BOUNDED AND HOMOCLINIC LIKE SOLUTIONS OF A SECOND ORDER SINGULAR DIFFERENTIAL EQUATION

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Abstract. We study the existence of positive solutions for the model scalar second order boundary value problem

$$\begin{cases} -u'' + c(x)u' + a(x)u = \frac{b(x)}{u(x)^p}, & x \in I\!R, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where a, b, c are locally bounded coefficients and $p > 0$.

1. Introduction

This note is devoted to the study of the existence of a decaying nontrivial positive solution for the model equation

$$-u'' + c(x)u' + a(x)u = \frac{b(x)}{u(x)^p}, \tag{1}$$

where $a, b, c \in C(I\!R; I\!R)$, that is, the existence of a positive function u that solves (1) for every $x \in I\!R$ as well as the boundary conditions

$$\lim_{|x|\to\infty} u(x) = 0. \tag{2}$$

When such a solution satisfies in addition

$$\lim_{|x|\to\infty} u'(x) = 0,$$

then it is usually called a homoclinic solution or a pulse though here, 0 is not a stationary solution of equation (1).

Equation (1) is a particular case of a more general class of Sturm equations of the type

$$-(P(x)u')' + Q(x)u = R(x)f(u), \tag{3}$$

where P is a strictly positive absolutely continuous function.

Such equations, even in the case $P \equiv 1$ where they are referred as of Schrödinger or Klein-Gordon type, appear in many scientific areas including quantum field theory, gas dynamics, fluid mechanics and chemistry. For instance, the study of pulse propagation is of primary importance in nonlinear optics and plasma physics, where homoclinic solutions of the corresponding model equations are sometimes referred as bound states or bright solitons. A typical example is the study of the ability of layered media to support the propagation of electromagnetic guided waves.

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Observe that equation (1) with constant c arises also when studying traveling wave fronts for parabolic reaction-diffusion equations with a singular local reaction term.

One can study (1) with $c = 0$ via a variational procedure, see e.g. the surveys [3, 4, 13]. Actually, one could also treat the case where $c \in L^1(I\!R)$ in the same way with only minor changes. Indeed, in this case the self-adjoint form of the equation is non-degenerate. In the case where c is non- integrable, we prefer to tackle the problem via a topological approach as the variational formulation of the problem would require working with weighted Sobolev spaces with unbounded or vanishing weight functions. Without being insurmountable, see for instance [7] for related problems, this difficulty makes the variational approach more delicate than the topological one.

Our method of proof combines the method of upper and lower solutions [12, 14] with some fixed point theorems in cones, which are well known consequences of fixed point index theory [9]. This is not the first time that such type or results, extensively used in equations on compact intervals, have been employed for problems defined on the whole real line. However, the point of view of the few recent relevant papers is quite different to the one we employ here. In [2, 8], generalizations to Fréchet spaces of fixed point theorems of cone-compressing and cone-condensing type are proved and then employed. On the other hand, [17] needs to use a weighted norm in $BC(0, +\infty)$.

Our main result is as follows.

Theorem 1. *Let us assume that*

(A) $a \in C(I\!R; I\!R)$ *and there exists* $\underline{a} > 0$ *such that* $a(x) \geq \underline{a}$ *for all* x *;*
(B) $b \in C(I\!R; I\!R)$ *is a nonzero nonnegative function such that* $b/a \in L^\infty(I\!R)$ *;*
(C) $c \in C(I\!R; I\!R)$ *is such that* $c/a \in L^\infty(I\!R)$.

Then, there exists a unique positive solution $u \in BC(I\!R)$ *of eq. (1). Moreover, if* b *satisfies*

$$\lim_{|x|\to\infty} b(x)/a(x) = 0,$$

then u *satisfies* $\lim_{|x|\to\infty} u(x) = 0$.

For further references, we fix the following assumption

(B_0) $b \in C(I\!R; I\!R)$ is a nonzero nonnegative function such that

$$\lim_{|x|\to\infty} b(x)/a(x) = 0.$$

2. The linear equation.

Let us consider the linear equation

$$Lu := -u'' + c(x)u' + a(x)u = 0. \tag{4}$$

We first observe that under assumptions (A) and (C), the equation is disconjugate at $\pm\infty$ and presents a dichotomy, see [10] for precise definitions. This facts are the basis of our approach as it ensures the existence of a nice Green kernel to represents L^{-1} in an integral form.

Lemma 1. *Assume that* a *satisfies (A) and* c *fulfills (C). Then the linear equation* (4) *possesses a positive (increasing) solution* u_1 *and a linearly independent positive* *(decreasing) solution* u_2 *such that*

$$\lim_{x\to-\infty} u_1(x) = \lim_{x\to+\infty} u_2(x) = 0$$

and

$$\lim_{x\to+\infty} u_1(x) = \lim_{x\to-\infty} u_2(x) = +\infty.$$

Moreover, u_1, u_2 *can be chosen in such a way that*

$$u_1'(0)u_2(0) - u_1(0)u_2'(0) = 1. \tag{5}$$

Proof. The proof follows from classical arguments of the theory of linear second order equations. The assumptions imply that the origin is a saddle point, hence u_2 (resp. u_1) is taken as a solution of the stable (resp. unstable) manifold. Condition (A) also implies that a given solution can not have positive maxima or negative minima. A direct consequence is the disconjugacy of the equation and the monotone behavior of u_1, u_2. Condition (5) is easily obtained multiplying u_1, u_2 by a suitable constant. □

For every right-hand side h such that h/a is bounded, the non-homogeneous equation

$$-u'' + c(x)u' + a(x)u = h(x)$$

has a unique bounded solution (one may argue for instance with the classical theory of lower and upper-solutions). This solution can be computed by variation of constants leading to the Green function for bounded solutions

$$G(x,s) = \begin{cases} u_1(x)u_2(s)\mathrm{e}^{-\int_0^s c(\tau)d\tau}, & -\infty < x \le s < +\infty \\ u_2(x)u_1(s)\mathrm{e}^{-\int_0^s c(\tau)d\tau}, & -\infty < s \le x < +\infty \end{cases} \tag{6}$$

where u_1, u_2 are the solutions described in Lemma 1. Note that by Lemma 1, u_1, u_2 intersect at a unique point x_0, so that we can define a function $p \in BC(\mathbb{R})$ by

$$p(x) = \begin{cases} \dfrac{1}{u_2(x)}, & x \le x_0, \\[2ex] \dfrac{1}{u_1(x)}, & x > x_0. \end{cases} \tag{7}$$

It follows from the monotonicity of u_1 and u_2 that $p(x) = 1/\max(u_1(x), u_2(x))$. We conclude this section by collecting some properties of the Green function. Most of these have been stated in [15] for $c = 0$. We provide a short proof in order to keep our paper self-contained.

Proposition 1. *Assume that* a*,* c *satisfy* (A) *and* (B)*. Then one has*

$(P1)$ $G(x,s) > 0$ *for every* $(x,s) \in \mathbb{R}^2$ *;*

$(P2)$ $G(x,s) \le G(s,s)$ *for every* $(x,s) \in \mathbb{R}^2$ *;*

$(P3)$ *for any non-empty compact subset* $P \subset \mathbb{R}$*,*

$$G(x,s) \ge m_1(P)p(s)G(s,s), \text{ for all } (x,s) \in P \times \mathbb{R},$$

where

$$m_1(P) = \min\{u_1(\inf P), u_2(\sup P)\}; \tag{8}$$

$(P4)$ $G(s,s)p(s) \ge G(x,s)p(x)$ *for every* $(x,s) \in \mathbb{R}^2$ *;*

Proof. Properties $(P1)$ and $(P2)$ are trivial because of the positivity and monotonicity of u_1 and u_2.

We prove $(P3)$ for $(x,s) \in P \times I\!R$ with $x \leq s$ since the remaining possibility is analogous. Using the fact that u_2 is a positive non-decreasing function and that the function p satisfies

$$p(x) \leq \frac{1}{u_2(x)}, \qquad x \in I\!R, \tag{9}$$

we have

$$G(x,s) \geq u_2(\inf P) u_1(s) \mathrm{e}^{-\int_0^s c(\tau)} d\tau \geq m_1(P) \frac{G(s,s)}{u_2(s)} \geq m_1(P) p(s) G(s,s).$$

Finally, we consider $(P4)$. Again, we only consider the case $x \leq s$. Write

$$G(s,s)p(s) = G(x,s)p(x) \frac{u_1(s)}{u_1(x)} \frac{p(s)}{p(x)}.$$

If $p(s) = \frac{1}{u_2(s)}$, then $p(x) = \frac{1}{u_2(x)}$ too and the conclusion follows from the monotonicity of u_1 and u_2. If $p(s) = \frac{1}{u_1(s)}$, then either $p(x) = \frac{1}{u_1(x)}$ or $p(x) = \frac{1}{u_2(x)}$. In the first case, the conclusion is obvious while in the second case, one uses the fact that $u_2(x) \geq u_1(x)$. □

Finally, the following lemma will be useful.

Lemma 2. *Assume that a, b and c satisfy conditions (A), (B_0) and (C). Then the equation*

$$-u'' + c(x)u' + a(x)u = b(x), \tag{10}$$

has a unique bounded positive solution that can be written as $u(x) = \int_{I\!R} G(x,s)b(s)ds$. Moreover, we have $u(\pm\infty) = 0$.

Proof. It is clear that $\int_{I\!R} G(x,s)b(s)ds$ is a positive solution of (10). The boundedness is justified next. In self-adjoint form, the homogeneous linear equation (4) reads

$$-\left[P(x)u'\right]' + a(x)P(x)u = 0, \tag{11}$$

where $P(x) = \mathrm{e}^{-\int_0^x c(\tau)} d\tau$. Note that by the choice of u_1, the function $P(x)u_1'(x)$ is positive and moreover it is strictly increasing by means of the latter equation. Therefore, it has a non-negative limit at $-\infty$,

$$L_1 := \lim_{x \to -\infty} P(x)u_1'(x) \geq 0.$$

An analogous argument gives

$$L_2 := \lim_{x \to +\infty} P(x)u_2'(x) \leq 0.$$

Now, by integrating (11) over $]-\infty, x[$ and $]x, +\infty[$ respectively, one gets

$$P(x)u_1'(x) - L_1 = \int_{-\infty}^x a(s)P(s)u_1(s)ds, \qquad L_2 - P(x)u_2'(x) = \int_x^{+\infty} a(s)P(s)u_2(s)ds.$$

Then, by combining Liouville's formula and (5),

$$\begin{aligned}\int_{I\!R} G(x,s)a(s)ds =& u_2(x)\int_{-\infty}^{x} a(s)p(s)u_1(s)ds + u_1(x)\int_{x}^{+\infty} a(s)p(s)u_2(s)ds = \\ &= [u_2(x)u_1'(x) - u_1(x)u_2'(x)]\,P(x) - L_1u_2(x) + L_2u_1(x) \leq \\ &\leq [u_2(x)u_1'(x) - u_1(x)u_2'(x)]\,P(x) = 1.\end{aligned}$$

With this in mind,

$$\int_{I\!R} G(x,s)b(s)ds = \int_{I\!R} G(x,s)\frac{b(s)}{a(s)}a(s)ds \leq \left\|\frac{b}{a}\right\|_\infty \int_{I\!R} G(x,s)a(s)ds \leq \left\|\frac{b}{a}\right\|_\infty,$$

so the boundedness is proved.

Notice also that uniqueness is straightforward; on the contrary the difference of two bounded solutions would be a non-trivial bounded solution of the homogeneous equation (4), which is impossible. Hence, it remains to compute the limits at $\pm\infty$.

Let us prove that $\lim_{x\to+\infty} u(x) = 0$. We separate the argument in two cases. Assume first that u reaches a positive limit L monotonically. Then $\lim_{x\to+\infty} u'(x) = 0$. If $L > 0$, the equation shows that $u''(x) \geq \underline{a}L/2$ as $x \to \infty$ which is impossible. Therefore, we conclude that $L = 0$.

If u is not asymptotically monotone, there is a sequence of local maxima $u(t_n) > 0$ with $\lim_{n\to\infty} t_n = +\infty$. From the equation, we infer that

$$a(t_n)u(t_n) \leq b(t_n)$$

and, using assumptions $(A) - (B_0)$, we conclude that $u(t_n) \to 0$. □

3. An auxiliary result with a compactly supported potential.

In this section we assume the auxiliary condition

(B_C) $b \in C(I\!R; I\!R)$ is a nonzero nonnegative function and has compact support.

Of course, (B_C) is stronger than (B_0). This condition will be used as a first step in the proof of Theorem 1.

Let $BC^+(I\!R)$ be the set of positive bounded continuous functions defined on $I\!R$. We learned from the preceding section that we can look for a positive bounded solution of (1) as a fixed point of the operator $T\colon BC^+(I\!R) \to BC^+(I\!R)$ which can be written as

$$Tu(x) := \int_{I\!R} G(x,s)\frac{b(s)}{u(s)^p}ds.$$

Under condition (B_C), such operator is well-defined and a fixed point is indeed a positive solution of (1) satisfying (2).

In order to find a fixed point of T, we will use the following well known theorem on cones (see for instance [11, p.148] or [1]).

Theorem 2. *Let P be a cone in the Banach space B. Assume Ω_1, Ω_2 are open bounded subsets of B with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. If $T\colon P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a continuous and compact map such that one of the following conditions is satisfied*

(H1) $\|Tu\| \leq \|u\|$, *if* $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, *if* $u \in P \cap \partial\Omega_2$

(H2) $\|Tu\| \geq \|u\|$, *if* $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, *if* $u \in P \cap \partial\Omega_2$.

Then, T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Since our operator works on a space of functions defined on the whole real line, Ascoli-Arzela Theorem is not enough to ensure compactness. We will use a compactness criterion inspired by [16]. The proof is included for the convenience of the reader.

Lemma 3. *Let $\Omega \subset BC(\mathbb{R})$. Let us assume that the functions $u \in \Omega$ are equicontinuous in each compact interval of $\mathbb{R}$ and that for all $u \in \Omega$ we have*

$$|u(x)| \leq \xi(x), \qquad \forall x \in \mathbb{R} \tag{12}$$

where $\xi \in BC(\mathbb{R})$ satisfies

$$\lim_{|x|\to+\infty} \xi(x) = 0. \tag{13}$$

Then, Ω is relatively compact.

Proof: Given a sequence $(u_n)_n$ of functions of Ω, we have to prove that there exists a partial sequence which is uniformly convergent to a certain u. Note that the elements of Ω are uniformly bounded by $\|\xi\|_\infty$ and equicontinuous on compact intervals by hypothesis, therefore the Ascoli-Arzela theorem and a diagonal argument provide a partial sequence (we still denote it by $(u_n)_n$) which is uniformly convergent to a certain u on compact intervals. Of course, u satisfies also (12). Now, we have to prove that

$$\forall \varepsilon > 0,\ \exists n_0 \text{ s.t. } n \geq n_0 \implies \|u_n - u\|_\infty < \varepsilon.$$

By using (13), fix $k > 0$ such that $\max\limits_{x\in\mathbb{R}\setminus]-k,k[} |\xi(x)| < \dfrac{\varepsilon}{2}$. On the other hand, by using the uniform convergence on compact intervals, there exists n_0 such that $\max\limits_{x\in[-k,k]} |u_n(x) - u(x)| < \dfrac{\varepsilon}{2}$ for all $n \geq n_0$. Then,

$$\|u_n - u\|_\infty \leq \max_{x\in[-k,k]} |u_n(x) - u(x)| + \max_{x\in\mathbb{R}\setminus]-k,k[} |u_n(x) - u(x)| < \varepsilon,$$

and the proof is finished. ■

Proposition 2. *Let us assume (A), (B_C) and (C). Then, there exists a positive solution $u \in BC^+(I\!R)$ of (1) satisfying (2).*

In order to apply Theorem 2, we consider the Banach space $BC(I\!R)$ endowed with the supremum norm and look for an invariant cone for T. Let us consider the set

$$P = \left\{ u \in BC(I\!R) \,:\, u(x) \geq 0,\ \min_{y\in Supp(b)} u(y) \geq m_1 p_0 \|u\| \right\}, \tag{14}$$

where $p_0 = \inf_{Supp(b)} p(x)$, $p(x)$ being defined by (7), and the constant $m_1 \equiv m_1(Supp(b))$ is defined by (8). Note that the compactness of $Supp(b)$ implies that $p_0 > 0$. Also, it is easy to see, by definition, that $m_1 p_0 \leq 1$, and hence this cone is non-empty.

Proof of Proposition 2. We define the open bounded sets Ω_1, Ω_2 as the open balls in $BC(I\!R)$ of radius r and R, to be fixed later.

Step 1. - We claim that $T(P\cap(\overline{\Omega}_2\setminus\Omega_1))\subset P$. Take $u\in P\cap(\overline{\Omega}_2\setminus\Omega_1)$. Property $(P1)$ of the Green function and the sign of b imply that $Tu(x)\geq 0$ for all x. Besides, for all $\tau\in I\!\!R$, we have

$$\min_{x\in Supp(b)} Tu(x) = \min_{x\, Supp(b)} \int_{-\infty}^{+\infty} G(x,s)\frac{b(s)}{u(s)^p}ds \geq m_1 \int_{-\infty}^{+\infty} p(s)G(s,s)\frac{b(s)}{u(s)^p}ds$$
$$\geq m_1 \int_{Supp(b)} p(\tau)G(\tau,s)\frac{b(s)}{u(s)^p}ds = m_1 p_0 (Tu)(\tau),$$

where we have used $(P3)$ and $(P4)$. Therefore $T(P\cap(\overline{\Omega}_2\setminus\Omega_1))\subset P$.

Step 2. - Compactness. The continuity of T is trivial so that we focus on the compactness property. We will use Lemma 3. Let $(u_n)_n\subset P\cap(\overline{\Omega_2}\setminus\Omega_1)$ be a bounded sequence. Define the sequence $(v_n)_n\subset P$ by $v_n(x)=Tu_n(x)$. We just need to prove that, up to a subsequence, v_n converges uniformly in $I\!\!R$. Notice that the u_n^p's are uniformly bounded, say by M. Therefore, we compute

$$|v_n(x)| = |\int_{I\!\!R} G(x,s)\frac{b(s)}{u_n(s)^p}ds| \leq \frac{1}{(rm_1p_0)^p}\int_{I\!\!R} G(x,s)b(s)ds.$$

Now, observe that by Lemma 2, the function $\int_{I\!\!R} G(x,s)b(s)ds$ goes to zero at $\pm\infty$. Moveover, the equicontinuity of the v_n sequence is clear as it follows directly from the continuity of the Green function. Hence, we are able to apply the previous compactness criterion.

Step 3. - We claim that $\|Tu\|\geq\|u\|$, if $u\in P\cap\partial\Omega_1$. For $u\in P\cap\partial\Omega_1$,

$$m_1p_0r\leq u(x)\leq r \text{ for all } x\in Supp(b).$$

If $u\in P\cap\partial\Omega_1$ and r is small enough,

$$\|Tu(x)\|\geq\frac{1}{r^p}\sup_{x\in I\!\!R}\int_{Supp\, b} G(x,s)b(s)ds\geq r=\|u\|.$$

Step 4. - We claim that $\|Tu\|\leq\|u\|$, if $u\in P\cap\partial\Omega_2$. For $u\in P\cap\partial\Omega_2$,

$$m_1p_0R\leq u(x)\leq R \text{ for all } x\in Supp(b).$$

Taking R big enough, we obtain

$$\|Tu\|\leq\frac{1}{(Rm_1p_0)^p}\sup_{x\in I\!\!R}\int_{Supp(b)} G(x,s)b(s)ds\leq R=\|u\|.$$

It follows from the previous steps that we can apply Theorem 2 and the proof is complete. □

4. Proof of Theorem 1.

The existence of a bounded solution is proved by using the theory of upper and lower solutions (see [12, 14] for more details). Let $\underline{b}$ satisfy (B_C) and be such that $\underline{b}(x)\leq b(x)$ for every $x\in I\!\!R$. By Proposition 2, the equation

$$-u''+c(x)u'+a(x)u=\frac{\underline{b}(x)}{u^p}$$

has a positive bounded solution $\alpha(x)$. This function is a lower solution of (1). A well-ordered upper solution is easily found as a constant

$$\beta > \max\left\{ \|\alpha\|_\infty \,, \left(\left\| \frac{b}{a} \right\|_\infty \right)^{\frac{1}{1+p}} \right\}.$$

Then, the classical theory of upper and lower solutions provides a bounded solution of the original equation (1) between α and β.

Suppose now (B_0) holds and let us prove the convergence to 0 of the bounded solution u. The argument is similar to that employed in Lemma 2. Let us prove that $\lim_{x\to+\infty} u(x) = 0$, the limit at $-\infty$ follows in the same way. There are two possibilities

If $\lim_{x\to+\infty} u(x) = L > 0$ monotonically, we easily reach a contradiction as in Lemma 2.

If there is a sequence a local maxima $u(x_n) > 0$ with $(x_n)_n \to +\infty$, then we infer from the equation that

$$u(x_n)^{1+p} \leq \frac{b(x_n)}{a(x_n)}.$$

Passing to the limit, we conclude that $u(x_n) \to 0$ and the proof is done.

It remains to prove the uniqueness. In fact, we can write the equation as

$$u'' - c(x)u' = f(x,u) \tag{15}$$

with $f(x,u) = a(x)u - \frac{b(x)}{u^p}$ uniformly increasing in the second variable and apply comparison arguments like those employed in [6]. The argument is as follows. Assume that there are two positive bounded solutions u_1, u_2 and define the difference $d(x) = u_1(x) - u_2(x)$. Then,

$$d'' - c(x)d' = f(x,u_1) - f(x,u_2).$$

Since $f(x,u)$ is increasing, d is a bounded function without positive maxima or negative mimina. Therefore it must have a limit at $\pm\infty$. A simple limiting argument as those employed before shows that this limit must be zero. Combining this information with the fact that d can not have positive maxima or negative minima, one deduces that $d(x) \equiv 0$.

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