|', |y_n|'$ their weak derivatives, as in equation (2.5). Our claim is to prove that for any $\psi \in H_0^1(0, 1)$

$$\langle |y_n|, \psi \rangle_{H_0^1(0,1)} \rightarrow \langle |y_\infty|, \psi \rangle_{H_0^1(0,1)}, \quad \text{as } n \rightarrow +\infty,$$

or, equivalently, that given any $\varphi \in L^2(0, 1)$ and $\varepsilon > 0$,

$$\limsup_{n \rightarrow +\infty} \left| \int_0^1 |y_n|' \varphi - \int_0^1 |y_\infty|' \varphi \right| < \varepsilon.$$

Since the measure of the set Z is zero, we can find a compact set $K_\varepsilon \subset (0, 1)$ such that

$$K_\varepsilon \cap Z = \emptyset \quad \text{and} \quad |(0, 1) \setminus K_\varepsilon| < \varepsilon.$$

The uniform convergence of $(y_n)_n$ to y_∞ implies that there exists $N = N(\varepsilon)$ such that for any $n \geq N$

$$y_n(\tau) \neq 0, \quad \forall \tau \in K_\varepsilon,$$

and hence

$$\frac{y_n(\tau)}{|y_n(\tau)|} \rightarrow \frac{y_\infty(\tau)}{|y_\infty(\tau)|}, \quad \forall \tau \in K_\varepsilon.$$

The pointwise convergence of the sequence $\frac{y_n}{|y_n|}$ to $\frac{y_\infty}{|y_\infty|}$ and Egorov's Theorem guarantee the existence of a measurable set $S_\varepsilon \subset K_\varepsilon$ such that

$$|K_\varepsilon \setminus S_\varepsilon| < \varepsilon$$

and

$$\frac{y_n}{|y_n|} \rightarrow \frac{y_\infty}{|y_\infty|}, \quad \text{uniformly on } S_\varepsilon. \quad (4.16)$$

Since weakly convergent sequences are bounded, there exists $C > 0$ such that $\|y_n\| \leq C$ for each n . We deduce that, given $\varphi \in L^2(0, 1)$,

$$\left| \int_0^1 |y_n|' \varphi - \int_{S_\varepsilon} |y_n|' \varphi \right| \leq \int_{(0,1) \setminus S_\varepsilon} \| |y_n|' \| |\varphi| \leq C \|\varphi\|_{L^2((0,1) \setminus S_\varepsilon)},$$

and the continuity of the integral of a measurable function with respect to the measure of the domain implies that

$$\|\varphi\|_{L^2((0,1) \setminus S_\varepsilon)} \leq C(\varepsilon),$$

where $C(\varepsilon)$ (which actually depends on φ) vanishes as $\varepsilon \rightarrow 0$.

Since $|y_\infty|' \varphi$ is integrable on $(0, 1)$, there exists $c(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that

$$\left| \int_0^1 |y_\infty|' \varphi - \int_{S_\varepsilon} |y_\infty|' \varphi \right| \leq c(\varepsilon).$$

We deduce

$$\left| \int_0^1 |y_n|' \varphi - \int_0^1 |y_\infty|' \varphi \right| \leq CC(\varepsilon) + c(\varepsilon) + \left| \int_{S_\varepsilon} (|y_n|' - |y_\infty|') \varphi \right|.$$

In order to estimate the last term we compute the weak derivative of the absolute value as

$$\int_{S_\varepsilon} (|y_n|' - |y_\infty|') \varphi = \underbrace{\int_{S_\varepsilon} \left\langle \frac{y_n}{|y_n|} - \frac{y_\infty}{|y_\infty|}, y_n' \right\rangle \varphi}_{I_1} + \underbrace{\int_{S_\varepsilon} \left\langle \frac{y_\infty}{|y_\infty|}, y_n' - y_\infty' \right\rangle \varphi}_{I_2}.$$

By (4.16),

$$|I_1| \leq \left\| \frac{y_n}{|y_n|} - \frac{y_\infty}{|y_\infty|} \right\|_{L^\infty(S_\varepsilon)} C \|\varphi\|_{L^2} \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Furthermore, denoting with χ_{S_ε} the characteristic function of S_ε , we write

$$I_2 = \int_0^1 \langle y_n' - y_\infty', \psi \rangle, \quad \text{where } \psi = \chi_{S_\varepsilon} \varphi \frac{y_\infty}{|y_\infty|}$$

and this quantity tends to 0 by weak convergence of $(y_n)_n$ to y_∞ . \square

Proof of Theorem 4.1. Let $(y_n)_n \subset H_0^1(0, 1; \mathbb{R}^d)$ be a minimizing sequence for \mathcal{B} such that y_n tends to y_∞ weakly in $H_0^1(0, 1; \mathbb{R}^d)$. By uniform convergence, we deduce that $\mathcal{L}(y_n) \rightarrow \mathcal{L}(y_\infty)$ and, by dominated convergence, $\mathcal{R}(y_n) \rightarrow \mathcal{R}(y_\infty)$. Furthermore, since $\mathcal{B}(y_n) \leq M$ for some M , by Corollary 4.4, $\mathcal{L}(y_\infty) \neq 0$, and we also obtain, by the w.l.s.c. of the norm

$$\frac{1}{\mathcal{L}(y_\infty)} \|y'_\infty\|_2^2 \leq \liminf_{n \rightarrow +\infty} \frac{1}{\mathcal{L}(y_n)} \|y'_n\|_2^2.$$

To conclude we need to show that

$$\frac{1}{\mathcal{L}(y_\infty)} \int_0^1 (|y_\infty'|)^2 \leq \liminf_{n \rightarrow +\infty} \frac{1}{\mathcal{L}(y_n)} \int_0^1 (|y_n'|)^2.$$

or, equivalently, that $\| |y_\infty'| \|_2 \leq \liminf \| |y_n'| \|_2$; we conclude applying Lemma 4.7, Proposition 4.6, Lemma 4.7, Proposition 3.1 (and recalling the discussion after Definition 1.1). \square

5 Periodic generalized solutions in dimension 2

In this section we assume that p is C^1 on \mathbb{R} and T -periodic, and we look for periodic generalized solutions to (1.1) in dimension $d = 2$. To do this, one would be tempted to look for critical points of the functional $\mathcal{B}_{\text{LC}}: W_1 \rightarrow \mathbb{R} \cup \{+\infty\}$, where W_1 is the space of 1-periodic loops:

$$W_1 := \{z \in H^1([0, 1]; \mathbb{C}) : z(1) = z(0)\}.$$

As mentioned in the introduction, the main obstruction in this direction is that the Palais-Smale condition is not satisfied in this setting.

We recall that a sequence $(z_n)_n$ is a (PS) sequence at level σ for \mathcal{B}_{LC} if

$$\mathcal{B}_{\text{LC}}(z_n) = \sigma + o(1), \quad \|\mathcal{B}_{\text{LC}}'(z_n)\| = o(1) \text{ as } n \rightarrow \infty,$$

and that \mathcal{B}_{LC} satisfies the (PS) condition at level σ if any such a sequence admits a strongly convergent subsequence.

To show that $\mathcal{B}_{\text{LC}}: W_1 \rightarrow \mathbb{R} \cup \{+\infty\}$ does not satisfy the (PS) condition, we take $p \equiv 0$ and observe that any sequence $(z_n)_n$ of constant functions with $|z_n| \rightarrow +\infty$ satisfies the (PS) condition. Indeed, from the definition of \mathcal{B}_{LC} and Remark 3.14,

$$\mathcal{B}_{\text{LC}}(z_n) = \mathcal{L}(z_n), \quad \mathcal{B}_{\text{LC}}'(z_n)[v] = -\frac{2}{T} \mathcal{L}(z_n)^2 \int_0^1 \langle z_n, v \rangle.$$

In particular, $\|\mathcal{B}_{\text{LC}}'(z_n)\| \leq 2T/|z_n|^3$.

To recover the Palais-Smale property, we will search for critical points of \mathcal{B}_{LC} in the space of anti-periodic functions

$$W_{-1} := \{z \in H^1([0, 1]; \mathbb{C}) : z(1) = -z(0)\}.$$

Notice that if $z \in W_{-1}$ then $|z|^2$ can be extended as a 1-periodic function; as a consequence, the function t_z defined in (3.15) is such that

$$t_z(\tau + k) = t_z(\tau) + kT, \quad \forall k \in \mathbb{Z},$$

and finally the function $\tau \mapsto p \circ t_z$ is 1-periodic and $C^1(\mathbb{R})$.

We will show the following result.

Theorem 5.1. *Let $p \in C^1(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^2)$. Then there exist infinitely many distinct critical points of \mathcal{B}_{LC} in the space W_{-1} of antiperiodic orbits, corresponding to infinitely many T -periodic generalized solutions of the perturbed Kepler problem (1.1) in dimension $d = 2$.*

Notice that the functional \mathcal{B}_{LC} is even. In order to prove the theorem, we will show that it satisfies the Palais-Smale (PS) condition at every level, and that it is bounded below. This will allow to exploit the theory of Krasnoselskii's genus, which we briefly recall here below (we follow [3, Ch. 10]).

Let $\mathcal{A} := \{A \in W_{-1} \setminus \{0\} : A = -A, A \text{ is closed}\}$. The *genus* of A is defined as

$$\gamma(A) := \inf\{n : \exists \phi \in C(A; \mathbb{R}^n \setminus \{0\}), \phi \text{ odd}\};$$

if such a ϕ does not exist we define $\gamma(A) = +\infty$, while $\gamma(\emptyset) = 0$. Moreover, let

$$\mathcal{A}_m := \{A \subset \mathcal{A} : A \text{ is compact and } \gamma(A) \geq m\}, \quad \sigma_m := \inf_{A \in \mathcal{A}_m} \sup_A \mathcal{B}_{LC}.$$

We are going to exploit the following well-known result.

Proposition 5.2 ([3, Prop. 10.8]). *Each finite σ_m is a critical level for \mathcal{B}_{LC} provided the (PS) condition holds at level σ_m . Moreover, if $\sigma_m = \sigma_{m+1}$ for some m , then there exist infinitely many critical points at level σ_m .*

The proof of this result relies on the fact that if a functional satisfies the Palais-Smale condition at some non-critical level σ , then it is possible to continuously deform a $(\sigma + \varepsilon)$ -sublevel into a $(\sigma - \varepsilon)$ -one (and this deformation can be done preserving symmetry). In the previous context this would contradict the minimax definition of the levels σ_m , since the genus of a set is not decreasing under continuous deformations.

Actually, even though the previous result was originally stated for functionals J which are C^1 in the whole space and such that 0 is not a critical point of J at level σ_m , it readily applies to \mathcal{B}_{LC} , which is C^1 only outside the origin but has the strong property of continuity at $z = 0$,

$$\mathcal{B}_{LC}(z) \rightarrow +\infty \quad \text{as } \|z\|_2 \rightarrow 0.$$

Therefore, for each $a < b < +\infty$, the sets $\{z \in W_{-1} : a \leq \mathcal{B}_{LC}(z) \leq b\}$ are closed in W_{-1} and the deformation argument applies with no restriction.

In order to apply Proposition 5.2 we need some preliminary lemmas.

Lemma 5.3. *For every $z \in W_{-1}$,*

$$\|z\|_\infty^2 \leq 2\|z\|_2 \|z'\|_2.$$

Proof. To start with we notice that, for every $\tau_1 \leq \tau_2$,

$$\left| |z(\tau_2)|z(\tau_2) - |z(\tau_1)|z(\tau_1) \right| \leq \int_{\tau_1}^{\tau_2} \left| \frac{d}{d\tau} (|z|z) \right| d\tau \leq \int_{\tau_1}^{\tau_2} 2|z||z'| d\tau \leq 2\|z\|_2 \|z'\|_2$$

(recall (2.5)). Since $|z(1)|z(1) + |z(0)|z(0) = 0$ we obtain, for every $\tau \in [0, 1]$,

$$2|z(\tau)|^2 \leq \left| |z(\tau)|z(\tau) - |z(0)|z(0) \right| + \left| |z(1)|z(1) - |z(\tau)|z(\tau) \right| \leq 4\|z\|_2 \|z'\|_2,$$

and the lemma follows. \square

Lemma 5.4. *Let $z \in W_{-1} \setminus \{0\}$, $\alpha := \|z\|_2 \|z'\|_2$, $\beta := \|z\|_2^2$. Then*

$$\frac{1}{T} \mathcal{B}_{LC}(z) \geq \frac{2}{T^2} \alpha^2 - 2\|p\|_\infty \alpha + \frac{1}{\beta}.$$

In particular,

$$\inf_{W_{-1}} \mathcal{B}_{LC} \geq -\frac{\|p\|_\infty^2 T^2}{2}.$$

Proof. By Lemma 5.3 we have that $\|z\|_4^4 \leq \|z\|_\infty^2 \|z\|_2^2 \leq 2\|z\|_2^3 \|z'\|_2$. Then

$$|\mathcal{R}(z)| = \left| \int_0^1 \langle p \circ t_z, z^2 |z|^2 \rangle \right| \leq \int_0^1 |p \circ t_z| |z|^4 \leq 2\|p\|_\infty \|z\|_2^3 \|z'\|_2,$$

and the lemma follows, since

$$\frac{1}{T} \mathcal{B}_{LC}(z) = \frac{2}{T^2} \|z\|_2^2 \|z'\|_2^2 + \frac{1}{\|z\|_2^2} (1 + \mathcal{R}(z)). \quad \square$$

Lemma 5.5. *For every $a \geq \inf_{W_{-1}} \mathcal{B}_{LC}$ there exist positive constants $C_i = C_i(a)$, $i = 1, 2, 3$, such that*

$$\mathcal{B}_{LC}(z) < a \quad \implies \quad \begin{cases} \|z'\|_2 \leq C_1(a) \\ C_2(a) \leq \|z\|_2 \leq C_3(a). \end{cases}$$

Proof. With the notations of the previous lemma, we have

$$\frac{2}{T^2} \alpha^2 - 2\|p\|_\infty \alpha \leq a \quad \text{and} \quad \frac{1}{\beta} \leq a + \frac{\|p\|_\infty^2 T^2}{2},$$

so that α is bounded above and $\beta = \|z\|_2^2$ is bounded away from 0, and the existence of C_2 follows. Then

$$\|z'\|_2^2 = \frac{\alpha^2}{\beta}$$

is bounded above, and also the existence of C_1 follows. Finally, since $\|z\|_2^2 \leq \|z\|_\infty^2$, Lemma 5.3 implies the Poincaré inequality $\|z\|_2 \leq 2\|z'\|_2$, and also the existence of C_3 follows. \square

The previous results allow to prove the Palais-Smale property for \mathcal{B}_{LC} .

Lemma 5.6. *The functional \mathcal{B}_{LC} satisfies the Palais-Smale condition in W_{-1} at any level $a \geq \inf_{W_{-1}} \mathcal{B}_{LC}$.*

Proof. Let $(z_n)_n \subset W_{-1}$ be a (PS) sequence for \mathcal{B}_{LC} at level a . By Lemma 5.5 we have that $\|z'_n\|_2 \leq C_1$, $0 < C_2 \leq \|z_n\|_2 \leq C_3$, where the constants are independent of n . Then, up to a subsequence, $z_n \rightharpoonup z \neq 0$ weakly in W_{-1} and uniformly. Since $(z_n - z)_n$ is bounded, using Remark 3.14 we have that

$$\mathcal{B}_{LC}'(z_n)[z_n - z] = \frac{4}{\mathcal{L}_n} \int_0^1 \left[\langle z'_n, z'_n - z' \rangle + \left\langle \frac{\mathcal{L}_n}{2T} \left(\mathcal{Q}_n - \mathcal{L}_n^2 (1 + \mathcal{R}_n) \right) z_n + \delta_{z_n}, z_n - z \right\rangle \right] = o(1)$$

(recall that δ_z does not depend on z'). Since $z_n - z \rightarrow 0$ strongly in L^2 , and all the terms are bounded, we have

$$\int_0^1 \left\langle \frac{\mathcal{L}_n}{2T} \left(\mathcal{Q}_n - \mathcal{L}_n^2 (1 + \mathcal{R}_n) \right) z_n + \delta_{z_n}, z_n - z \right\rangle = o(1),$$

and thus

$$o(1) = \int_0^1 \langle z'_n, z'_n - z' \rangle = \int_0^1 \langle z'_n, z'_n \rangle - \int_0^1 \langle z'_n, z' \rangle = \int_0^1 \langle z'_n, z'_n \rangle - \int_0^1 \langle z', z' \rangle + o(1).$$

Then $\|z'_n\|_2 \rightarrow \|z'\|_2$, which, together with the weak convergence, yields the strong one, concluding the proof. \square

Proposition 5.7. *The functional \mathcal{B}_{LC} admits infinitely many critical points in W_{-1} .*

Proof. The proposition follows from Proposition 5.2: indeed, each $\sigma_m > -\infty$ because of Lemma 5.4; each $\sigma_m < +\infty$ because \mathcal{A}_m is not empty (for instance, it contains homeomorphic symmetric images of \mathbb{S}^{m-1}); (PS) holds at any σ_m , by Lemma 5.6. \square

Of course, critical points of \mathcal{B}_{LC} in W_{-1} are solutions of a boundary value problem for the corresponding Euler-Lagrange equation.

Lemma 5.8. *Let z be a critical point of \mathcal{B}_{LC} in W_{-1} . Then $z \in C^3([0, 1])$ satisfies*

$$\begin{cases} z'' = \frac{\mathcal{L}}{2T} (Q - \mathcal{L}^2(1 + \mathcal{R})) z + \delta_z, & \tau \in (0, 1), \\ z(1) = -z(0), & z'(1) = -z'(0), \end{cases}$$

where δ_z is defined in Lemma 3.13.

Proof. Since $\mathcal{D}(0, 1; \mathbb{C}) \subset W_{-1}$, the regularity of z and equation (3.16) follow by Lemma 3.13. As a consequence, for every $v \in W_{-1}$ we can integrate by parts in Remark 3.14, obtaining

$$\begin{aligned} 0 = \mathcal{B}_{LC}'(z)[v] &= \frac{4}{\mathcal{L}} \int_0^1 \left[\langle z', v' \rangle + \left\langle \frac{\mathcal{L}}{2T} (Q - \mathcal{L}^2(1 + \mathcal{R})) z + \delta_z, v \right\rangle \right] \\ &= \frac{4}{\mathcal{L}} [\langle z'(1), v(1) \rangle - \langle z'(0), v(0) \rangle] = \frac{4}{\mathcal{L}} \langle z'(1) + z'(0), v(1) \rangle, \end{aligned}$$

and the lemma follows choosing $v(\tau) = [z'(1) + z'(0)] \cos(\pi\tau) \in W_{-1}$. \square

End of the proof of Theorem 5.1. By Proposition 5.7, we are left to show that if z is a critical point of \mathcal{B}_{LC} in W_{-1} then

$$x(t) = z^2(\tau_z(t))$$

is a generalized T -periodic solution of (1.1).

Again, since $\mathcal{D}(0, 1; \mathbb{C}) \subset W_{-1}$, Proposition 3.12 applies, so that x is a generalized solution of (1.1). To prove that it is T -periodic we observe that, on the one hand,

$$x(T) = z^2(\tau_z(T)) = z^2(1) = (-z(0))^2 = z^2(\tau_z(0)) = x(0).$$

On the other hand, recalling Lemma 3.17, we know that

$$z(\tau_z(t))z'(\tau_z(t)) = \frac{\mathcal{L}}{2} |x(t)|\dot{x}(t) \text{ outside collisions, and } |\dot{x}(t)|^2|x(t)| = \frac{4}{\mathcal{L}^2} |z'(\tau_z(t))|^2 \text{ on } [0, T].$$

Therefore, if $z(0) \neq 0$, we can use the first equality to show $\dot{x}(T) = \dot{x}(0)$; in case $z(0) = 0$, we can use the second one and argue as in the end of the proof of Proposition 3.12 to show that

$$\lim_{t \rightarrow 0^+} \left[\frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|} \right] = \lim_{\tau \rightarrow 0^+} \frac{2|z'|^2 - \mathcal{L}^2}{\mathcal{L}^2|z|^2} = \lim_{\tau \rightarrow 1^-} \frac{2|z'|^2 - \mathcal{L}^2}{\mathcal{L}^2|z|^2} = \lim_{t \rightarrow T^-} \left[\frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|} \right]$$

and

$$\lim_{t \rightarrow 0^+} \frac{x}{|x|} = \lim_{\tau \rightarrow 0^+} \frac{z^2}{|z|^2} = \frac{2}{\mathcal{L}^2} z'(0)^2 = \frac{2}{\mathcal{L}^2} z'(1)^2 = \lim_{\tau \rightarrow 1^-} \frac{z^2}{|z|^2} = \lim_{t \rightarrow T^-} \frac{x}{|x|}.$$

To conclude, we show that if z_1, z_2 are critical points of \mathcal{B}_{LC} in W_{-1} , with $z_1 \neq \pm z_2$, then $x_1 \neq x_2$ with $x_j(t) = z_j^2(\tau_{z_j}(t))$, $j = 1, 2$. Indeed, if $x_1 = x_2 = x$ then $|z_1(\tau)| = |z_2(\tau)|$, implying that $\tau_{z_1} = \tau_{z_2}$.

Moreover $z_1^2 = z_2^2$ and therefore $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, $Q_1 = Q_2$, $\mathcal{R}_1 = \mathcal{R}_2$, $\frac{1}{z_1} \delta_{z_1} = \frac{1}{z_2} \delta_{z_2}$. In view of Lemma 5.8, both z_1 and z_2 are solutions of the same second order linear differential equation. Let $\tau_* \in [0, 1]$ be such that $z_1(\tau_*) \neq 0$. Then $z_2(\tau_*) = \pm z_1(\tau_*)$ and, writing $\tau_* = \tau_{z_j}(t_*)$

$$z_1(\tau_*)z_1'(\tau_*) = \frac{\mathcal{L}}{2} |x(t_*)|\dot{x}(t_*) = z_2(\tau_*)z_2'(\tau_*).$$

In consequence, $z_2'(\tau_*) = \pm z_1'(\tau_*)$. By the uniqueness of the initial value problem, either $z_1 = z_2$ or $z_1 = -z_2$ on $[0, 1]$. \square

6 Periodic generalized solutions in dimension 3

In order to obtain periodic solutions $x = x(t)$ to the perturbed Kepler problem in dimension 3, one would like to adapt the arguments of Section 5 to the functional \mathcal{B}_{KS} . As a further difficulty, we know from Example 3.27 that the variational principle for $\mathcal{B}_{\text{KS}}: H^1(\mathbb{R}/\mathbb{Z}; \mathbb{H}) \rightarrow \mathbb{R} \cup \{+\infty\}$ is not consistent with the periodic problem for (1.1). To overcome this difficulty we will consider the manifold \mathcal{M} composed by all non-trivial H^1 functions satisfying a condition of Floquet type, namely

$$z(\tau + 1) = \xi z(\tau), \quad \xi \in \mathbb{S}^1, \quad (6.1)$$

where $\mathbb{S}^1 := \{e^{i\theta} = \cos \theta + i \sin \theta : \theta \in \mathbb{R}\}$ is the unit circle in the plane $\mathbb{C} \subset \mathbb{H}$. The variational principle for $\mathcal{B}_{\text{KS}}: \mathcal{M} \rightarrow \mathbb{R} \cup \{+\infty\}$ will be consistent with the periodic problem.

Theorem 6.1. *Let $p \in C^1(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^3)$. Then there exist infinitely many T -periodic generalized solutions of the perturbed Kepler problem (1.1) in dimension $d = 3$.*

Remark 6.2. The use of different domains, W_{-1} or \mathcal{M} , in dimensions $d = 2$ and $d = 3$, is related to the different topology of the corresponding regularization maps. For $d = 2$, the Levi-Civita map $\Phi_{\text{LC}}: z \in \mathbb{C} \setminus \{0\} \rightarrow x = z^2 \in \mathbb{C} \setminus \{0\}$ has finite fibers $\Phi_{\text{LC}}^{-1}(x) = \{z, -z\}$. For $d = 3$, the Kustaanheimo-Stiefel map $\Phi_{\text{KS}}: z \in \mathbb{H} \setminus \{0\} \rightarrow x = \bar{z}iz \in \mathbb{IH} \setminus \{0\}$ has \mathbb{S}^1 -fibers $\Phi_{\text{KS}}^{-1}(x) = \{\xi z : \xi \in \mathbb{S}^1\}$.

We recall that, when the forcing term p takes values in the plane, the functional $\mathcal{B}_{\text{KS}}: \mathcal{M} \rightarrow \mathbb{R} \cup \{+\infty\}$ can be seen as an extension of $\mathcal{B}_{\text{LC}}: W_1 \rightarrow \mathbb{R} \cup \{+\infty\}$, see Remark 2.2. Recalling the discussion at the beginning of Section 5.1, this implies that the Palais-Smale property will not hold in general for \mathcal{B}_{KS} on \mathcal{M} (notice that, taking $\xi = 1$ in (6.1), we have that \mathcal{M} contains also the 1-periodic loops in \mathbb{H}). Nonetheless, we will show that the weaker Palais-Smale-Cerami property holds at positive levels of \mathcal{B}_{KS} . This will provide enough compactness to obtain the existence of infinitely many critical points.

The proof of Theorem 6.1 is divided in two parts: first we will specify the suitable manifold \mathcal{M} , encoding (6.1), and provide differential and Riemannian structures; secondly, we will prove the existence of critical points of \mathcal{B}_{KS} on \mathcal{M} .

6.1 The manifold \mathcal{M}

In principle, functions z such that $\bar{z}iz$ is periodic, are defined for all $\tau \in \mathbb{R}$, hence they can be seen as elements of the space $H_{\text{loc}}^1(\mathbb{R}; \mathbb{H})$. In order to set the problem in a more convenient functional space, one would be tempted to consider restrictions of such functions to the interval $(0, 1)$ (as we did in the 2-dimensional case). As a matter of fact, as we will show in the following, the right choice is to consider functions defined on the interval $(0, 2)$. For easier notation, in the following we denote

$$X := H^1(0, 2; \mathbb{H}).$$

To start our construction, for every $\xi \in \mathbb{S}^1 \subset \mathbb{C} \subset \mathbb{H}$ (i.e. $\xi = e^{i\alpha}$, with $\alpha \in \mathbb{R}$) we consider the vector space

$$W_\xi = \{z \in X : z(\tau + 1) = \xi z(\tau), \tau \in (0, 1)\},$$

endowed with the inner product induced by X . In particular, we write $W := W_1$ for the space of 1-periodic loops in X , while W_{-1} denotes the space of anti-periodic ones. It is easy to check that

$$W_{\xi_1} \cap W_{\xi_2} = \{0\} \quad \text{whenever } \xi_1 \neq \xi_2. \quad (6.2)$$

Given $\alpha \in \mathbb{R}$ and $z \in X$, we define a new function $E_\alpha z \in X$ by the formula

$$E_\alpha z(\tau) = e^{i\alpha\tau} z(\tau).$$

Then E_α induces a linear isomorphism between W_{ξ_1} and W_{ξ_2} , with $\xi_2 = e^{i\alpha}\xi_1$. The inverse operator is $E_{-\alpha}$. By direct computations,

$$\|E_\alpha z\|^2 = \|z\|_2^2 + \|z' + i\alpha z\|_2^2 \leq \|z\|_2^2 + 2(\|z'\|_2^2 + \alpha^2\|z\|_2^2) \leq \mu^2(\alpha)\|z\|^2,$$

where $\|\cdot\| = \|\cdot\|_X$, $\|\cdot\|_2 = \|\cdot\|_{L^2(0,2;\mathbb{H})}$ and $\mu^2(\alpha) = \max\{2, 1 + 2\alpha^2\}$. Summing up

$$\frac{1}{\mu(\alpha)}\|z\| \leq \|E_\alpha z\| \leq \mu(\alpha)\|z\| \quad \text{and} \quad \mu(\alpha) = \max\{\sqrt{2}, \sqrt{1 + 2\alpha^2}\}. \quad (6.3)$$

In particular, each W_ξ is isomorphic to W .

Define

$$\mathcal{M} := \bigcup_{\xi \in \mathbb{S}^1} (W_\xi \setminus \{0\}) = \{E_\alpha w : w \in W \setminus \{0\}, \alpha \in \mathbb{R}\}. \quad (6.4)$$

This is a disjoint union in view of (6.2).

Lemma 6.3. *Let $z \in \mathcal{M}$. Then the functions*

$$\bar{z}iz, \quad \bar{z}iz', \quad |z|, \quad |z'|,$$

can be extended to 1-periodic functions.

Proof. By assumption, $z \in W_\xi \setminus \{0\}$, for some $\xi = e^{i\alpha}$. Thus, for every $\tau \in (0, 1)$,

$$\bar{z}(\tau + 1)iz(\tau + 1) = \bar{z}(\tau)e^{-i\alpha}ie^{i\alpha}z(\tau) = \bar{z}(\tau)iz(\tau).$$

In particular, by continuity, $\bar{z}(2)iz(2) = \bar{z}(1)iz(1) = \bar{z}(0)iz(0)$, and the property of $\bar{z}iz$ follows. The other properties follow from analogous computations, possibly in a.e. sense when z' is involved. \square

Remark 6.4. The previous lemma shows that the correspondence $z \mapsto \bar{z}iz$ maps $\mathcal{M} \subset X$ into $H^1(0, 1; \mathbb{H})$. Notice that, in principle, different elements of \mathcal{M} may have the same restriction to $(0, 1)$, as for instance

$$z_1(\tau) = |\sin(\pi\tau)| \in W, \quad z_2(\tau) = \sin(\pi\tau) \in W_{-1}.$$

This explains the choice to work in X .

We are going to show the following results.

Proposition 6.5. *Under the previous notation:*

1. \mathcal{M} is a C^∞ submanifold of X , modelled on $W \times \mathbb{R}$;
2. the tangent space at $\mathcal{M} \ni z = E_\alpha w$, $w \in W$, is

$$\begin{aligned} T_{E_\alpha w} \mathcal{M} &= \{E_\alpha(\Delta + i\delta I \cdot w) : \Delta \in W, \delta \in \mathbb{R}\} \\ &= E_\alpha(W \oplus (iI \cdot w)\mathbb{R}), \end{aligned}$$

where $I(\tau) = \tau$ is the identity on $[0, 2]$;

3. the geodesic distance on \mathcal{M} (induced by the embedding $\mathcal{M} \hookrightarrow X$) satisfies

$$\text{dist}_{\mathcal{M}}(z_1, z_2) \geq \|z_1 - z_2\|.$$

Remark 6.6. It is worth noticing that both the dimension and the codimension of \mathcal{M} are infinite: this follows from the splitting

$$X = W \oplus W_{-1} \oplus V_4,$$

where V_4 is the 4-dimensional space

$$V_4 = \{qI : q \in \mathbb{H}\}.$$

Indeed, given z in X , we have the decomposition

$$z = P_W z + P_{W_{-1}} z + z_{**},$$

where $z_{**} = \frac{z(2) - z(0)}{2} I \in V_4$, $z = z_* + z_{**}$, and $P_{W_{\pm 1}} z = \frac{1}{2} [z_*(\tau) \pm z_*(\tau + 1)]$. Notice that z_* can be extended to a 2-periodic function, so that the previous formulas are well-defined.

Remark 6.7. Notice that, as a submanifold, \mathcal{M} inherits the Riemannian structure of X . In particular, for every $v \in T_z \mathcal{M}$,

$$\|v\|_{T_z \mathcal{M}} = \|v\|_X = \|v\|.$$

Moreover the geodesic distance on \mathcal{M} is defined as

$$\text{dist}_{\mathcal{M}}(z_0, z_1) = \inf \left\{ \int_0^1 \left\| \frac{d}{ds} \gamma(s) \right\| ds : \gamma : [0, 1] \rightarrow \mathcal{M} \text{ smooth, } \gamma(i) = z_i \right\},$$

where as usual $\|\cdot\|$ denotes the norm in X .

The rest of the section is devoted to the proof of the above proposition. To start with we notice that, since \mathcal{M} is a subset of X , it is a metric space with the induced distance.

Lemma 6.8. *The functional*

$$\mathcal{M} \ni z \mapsto \xi \in \mathbb{S}^1,$$

where $z \in W_\xi$, satisfies

$$|\xi(z_1) - \xi(z_2)| \leq \frac{4}{\max_i \|z_i\|_2} \|z_1 - z_2\|.$$

In particular, ξ is continuous.

Proof. We multiply the identity $z(\tau + 1) = \xi z(\tau)$ by $\bar{z}(\tau)$ to the right and integrate over $[0, 1]$. We obtain

$$\xi = \frac{2}{\|z\|_2^2} \int_0^1 z(\tau + 1) \bar{z}(\tau) d\tau$$

(recall that, by Lemma 6.3, $\|z\|_2^2 = 2 \int_0^1 |z|^2 = 2 \int_1^2 |z|^2$). The functional $z \mapsto \xi$ is 0-homogeneous:

$$\xi(\lambda z) = \xi(z) \text{ if } \lambda > 0.$$

Assuming first that both $\|z_i\|_2^2 = 2$ we obtain

$$\begin{aligned} |\xi(z_1) - \xi(z_2)| &= \left| \int_0^1 [z_1(\tau + 1) \bar{z}_1(\tau) - z_2(\tau + 1) \bar{z}_2(\tau)] d\tau \right| \\ &\leq \int_0^1 |z_1(\tau + 1) - z_2(\tau + 1)| \cdot |\bar{z}_1(\tau)| d\tau + \int_0^1 |z_2(\tau + 1)| \cdot |\bar{z}_1(\tau) - \bar{z}_2(\tau)| d\tau \\ &\leq \|z_1\|_{L^2(0,1)} \|z_1 - z_2\|_{L^2(1,2)} + \|z_2\|_{L^2(1,2)} \|z_1 - z_2\|_{L^2(0,1)} \leq \sqrt{2} \|z_1 - z_2\|_{L^2(0,2)}. \end{aligned}$$

In the general case, assume for concreteness $\|z_2\|_2 \leq \|z_1\|_2$. Then

$$\begin{aligned} |\xi(z_1) - \xi(z_2)| &= \left| \xi \left(\frac{z_1 \sqrt{2}}{\|z_1\|_2} \right) - \xi \left(\frac{z_2 \sqrt{2}}{\|z_2\|_2} \right) \right| \leq 2 \left\| \frac{z_1}{\|z_1\|_2} - \frac{z_2}{\|z_2\|_2} \right\|_2 \\ &= \frac{2}{\|z_1\|_2 \|z_2\|_2} \left\| \|z_2\|_2 z_1 - \|z_2\|_2 z_2 + \|z_2\|_2 z_2 - \|z_1\|_2 z_2 \right\|_2 \leq \frac{4}{\|z_1\|_2} \|z_1 - z_2\|_2. \quad \square \end{aligned}$$

By now, \mathcal{M} is a metric space. Now we are going to induce on it a structure of smooth submanifold of X , modeled on the Hilbert space $W \times \mathbb{R}$. To this end, we consider the map

$$\Phi : W \times \mathbb{R} \rightarrow X, \quad (w, \alpha) \mapsto E_\alpha w.$$

Since $W \times \mathbb{R}$ is an Hilbert space, in the following we identify it with its tangent space. Moreover, we recall that W is a subspace of X , thus we use in it the norm $\|\cdot\| = \|\cdot\|_{H^1(0,2;\mathbb{H})}$.

Lemma 6.9. Φ is C^∞ . Moreover, for every $(w, \alpha), (\Delta, \delta) \in W \times \mathbb{R}$

$$\Phi'(w, \alpha)[\Delta, \delta] = E_\alpha (\Delta + i\delta I \cdot w).$$

Proof. The parameterized curve $\mathcal{E} : \mathbb{R} \rightarrow X, \alpha \mapsto E_\alpha 1$ can be differentiated and the velocity vector is $\dot{\mathcal{E}}(\alpha) = iI \cdot \mathcal{E}(\alpha)$. This is a direct consequence of the identity $\mathcal{E}(\alpha)(\tau) = e^{i\alpha\tau}$. It is now easy to deduce that \mathcal{E} is C^∞ . In fact the successive derivatives are easily obtained from the formula for the velocity vector. Furthermore, let the map T be defined as

$$W \times X \ni (w, \varphi) \mapsto T(w, \varphi) := \varphi \cdot w \in X.$$

We have that T is bilinear and continuous, thus it is smooth as well. Since $\Phi = T \circ (\text{Id} \times \mathcal{E})$, the chain rules implies that Φ is C^∞ . Moreover

$$\Phi'(w, \alpha)[\Delta, \delta] = E_\alpha \Delta + \delta \dot{\mathcal{E}}(\alpha) w = E_\alpha (\Delta + i\delta I \cdot w). \quad \square$$

Next we address the injectivity and surjectivity properties of Φ . Given an open interval $I \subset \mathbb{R}$, we denote with $|I|$ its length and with

$$\Phi_I := \Phi|_{(W \setminus \{0\}) \times I}, \quad \mathcal{U}_I := \Phi((W \setminus \{0\}) \times I).$$

Lemma 6.10. Under the previous notation,

- $\Phi(w, \alpha + 2\pi) = \Phi(E_{2\pi} w, \alpha)$, for every $(w, \alpha) \in W \times \mathbb{R}$;
- if $|I| > 2\pi$ then Φ_I is not injective and $\mathcal{U}_I = \mathcal{M}$;
- if $|I| \leq 2\pi$ then Φ_I is injective and $\mathcal{U}_I \subsetneq \mathcal{M}$.

Proof. The first part is trivial, since

$$\Phi(w, \alpha + 2\pi)(\tau) = e^{i(\alpha+2\pi)\tau} w(\tau) = e^{i\alpha\tau} e^{i2\pi\tau} w(\tau) = \Phi(E_{2\pi} w, \alpha).$$

Concerning the injectivity of Φ_I , we observe that

$$\Phi(w_1, \alpha_1) = \Phi(w_2, \alpha_2) \iff \exists k \in \mathbb{Z} : \alpha_2 = \alpha_1 + 2k\pi \text{ and } w_2 = E_{-2k\pi} w_1.$$

Finally, concerning the surjectivity, we have that

$$z \in W_\xi \iff \exists (w, \alpha) \in W \times \mathbb{R} : z = \Phi(w, \alpha) \text{ and } \xi = e^{i\alpha}. \quad \square$$

Lemma 6.11. Φ is open as a map from $(W \setminus \{0\}) \times \mathbb{R}$ onto \mathcal{M} .

Proof. Let us fix $(w, \alpha) \in W \times \mathbb{R}$, $z = \Phi(w, \alpha)$ and $r > 0$. We will prove that there exists $\rho > 0$ such that if $z_1 \in \mathcal{M}$ and $\|z - z_1\| < \rho$ then there exists $(w_1, \alpha_1) \in W \times \mathbb{R}$ with $\|w - w_1\| + |\alpha - \alpha_1| < r$ such that $z_1 = \Phi(w_1, \alpha_1)$. Define $\xi = e^{i\alpha}$ and let \log denote the holomorphic branch of the logarithm defined on $\mathbb{C} \setminus \{\lambda\xi : \lambda < 0\}$ with $\log \xi = i\alpha$. In view of Lemma 6.8, we can find $\rho_1 > 0$ such that if $z_1 \in \mathcal{M}$ and $\|z - z_1\| < \rho_1$ then $\xi(z_1)$ belongs to the above domain and $\alpha_1 = -i \log \xi(z_1)$ is such that $|\alpha - \alpha_1| < r/2$. Define $w_1(\tau) = e^{-i\alpha_1\tau}z(\tau)$. Then $w_1 \in W$, and $\|w - w_1\| < r/2$ if z and z_1 are sufficiently close, say $\|z - z_1\| < \rho_2$. \square

As a consequence of the above lemma, \mathcal{U}_I is open in \mathcal{M} , for every open I .

Lemma 6.12. If $|I| \leq 2\pi$ then \mathcal{U}_I is a C^∞ submanifold in X , modeled on $W \times \mathbb{R}$.

Proof. The lemma follows by [1, p. 178-179], in particular by Thms. 3.5.7 and 3.5.9 there. According to such results, to prove the lemma we have to show that

1. Φ_I is injective;
2. Φ_I is open as a map from $(W \setminus \{0\}) \times I$ onto \mathcal{U}_I ;
3. Φ_I is an immersion, i.e. $\Phi'_I(w, \alpha)$ has trivial kernel and closed range for every $(w, \alpha) \in (W \setminus \{0\}) \times I$.

Notice that the above properties are sufficient because the manifolds are modeled over Hilbert spaces; for ones modeled on Banach spaces, we should ask for a closed split range Φ'_I .

The first property follows from Lemma 6.10. As for the second one, it is a consequence of Lemma 6.11.

To show 3., we recall from Lemma 6.9 that

$$\Phi'(w, \alpha)[\Delta, \delta] = E_\alpha(\Delta + i\delta I \cdot w).$$

Since E_α is an isomorphism, the points $(\Delta, \delta) \in \ker \Phi'_I(w, \alpha)$ should satisfy $\Delta + i\delta I \cdot w = 0$. Assume that $(\Delta, \delta) \neq 0$, then $\delta \neq 0$ and the previous identity is equivalent to

$$\Delta(\tau + 1) + i\delta(\tau + 1)w(\tau + 1) = \Delta(\tau) + i\delta\tau w(\tau) = 0,$$

for every $\tau \in [0, 1]$. Since Δ and w are 1-periodic, we are led to $w \equiv 0$. This is a contradiction, since $w \in W \setminus \{0\}$. Once we have proved that the kernel is trivial, we observe that the range of $\Phi'(w, \alpha)$ is $E_\alpha(W \oplus (iI \cdot w)\mathbb{R})$. Since $W \oplus (iI \cdot w)\mathbb{R}$ is closed in X and E_α is an isomorphism, we deduce that the range is also closed in X . \square

Proof of Proposition 6.5. We first show that \mathcal{M} is a submanifold of X . Let us take the intervals $I_1 = (-\pi, \pi)$ and $I_2 = (0, 2\pi)$. By Lemma 6.10 we have that $\mathcal{U}_{I_1} \cup \mathcal{U}_{I_2} = \mathcal{U}_{(-\pi, 2\pi)} = \mathcal{M}$, where each \mathcal{U}_{I_i} is open in \mathcal{M} by Lemma 6.11. In view of Lemma 6.12, we are left to show that the diffeomorphisms Φ_{I_i} from $W \times I_i$ onto \mathcal{U}_{I_i} , $i = 1, 2$, are compatible. This is again a consequence of Lemma 6.10, since

$$\Phi_{I_2}^{-1} \circ \Phi_{I_1}(w, \alpha) = \begin{cases} (E_{-2\pi}w, \alpha + 2\pi) & \alpha \in (-\pi, 0) \\ (w, \alpha) & \alpha \in (0, \pi), \end{cases}$$

and a similar formula also holds for $\Phi_{I_1} \circ \Phi_{I_2}^{-1}$.

The expression of $T_{E_\alpha w} \mathcal{M}$ follows from Lemma 6.9, since $\Phi'(w, \alpha)$ is an isomorphism between $W \times \mathbb{R}$ and such tangent space.

Finally, once \mathcal{M} is a submanifold of an Hilbert space, it inherits the corresponding Riemmanian structure. Then the geodesic distance $\text{dist}_{\mathcal{M}}$ is well defined, and of course

$$\text{dist}_{\mathcal{M}}(z_1, z_2) \geq \|z_1 - z_2\|.$$

\square

6.2 Variational principles for \mathcal{B}_{KS} on \mathcal{M}

In the following, we assume that p is C^1 and T -periodic. Moreover, for any $z \in \mathcal{M}$, when we write $z = E_\alpha w$ we understand that w is 1-periodic.

By definition, functions in the manifold \mathcal{M} are defined in the interval $0 < \tau < 2$, while in Section 3.3 we developed the theory for the functional \mathcal{B}_{KS} on functions defined in $0 < \tau < 1$. To proceed, we have two possibilities: either we have to restrict the functions of \mathcal{M} on $(0, 1)$, or to extend the definition of \mathcal{B}_{KS} to functions defined on $(0, 2)$. Actually, the two points of view turn out to be equivalent: indeed, retracing the arguments in Section 2 and 3.3, it is possible to see that the natural definition for doubled intervals is

$$\begin{aligned} \mathcal{B}_{\text{KS}_2} &: H^1(0, 2; \mathbb{H}) \rightarrow \mathbb{R} \cup \{+\infty\} \\ \mathcal{B}_{\text{KS}_2}(z) &:= \frac{1}{\mathcal{L}_2(z)} \mathcal{Q}_2(z) + \mathcal{L}_2(z) \left[2 + \mathcal{R}_2(z) \right], \quad \mathcal{B}_2(0) := +\infty, \end{aligned}$$

where

$$\mathcal{L}_2(z) := \frac{2T}{\int_0^2 |z|^2}, \quad \mathcal{Q}_2(z) := 2 \int_0^2 |z'|^2, \quad \mathcal{R}_2(z) := \int_0^2 |z|^2 \langle p \circ t_z, \bar{z}i z \rangle, \quad t_{z_2}(\tau) := \mathcal{L}_2(z) \int_0^\tau |z|^2$$

(in such a way that $t_{z_2}(2) = 2T$). Then, since p is T -periodic, direct computations show that

$$\mathcal{B}_{\text{KS}_2}(z) = 2\mathcal{B}_{\text{KS}}(z|_{(0,1)}) \quad \text{for every } z \in \mathcal{M},$$

where \mathcal{B}_{KS} is defined as usual (recall Lemma 6.3). For this reason we will work directly with $\widetilde{\mathcal{B}}_{\text{KS}}(z) = \mathcal{B}_{\text{KS}}(z|_{(0,1)})$. We first check that critical points of $\widetilde{\mathcal{B}}_{\text{KS}}$ correspond to generalized solutions of the perturbed Kepler problem; next, we will address the existence of such critical points.

Lemma 6.13. *Let $z \in \mathcal{M}$, $z = E_\alpha w$, be a critical point for $\widetilde{\mathcal{B}}_{\text{KS}}$ on \mathcal{M} . Then $z|_{(0,1)}$ satisfies assumption (3.28) of Proposition 3.19, namely*

$$\frac{d}{d\varepsilon} \left[\mathcal{B}_{\text{KS}}(z|_{(0,1)} + \varepsilon\varphi) \right]_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in \mathcal{D}(0, 1; \mathbb{H}).$$

As a consequence, $z|_{(0,1)} \in C^3([0, 1])$ satisfies the Euler-Lagrange equation (3.30),

$$\langle z'(1^-), v(1) \rangle - \langle z'(0^+), v(0) \rangle = 0 \quad \text{for every } v \in W_\xi, \text{ where } \xi = e^{i\alpha}, \quad (6.5)$$

and

$$\langle z'(1), iz(1) \rangle = 0. \quad (6.6)$$

Proof. First, let us take $v \in W_\xi$, where $\xi = e^{i\alpha}$. Then $z + \varepsilon v \in W_\xi \subset \mathcal{M}$, for every ε small. Since $z = E_\alpha w$ is a critical point of $\widetilde{\mathcal{B}}_{\text{KS}}$ on \mathcal{M} we have that

$$\frac{d}{d\varepsilon} \left[\widetilde{\mathcal{B}}_{\text{KS}}(z + \varepsilon v) \right]_{\varepsilon=0} = 0 \quad \text{for every } v \in W_\xi.$$

Since the restriction $z \mapsto z|_{(0,1)}$ is linear, this is equivalent to

$$\frac{d}{d\varepsilon} \left[\mathcal{B}_{\text{KS}}(z|_{(0,1)} + \varepsilon v|_{(0,1)}) \right]_{\varepsilon=0} = 0 \quad \text{for every } v \in W_\xi. \quad (6.7)$$

Given $\varphi \in \mathcal{D}(0, 1; \mathbb{H})$, we extend it to a function $\tilde{\varphi} \in W_\xi$:

$$\tilde{\varphi}(\tau) := \begin{cases} \varphi(\tau) & \text{if } \tau \in [0, 1] \\ e^{i\alpha}\varphi(\tau - 1) & \text{if } \tau \in [1, 2]. \end{cases}$$

From the previous identity we deduce that

$$\frac{d}{d\varepsilon} [\mathcal{B}_{\text{KS}}(z|_{(0,1)} + \varepsilon\varphi)]_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in \mathcal{D}(0, 1; \mathbb{H}).$$

Once (3.28) is satisfied, Lemma 3.20 implies that $z|_{(0,1)}$ is $C^3([0, 1])$ and satisfies the Euler-Lagrange equation (3.30). Then, reasoning as in Lemma 5.8, we can integrate by parts in (6.7) and use the equation to obtain (6.5).

It remains to prove the identity (6.6). We consider the path $[0, 1] \ni \varepsilon \mapsto \gamma_\varepsilon \in \mathcal{M}$, $\gamma_\varepsilon(\tau) = e^{i\varepsilon\tau}z(\tau)$. Then

$$\frac{d}{d\varepsilon} [\widetilde{\mathcal{B}}_{\text{KS}}(\gamma_\varepsilon)]_{\varepsilon=0} = 0.$$

Using again that z solves the Euler-Lagrange equation, we conclude that

$$0 = \widetilde{\mathcal{B}}_{\text{KS}}'(z)[\dot{\gamma}_0] = \frac{4}{\mathcal{L}} [\langle z'(1^-), \dot{\gamma}_0(1) \rangle - \langle z'(0^+), \dot{\gamma}_0(0) \rangle],$$

$$\text{where } \dot{\gamma}_0(\tau) = \left. \frac{d\gamma_\varepsilon(\tau)}{d\varepsilon} \right|_{\varepsilon=0} = i\tau z(\tau). \quad \square$$

We have all the ingredients to prove the following result.

Proposition 6.14. *Let p be C^1 and T -periodic, and let $z \in \mathcal{M}$ be a critical point for $\widetilde{\mathcal{B}}_{\text{KS}}$ on \mathcal{M} . Then*

$$x(t) = \bar{z}(\tau_z(t))iz(\tau_z(t)),$$

where τ_z is defined as in Proposition 3.19, is a T -periodic generalized solution of equation (1.1).

Proof. In view of Lemma 6.13 and Proposition 3.19, we have that τ_z and x are well defined, of class C^2 , and that x is a generalized solution of equation (1.1) in $(0, T)$. The fact that x is (extendable as) a T -periodic function follows from Lemma 6.3 and Remark 3.23, in case $x(0) = x(T) \neq 0$. Hence we are left to prove that, in case $x(0) = x(T) = 0$,

$$\lim_{t \rightarrow 0^+} \frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|} = \lim_{t \rightarrow T^-} \frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|} \quad \text{and} \quad \lim_{t \rightarrow 0^+} \frac{x}{|x|} = \lim_{t \rightarrow T^-} \frac{x}{|x|}.$$

This can be done as for collisions in $(0, T)$, which were treated in Proposition 3.19. More precisely, we exploit the fact that the map defined in equation (3.22), namely

$$t \mapsto -\dot{x}^2(t)|x(t)| = |\dot{x}(t)|^2|x(t)| = \frac{4}{\mathcal{L}^2} |z'(\tau_z(t))|^2$$

tends to the same limit as $t \rightarrow 0^+$ and $t \rightarrow T^-$ (see also the end of the proof of Theorem 5.1). \square

Once the role of $\widetilde{\mathcal{B}}_{\text{KS}}$ is clarified, we turn to the variational framework. Since \mathcal{M} is symmetric, i.e. $-\mathcal{M} = \mathcal{M}$, we can follow the lines of Section 5, defining the genus of $A \in \mathcal{A} := \{A \in \mathcal{M} \setminus \{0\} : A = -A, A \text{ is closed}\}$ as

$$\gamma(A) := \inf\{n : \exists \phi \in C(A; \mathbb{R}^n \setminus \{0\}), \phi \text{ odd}\},$$

and, accordingly,

$$\mathcal{A}_m := \{A \subset \mathcal{A} : A \text{ is compact and } \gamma(A) \geq m\}, \quad \sigma_m := \inf_{A \in \mathcal{A}_m} \sup_A \widetilde{\mathcal{B}}_{\text{KS}}.$$

Since $W \subset \mathcal{M}$, also in this case we have that $\mathcal{A}_m \neq \emptyset$, for every m . In particular, $\sigma_m < +\infty$ for every m .

As we already mentioned, the main difference with the planar case is that now the Palais-Smale condition can not hold true: for instance, if p has non-zero average on $[0, T]$, it is easy to construct diverging (PS) sequences at any level of $\widetilde{\mathcal{B}}_{\text{KS}}$, by considering constant loops. To overcome this difficulty, we show that a weaker compactness property holds true, at least at positive levels. A sequence $(z_n)_n \subset \mathcal{M}$ is said to be a Palais-Smale-Cerami (PSC) sequence at level σ for $\widetilde{\mathcal{B}}_{\text{KS}}$ if, for some fixed $\hat{z} \in \mathcal{M}$,

$$\widetilde{\mathcal{B}}_{\text{KS}}(z_n) = \sigma + o(1), \quad \|\nabla \widetilde{\mathcal{B}}_{\text{KS}}(z_n)\| (1 + \text{dist}_{\mathcal{M}}(z_n, \hat{z})) = o(1) \text{ as } n \rightarrow \infty \quad (6.8)$$

(recall Remark 6.7). Accordingly, $\widetilde{\mathcal{B}}_{\text{KS}}$ satisfies the (PSC) condition at level σ if any such a sequence admits a strongly convergent subsequence. Such condition was introduced in [8], see also [9, 5]. The (PSC) condition is slightly weaker than the (PS) one, while the most important implications are retained, see [19, Ch. II, Rmk. 2.5]. In particular, it is possible to show the following result.

Proposition 6.15. *Each finite σ_m is a critical level for $\widetilde{\mathcal{B}}_{\text{KS}}$ provided the (PSC) condition holds at level σ_m .*

Sketch of the proof. Actually, this proposition is a version of Proposition 5.2, with (PS) replaced by (PSC). Again, the argument is based on the deformation lemma, therefore it is enough to assume the regularity of $\widetilde{\mathcal{B}}_{\text{KS}}$ and the completeness of \mathcal{M} only on sublevels of $\widetilde{\mathcal{B}}_{\text{KS}}$, see [19, Ch. II, Remarks after Thm. 5.7]. The fact that (PSC) is enough to define a pseudo-gradient flow, and hence to prove a deformation lemma, is very well known in the literature, see e.g. [16, Thm. 4.7]. Actually, our proposition can be proved also applying directly Teorema (*) in the original papers by Cerami [8, 9] to the quotient manifold \mathcal{M}/\sim , where \sim is the equivalence relation induced by the involution $z \mapsto -z$. \square

In order to prove the (PSC) condition for $\widetilde{\mathcal{B}}_{\text{KS}}$ on \mathcal{M} we need some preliminary lemmas.

Lemma 6.16. *For every positive a, b there exist positive constants $C_i = C_i(a, b)$, $i = 1, 2$, such that, for every $y \in H^1(0, 1; \mathbb{H})$,*

$$\begin{cases} \mathcal{B}_{\text{KS}}(y) \leq a \\ \|y\|_2^2 \leq b \end{cases} \implies \begin{cases} \|y'\|_2 \leq C_1(a, b) \\ C_2(a, b) \leq \|y\|_2 \leq b^{1/2}. \end{cases}$$

Proof. The lemma follows in three steps, by reasoning as in Lemmas 5.3, 5.4 and 5.5, respectively. We refer to such lemmas for further details. For the sake of simplicity, we write $\alpha := \|y\|_2 \cdot \|y'\|_2$, $\beta := \|y\|_2^2$.

Step 1. To start with, we claim that, for any $y \in H^1(0, 1; \mathbb{H})$, the following Gagliardo-Nirenberg inequality holds:

$$\|y\|_\infty^2 \leq 2\alpha + \beta.$$

Indeed, this follows by integrating with respect to τ_0 the elementary inequality

$$|y(\tau)|^2 \leq |y(\tau_0)|^2 + \int_0^1 2|y||y'| d\tau.$$

Step 2. As a consequence of Step 1, we have that

$$|\mathcal{R}(y)| \leq \int_0^1 |p \circ t_y||y|^4 \leq \|p\|_\infty \|y\|_\infty^2 \|y\|_2^2 \leq \|p\|_\infty (2\alpha + \beta) \beta,$$

and therefore

$$\frac{1}{T}\mathcal{B}_{\text{KS}}(y) = \frac{2}{T^2}\alpha^2 + \frac{1}{\beta}(1 + \mathcal{R}(z)) \geq \frac{2}{T^2}\alpha^2 - 2\|p\|_\infty\alpha + \frac{1}{\beta} - \|p\|_\infty\beta \geq -\frac{\|p\|_\infty^2 T^2}{2} - \|p\|_\infty\beta. \quad (6.9)$$

Step 3. Finally, using the assumptions, the last estimate implies

$$\frac{2}{T^2}\alpha^2 - 2\|p\|_\infty\alpha \leq \frac{1}{T}a + \|p\|_\infty b \quad \text{and} \quad \frac{1}{\beta} \leq \frac{1}{T}a + \|p\|_\infty b + \frac{\|p\|_\infty^2 T^2}{2},$$

so that α is bounded above and $\beta = \|y\|_2^2$ is bounded away from 0, and the existence of C_2 follows. Then

$$\|y'\|_2^2 = \frac{\alpha^2}{\beta}$$

is bounded above, and also the existence of C_1 follows. \square

Lemma 6.17. *Let $(z_n)_n \subset \mathcal{M}$ be a (PSC) sequence for $\widetilde{\mathcal{B}}_{\text{KS}}$ at level $\sigma > 0$. Then there exists $z \in \mathcal{M}$ such that, up to subsequences, $z_n \rightharpoonup z$ weakly in X and uniformly. In particular, z_n is both bounded and bounded away from 0.*

Proof. We notice that $z_n \in T_{z_n}\mathcal{M}$ (indeed, writing $z_n = E_{\alpha_n}w_n$, then it corresponds to the choice $(\Delta, \delta) = (w, 0)$ in Proposition 6.5, 2). We deduce that

$$\begin{aligned} \left| \widetilde{\mathcal{B}}_{\text{KS}}'(z_n)[z_n] \right| &\leq \|\nabla \widetilde{\mathcal{B}}_{\text{KS}}(z_n)\| \|z_n\| \leq \|\nabla \widetilde{\mathcal{B}}_{\text{KS}}(z_n)\| (\|\hat{z}\| + \|z_n - \hat{z}\|) \\ &\leq \|\nabla \widetilde{\mathcal{B}}_{\text{KS}}(z_n)\| (\|\hat{z}\| + \text{dist}_{\mathcal{M}}(z_n, \hat{z})) = o(1), \end{aligned}$$

as $n \rightarrow +\infty$, where $\hat{z} \in \mathcal{M}$ is such that (6.8) holds. For easier notation we write $y_n = z_n|_{(0,1)}$, so that

$$\widetilde{\mathcal{B}}_{\text{KS}}(z_n) = \mathcal{B}_{\text{KS}}(y_n), \quad \widetilde{\mathcal{B}}_{\text{KS}}'(z_n)[z_n] = \mathcal{B}_{\text{KS}}'(y_n)[y_n].$$

We deduce

$$\begin{cases} \mathcal{B}_{\text{KS}}(y_n) = \frac{\mathcal{Q}(y_n)}{\mathcal{L}(y_n)} + \mathcal{L}(y_n) + \mathcal{L}(y_n)\mathcal{R}(y_n) = \sigma + o(1) \\ \frac{1}{2}\mathcal{B}_{\text{KS}}'(y_n)[y_n] = 2\frac{\mathcal{Q}(y_n)}{\mathcal{L}(y_n)} - \mathcal{L}(y_n) + \mathcal{L}(y_n)\mathcal{R}(y_n) = o(1), \end{cases}$$

as $n \rightarrow +\infty$ (the expression of $\mathcal{B}_{\text{KS}}'(y_n)[y_n]$ can be obtained by using Lemma 3.20 or, in an easier way, by noticing that the functionals \mathcal{Q} , \mathcal{L} and \mathcal{R} are homogeneous of degree 2, -2 and 4 respectively). Subtracting the above relations we infer

$$\frac{T}{\|y_n\|_2^2} = \mathcal{L}(y_n) \geq \mathcal{L}(y_n) - \frac{\mathcal{Q}(y_n)}{2\mathcal{L}(y_n)} = \frac{1}{2}\sigma + o(1).$$

Resuming, we have that

$$\begin{cases} \mathcal{B}_{\text{KS}}(y_n) \leq \sigma + 1 \\ \|y_n\|_2^2 \leq \frac{2T}{\sigma} + 1, \end{cases}$$

for n large, and Lemma 6.16 yields $\|y_n'\|_2 \leq C_1$, $0 < C_2 \leq \|y_n\|_2 \leq C_3$, where the constants are independent of n . Then, up to a subsequence, $y_n \rightharpoonup y \neq 0$ weakly in $H^1(0, 1; \mathbb{H})$ and uniformly. To conclude the proof of the lemma, we are left to show that y can be extended to a function in $z \in \mathcal{M}$, and that $z_n \rightharpoonup z$ (again possibly up to subsequences).

Let $(\xi_n)_n \subset \mathbb{S}^1$ be such that $z_n \in W_{\xi_n}$. By compactness of \mathbb{S}^1 we can assume that, up to subsequences, $\xi_n \rightarrow \xi$. In particular, by uniform convergence,

$$y(1) \leftarrow y_n(1) = z_n(1) = \xi_n z_n(0) = \xi_n y_n(0) \rightarrow \xi y(0).$$

We deduce that

$$z_n \rightarrow z := \begin{cases} y(\tau) & \tau \in [0, 1], \\ \xi y(\tau - 1) & \tau \in [1, 2]. \end{cases} \in \mathcal{M} \quad \square$$

Lemma 6.18. *The functional $\widetilde{\mathcal{B}}_{KS}$ satisfies the (PSC) condition in \mathcal{M} at any level $\sigma > 0$.*

Proof. Let $(z_n)_n \subset \mathcal{M}$ be a (PSC) sequence at level $\sigma > 0$ and, say, $z_n \in \mathcal{M}$. By Lemma 6.17 $z_n \rightarrow z$ in X , and we can assume $z \in W_\xi$. Moreover, we can choose $\alpha_n \rightarrow \alpha$ in such a way that $\xi_n = e^{i\alpha_n}$, $\xi = e^{i\alpha}$. In particular, $E_{-\alpha} z \in W$. We notice that, by direct computations,

$$E_{\alpha_n - \alpha} z \rightarrow z \text{ strongly in } X. \quad (6.10)$$

Let us consider the sequence $(z_n - E_{\alpha_n - \alpha} z)_n$: on the one hand, it is weakly convergent, and hence bounded; on the other hand, $z_n - E_{\alpha_n - \alpha} z \in T_{z_n} \mathcal{M}$ (indeed, writing $z_n = E_{\alpha_n} w_n$, then it corresponds to the choice $(\Delta, \delta) = (w_n - E_{-\alpha} z, 0)$ in Proposition 6.5, 2). Since $(z_n)_n$ is a (PSC) sequence we deduce that

$$\widetilde{\mathcal{B}}_{KS}'(z_n)[z_n - E_{\alpha_n - \alpha} z] = o(1) \quad \text{as } n \rightarrow +\infty.$$

Let us write again $y_n = z_n|_{(0,1)}$, $y = z|_{(0,1)}$, and, with some abuse, $E_{\alpha_n - \alpha} y = E_{\alpha_n - \alpha} z|_{(0,1)}$; we can reason as in the proof of Lemma 5.6, using the appropriate expression of \mathcal{B}_{KS}' , to infer

$$o(1) = \frac{\mathcal{L}(y_n)}{4} \mathcal{B}_{KS}'(y_n)[y_n - E_{\alpha_n - \alpha} y] = \int_0^1 \langle y_n', y_n' - (E_{\alpha_n - \alpha} y)' \rangle + o(1) = \|y_n'\|_2^2 - \|y'\|_2^2 + o(1),$$

where we used (6.10), $y_n \rightarrow y$, and the fact that $\mathcal{L}(y_n)$ is bounded since we have a positive lower bound of $\|y_n\|_2$. We deduce that $y_n \rightarrow y$ strongly in $H^1(0, 1; \mathbb{H})$ and thus, reasoning as in Lemma 6.17, that $z_n \rightarrow z$ strongly in \mathcal{M} . \square

To conclude, we have to show that $\sigma_m > 0$, for some m . Since $\sigma_{m+1} \geq \sigma_m$, for every m , this will imply the existence of infinitely many positive critical points.

Lemma 6.19. *Let $k \in \mathbb{N}^+$ and $A \in \mathcal{A}_{4k+1}$. Then there exists $z_A \in A$ such that*

$$z_A(i/k) = 0 \quad \text{for every } i = 0, 1, \dots, k-1 \quad (6.11)$$

(since $z_A \in \mathcal{M}$, $z_A(i/k) = 0$, $i = k, k+1, \dots, 2k$, as well). In particular,

$$\|z_A|_{(0,1)}\|_2 \leq \frac{1}{k\pi} \|z_A'|_{(0,1)}\|_2. \quad (6.12)$$

Proof. Let $\phi : A \rightarrow \mathbb{H}^k \cong \mathbb{R}^{4k}$ be defined as

$$\phi(z) = (z(0), z(1/k), z(2/k), \dots, z((k-1)/k)).$$

Since ϕ is odd and continuous, and $\gamma(A) \geq 4k+1$, by definition of genus we infer the existence of z_A such that $\phi(z_A) = 0$, and (6.11) follows. Then $z_A|_{((i-1)/k, i/k)} \in H_0^1((i-1)/k, i/k)$. As a consequence

$$\int_{(i-1)/k}^{i/k} |z_A|^2 d\tau \leq \frac{1}{k^2 \pi^2} \int_{(i-1)/k}^{i/k} |z_A'|^2 d\tau,$$

and also (6.12) follows. \square

Lemma 6.20. $\sigma_m \rightarrow +\infty$ as $m \rightarrow +\infty$.

Proof. Let A be such that $\gamma(A) \geq 4k + 1$, and let $z_A \in A$ be such that Lemma 6.19 holds true. As usual, we write $y_A = z_A|_{(0,1)}$, $\alpha := \|y_A\|_2 \cdot \|y'_A\|_2$, $\beta := \|y_A\|_2^2$. Then Lemma 6.19 implies

$$\beta \leq \frac{1}{k\pi} \alpha.$$

Using (6.9) we obtain

$$\begin{aligned} \frac{1}{T} \mathcal{B}_{\text{KS}}(y_A) &\geq \frac{2}{T^2} \alpha^2 - 2\|p\|_\infty \alpha + \frac{1}{\beta} - \|p\|_\infty \beta \geq \frac{2}{T^2} \alpha^2 - \|p\|_\infty \left(2 + \frac{1}{k\pi}\right) \alpha + \frac{k\pi}{\alpha} \\ &\geq \frac{2}{T^2} \alpha^2 - 3\|p\|_\infty \alpha + \frac{k}{\alpha} =: g_k(\alpha). \end{aligned}$$

Now, by direct computation one checks that g_k has a unique critical point $\alpha_k > 0$, with $\min_{\alpha > 0} g_k(\alpha) = g_k(\alpha_k)$. Moreover, $\alpha_k \sim \left(\frac{T^2 k}{4}\right)^{1/3}$ as $k \rightarrow +\infty$, and therefore $g_k(\alpha_k) \rightarrow +\infty$ too. Then, we have that

$$A \in \mathcal{A}_{4k+1} \quad \implies \quad \sup_A \widetilde{\mathcal{B}}_{\text{KS}} \geq \widetilde{\mathcal{B}}_{\text{KS}}(z_A) \geq T g_k(\alpha_k),$$

so that $\sigma_m \geq T g_k(\alpha_k)$ whenever $m \geq 4k + 1$ and the lemma follows. \square

Proof of Theorem 6.1. By Lemma 6.20 we know that infinitely many values σ_m are positive. Then, Proposition 6.15 and Lemma 6.18 insures that such σ_m are critical values for $\widetilde{\mathcal{B}}_{\text{KS}}$, with (distinct) critical points, say, z_m . By Proposition 6.14 we obtain that the corresponding $x_m(t) = \bar{z}_m(\tau_{z_m}(t)) i z_m(\tau_{z_m}(t))$ are periodic generalized solutions of (1.1). Finally, since $\mathcal{A}(x_m) = \widetilde{\mathcal{B}}_{\text{KS}}(z_m) = \sigma_m \rightarrow +\infty$, there exist infinitely many different solutions x_m . \square

A Some auxiliary results

A.1 Proof of Lemma 2.1

Let us observe that the typical example for Lemma 2.1 is $a(t) = \frac{3}{2}t^{1/3}$, with $A(t) = t^{2/3}$ and $B(\tau) = \tau^{3/2}$.

Proof of Lemma 2.1. Since $\frac{1}{a(t)}$ is integrable, the set

$$Z := \{t \in [0, T] : a(t) = 0\}$$

has zero measure. Hence, the function $A : [0, T] \rightarrow [0, \Xi]$ is continuous and strictly increasing. Therefore the inverse $B(\tau)$ is continuous. To prove that B has a derivative we first observe that the quotient

$$\Delta(\tau, h) = \frac{B(\tau + h) - B(\tau)}{h}, \quad \tau \in [0, \Xi], h \neq 0$$

satisfies

$$\frac{1}{\Delta(\tau, h)} = \frac{1}{B(\tau + h) - B(\tau)} \int_{B(\tau)}^{B(\tau+h)} \frac{d\xi}{a(\xi)}.$$

We now distinguish two cases; first, we take $\tau \in [0, \Xi]$ such that $a(B(\tau)) = 0$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $a(\zeta) = |a(\zeta)| < \varepsilon$ if $|\zeta - B(\tau)| < \delta$. Then, if $|h|$ is so small that $|B(\tau + h) - B(\tau)| < \delta$, the inequality

$$\left| \int_{B(\tau)}^{B(\tau+h)} \frac{d\xi}{a(\xi)} \right| > \frac{|B(\tau + h) - B(\tau)|}{\varepsilon}$$

holds. As a consequence, for any $\varepsilon > 0$ we can choose h sufficiently small such that $|\Delta(\tau, h)| < \varepsilon$. This implies that $B'(\tau) = 0$.

Second, let us take $\tau \in [0, \Xi]$ be such that $a(B(\tau)) > 0$. The continuity of $\frac{1}{a(t)}$ at $t = B(\tau)$ implies that

$$\frac{1}{\Delta(\tau, h)} \rightarrow \frac{1}{a(B(\tau))}, \quad \text{as } h \rightarrow 0,$$

therefore $B'(\tau) = a(B(\tau))$.

Summing up, the derivative of B exists everywhere and $B'(\tau) = a(B(\tau))$ is a continuous function. \square

A.2 Details of Remark 3.26

By assumption, $y \in H_0^1((0, 1); \mathbb{R}^d)$ satisfies $|y(\tau)| > 0$ for $\tau \in (0, 1)$ and

$$\frac{d}{d\varepsilon} [\mathcal{B}(y + \varepsilon\psi)]_{\varepsilon=0} = 0 \quad \text{for every } \psi \in \mathcal{D}(0, 1), \quad (\text{A.1})$$

where

$$\mathcal{B}(y) := \frac{1}{\mathcal{L}(y)} \mathcal{Q}(y) + \mathcal{L}(y).$$

Moreover, by Proposition 3.1, $x_y = x_y(t)$ satisfies

$$\ddot{x}_y = -\frac{x_y}{|x_y|^3} \quad \text{in } (0, T). \quad (\text{A.2})$$

Now, let $\varphi \in \mathcal{D}(0, 1/2)$. By definition of w , we obtain

$$\begin{aligned} \int_0^{1/2} |w(\tau) + \varepsilon\varphi(\tau)|^2 d\tau &= h_1^2 \int_0^{1/2} \left| y(2\tau) + \frac{\varepsilon}{h_1^2} \varphi(\tau) \right|^2 d\tau = \frac{h_1^2}{2} \int_0^1 |y + \varepsilon\psi|^2 \\ \int_{1/2}^1 |w(\tau) + \varepsilon\varphi(\tau)|^2 d\tau &= h_2^2 \int_{1/2}^1 |y(2-2\tau)|^2 d\tau = \frac{h_2^2}{2} \int_0^1 |y|^2, \end{aligned}$$

where we wrote $\varphi(\tau) = h_1^2\psi(2\tau)$, $\psi \in \mathcal{D}(0, 1)$. Analogously,

$$\mathcal{Q}(w + \varepsilon\varphi) = 2h_1^2\mathcal{Q}(y + \varepsilon\psi) + 2h_2^2\mathcal{Q}(y).$$

Direct calculations yield

$$\frac{d}{d\varepsilon} [\mathcal{B}(w + \varepsilon\varphi)]_{\varepsilon=0} = 2^{1/3} h_1^2 \frac{d}{d\varepsilon} [\mathcal{B}(y + \varepsilon\psi)]_{\varepsilon=0} = 0$$

by (A.1). Since analogous arguments hold when $\varphi \in \mathcal{D}(1/2, 1)$, (3.37) follows.

Taking $\varepsilon = 0$ on the previous computations, and recalling that $h_1^2 + h_2^2 = 2^{1/3}$, we obtain

$$\mathcal{L}(w) = 2^{2/3} \mathcal{L}(y), \quad \mathcal{Q}(w) = 2^{4/3} \mathcal{Q}(y).$$

Now, let $0 < \tau < 1/2$. We have

$$t_w(\tau) := \mathcal{L}(w) \int_0^\tau |w(\xi)|^2 d\xi = 2^{-1/3} h_1^2 t_y(2\tau) \quad \iff \quad \tau_w(t) = \frac{1}{2} \tau_y \left(2^{1/3} h_1^{-2} t \right),$$

whenever $0 < t < 2^{-1/3} h_1^2 T$, and

$$x_w(t) := |w(\tau_w(t))| w(\tau_w(t)) = h_1^2 x_y \left(2^{1/3} h_1^{-2} t \right).$$

Substituting into (A.2) we finally obtain

$$\ddot{x}_w = -2^{2/3} h_1^2 \frac{x_w}{|x_w|^3} \quad \text{in } (0, 2^{-1/3} h_1^2 T).$$

Performing similar calculations in $(1/2, 1)$ (or changing variable as $\tau \leftrightarrow 1 - \tau$) we have

$$\ddot{x}_w = -2^{2/3} h_2^2 \frac{x_w}{|x_w|^3} \quad \text{in } (2^{-1/3} h_1^2 T, T).$$

In particular, this implies (3.38) with $\mu_i = 2^{2/3} h_i^2$, and this holds for any choice of positive μ_i satisfying

$$\mu_1 + \mu_2 = 2,$$

in agreement with (3.36).

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