Lusternik-Schnirelman theory for the action integral of the Lorentz force equation

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Abstract

In this paper we introduce new Lusternik-Schnirelman type methods for nonsmooth functionals including the action integral associated to the relativistic Lagrangian of a test particle under the action of an electromagnetic field

$$\mathcal{L}(t,q,q') = 1 - \sqrt{1-|q'|^2} + q' \cdot W(t,q) - V(t,q),$$

where $V : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are two C^1-functions with V even and W odd in the second variable. By applying them, we obtain various multiplicity results concerning T-periodic solutions of the relativistic Lorentz force equation in Special Relativity,

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = E(t,q) + q' \times B(t,q),$$

where $E = -\nabla_q V - \frac{\partial W}{\partial t}, B = \operatorname{curl}_q W$. The zero Dirichlet boundary value conditions are considered as well.

Keywords: Poincaré relativistic Lagrangian, Lorentz force equation, Lusternik-Schnirelman, periodic solution, Dirichlet problem.

1 Introduction

1.1 The Lorentz force equation and the Poincaré relativistic Lagrangian - A brief history

The Lorentz force equation (LFE) rules the quasi-stationary motion of a charged particle in a electromagnetic field and constitutes, together with Maxwell equations, one of the pillars of Electromagnetism. For a fixed period $T > 0$, if it is assumed without loss of generality that the mass-to-charge ratio and the speed of light are equal to one, and that $V : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric and magnetic potentials respectively, the LFE reads

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' + (W(t,q))' = \mathcal{E}(t,q,q') - \nabla_q V(t,q), \tag{1}$$

where $\mathcal{E} : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathcal{E}(t,q,p) = (p \cdot D_{q_1} W(t,q), p \cdot D_{q_2} W(t,q), p \cdot D_{q_3} W(t,q)).$$

By a solution q of the LFE we mean a function $q = (q_1, q_2, q_3)$ of class C^2 such that $|q'(t)| < 1$ for all t, and which verifies the equation. In terms of the electric and magnetic fields given by

$$E = -\nabla_q V - \frac{\partial W}{\partial t}, \qquad B = \operatorname{curl}_q W,$$

the equation above is written in the more familiar form

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = E(t,q) + q' \times B(t,q).$$

While the force field $E(t,q) + q' \times B(t,q)$ acting on the particle was discovered by H.A. Lorentz and dates back to 1895 [19], the equation itself requires the notion of relativistic momentum and can be attributed to H. Poincaré, who introduced it with a different notation in his fundamental 1906's paper on Special Relativity, see [26, equation (5) in Section 7] (see also [13, 20] for a detailed description of this Poincaré pioneering work). Independently, and motivated by the results in the celebrated paper by Einstein [15, Section 10], the LFE was introduced in the same year 1906 by Planck in [25, equation (6)].

In those papers by Poincaré [26, Sections 2 and 7] and Planck [25, equation (5)], the authors identify formally the LFE as the Euler-Lagrange equation associated to the relativistic Lagrangian

$$\mathcal{L}(t,q,q') = 1 - \sqrt{1-|q'|^2} + q' \cdot W(t,q) - V(t,q).$$

Actually, these authors do not include the addend 1. Adding this constant 1 gives the positiveness of the relativistic part of $\mathcal{L}$ which make easier the reading of the paper. Clearly, it has neither mathematical nor physical relevance: both Lagrangians give the same (Euler-Lagrange) equations of motion. Summing up, from the hystorical point of view the non-smooth part (kinetic energy) of the Lagrangian is due to Poincaré, while the smooth (potential energy) was deduced earlier by Lorentz. In the sequel, we will refer to $\mathcal{L}(t,q,q')$ as the Poincaré relativistic Lagrangian or the Lagrangian of the LFE.

We point out that the formal deduction the the LFE from its Lagrangian is widely accepted in the physical community (see for instance, [18, Section 17 of Chapter 3] or [17]). In particular, we can read in the 19th lecture of the well-known ones by Feynman [17] the following (Feynman does not assume that the mass (m_0) -to- charge (q) ratio and the speed of light c are equal to one and he denotes the velocity ($|q'|$ above) of the particle by v, the electric potential V by the letter ϕ and the magnetic potential W by A and $\mathrm{v} = q'$):

> "The question is: Is there a corresponding principle of least action for the relativistic case? There is. The formula in this case of relativity is the following:
>
> $$S = -m_0c^2 \int_{t_1}^{t_2} \sqrt{1-\frac{v^2}{c^2}}dt - q\int_{t_1}^{t_2} [\phi(x,y,z,t) - \mathrm{v}\cdot A(x,y,z,t)]dt.$$
>
> The first part of the action integral is the rest mass m_0 times c^2 times the integral of a function of velocity $\sqrt{1-v^2/c^2}$. Then instead of just the potential energy, we have an integral over the scalar potential ϕ and over v times the vector potential A. Of course, we are then including only electromagnetic forces. All electric and magnetic fields are given in term of ϕ and A. This action function gives the complete theory of relativistic motion of a single particle in an electromagnetic field."

The proof that this action formula gives the correct equations of motion is left by Feynman to his students.

Besides the Lagrangian approach, there are other mathematical methods that have been used for the analytical study of the LFE. A first alternative is to use topological degree arguments like in [5]. It is also possible to use action integral associated to the Hamiltonian, see [30, 31] and also [2, Section 5]. In this case, the action functional is smooth but strongly indefinite. A different approach to study the LFE is to formulate the problem in the 4-vector formalism. The connection between the 3-vector and 4-vector formulations is well-known, one can find a nicely exposition for instance in [4] (see also [27, §3.8.]). If q is a T-periodic solution of the LFE, the proper time is defined as $\alpha(t) = \int_0^t \sqrt{1-|q'(\tau)|^2}d\tau$. Then, the 4-vector curve $(r,\delta) := (q\circ\alpha^{-1}, \alpha^{-1})$ in $M_0 \times \mathbb{R} = \mathbb{R}^3 \times \mathbb{R}$ satisfies the Lorentz force equation in 4-dimensions

$$\begin{aligned} r''(s) &= \delta'(s)E(\delta(s), r(s)) + r'(s) \times B(\delta(s), r(s)), \\ \delta''(s) &= r'(s)\cdot E(\delta(s), r(s)),\ \ s \in [0, \alpha(T)]. \end{aligned} \tag{2}$$

The Lorentz force equation in 4-vector form (2) is generalized to a general Lorentz manifold $M = M_0 \times \mathbb{R}$, where M_0 is a Riemannian manifold. Through this approach, existence results for the generalized Lorentz force equation are discussed in many papers for the case of inhomogeneous Dirichlet conditions (the so-called connecting orbits, see for instance [9, 10, 11, 22] and the references). Comparatively, the number of papers using the 4-vector formulation to find periodic solutions is considerably smaller [3, 8]. We observe that the periodic solutions of the 4-vector LFE found in those papers have a non-prescribed rest mass m_0.

1.2 Critical point theory framework for the Poincaré relativistic Lagrangian

In a recent paper [2], we have studied the differentiability of the action integral associated to the relativistic Lagrangian $\mathcal{L}(t, q, q')$. Note for instance that it is not Fréchet differentiable due to the square root appearing in the Lagrangian. However, it is differentiable in the sense of the theory of Szulkin [29] and we have proved for the first time that LFE is indeed the Euler-Lagrange equation of the relativistic Lagrangian $\mathcal{L}$; i.e., the equation satisfied by the critical points in the sense of Szulkin of the action integral. Denoting by $W^{1,\infty}(0,T)$ the space of all Lipschitz functions in $[0,T]$ (or equivalently the absolutely continuous functions in $[0,T]$ with bounded derivatives), we consider the Banach space

$$W_*^{1,\infty} = \{q \in [W^{1,\infty}(0,T)]^3 \,:\, q(0) = q(T)\}$$

endowed with the usual norm $\|\cdot\|_{1,\infty}$ given by

$$\|q\|_{1,\infty} = \|q\|_\infty + \|q'\|_\infty,$$

where $\|q\|_\infty = \max_{t\in[0,T]} |q(t)|$ and $\|q'\|_\infty = \max_{t\in[0,T]} |q'(t)|$. If

$$\mathcal{K}_* = \{q \in W_*^{1,\infty} : \|q'\|_\infty \leq 1\},$$

the keystone in our approach is that a function q is a T-periodic solution for LFE, that is, a solution for LFE satisfying

$$q(0) = q(T), \qquad q'(0) = q'(T),$$

if and only if $q \in \mathcal{K}_*$ and

$$\int_0^T [\sqrt{1-|q'|^2} - \sqrt{1-|\varphi'|^2}]dt + \int_0^T [\mathcal{E}(t,q,q') - \nabla_q V(t,q)] \cdot (\varphi - q)dt + \int_0^T W(t,q)\cdot(\varphi' - q')dt \geq 0, \quad \text{for all } \varphi \in \mathcal{K}_*.$$

Now the relation between the Poincaré relativistic Lagrangian and the Critical Point Theory is straightforward. Consider the action functional associated to the Poincaré relativistic Lagrangian $\mathcal{L}$ with periodic boundary conditions

$$\mathcal{I}_* : W_*^{1,\infty} \to (-\infty, +\infty], \quad \mathcal{I}_* = \Psi_* + \mathcal{F},$$

where Ψ_* is given by

$$\Psi_*(q) = \begin{cases} \displaystyle\int_0^T [1 - \sqrt{1 - |q'|^2}]dt, & \text{if } q \in \mathcal{K}_*, \\ \\ +\infty, & \text{if } q \in W_*^{1,\infty} \setminus \mathcal{K}_*, \end{cases}$$

and $\mathcal{F}$ is defined by

$$\mathcal{F}(q) := \int_0^T [q' \cdot W(t,q) - V(t,q)]dt, \qquad \text{for all } q \in W_*^{1,\infty}.$$

Observe that Ψ_* is continuous on $\mathcal{K}_*$ and that $\mathcal{I}_*$ is the sum of the proper convex lower semicontinuous functional Ψ_* and of the C^1-functional $\mathcal{F}$ with

$$\mathcal{F}'(q)[\varphi] = \int_0^T (\mathcal{E}(t,q,q') - \nabla_q V(t,q)) \cdot \varphi dt + \int_0^T W(t,q) \cdot \varphi' dt, \ (q, \varphi \in W_*^{1,\infty}).$$

Hence the above key result means that a function q is a T-periodic solution of LFE if and only if q is a critical point for $\mathcal{I}_*$ in the Szulkin sense [29], that is $q \in \mathcal{K}_*$ and

$$\Psi_*(\varphi) - \Psi_*(q) + \mathcal{F}'(q)[\varphi - q] \geq 0, \quad \text{for all } \varphi \in W_*^{1,\infty}.$$

Moreover, in [2], we provide sufficient conditions for the existence of at least one critical point of $\mathcal{I}_*$ that corresponds either to a minimum or to a saddle point, and consequently also a T-periodic solution of the LFE by the discussion above. As usual these sufficient conditions imply some geometrical and compactness properties of the functional $\mathcal{I}_*$ associated to LFE. We have to point out that the geometry of $\mathcal{I}_*$ is not standard since the nonsmooth and main part Ψ_* of it is bounded from above in $\mathcal{K}_*$. With respect to the compactness of $\mathcal{I}_*$, we also have to remark that $\mathcal{I}_*$ does not satisfy the generalized (PS) condition of [29, page 80]. Thus, we use in [2] the following weak Palais-Smale condition, (wPS) in the sequel, of the Poincaré action functional: *If $(q_n) \subset K_*$ is a bounded Palais-Smale sequence of $\mathcal{I}_*$ at the level $c \in \mathbb{R}$, i.e., if $\mathcal{I}_*(q_n) \to c$ and there is $\varepsilon_n \to 0$ such that for each integer $n \geq 1$*

$$\Psi_*(\varphi) - \Psi_*(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\|_{1,\infty}, \quad \textit{for all } \varphi \in \mathcal{K}_*,$$

then there exists a subsequence (q_{n_k}) of (q_n) converging in $(C([0,T], \mathbb{R}^3), \|\cdot\|_\infty)$ to a critical point $q \in \mathcal{K}_*$ of $\mathcal{I}_*$ with level $\mathcal{I}_*(q) = c$.

1.3 Even Poincaré action functional and the Lusternik-Schnirelman method

A natural continuation of the research initiated in [2] is to study whether the symmetry properties, like e.g. if $\mathcal{I}_*$ is even, of the Poincaré action functional $\mathcal{I}_*$ imply multiple T-periodic solutions of LFE.

In the pioneering work [21], the authors combine a topological invariant of the domain (the Lusternik-Schnirelman category) with the symmetry properties of the action functional to obtain multiple critical points. The original result is formulated for action functionals defined on finite-dimensional manifolds, but was extended to the infinite-dimensional Hilbert manifolds by Schwartz [28] and to the infinite-dimensional Banach manifolds by Palais [24]. An important variant of the Lusternik-Schnirelman theory for smooth even functional in Banach spaces is due to Clark [12], who applies the notion of genus instead of the category tool. The first paper that applies Lusternik-Schnirelman methods for Euler-Lagrange action functionals associated to differential equations is due to Browder [7].

Next, in their seminal paper [1], where the well known Mountain Pass theorem was proved, Ambrosetti and Rabinowitz introduced also a version [1, Theorem 2.23] of the Mountain Pass Theorem for even C^1-functionals unbounded from below. This gives the existence of a sequence of positive critical values converging to infinity.

Some years later, Szulkin generalized both the Lusternik-Schnirelman theory (more precisely, [12, Theorem 8]) and the symmetric Mountain Pass theorem to nonsmooth even functionals like $\mathcal{I}_*$. These generalizations are given in Theorems 4.3 and 4.4 in [29]. A fundamental hypothesis in the latter generalized theorem is that the restriction of the functional to finite dimensional subspaces must tend to $-\infty$ as q goes to infinity (see hypothesis (ii) in [29, Theorem 4.4]). It is easy to observe that the Poincaré action functional $\mathcal{I}_*$ does not satisfy this fundamental hypothesis, given that it is bounded from below modulo a finite codimesional subspace. In fact, $\mathcal{I}_*$ is bounded from below on the three codimensional subspace of those functions in $W^{1,\infty}_*$ having zero mean value.

1.4 Main contributions

We develop two abstract main tools (Theorems 1 and 2 of Sections 2 and 3) from which various multiplicity results concerning T-periodic solutions of LFE can be deduced. A particular version of the first one reads as follows (see Corollary 1 and Remark 1-ii) below) for even Poincaré action functionals which are bounded from below modulo a finite codimensional subspace:

Theorem A *Assume that W is odd and V is even in the second variable. If $\mathcal{I}_*$ is bounded from below in a subspace $\widetilde{X}_{l-1}$ of codimension $l-1$ with $l \geq 1$ and satisfies (wPS)-condition and for some $k \geq l$*

($\mathcal{I}^1_*$) *there exist a subspace X_k of $W^{1,\infty}_*$ with $\dim X_k = k$ and $r > 0$ such that $\mathcal{I}_*(q) < \mathcal{I}_*(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$,*

then $\mathcal{I}_*$ possesses at least $k-l+1$ distinct pairs of nontrivial critical points with negative levels.

As it has been mentioned, the above result follows from our abstract Theorem 1, which is a generalization of Theorem 4.3 in [29] where the author assumes that $l = 1$ and, instead of (wPS), that the stronger Szulkin' version of the Palais-Smale (PS) holds true.

The second tool is a Mountain Pass Theorem for non-smooth functionals having $\mathbb{Z}_2$-symmetry. In particular, we have the following Mountain Pass Theorem for the even Poincaré action functional.

Theorem B *Assume that W is odd and V is even in the second variable with $V(t, 0) = 0$ for every $t \in [0, T]$. If $\mathcal{I}_*$ satisfies (wPS)-condition, hypothesis $(\mathcal{I}_*^1)$ and*

$(\mathcal{I}_*^2)$ *there exist a subspace $\widetilde{X}_{\overline{k}}$ of $W_*^{1,\infty}$ with codim $\widetilde{X}_{\overline{k}} = \overline{k} < k$ and constants $\rho \in (0, r)$ and $\alpha > 0$ such that $\mathcal{I}_*(q) \geq \alpha$ for all $q \in \widetilde{X}_{\overline{k}}$ with $\|q\|_\infty = \rho$,*

then $\mathcal{I}_*$ has at least $k - \overline{k}$ distinct pairs of nontrivial critical points with positive levels.

We observe explicitly that, contrary to [29, Theorem 4.4], it is not required that $\mathcal{I}_*(q)$ is tending to $-\infty$ when $q \in X_k$ is tending to infinity. Moreover, we only impose the (wPS) condition which is weaker than the (PS) condition asumed in the previously cited paper. Hence, the above result, or more precisely, the general Theorem 2 below improves the Theorem 4.4 in [29]. Even more, in contrast with the *"not fully satisfactory"* minimax characterization given in the cited theorem (see [29, Remark 4.7]), we will see in Remark 2 that Theorem 2 characterizes the critical values associated to the pairs of nontrivial critical points. In this way, Theorem 2 is a complete generalization of [1, Theorem 2.23] under the weaker hypothesis (wPS) than the standard Palais-Smale condition.

One advantage of the right variational characterization given in our theorems is that we can combined both to obtain pairs of nontrivial critical points either with negative levels or with positive ones. See Theorem 3 below for an example.

To prove the general Theorems 1 and 2 (which imply the above results) we use Ekeland variational principle (see [16]) together with some ideas from the proof of Theorem 4.4 in [29].

Specifically, we have to use new ideas based on the continuity with respect to the $W^{1,\infty}$-norm of the functional on its domain of definition to overcome the corresponding difficulties observed for the Poincaré action functional.

Next, we present an example of applicability of Theorem A. The proof follows from Theorems 5 and 6 below.

Example A *Assume that W is odd and V is even in their second variable with $V(t, 0) = 0$ for every $t \in [0, T]$. Suppose also that there exist $r_1 \in (0, 1)$, $c, d > 0$ and $\mu, \nu > 0$ with $\mu < \min\{2, \nu + 1\}$ such that*

$$|W(t, q)| \leq c|q|^\nu, \ V(t, q) \geq d|q|^\mu \quad \text{for } t \in [0, T], |q| \leq r_1.$$

If, in addition, there exist $\overline{\mu} > 1$, $C > 0$ and a sufficiently large $R > 0$ such that either

$$|W(t, q)| + V(t, q) \leq -C|q|^{\overline{\mu}}, \quad \text{for } t \in [0, T], |q| \geq R,$$

or

$$|W(t,q)| - V(t,q) \leq -C|q|^{\overline{\mu}}, \quad \textit{for } t \in [0,T], |q| \geq R,$$

then the Lorentz force equation (1) *has infinitely many pairs of nontrivial T-periodic solutions.*

The symmetric mountain pass result presented as Theorem B also can be applied to LFE together with Theorem A (see Remark 5 in Section 4), as in the next example.

Example B *Let V be given by*

$$V(t,q) = \lambda\beta(t)|q|^{\mu} \quad \textit{for all } (t,q) \in [0,T] \times \mathbb{R}^3,$$

where $\mu > 2$ and $\beta : [0,T] \to \mathbb{R}$ is a positive, continuous function. Assume that the magnetic potential W is such that $W(t,\cdot)$ is odd for all $t \in [0,T]$, and

$$\lim_{|q|\to 0} \frac{|W(t,q)|}{|q|^2} = 0, \qquad \limsup_{|q|\to\infty} \frac{|W(t,q)|}{|q|^{\mu}} < \infty,$$

uniformly in $t \in [0,T]$. Then, for any integer $m \geq 1$, there is $\Lambda_m > 0$ such that the Lorentz force equation has at least $2m$ pairs of nontrivial T-periodic solutions (m pairs corresponding to negative and m pairs to positive critical values of the relativistic Poincaré action functional) for any $\lambda \geq \Lambda_m$.

More results on multiplicity of periodic solutions for LFE can be given by imposing suitable hypotheses on the behavior of the potentials V and W at infinity and at zero as in the preceding example, see Theorems 7, 8, 9, 10 below. The case of zero Dirichlet boundary conditions will be also consider on the last section of this paper.

1.5 Organization of the paper

The paper is organized as follows. The second section develops and improves the abstract Lusternik-Schnirelman method for the Szulkin's nonsmooth functionals which are continuous in their domain and satisfy only the weak compactness condition (wPS) (Subsection 2.1) and the symmetric Mountain Pass Theorem (Subsection 2.2). Our abstract results are tailored for the Poincaré relativistic Lagrangian in the next sections. Specifically, in Section 3 we study the existence of multiple periodic solutions for the Lorentz force equation. The last section is devoted to the extension of the previous results to the case of zero Dirichlet boundary conditions (Subsection 4.1) and to some remarks about the related generalized Lorentz force equation in 4-vector form (Subsection 4.2).

2 Existence of critical points for abstract even functionals

2.1 The Lusternik-Schnirelman method

Let $\mathcal{X}$ be a Banach space with a norm $\|\cdot\|$ which is continuously embedded into a Banach space Y, whose norm is denoted by $\|\cdot\|_Y$. Let now $\mathcal{I}:\mathcal{X}\to(-\infty,\infty]$ be a functional given by

$$\mathcal{I}=\Psi+\mathcal{F},$$

where $\mathcal{F}\in C^1(\mathcal{X},\mathbb{R})$ and $\Psi:\mathcal{X}\to(-\infty,\infty]$ is convex and continuous on the non-empty and closed domain $D(\Psi)=\{q\in\mathcal{X}:\Psi(q)\neq\infty\}$. Recall the Szulkin's definition of critical point:

Definition 1 ([29]) *A function $q\in\mathcal{X}$ is called a* **critical point** *of $\mathcal{I}$ if $q\in D(\Psi)$ and it satisfies the following variational inequality*

$$\Psi(\varphi)-\Psi(q)+\mathcal{F}'(q)[\varphi-q]\geq 0,\quad \textit{for all } \varphi\in D(\Psi).$$

In what follows we assume that $\mathcal{I}$ satisfies the **weak Palais-Smale condition** introduced in [2]:

(wPS) For every $c\in\mathbb{R}$, $(q_n)\subset\mathcal{X}$ and $\varepsilon_n\to 0$ such that $\mathcal{I}(q_n)\to c$ and

$$\Psi(\varphi)-\Psi(q_n)+\mathcal{F}'(q_n)[\varphi-q_n]\geq-\varepsilon_n\|\varphi-q_n\|\quad\text{for all } \varphi\in D(\Psi), n\geq 1.$$

there exists a subsequence (q_{n_k}) of (q_n) converging in Y to a critical point $q\in\mathcal{D}(\Psi)$ of $\mathcal{I}$ with level $\mathcal{I}(q)=c$.

We will consider even functionals $\mathcal{I}$, i.e., we assume that

$(\mathcal{I}_0)$ Ψ and $\mathcal{F}$ are even.

It is advisable in this case to use the concept of *genus* of a nonempty, closed, symmetric (w.r.t. the origin) subset A of $\mathcal{X}$. Recall that the genus $\gamma(A)$ of A is the smallest integer k for which there exists some odd, continuous mapping $f:A\to\mathbb{R}^k\setminus\{0\}$; with $\gamma(A)=\infty$ if no such mapping exists. Set $\gamma(\emptyset)=0$. In particular, since every odd continuous function f in a nonempty, closed, symmetric subset $A\subset\mathcal{X}$ containing the zero satisfies that $f(0)=0$, we have $\gamma(A)=\infty$ provided that $0\in A$. The genus is a $\mathbb{Z}_2$-index, i.e., for any closed, symmetric subsets A,A_1,A_2 of $\mathcal{X}$ one has that

(G_1) $\gamma(A)=0$ if and only if $A=\emptyset$;

(G_2) If $h:A_1\to A_2$ is odd and continuous, then $\gamma(A_1)\leq\gamma(A_2)$;

(G_3) $\gamma(A_1\cup A_2)\leq\gamma(A_1)+\gamma(A_2)$;

(G_4) If A is compact and $0\notin A$, then $\gamma(A)<\infty$ and there is an $\varepsilon>0$ such that $\gamma(A)=\gamma(\{q\in\mathcal{X}:\operatorname{dist}(q,A)\leq\varepsilon\})$.

In addition, the genus also satisfies the following properties:

(G_5) If there exists an odd homeomorphism of the $(k-1)$-sphere onto A, then $\gamma(A) = k$.

(G_6) If $\widetilde{X}_k$ is a subspace of $\mathcal{X}$ of codimension k and $A \cap \widetilde{X}_k = \emptyset$, then $\gamma(A) \leq k$.

We refer to the reader to [12, Lemma 6] for the proofs of (G_{1-6}).

For $q_0 \in \mathcal{X}$ and $a > 0$ we consider the open set in $\mathcal{X}$ given by $B_Y(q_0, a) = \{q \in \mathcal{X} : ||q - q_0||_Y < a\}$ and, by the continuous embedding of $\mathcal{X}$ into Y, its closure in $\mathcal{X}$ given by $\overline{B}_Y(q_0, a) = \{q \in \mathcal{X} : ||q - q_0||_Y \leq a\}$. Observe that the boundary $\partial B_Y(q_0, a)$ in $\mathcal{X}$ of $B_Y(q_0, a)$ (or $\overline{B}_Y(q_0, a)$) is just $\partial B_Y(q_0, a) = \{q \in \mathcal{X} : \|q - q_0\|_Y = a\}$. We will need another property of the genus:

(G_7) Consider $a > 0$ and n of pairs of points $\{\pm q_m : 1 \leq m \leq n\}$ such that the sets $\overline{B}_Y(\pm q_m, a)$, are mutually disjoint and do not contain the origin. Then the set $\cup_{m=1}^n [\overline{B}_Y(q_m, a) \cup \overline{B}_Y(-q_m, a)]$ is symmetric, closed in $\mathcal{X}$ and has genus one.

Indeed, to show (G_7) it suffices to observe that the function given by $f(q) = 1$ if $q \in \cup_{m=1}^n \overline{B}_Y(q_m, a)$ and $f(q) = -1$ if $q \in \cup_{m=1}^n \overline{B}_Y(-q_m, a)$, is continuous and odd.

Next, we consider the complete metric space $(\mathcal{M}, \delta)$ where

$$\mathcal{M} = \{A \subset \mathcal{X} : A \text{ is closed, bounded and nonempty}\},$$

and δ is the Hausdorff-Pompeiu distance given by

$$\delta(A, B) = \max\{\sup_{a \in A} \ \text{dist}(a, B), \sup_{b \in B} \ \text{dist}(b, A)\}.$$

For $j \geq 1$, we define

$$\mathcal{A}_j = \{A \subset \mathcal{X} : A \text{ is compact, symmetric and } \gamma(A) \geq j\}.$$

Since $\mathcal{A}_j$ is closed in $\mathcal{M}$, $(\mathcal{A}_j, \delta)$ is a complete metric space and $\mathcal{A}_j \subset \mathcal{A}_{j-1}$, for every $j \geq 2$. Observe also that if X_j is a j-dimensional subspace of $\mathcal{X}$, then, by (G_5), $\gamma(X_j \cap \partial B_{\mathcal{X}}(0, 1)) = j$ and $X_j \cap \partial B_{\mathcal{X}}(0, 1) \in \mathcal{A}_j$. Following the Lusternik-Schnirelman method [7, 12, 21, 24, 28, 29], we consider for every $j \geq 1$ the values

$$-\infty \leq c_j = \inf_{A \in \mathcal{A}_j} \sup_A \mathcal{I}.$$

The main properties of the sequence (c_j) are compiled in the next lemma.

Lemma 1 *(i) The sequence (c_j) is non-decreasing.*

(ii) If $0 \in \ \text{int}\, D(\Psi)$ (respectively, $0 \notin D(\Psi)$), then $c_j \leq \mathcal{I}(0)$ (respectively, $c_j < \mathcal{I}(0)$) for any $j \geq 1$.

(iii) If $-\infty < c_j$, then given $\varepsilon > 0$ and $B \in \mathcal{A}_j$ with

$$\sup_B \mathcal{I} \leq c_j + \varepsilon,$$

there exists $C \in \mathcal{A}_j$ such that

$$\sup_C \mathcal{I} \leq \sup_B \mathcal{I}, \quad \delta(B, C) \leq \sqrt{\varepsilon}, \tag{3}$$

$$\sup_D \mathcal{I} \geq \sup_C \mathcal{I} - \sqrt{\varepsilon}\delta(C, D) \text{ for all } D \in \mathcal{A}_j. \tag{4}$$

Proof. (i) From $\mathcal{A}_j \subset \mathcal{A}_{j-1}$ for every $j \geq 2$, it follows that $c_{j-1} \leq c_j$.

(ii) If $0 \in \operatorname{int} D(\Psi)$, using that $\mathcal{I}$ is continuous at $0 \in \operatorname{int} D(\Psi)$ we deduce that $c_j \leq \mathcal{I}(0)$ for any $j \geq 1$. Indeed, from $0 \in \operatorname{int} D(\Psi)$ it follows that there exists $\epsilon_0 > 0$ such that $B_{\mathcal{X}}(0, \epsilon_0) \subset \operatorname{int} D(\Psi)$ and, by (G_5), $\partial B_{\mathcal{X}}(0, \epsilon) \cap X_j$ has genus j for any $\epsilon \in (0, \epsilon_0)$ and any j-dimensional subspace X_j of $\mathcal{X}$. Then

$$c_j \leq \sup_{u \in \partial B_{\mathcal{X}}(0,\epsilon) \cap X_j} \mathcal{I}(u)$$

Taking limits as $\epsilon \to 0$, by the continuity of $\mathcal{I}$ at 0, we obtain $c_j \leq \mathcal{I}(0)$. On the other hand, in the case $0 \notin D(\Psi)$, we also have that $c_j < \mathcal{I}(0) = \infty$.

(iii) Observe that $c_j = \inf_{A \in \mathcal{A}_j} \Pi(A)$, where the mapping Π is defined in the complete metric space $(\mathcal{A}_j, \delta)$ by

$$\Pi(A) = \sup_A \mathcal{I} \in (-\infty, \infty], \quad \text{for all } A \in \mathcal{A}_j. \tag{5}$$

Using that Π is lower semicontinuous, the result follows now by the Ekeland variational principle. ∎

Theorem 1 *If the functional $\mathcal{I}$ satisfies $(\mathcal{I}_0)$, (wPS)-condition and*

$$-\infty < c_j < \mathcal{I}(0) \text{ for } j = l, ..., k,$$

then c_j is a critical value of $\mathcal{I}$ for every $l \leq j \leq k$. Moreover, if $c_j = c_i = c$ for some $l \leq j < i \leq k$, then $\mathcal{I}$ has infinitely many pairs of critical points at the level c. In particular, $\mathcal{I}$ has at least $k - l + 1$ distinct pairs of nontrivial critical points with critical levels below $\mathcal{I}(0)$.

Proof. We begin by proving the more difficult part, namely that if $c_i = c_j = c$ for some $l \leq j < i \leq k$, then $\mathcal{I}$ has infinitely many pairs of critical points at the level c. The proof that c_j are critical values is simpler and will be outlined at the end . Since $\mathcal{I}$ is even, assume by contradiction in this case that $\mathcal{I}$ has only a finite number n of pairs of critical points $\{\pm q_m : 1 \leq m \leq n\}$ at the level c. Using that $c_j < \mathcal{I}(0)$, it follows that all those points are different from zero. Thus, there exists a radius $a > 0$ such that for every $1 \leq m \leq n$ the sets

$\overline{B}_Y(\pm q_m, 2a)$, are mutually disjoint and do not contain the origin. For $\alpha = a$ and $\alpha = 2a$ we consider the open symmetric set N_α in $\mathcal{X}$ given by

$$N_\alpha = \bigcup_{m=1}^{n} [B_Y(q_m, \alpha) \cup B_Y(-q_m, \alpha)].$$

We claim that there exists $\varepsilon \in (0, a^2)$ such that $c + \varepsilon < \mathcal{I}(0)$ and given $q \in \mathcal{I}^{-1}([c - \varepsilon, c + \varepsilon]) \setminus N_a$, there is $\varphi_q \neq q$ with

$$\Psi(\varphi_q) - \Psi(q) + \mathcal{F}'(q)[\varphi_q - q] < -\sqrt{\varepsilon}\|\varphi_q - q\|.$$

Indeed, if it were not the case, there would be a sequence (p_m) satisfying

$$p_m \in \mathcal{X} \setminus N_a, \quad c - \frac{1}{m} \leq \mathcal{I}(p_m) \leq c + \frac{1}{m},$$

and

$$\Psi(\varphi) - \Psi(p_m) + \mathcal{F}'(p_m)[\varphi - p_m] \geq -\sqrt{\frac{1}{m}}\|\varphi - p_m\|, \quad \text{for all } \varphi \in D(\Psi),$$

for all $m \geq 1$. Then, using (wPS)-condition, going if necessary to a subsequence, one has that there exists a critical point $q \in D(\Psi)$ of $\mathcal{I}$ with $\mathcal{I}(q) = c$ and $\|p_m - q\|_Y \to 0$, which implies that $q \notin N_a$, a contradiction with the definition of N_a.

Moreover, using that $\Psi, \mathcal{F}$ are even it is easy to see that in the above claim we can take

$$\varphi_{-q} = -\varphi_q.$$

Next, using the definition of c_i (and of the function Π in (5)) we can pick $A \in \mathcal{A}_i$ such that

$$\Pi(A) = \sup_A \mathcal{I} \leq c + \varepsilon < \mathcal{I}(0).$$

Then $0 \notin A \subset D(\Psi)$, and using that Ψ is continuous on $D(\Psi)$ and that A is compact, it follows that the above "sup" is a "max". Since

$$B = A \setminus N_{2a}.$$

is closed in $\mathcal{X}$, it is compact, symmetric and we deduce from the inclusion $A \subset B \cup \left[\cup_{m=1}^{n} [\overline{B}_Y(q_m, 2a) \cup \overline{B}_Y(-q_m, 2a)]\right]$ and (G_7) that

$$j < i \leq \gamma(A) \leq \gamma(B) + \gamma\Big(\bigcup_{m=1}^{n} [\overline{B}_Y(q_m, 2a) \cup \overline{B}_Y(-q_m, 2a)]\Big) = \gamma(B) + 1.$$

Thus, $\gamma(B) > j - 1$ and $B \in \mathcal{A}_j$. Using that $B \subset A$ and the definition of Π, we notice that

$$c \leq \Pi(B) \leq \Pi(A) \leq c + \varepsilon.$$

By Lemma 1-(iii), there exists $C \in \mathcal{A}_j$ satisfying (3) and (4).

Using that $B \cap N_{2a} = \emptyset$ and that the Hausdorff-Pompeiu distance $\delta(B, C) \leq \sqrt{\varepsilon} \leq a$, it follows that $C \cap N_a = \emptyset$. In particular,

$$S := \{q \in C : c - \varepsilon \leq \mathcal{I}(q)\} \subset \mathcal{I}^{-1}([c - \varepsilon, c + \varepsilon]) \setminus N_a.$$

Moreover, using that $\mathcal{I}$ is continuous on $D(\Psi)$, it follows that S is compact in $\mathcal{X}$. On the other hand, using the continuity of $\mathcal{F}'$, the continuity of Ψ on $D(\Psi)$ and the above claim, for each $q \in S$, we can find $\delta_q > 0$ such that

i) $\delta_q = \delta_{-q}$,

ii) the closed balls $\overline{B}_{\mathcal{X}}(\pm q, \delta_q)$ in $\mathcal{X}$ (of centers $\pm q$ and radius δ_q with respect to the norm $\| \cdot \|$ in $\mathcal{X}$) are disjoint: $\overline{B}_{\mathcal{X}}(q, \delta_q) \cap \overline{B}_{\mathcal{X}}(-q, \delta_q) = \emptyset$,

iii) $0, \varphi_q \notin \overline{B}_{\mathcal{X}}(q, \delta_q)$,

iv) for every $h \in \overline{B}_{\mathcal{X}}(0, \delta_q)$ and $p \in \overline{B}_{\mathcal{X}}(q, \delta_q)$,

$$\Psi(\varphi_q) - \Psi(p) + \mathcal{F}'(q + h)[\varphi_q - p] < -\sqrt{\varepsilon}\|\varphi_q - p\|.$$

Using that S is compact in $\mathcal{X}$ and that the open balls $B_{\mathcal{X}}(\pm q, \delta_q)$ with $q \in S$ are a cover of it, it follows that there exist $p_1, ..., p_\ell \in S$ such that $\{B_{\pm\iota} = B_{\mathcal{X}}(\pm p_\iota, \delta_{p_\iota}) : 1 \leq \iota \leq \ell\}$ is a covering of S:

$$S \subset \bigcup_{\iota=1}^{\ell} (B_\iota \cup B_{-\iota}).$$

Using that $\varphi_{p_\iota} \notin \overline{B}_\iota$, (by iii)) we have

$$\operatorname{dist}(\varphi_{p_\iota}, \overline{B}_\iota \cap C) = \min_{p \in \overline{B}_\iota \cap C} \|\varphi_{p_\iota} - p\| > 0$$

for every $\iota = -\ell, \ldots, -1, 1, \ldots, \ell$.

Let us fix

$$0 < \delta \leq \min\{\delta(B, C), \delta_{p_\iota}, \operatorname{dist}(\varphi_{p_\iota}, \overline{B}_\iota \cap C) \,:\, \iota = -\ell, \ldots, -1, 1, \ldots, \ell\}$$

and consider a continuous and even function $\eta : C \to [0, 1]$ satisfying

$$\eta(q) = \begin{cases} 1, & \text{if } c \leq \mathcal{I}(q), \\ 0, & \text{if } \mathcal{I}(q) \leq c - \varepsilon, \end{cases}$$

and the function $\eta_\iota : C \to [0, 1]$ with η given by

$$\eta_\iota(q) = \begin{cases} \dfrac{\operatorname{dist}\,(q, C \setminus B_\iota)}{\displaystyle\sum_{\substack{m = -\ell \\ m \neq 0}}^{\ell} \operatorname{dist}(q, C \setminus B_m)}, & \text{if } q \in B_\iota \cap C, \\ 0, & \text{if } q \in C \setminus B_\iota. \end{cases}$$

Observe that

$$\sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \eta_\iota = 1 \quad \text{on } S$$

and that each η_ι is odd (since, by i) and ii), $\varphi_{-p_\iota} = -\varphi_{p_\iota}$ and $\overline{B}_\iota \cap \overline{B}_{-\iota} = \emptyset$ for every $1 \leq \iota \leq \ell$).

Now, let us define the function $\beta : [0,1] \times C \to \mathcal{X}$ given by

$$\beta(t,q) = \beta_t(q) := q + t\delta\eta(q) \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}(\varphi_{p_\iota} - q), \quad \forall t \in [0,1],\ \forall q \in C.$$

Clearly, each $\beta_t : C \to \mathcal{X}$ is continuous and odd with β_0 the identity in C. (Thus, β is called a deformation of C associated to the above covering). In particular, using that β_1 is continuous and odd, we deduce that

$$D = \beta_1(C)$$

is compact and symmetric with $\gamma(D) \geq \gamma(C)$, which implies that $D \in \mathcal{A}_j$.

Moreover, by taking into account that $\eta_\iota(q) = 0$ for all $q \notin B_\iota$ we deduce for $M := (\operatorname{dist}(\varphi_{p_\iota}, \overline{B}_\iota \cap C))^{-1}$ that

$$t\delta\eta(q) \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|} \leq \delta \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\operatorname{dist}(\varphi_{p_\iota}, \overline{B}_\iota \cap C)} \leq \delta M,$$

for every $(t,q) \in [0,1] \times C$. Hence, using $\delta \leq 1/M$, we observe that $\beta_t(q)$ is a convex combination of $q, \varphi_{p_1}, ..., \varphi_{p_\ell}$. Indeed, for any $q \in C$ it holds that

$$\beta_t(q) = \left[1 - t\delta\eta(q) \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\right] q + t\delta\eta(q) \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\varphi_{p_\iota}.$$

Therefore, by the convexity of Ψ, we obtain

$$\Psi(\beta_t(q)) \leq \left[1 - t\delta\eta(q) \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\right]\Psi(q) + t\delta\eta(q) \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\Psi(\varphi_{p_\iota}).$$

On the other hand, note also that by the mean value theorem, given $q \in C$, there exists $\tau \in (0,1)$ such that

$$\mathcal{F}(\beta_1(q)) - \mathcal{F}(q) = \delta\eta(q) \sum_{\substack{\iota=-\ell \\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\mathcal{F}'(\beta_\tau(q))[\varphi_{p_\iota} - q].$$

This together with the above inequality imply that

$$\begin{aligned}\mathcal{I}(\beta_1(q)) =&\Psi(\beta_1(q)) + \mathcal{F}(\beta_1(q))\\ &\leq \left[1 - \delta\eta(q) \sum_{\substack{\iota=-\ell\\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\right]\Psi(q) + \mathcal{F}(q)\\ &\quad + \delta\eta(q) \sum_{\substack{\iota=-\ell\\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\Psi(\varphi_{p_\iota})\\ &\quad + \delta\eta(q) \sum_{\substack{\iota=-\ell\\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}\mathcal{F}'(\beta_\tau(q))[\varphi_{p_\iota} - q],\end{aligned}$$

which implies that

$$\mathcal{I}(\beta_1(q)) \leq \mathcal{I}(q) + \delta\eta(q) \sum_{\substack{\iota=-\ell\\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}[\Psi(\varphi_{p_\iota}) - \Psi(q) + \mathcal{F}'(\beta_\tau(q))[\varphi_{p_\iota} - q]].$$

Observing that

$$\left\|\tau\delta\eta(q) \sum_{\substack{\iota=-\ell\\ \iota\neq 0}}^{\ell} \frac{\eta_\iota(q)}{\|\varphi_{p_\iota} - q\|}(\varphi_{p_\iota} - q)\right\| \leq \delta \leq \delta_{p_\iota},$$

it follows that

$$\Psi(\varphi_{p_\iota}) - \Psi(q) + \mathcal{F}'(\beta_\tau(q))[\varphi_{p_\iota} - q] < -\sqrt{\varepsilon}\|\varphi_{p_\iota} - q\|,$$

for all $q \in \overline{B}_\iota$ and $\iota = -\ell, \dots, -1, 1, \dots, \ell$. We deduce that

$$\mathcal{I}(\beta_1(q)) < \mathcal{I}(q) - \delta\eta(q)\sqrt{\varepsilon} \sum_{\substack{\iota=-\ell\\ \iota\neq 0}}^{\ell} \eta_\iota(q) \quad \text{for all } q \in C.$$

In particular,

$$\mathcal{I}(\beta_1(q)) < \mathcal{I}(q) - \delta\eta(q)\sqrt{\varepsilon}, \quad \forall q \in S.$$

Notice that if $q \in C \setminus S$, then $\eta(q) = 0$ and thus $\beta_1(q) = q$. Consequently,

$$\mathcal{I}(\beta_1(q)) = \mathcal{I}(q) < c - \varepsilon \quad \text{for all } q \in C \setminus S.$$

In particular, since $D \in \mathcal{A}_j$, we have $\max_C \mathcal{I} \circ \beta_1 \geq c$, and hence there exists $q_0 \in S$ such that

$$\max_C \mathcal{I} \circ \beta_1 = \mathcal{I}(\beta_1(q_0)).$$

Therefore, we deduce that

$$c \leq \Pi(D) = \mathcal{I}(\beta_1(q_0)) < \mathcal{I}(q_0) - \delta\eta(q_0)\sqrt{\varepsilon} \leq \mathcal{I}(q_0),$$

which implies that $\eta(q_0) = 1$ and then that

$$\Pi(D) = \mathcal{I}(\beta_1(q_0)) < \mathcal{I}(q_0) - \delta\sqrt{\varepsilon} \leq \Pi(C) - \delta\sqrt{\varepsilon} \leq \Pi(C) - \delta(C, D)\sqrt{\varepsilon},$$

a contradiction with the choice of the set C, showing that necessarily that $\mathcal{I}$ has infinitely many pairs of critical points at the level c when $c_i = c_j = c$ for some $l \leq j < i \leq k$.

With respect to the proof of the first part of theorem, that is, that c_j are critical values, we have only to follow the same argument like above just replacing in the above proof N_a and N_{2a} with the empty set. ■

Recalling (G_5), any symmetric set $K \subset \mathcal{X}$ homeomorphic to the unit $(k-1)$-sphere S^{k-1} by an odd map has genus k; i.e., $K \in \mathcal{A}_k$. In addition, if $\mathcal{I}$ is bounded from below in a subspace $\widetilde{X}_{l-1}$ of codimension $l-1$, by (G_6), $A \cap \widetilde{X}_{l-1} \neq \emptyset$ for every $A \in \mathcal{A}_l$ and then $c_l \geq \inf_{\widetilde{X}_{l-1}} \mathcal{I} > -\infty$. Therefore, we deduce the following consequence.

Corollary 1 *Assume $(\mathcal{I}_0)$. Suppose also that $\mathcal{X}$ is continuously embedded into a Banach space Y and that $\mathcal{I} = \Psi + \mathcal{F}$ is bounded from below in a subspace $\widetilde{X}_{l-1}$ of codimension $l-1$ and satisfies (wPS)-condition. If there exist $k \geq l$ and a symmetric set $K \subset \mathcal{X}$ homeomorphic to the unit $(k-1)$-sphere S^{k-1} by an odd map such that $\sup_K \mathcal{I} < \mathcal{I}(0)$, then $\mathcal{I}$ possesses at least $k-l+1$ distinct pairs of nontrivial critical points with negative levels.*

Remark 1 (i) A particular case is when $\mathcal{I}$ is bounded from below in all $\mathcal{X}$. In this case, we can take in the above corollary any subspace $\widetilde{X}_{l-1}$ of codimension $l-1$ with $1 \leq l \leq k$ and therefore we obtain k distinct pairs of nontrivial critical points with negative levels.

(ii) Notice that the hypothesis

$(\mathcal{I}_1)$ There exist a subspace X_k of $\mathcal{X}$ with $\dim X_k = k$ and $r > 0$ such that $\mathcal{I}(q) < \mathcal{I}(0)$ for all $q \in X_k$ with $\|q\|_Y = r$.

allows to take $K = \{q \in X_k \,:\, \|q\|_Y = r\}$ in the above corollary.

2.2 The symmetric Mountain Pass Theorem

In addition to $(\mathcal{I}_0)$-$(\mathcal{I}_1)$, assume also that $\mathcal{I}(0) = 0$ and the following hypotheses concerning the geometry of the action functional $\mathcal{I}$ for some integer $\overline{k} < k$:

$(\mathcal{I}_2)$ There exist a subspace $\widetilde{X}_{\overline{k}}$ of $\mathcal{X}$ with $\operatorname{codim} \widetilde{X}_{\overline{k}} = \overline{k}$ and constants $\rho \in (0, r)$ and $\alpha > 0$ such that $\mathcal{I}(q) \geq \alpha$ for all $q \in \widetilde{X}_{\overline{k}}$ with $\|q\|_Y = \rho$.

Take

$$Q = \overline{B}_Y(0,r) \cap X_k = \{q \in X_k : ||q||_Y \le r\}.$$

Thus, the boundary of Q in X_k is

$$\partial Q = \partial B_Y(0,r) \cap X_k = \{q \in X_k : ||q||_Y = r\}.$$

Consider

$$\mathcal{H} = \{h \in C(Q, \mathcal{X}) : h \text{ odd},\ h = id \text{ on } \partial Q\}.$$

Notice that $id \in \mathcal{H}$ and $\mathcal{H} \ne \emptyset$. Next, let $1 \le j \le k$ be fixed and consider the family Γ_j of all subsets $A \subset \mathcal{X}$ such that $A = h(Q \setminus V)$ with $h \in \mathcal{H}$ and V is a symmetric and open set in $B_Y(0,r) \cap X_k$ satisfying that each closed and symmetric (possibly empty) $Z \subset V \setminus \{0\}$ verifies $\gamma(Z) \le k - j$. Consider the min-max family $\mathcal{B}_j$ $(1 \le j \le k)$ of all compact, symmetric, nonempty subsets $A \subset \mathcal{X}$ such that for every open set U in $\mathcal{X}$ containing A, there exists $A_0 \subset U$ with $A_0 \in \Gamma_j$. Notice that the compactness of Q implies that $h(Q \setminus V)$ is compact for every $h \in \mathcal{H}$ and every symmetric and open V in $B_Y(0,r)$. Hence, $\Gamma_j \subset \mathcal{B}_j$.

Lemma 2 *(i) $\mathcal{B}_j$ is nonempty and $(\mathcal{B}_j, \delta)$ is a complete metric space for all $1 \le j \le k$.*

(ii) $\mathcal{B}_{j+1} \subset \mathcal{B}_j$ for all $1 \le j \le k-1$.

(iii) If $1 \le j \le k$ and $A \in \mathcal{B}_j$, then $\partial Q \subset A$. Moreover, if $\beta : A \to \mathcal{X}$ is a continuous, odd function such that $\beta(q) = q$ for all $q \in \partial Q$, then $\beta(A) \in \mathcal{B}_j$.

(iv) If $q_1, q_2, \dots q_n$ are points in $\mathcal{X} \setminus \{0\}$ satisfying

$$q_m \ne \pm q_l \text{ and } \|q_m\|_Y \ne r, \quad \text{for all } m, l = 1, 2 \dots, n,\ l \ne m,$$

and we denote $N_\alpha = \bigcup_{m=1}^{n} [B_Y(q_m, \alpha) \cup B_Y(-q_m, \alpha)]$, then for $\alpha > 0$ small enough and $2 \le j \le k$, we have

$$A \setminus N_\alpha \in \mathcal{B}_{j-1}, \quad \text{for all } A \in \mathcal{B}_j.$$

(v) If $\overline{k} < j \le k$, then one has the intersection property

$$A \cap \partial B_Y(0,\rho) \cap \widetilde{X}_{\overline{k}} \ne \emptyset \quad \text{for all } A \in \mathcal{B}_j.$$

Proof. (i) It is clear that $Q \in \Gamma_j \subset \mathcal{B}_j$ and $\mathcal{B}_j \ne \emptyset$. We show that $\mathcal{B}_j$ is closed in the complete metric space $(\mathcal{M}, \delta)$. Consider $(B_n) \subset \mathcal{B}_j$ such that $\delta(B_n, B) \to 0$ for some $B \subset \mathcal{X}$ nonempty, compact and symmetric set. Let $U \supset B$ be open in $\mathcal{X}$. Take n big enough such that $B_n \subset U$. Using that $B_n \in \mathcal{B}_j$, there exists $A_0 \in \Gamma_j$ with $A_0 \subset U$. Hence one has that $B \in \mathcal{B}_j$ and $(\mathcal{B}_j, \delta)$ is a complete metric space.

(ii) It is clear that $\Gamma_{j+1} \subset \Gamma_j$, hence $\mathcal{B}_{j+1} \subset \mathcal{B}_j$ for all $1 \le j \le k-1$.

(iii) Let $A \in \mathcal{B}_j$ for $1 \le j \le k$. Consider an open set U in $\mathcal{X}$ such that $A \subset U$. Using that $A \in \mathcal{B}_j$, it follows that there exists $A_0 = h(Q \setminus V) \in \Gamma_j$ such that $A_0 \subset U$. Since $V \cap \partial Q = \emptyset$ and $h = id$ on ∂Q, it is clear that $\partial Q \subset A_0 \subset U$. Hence, if $\mathcal{U} = \{U : U \text{ is open in } \mathcal{X},\ A \subset U\}$, then $\partial Q \subset \cap_{U \in \mathcal{U}} U$; i.e., ∂Q is a subset of the closure in $\mathcal{X}$ of A. Using that A is compact, this means that $\partial Q \subset A$.

In addition, if $\beta : A \to \mathcal{X}$ is a continuous, odd function such that $\beta(q) = q$ for all $q \in \partial Q$, then by the Dugundji's extension theorem (see [14, Theorem 4.1]), there exists a continuous, odd mapping $\widetilde{\beta} : \mathcal{X} \to \mathcal{X}$ such that $\widetilde{\beta}(q) = \beta(q)$ for all $q \in A$. In particular, $\widetilde{\beta}(q) = \beta(q) = q$ for every $q \in \partial Q$, which implies that $\widetilde{\beta} \circ h \in \mathcal{H}$ and thus, $\widetilde{\beta}(A_0) \in \Gamma_j$, for every $A_0 \in \Gamma_j$. Moreover, for each open set U in $\mathcal{X}$ such that $\beta(A) \subset U$, one has that $G_1 = \widetilde{\beta}^{-1}(U)$ is open in $\mathcal{X}$ with $A \subset G_1$. Using that $A \in \mathcal{B}_j$, it follows that there exists $A_0 = h(Q \setminus V) \in \Gamma_j$ with $A_0 \subset G_1$. Since $\widetilde{\beta}(A_0) \subset \widetilde{\beta}(G_1) \subset U$, we deduce that $\beta(A) \in \mathcal{B}_j$.

(iv) Choose $\alpha > 0$ small enough to satisfy that every two balls in the family $\{\overline{B}_Y(\pm q_m, \alpha) \, : \, m = 1, 2, \ldots n\}$ are disjoint and none of these balls contains points of $\partial B_Y(0, r)$. Thus, $N_\alpha = \bigcup_{m=1}^{n} [B_Y(q_m, \alpha) \cup B_Y(-q_m, \alpha)]$ satisfies

$$N_\alpha \cap \partial Q = \emptyset. \tag{6}$$

Fix $A \in \mathcal{B}_j$ with $2 \leq j \leq k$ and let $U \supset A \setminus N_\alpha$ be open in $\mathcal{X}$. Hence, $U_0 = U \cup N_\alpha$ is open in $\mathcal{X}$ with $A \subset U_0$. Since $A \in \mathcal{B}_j$ there exists $A_0 \in \Gamma_j$ such that $A_0 \subset U_0$. By definition of Γ_j, this means that $A_0 = h(Q \setminus V)$ with $h \in \mathcal{H}$ and V a symmetric and open in $B_Y(0, r) \cap X_k$ such that each closed and symmetric (possibly empty) $Z \subset V \setminus \{0\}$ verifies $\gamma(Z) \leq k - j$. Note that

$$A_0 \setminus N_\alpha = h(Q \setminus V) \setminus N_\alpha = h(Q \setminus [V \cup h^{-1}(N_\alpha)]).$$

Using that $V \cup h^{-1}(N_\alpha)$ is symmetric, open in Q, that $h = id$ on ∂Q and (6), we deduce that $V \cup h^{-1}(N_\alpha)$ does not intersect ∂Q. Moreover, for any closed and symmetric (possibly empty) $P \subset [V \cup h^{-1}(N_\alpha)] \setminus \{0\}$, there exist closed and symmetric subsets P_1, P_2 in $\mathcal{X}$ such that $P = P_1 \cup P_2$ and $P_1 \subset V, P_2 \subset h^{-1}(N_\alpha)$ (for instance, take $P_1 = \{x \in P \, : \, \mathrm{dist}\,(x, Q \setminus V) \geq \mathrm{dist}\,(x, Q \setminus h^{-1}(N_\alpha))\}$ and P_2 defined similarly by reversing the inequality). Since $A_0 \in \Gamma_j$, it follows that

$$\gamma(P_1) \leq k - j.$$

On the other hand, using that h is odd, we deduce by (G_2) and (G_7) that

$$\gamma(P_2) \leq \gamma(h(P_2)) \leq \gamma(\cup_{m=1}^{n} [\overline{B}_Y(q_m, \alpha) \cup \overline{B}_Y(-q_m, \alpha)]) = 1.$$

Hence, by (G_3),

$$\gamma(P) \leq \gamma(P_1) + \gamma(P_2) \leq k - j + 1.$$

Therefore, we have $A_0 \setminus N_\alpha \in \Gamma_{j-1}$ and using that $A_0 \setminus N_\alpha \subset U$, it follows that $A \setminus N_\alpha \in \mathcal{B}_{j-1}$.

(v) Let $A \in \mathcal{B}_j$. If it would be

$$A \cap \partial B_Y(0, \rho) \cap \widetilde{X}_{\overline{k}} = \emptyset,$$

then $A \subset \mathcal{X} \setminus [\partial B_Y(0, \rho) \cap \widetilde{X}_{\overline{k}}]$, which together to the fact that $\mathcal{X} \setminus [\partial B_Y(0, \rho) \cap \widetilde{X}_{\overline{k}}]$ is an open set, implies that there exists $A_0 = h(Q \setminus V) \in \Gamma_j$ which does not intersect $\partial B_Y(0,\rho)\cap\widetilde{X}_{\overline{k}}$. By definition of Γ_j, $A_0 = h(Q\setminus V)$ where $h \in \mathcal{H}$ and V is a symmetric and open in $B_Y(0,r)\cap X_k$ such that each closed and symmetric (possibly empty) $Z \subset V\setminus\{0\}$ verifies $\gamma(Z) \leq k-j$.

Since $Q\setminus V$ is compact, $h(Q\setminus V)$ is also compact in the open set $\mathcal{X}\setminus[\partial B_Y(0,\rho)\cap\widetilde{X}_{\overline{k}}]$. It follows that there exists an open symmetric set U with $0\in U$ such that $h(Q\setminus V)\subset U$ and $\overline{U}\subset\mathcal{X}\setminus[\partial B_Y(0,\rho)\cap\widetilde{X}_{\overline{k}}]$ (take U as a finite union of open balls, one of them with center at zero). In particular, $h^{-1}(U)$ is symmetric, open in Q and $Q\setminus V\subset h^{-1}(U)$. Therefore the symmetric and compact set $Z := Q\setminus h^{-1}(U)\subset V\setminus\{0\}$ (because $h(0)=0$ and $0\in U$) satisfies $h(\overline{Q\setminus Z})\subset\overline{h(Q\setminus Z)}\subset\overline{U}\subset\mathcal{X}\setminus[\partial B_Y(0,\rho)\cap\widetilde{X}_{\overline{k}}]$; i.e.,

$$h(\overline{Q\setminus Z})\cap\partial B_Y(0,\rho)\cap\widetilde{X}_{\overline{k}}=\emptyset.$$

From the definition of Γ_j we have $\gamma(Z)\leq k-j$. Consider now the open set Ω in Q given by

$$\Omega := h^{-1}(B_Y(0,\rho)).$$

From $h = id$ on ∂Q and $r>\rho$ we deduce that $\Omega\cap\partial Q=\emptyset$. Moreover, since h is odd, Ω is bounded and symmetric in X_k with $0\in\Omega$. Hence, by (G_5),

$$\gamma(\partial\Omega)=k.$$

In particular one has that $\partial\Omega\neq\emptyset$. Moreover, we have

$$h(\partial\Omega)\subset\partial B_Y(0,\rho).$$

Indeed, for every fixed $x\in\partial\Omega$ there exists a sequence (x_n) in Ω converging to x. Hence, the sequence $(h(x_n))$ is contained in $B_Y(0,\rho)$ and converges to $h(x)$. This means that $h(x)\in\overline{B}_Y(0,\rho)$. Using that $x\notin\Omega$ we also have $h(x)\notin B_Y(0,\rho)$ and consequently, $h(x)\in\partial B_Y(0,\rho)$.

Next, we consider the compact symmetric set

$$P=h^{-1}(\partial B_Y(0,\rho)).$$

Since $\partial\Omega\subset P$, by (G_2), $\gamma(P)\geq\gamma(\partial\Omega)=k$. It follows from (G_3) that

$$\gamma(\overline{P\setminus Z})\geq\gamma(P)-\gamma(Z)\geq k-(k-j)=j,$$

which implies by (G_2) that

$$\gamma(h(\overline{P\setminus Z}))\geq\gamma(\overline{P\setminus Z})\geq j>\overline{k}.$$

By (G_6) it follows that $h(\overline{P\setminus Z})\cap\widetilde{X}\neq\emptyset$. In consequence, using that $h(\overline{P\setminus Z})\subset h(\overline{Q\setminus Z})\cap\partial B_Y(0,\rho)$, we get a contradiction with $h(\overline{Q\setminus Z})\cap\partial B_Y(0,\rho)\cap\widetilde{X}=\emptyset$ and property (v) is proved. ■

Now, like in the Lusternik-Schnirelman method we consider for $1\leq j\leq k$ the values

$$b_j=\inf_{A\in\mathcal{B}_j}\sup_A\mathcal{I}$$

and we compile its properties in the next lemma.

Lemma 3 *(i)* $\alpha \le b_{\overline{k}+1} \le \cdots \le b_k$.

(ii) Given $\varepsilon > 0$ *and* $B \in \mathcal{B}_j$ *with*

$$\max_B \mathcal{I} \le b_j + \varepsilon,$$

there exists $C \in \mathcal{B}_j$ *such that*

$$\max_C \mathcal{I} \le \max_B \mathcal{I}, \quad \delta(B, C) \le \sqrt{\varepsilon},$$

$$\sup_D \mathcal{I} \ge \max_C \mathcal{I} - \sqrt{\varepsilon}\delta(C, D) \text{ for all } D \in \mathcal{B}_j.$$

Proof. (i) It is a consequence of $(\mathcal{I}_2)$ together with (ii) and (v) from Lemma 2.

(ii) The mapping Λ defined in the complete metric space $(\mathcal{B}_j, \delta)$ (see (i) in the Lemma 2) by

$$\Lambda(A) = \sup_A \mathcal{I} \in (-\infty, \infty], \quad \text{for all } A \in \mathcal{B}_j$$

is lower semicontinuous. Indeed, if (A_n) is a sequence in $\mathcal{B}_j$ and $A \in \mathcal{B}_j$ is such that $\delta(A_n, A) \to 0$, then for any $q \in A$, there exists a sequence (q_n) with $q_n \in A_n$ and $q_n \to q$. Since $\mathcal{I}$ is continuous on $D(\Psi)$ and $\mathcal{I} = \infty$ on $\mathcal{X} \setminus D(\Psi)$, we have

$$\mathcal{I}(q) \le \liminf_{n\to\infty} \mathcal{I}(q_n) \le \liminf_{n\to\infty} \Lambda(A_n).$$

The arbiteriness of $q \in A$ implies that $\Lambda(A) \le \liminf_{n\to\infty} \Lambda(A_n)$.

Notice that

$$\alpha \le b_j = \inf_{\mathcal{B}_j} \Lambda \quad (\overline{k} + 1 \le j \le k).$$

Consequently, the result is deduced by the Ekeland variational principle applied to the lower semicontinuous, bounded function Λ. ∎

The main result of this section is the following Ambrosetti-Rabinowitz type result [1] for the Poincaré type action functionals.

Theorem 2 *If the functional* $\mathcal{I}$ *satisfies* (wPS)*-condition,* $\mathcal{I}(0) = 0$ *and conditions* $(\mathcal{I}_{0,1,2})$ *hold true with* $\overline{k} < k$*, then* b_j *is a critical value of* $\mathcal{I}$ *for every* $\overline{k} + 1 \le j \le k$*. Moreover, if* $b_i = b_j = b$ *for some* $j < i$*, then* $\mathcal{I}$ *has infinitely many pairs of critical points at the level* b*. In particular,* $\mathcal{I}$ *has at least* $k - \overline{k}$ *distinct pairs of nontrivial critical points with positive levels.*

Remark 2 The above theorem improves Theorem 4.4 of [29]. Indeed, first we are not imposing that $\mathcal{I}(q)$ is tending to $-\infty$ as $q \in X_k$ is going to infinity and moreover, we only impose the condition (wPS) which is weaker that the condition (PS) assumed in [29]. Secondly and more important, we prove a minimax characterization of the critical values (b_j) of $\mathcal{I}$ while in [29] (see Remark 4.7 in that paper) the author needs to add the assumption that $\mathcal{I}$ has not critical values below a certain negative level in order to state that his values (called c_j in [29]) are critical values of $\mathcal{I}$.

Proof. We show that if $b_i = b_j = b$ for some $\overline{k}+1 \le j < i \le k$, then $\mathcal{I}$ has infinitely many pairs of critical points at the level b. The proof of the fact that b_j ($\overline{k}+1 \le j \le k$) are critical values follows analogously, even more easier. Since $\mathcal{I}$ is even, assume by contradiction in this case that $\mathcal{I}$ has only a finite number n of pairs of critical points $\{\pm q_m : 1 \le m \le n\}$ at the level b. Notice that $b \ge \alpha$ and using $(\mathcal{I}_1)$ it follows that $q_m \notin \partial B_Y(0,r)$ $(1 \le m \le n)$. Moreover, using that $\mathcal{I}(0) = 0$, we deduce that $q_m \ne 0$ for all m and there exists $a > 0$ such that $\overline{B}_Y(q_m, 2a) \cap \overline{B}_Y(-q_m, 2a) = \emptyset$. For $\alpha = a$ and $\alpha = 2a$ we consider the open symmetric set N_α in $\mathcal{X}$ given by

$$N_\alpha = \bigcup_{m=1}^{n} [B_Y(q_m, \alpha) \cup B_Y(-q_m, \alpha)].$$

Arguing similarly as in the proof of Theorem 1, by the (wPS)-condition, we have the claim: there exists $\varepsilon \in (0, a^2)$ such that $b - \varepsilon > 0$ and given $q \in \mathcal{I}^{-1}([b-\varepsilon, b+\varepsilon]) \setminus N_a$, there is $\varphi_q \ne q$ with $\varphi_{-q} = -\varphi_q$ and

$$\Psi(\varphi_q) - \Psi(q) + \mathcal{F}'(q)[\varphi_q - q] < -\sqrt{\varepsilon}\|\varphi_q - q\|.$$

Next, using the definition of b_i, we can pick $A \in \mathcal{B}_i$ such that

$$0 < b - \varepsilon < b \le \sup_A \mathcal{I} \le b + \varepsilon.$$

Then $0 \notin A \subset D(\Psi)$, and using that $\mathcal{I}$ is continuous on $D(\Psi)$ and that A is compact, it follows that the above "sup" is a "max". By case (iv) of Lemma 2 we deduce that

$$B := A \setminus N_{2a} \in \mathcal{B}_{i-1}.$$

Since $B \subset A$ we notice that

$$b \le \max_B \mathcal{I} \le \max_A \mathcal{I} \le b + \varepsilon.$$

with $B \in \mathcal{B}_j$ because $i-1 \ge j$ and case (ii) of Lemma 2. By case (ii) of Lemma 3, there exists $C \in \mathcal{B}_j$ such that

$$\max_C \mathcal{I} \le \max_B \mathcal{I}, \quad \delta(B, C) \le \sqrt{\varepsilon},$$

$$\sup_D \mathcal{I} \ge \max_C \mathcal{I} - \sqrt{\varepsilon}\delta(C, D) \text{ for all } D \in \mathcal{B}_j.$$

The proof follows in a similar way to this one of Theorem 1 (using case (iii) of Lemma 2 to prove for the corresponding function β_1 that $D := \beta_1(C) \in \mathcal{B}_j$). ∎

Combining Corollary 1 with Theorem 2 we obtain the following theorem

Theorem 3 *Assume that $\mathcal{I} = \Psi + \mathcal{F}$ is bounded from below, $\mathcal{I}(0) = 0$ and satisfies (wPS)-condition. If it also satisfies hypotheses $(\mathcal{I}_0)$, $(\mathcal{I}_1)$ and $(\mathcal{I}_2)$ with $\widetilde{X}_{\overline{k}} = \mathcal{X}$ ($\overline{k} = 0$), then $\mathcal{I}$ possesses at least $2k$ distinct pairs of nontrivial critical points, k with negative levels and k with positive levels.*

3 Multiple periodic solutions for the Lorentz force equation

3.1 Functional framework

In what follows $\mathbb{R}^3$ is endowed with the Euclidean scalar product " $\cdot$ " and the Euclidean norm "$|\ \ |$". Let $T > 0$ be fixed. If $W^{1,\infty}(0,T)$ denotes the space of all Lipschitz functions in $[0,T]$ (or equivalently the absolutely continuous functions in $[0,T]$ with bounded derivatives), we consider the Banach space

$$W_*^{1,\infty} = \{q \in [W^{1,\infty}(0,T)]^3 \,:\, q(0) = q(T)\}$$

endowed with the usual norm $\|\cdot\|_{1,\infty}$ given by

$$\|q\|_{1,\infty} = \|q\|_\infty + \|q'\|_\infty,$$

where $\|q\|_\infty = \max_{t\in[0,T]} |q(t)|$ and $\|q'\|_\infty = \max_{t\in[0,T]} |q'(t)|$.

Consider also the Euler-Lagrange action functional associated to the Poincaré relativistic Lagrangian $\mathcal{L}$ with periodic boundary conditions, i.e.,

$$\mathcal{I}_* : W_*^{1,\infty} \to (-\infty, +\infty], \quad \mathcal{I}_* = \Psi_* + \mathcal{F},$$

where Ψ_* (respectively, $\mathcal{F}$) is associated to the "nonsmooth" part (respectively, the "smooth" part) of the relativistic Lagrangian. Specifically, if

$$\mathcal{K}_* = \{q \in W_*^{1,\infty} : \|q'\|_\infty \leq 1\},$$

Ψ_* is given by

$$\Psi_*(q) = \begin{cases} \displaystyle\int_0^T [1 - \sqrt{1 - |q'|^2}]dt, & \text{if } q \in \mathcal{K}_*, \\ +\infty, & \text{if } \ q \notin W_*^{1,\infty} \setminus \mathcal{K}_*, \end{cases} \tag{7}$$

while $\mathcal{F}$ is defined by

$$\mathcal{F}(q) := \int_0^T [q' \cdot W(t,q) - V(t,q)]dt, \qquad \text{for all } q \in W_*^{1,\infty}.$$

It is standard that $\mathcal{F}$ is of class C^1 in $W_*^{1,\infty}$ with

$$\mathcal{F}'(q)[\varphi] = \int_0^T (\mathcal{E}(t,q,q') - \nabla_q V(t,q)) \cdot \varphi dt + \int_0^T W(t,q) \cdot \varphi' dt,$$

for every $q, \varphi \in W_*^{1,\infty}$, where the function $\mathcal{E} : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is given for each $t \in [0,T]$ and $p, q \in \mathbb{R}^3$ by

$$\mathcal{E}(t,q,p) = (p \cdot D_{q_1} W(t,q), p \cdot D_{q_2} W(t,q), p \cdot D_{q_3} W(t,q)).$$

In addition, we have the following properties of Ψ_* and its domain $\mathcal{K}_*$ (see [2, 6]):

Lemma 4 *(i) The set $\mathcal{K}_*$ is convex and closed in $C([0,T],\mathbb{R}^3)$ and thus in $W^{1,\infty}_*$. Moreover, if (q_n) is a sequence in $\mathcal{K}_*$ converging pointwise in $[0,T]$ to a continuous function $q:[0,T]\to\mathbb{R}^3$, then $q\in\mathcal{K}_*$ and $q_n'\to q'$ in the w^*-topology $\sigma(L^\infty,L^1)$.*

(ii) Each $q\in\mathcal{K}_*$ satisfying $\int_0^T q\,dt=0$ fulfills

$$\|q\|_\infty \le T.$$

(iii) If (q_n) is a sequence in $\mathcal{K}_*$ converging in $C([0,T],\mathbb{R}^3)$ to q, then

$$\Psi_*(q)\le \liminf_{n\to\infty}\Psi_*(q_n).$$

In particular, the functional Ψ_* is weakly lower semicontinuous and convex in $W^{1,\infty}_*$.

(iv) The restriction of the functional Ψ_* to its domain $\mathcal{K}_*$ is continuous.

In conclusion, since $\mathcal{I}_*$ is the sum of the proper convex lower semicontinuous functional Ψ_* and of the C^1-functional, a function $q\in W^{1,\infty}_*$ is a **critical point** of $\mathcal{I}_*$ if $q\in\mathcal{K}_*$ and

$$\int_0^T[\sqrt{1-|q'|^2}-\sqrt{1-|\varphi'|^2}]dt+\int_0^T[\mathcal{E}(t,q,q')-\nabla_q V(t,q)]\cdot(\varphi-q)dt$$
$$+\int_0^T W(t,q)\cdot(\varphi'-q')dt\ge 0,\quad \text{for all } \varphi\in\mathcal{K}_*.$$

In [2] it is proved that the critical points of $\mathcal{I}_*$ are just the T-periodic solutions of the Lorentz force equation (1). By a T-periodic solution q we mean a function $q=(q_1,q_2,q_3)$ of class C^2 such that $|q'(t)|<1$ for all t, and which verifies the equation (1) and

$$q(0)=q(T),\quad q'(0)=q'(T).$$

Specifically, it is shown the following result.

Theorem 4 ([2]) *A given $q\in W^{1,\infty}_*$ is a critical point of $\mathcal{I}_*$ if and only if q is a T-periodic solution of the Lorentz force equation (1).*

The following result in [2] will be essential to prove the weak Palais-Smale condition for the functional $\mathcal{I}_*$ with $Y=C([0,T],\mathbb{R}^3)$.

Lemma 5 *If $c\in\mathbb{R}$, (ε_n) is a sequence of positive numbers converging to zero and (q_n) is a bounded sequence in $W^{1,\infty}_*$ satisfying that*

$$\lim_{n\to\infty}\mathcal{I}_*(q_n)=c,$$

and for each integer $n\ge 1$,

$$\Psi_*(\varphi)-\Psi_*(q_n)+\mathcal{F}'(q_n)[\varphi-q_n]\ge -\varepsilon_n\|\varphi-q_n\|_{1,\infty},\quad \textit{for all } \varphi\in\mathcal{K}_*,$$

then there exists a subsequence (q_{n_k}) of (q_n) which is converging in $C([0,T],\mathbb{R}^3)$ to a critical point $q\in\mathcal{K}_*$ of $\mathcal{I}_*$ with level $\mathcal{I}_*(q)=c$.

In the rest of the section we apply the results proved in Sections 2 and 3 to state the existence of multiple periodic solutions for the Lorentz force equation under different hypotheses on the electric potential V and the magnetic potential W.

3.2 Infinitely many periodic solutions

As a first application, we have the following result.

Theorem 5 *Assume that V and W satisfy*

(H_1) *$V(t,\cdot)$ is even and $W(t,\cdot)$ is odd for all $t \in [0,T]$.*

(H_2) *There exist $\overline{\mu} > 1$, $C > 0$ and a sufficiently large $R > 0$ such that*

$$|W(t,q)| + V(t,q) \leq -C|q|^{\overline{\mu}}, \quad \textit{for } t \in [0,T], |q| \geq R.$$

(H_3) *$V(t,0) = 0$ for every $t \in [0,T]$ and there exist $r_1 \in (0,1)$, $c, d > 0$ and $\mu, \nu > 0$ with $\mu < \min\{2, \nu+1\}$ such that*

$$|W(t,q)| \leq c|q|^{\nu}, \; V(t,q) \geq d|q|^{\mu} \quad \textit{for } t \in [0,T], |q| \leq r_1.$$

Then the Lorentz force equation (1) has infinitely many pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the action funcional).

Remark 3 A sufficient condition for (H_2) is that there exist $R > 0$, $C > 0$ and $1 \leq \overline{\nu} < \overline{\mu}$ such that

$$|W(t,q)| \leq C|q|^{\overline{\nu}}, \; V(t,q) \leq -C|q|^{\overline{\mu}} \quad \text{for } t \in [0,T], |q| \geq R.$$

Proof. Our aim is to apply Corollary 1 with $l = 1$ and any $k \geq 1$. Observe that using (H_1) one has that $\mathcal{F}$, and thus $\mathcal{I}_*$, is even. For $q \in \mathcal{K}_*$ we define

$$q = \overline{q} + \widetilde{q}, \qquad \overline{q} = \frac{1}{T}\int_0^T q dt.$$

By Lemma 4-(ii),

$$\|\widetilde{q}\|_\infty \leq T,$$

and using $\|q'\|_\infty \leq 1$, (H_2) and that $|\overline{q}|^{\overline{\mu}} \leq 2^{\overline{\mu}-1}[|q|^{\overline{\mu}} + |\widetilde{q}|^{\overline{\mu}}]$, we obtain constants $C_1, C_2 > 0$ such that

$$\begin{aligned}\mathcal{I}_*(q) &\geq \int_0^T [|q'||W(t,q)| - V(t,q)]dt \geq -\int_0^T [|W(t,q)| + V(t,q)]dt \\ &\geq C\int_0^T |q|^{\overline{\mu}} dt - C_1 \geq 2^{1-\overline{\mu}} CT|\overline{q}|^{\overline{\mu}} - C_2, \quad \text{for all } q \in W^{1,\infty}_*.\end{aligned}$$

Therefore, we deduce that $\mathcal{I}_*$ is bounded from below and by Remark 1-i) we can take $l = 1$ in Corollary 1.

We claim also that $\mathcal{I}_*$ satisfies (wPS)-condition with $Y = C([0,T],\mathbb{R}^3)$. Indeed, let $(q_n) \subset W_*^{1,\infty}$ be a sequence such that $(\mathcal{I}_*(q_n))$ is bounded. Then the above inequality implies that $(\overline{q}_n)$, and thus (q_n) by Lemma 4-(ii), is a bounded sequence in $W_*^{1,\infty}$, hence Lemma 5 implies our claim.

Next, we deduce from (H_3) that $\mathcal{I}_*(0) = 0$. Let us fix $k \geq 1$ and take X_k a k-dimensional subspace of $W_*^{1,\infty}$. Consider the $(k-1)$-sphere of radius r with respect to the norm $\|\cdot\|_{1,\infty}$:

$$K_r = \{q \in X_k : \|q\|_{1,\infty} = r\}$$

with $r \leq r_1$. Since $r_1 < 1$, one has that $K_r \subset \mathcal{K}_*$ and hence

$$\begin{aligned}
\mathcal{I}_*(q) &= \int_0^T [1 - \sqrt{1-|q'|^2}]dt + \int_0^T [q' \cdot W(t,q) - V(t,q)]dt \\
&\leq \int_0^T |q'|^2 dt + \int_0^T [|q'||W(t,q)| - V(t,q)]dt \\
&\leq T\|q\|_{1,\infty}^2 + cr\|q\|_{L^\nu}^\nu - d\|q\|_{L^\mu}^\mu, \quad \text{for all } q \in K_r.
\end{aligned}$$

Now, since X_k is finite-dimensional all the norms are equivalent and, using that $\mu < \min\{2, \nu+1\}$, this means that there exists some constants $C_3, C_4 > 0$ such that

$$\begin{aligned}
\mathcal{I}_*(q) &\leq T\|q\|_{1,\infty}^2 + C_3 r\|q\|_{1,\infty}^\nu - C_4\|q\|_{1,\infty}^\mu = Tr^2 + C_3 r^{\nu+1} - C_4 r^\mu \\
&< 0 = \mathcal{I}_*(0), \quad \text{for all } q \in K_r,
\end{aligned}$$

for r small enough. Corollary 1 implies the existence of at least k pairs of nontrivial T-periodic solutions of (1). Taking into account that k is any positive integer, the proof is completed. ■

The proof of the latter result relies on Corollary 1 in the case that the action functional $\mathcal{I}_*$ is bounded from below. Now, we give a variant with conditions that do not imply $\mathcal{I}_*$ to be bounded from below. To prove it we apply Corollary 1 with $l = 4 \leq k$ for any k.

Theorem 6 *Assume that V and W satisfy the conditions of Theorem 5 with condition (H_2) replaced by*

(H_2') *There exist $\overline{\mu} > 1$, $C > 0$ and a sufficiently large $R > 0$ such that*

$$|W(t,q)| - V(t,q) \leq -C|q|^{\overline{\mu}}, \quad \textit{for } t \in [0,T], |q| \geq R.$$

Then the Lorentz force equation (1) has infinitely many pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the action functional).

Remark 4 A sufficient condition for (H_2') is that there exist $R > 0$, $C > 0$ and $1 \leq \overline{\nu} < \overline{\mu}$ such that

$$|W(t,q)| \leq C|q|^{\overline{\nu}}, \ V(t,q) \geq C|q|^{\overline{\mu}} \quad \text{for } t \in [0,T], |q| \geq R.$$

Proof. The functional $\mathcal{I}_*$ satisfies $(\mathcal{I}_0)$ by (H_1). Meanwhile, similarly to the proof of the previous theorem, (H_2') implies that for every $q \in \mathcal{K}_*$,

$$\begin{aligned}\mathcal{I}_*(q) &\leq T + \int_0^T [|q'||W(t,q)| - V(t,q)]dt \leq T + \int_0^T [|W(t,q)| - V(t,q)]dt \\ &\leq -C\int_0^T |q|^{\overline{\mu}}dt + C_1 \leq -C_2|\overline{q}|^{\overline{\mu}} + C_3,\end{aligned}$$

for some constants $C_1, C_2, C_3 > 0$. Thus, $\mathcal{I}_*$ satisfies the (wPS)-condition.

On the other hand, consider the subspace of $W_*^{1,\infty}$ of codimension 3 defined by

$$\widetilde{W}_*^{1,\infty} = \{q \in W_*^{1,\infty} \ : \ \overline{q} = 0\}.$$

It is easy to check that $\mathcal{I}_*$ is bounded from below in $\widetilde{W}_*^{1,\infty}$; i.e.,

$$\inf_{q \in \widetilde{W}_*^{1,\infty}} \mathcal{I}_*(q) > -\infty.$$

Besides, using (H_3) as in the proof of Theorem 5 for every subspace X_k of $W_*^{1,\infty}$ with dimension k, there exists a $(k-1)$-sphere of radius r with respect to the norm $\|\cdot\|_{1,\infty}$ such that the supremum on this sphere of the functional $\mathcal{I}_*$ is negative.

Therefore, by applying Corollary 1, we conclude the existence of at least k pairs of nontrivial T-periodic solutions of (1) for every $k \geq 4$ and the proof is completed. ∎

3.3 Nonlinear eigenvalue problems

In the next application, we consider the Lorentz force equation with the electric potential depending on a parameter $\lambda > 0$

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' + (W(t,q))' = \mathcal{E}(t,q,q') - \lambda \nabla_q V(t,q), \tag{8}$$

where V, W and $\mathcal{E}$ are defined above. We have the following result.

Theorem 7 *Assume that V and W satisfy (H_1), (H_2) (or (H_2')) and*

(H_4) *There is $1 \geq r_1 > 0$ such that*

$$V(t,q) > 0 = V(t,0) \quad \text{for } t \in [0,T], \ 0 < |q| \leq r_1.$$

Then for any integer $m \geq 1$, there is $\Lambda_m > 0$ such that the Lorentz force equation (8) has at least m pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the action funcional) for any $\lambda \geq \Lambda_m$.

Proof. Let us take

$$\mathcal{F}_\lambda(q) = \int_0^T [q' \cdot W(t,q) - \lambda V(t,q)]dt, \qquad \text{for all } q \in W_*^{1,\infty}$$

and the action

$$\mathcal{I}_*^\lambda = \Psi_* + \mathcal{F}_\lambda$$

corresponding to the problem associated to the equation (8) with periodic boundary conditions. From (H_1) it follows that $\mathcal{F}_\lambda$ is even and recall that Ψ_* is even. Thus $\mathcal{I}_*^\lambda$ is even. On the other hand, since $V(t,0) = 0$, one has that $\mathcal{I}_*^\lambda(0) = 0$.

In the case that (H_2) holds true, using same arguments as in the proof of Theorem 5 it follows that there are positive constants C_1, C_2 depending on λ such that

$$\mathcal{I}_*^\lambda(q) \geq C_1 |\overline{q}|^{\overline{\mu}} - C_2 \quad \text{for all } q \in W_*^{1,\infty}.$$

This implies, as in the proof of Theorem 5, that $\mathcal{I}_*^\lambda$ is bounded from below and satisfies (wPS)-condition. In particular, we can take $l \geq 1$ in Corollary 1.

If hypothesis (H_2') is satisfied instead of (H_2), by same arguments used in the proof of Theorem 6 there are positive constants C_3, C_4 depending on λ such that

$$\mathcal{I}_*^\lambda(q) \leq -C_3 |\overline{q}|^{\overline{\mu}} + C_4 \quad \text{for all } q \in W_*^{1,\infty},$$

and, as in the proof of Theorem 6, we deduce that $\mathcal{I}_*^\lambda$ satisfies (wPS)-condition. Moreover, note that $\mathcal{I}_*^\lambda$ is bounded from below on $\widetilde{W}_*^{1,\infty}$. Hence, in this case, we can take $l \geq 4$ in Corollary 1.

Next, consider a k-dimensional subspace X_k of $W_*^{1,\infty}$ with $k \geq l = 1$ if (H_2) is satisfied and with $k \geq l = 4$ if condition (H_2') holds true. If

$$K = \{q \in X_k : \|q\|_{1,\infty} = r_1\},$$

observe that (H_4) implies that $K \subset \mathcal{K}_*$ and

$$\inf_{q \in K} \int_0^T V(t,q)dt > 0.$$

On the other hand,

$$\begin{aligned}
\mathcal{I}_*^\lambda(q) &= \int_0^T [1 - \sqrt{1 - |q'|^2}]dt + \int_0^T q' \cdot W(t,q)dt - \lambda \int_0^T V(t,q)dt \\
&\leq T(1 + \max_{[0,T] \times B_{\mathbb{R}^3}[0,r_1]} |W|) - \lambda \inf_{q \in K} \int_0^T V(t,q)dt,
\end{aligned}$$

for any $q \in K$. Thus, there is $\Lambda_m > 0$ such that $\sup_K \mathcal{I}_*^\lambda < 0$ for all $\lambda \geq \Lambda_m$. Now the result follows from Corollary 1 with $m = k \geq l = 1$ if (H_2) is satisfied and with $m = k - 3$, $k \geq l = 4$ if condition (H_2') holds true. ■

Theorem 8 *(i) If V and W satisfy (H_1), (H_2) and*

(H_5) *there is $r_1 \in (0,1]$ such that*

$$V(t,q) > 0 = V(t,0) \quad \textit{for } 0 < |q| \le r_1, \textit{ and}$$

$$\lim_{|q|\to 0} \frac{|W(t,q)| + V(t,q)}{|q|^2} = 0, \qquad \textit{uniformly in } [0,T],$$

then for any integer $m \ge 1$, there is $\Lambda_m > 0$ such that

- *the Lorentz force equation (8) has at least m pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the action functional) for any $\lambda \ge \Lambda_m$,*
- *if $m \ge 4$, then (8) has at least $m-3$ pairs of nontrivial T-periodic solutions (corresponding to positive critical values of the action functional) for any $\lambda \ge \Lambda_m$.*

(ii) If V and W satisfy (H_1), (H_2') and (H_5), then for any integer $m \ge 1$, there is $\Lambda_m > 0$ such that the Lorentz force equation (8) has

- *at least m pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the action functional) for any $\lambda \ge \Lambda_m$,*
- *at least m pairs of nontrivial T-periodic solutions (corresponding to positive critical values of the action functional) for any $\lambda \ge \Lambda_m$.*

Proof. We keep the notation introduced in the proof of Theorem 7. Let $m \ge 1$ be a fixed integer.

From the proof of Theorem 7 we know that $\mathcal{I}_*^\lambda$ satisfies (wPS)-condition for each $\lambda > 0$. In addition, $\mathcal{I}_*^\lambda$ is bounded from below if (H_2) holds true, while when (H_2') is satisfied, $\mathcal{I}_*^\lambda$ is bounded from below on $\widetilde{W}_*^{1,\infty}$. Therefore, we can choose $l \ge 1$ if (H_2) holds true and $l \ge 4$ when (H_2') is satisfied.

Now, for $k = m$ if (H_2) holds true and $k = m + 3$ if it is satisfied (H_2'), consider a k-dimensional subspace X_k of $W_*^{1,\infty}$. Let us fix $r > 0$ and consider

$$K = \{q \in X_k : \|q\|_\infty = r\}.$$

Note that there exists $\alpha_k > 0$ such that

$$\|q\|_{1,\infty} \le \alpha_k \|q\|_\infty \quad \text{for all } q \in X_k.$$

Thus, taking $0 < r < r_1$ sufficiently small and using that V is positive when $0 < |q| \le r_1$, it follows with same arguments as in the proof of Theorem 7 that there is $\Lambda_m > 1$ such that $\sup_K \mathcal{I}_*^\lambda < 0$ for all $\lambda \ge \Lambda_m$. Corollary 1 implies that $\mathcal{I}_*^\lambda$ has at least m distinct pairs of non-trivial critical points with negative levels for all $\lambda \ge \Lambda_m$.

To look for the critical points with positive level suppose that $k \geq 4$, i.e. that $m \geq 4$ if it is satisfied condition (H_2) and that $m \geq 1$ when (H_2') holds true. Next, by the Sobolev inequality there exists $\alpha > 0$ such that

$$\|q\|_\infty \leq \alpha \|q'\|_{L^2} \quad \text{for all } q \in \widetilde{W}_*^{1,\infty},$$

which together with the elementary inequality

$$1 - \sqrt{1 - s^2} \geq \frac{s^2}{2} \quad \text{for all } s \in [0, 1],$$

implies that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
\mathcal{I}_*^\lambda(q) &= \int_0^T [1 - \sqrt{1 - |q'|^2}] dt + \int_0^T q' \cdot W(t, q) dt - \lambda \int_0^T V(t, q) dt \\
&\geq C_1 \|q\|_\infty^2 - \int_0^T \big[|W(t, q)| + \lambda V(t, q) \big] dt \\
&\geq C_1 \|q\|_\infty^2 - \lambda \int_0^T \big[|W(t, q)| + V(t, q) \big] dt + (\lambda - 1) \int_0^T |W(t, q)| dt \\
&\geq C_1 \|q\|_\infty^2 - \lambda \int_0^T \big[|W(t, q)| + V(t, q) \big] dt,
\end{aligned}$$

for all $\lambda \geq \Lambda_m > 1$ and $q \in \widetilde{W}_*^{1,\infty}$. This together with (H_5) implies that there exists $r > \rho > 0$ and $C_2 > 0$, depending on C_1 and λ, such that

$$\mathcal{I}_*^\lambda(q) \geq C_2 \|q\|_\infty^2 = C_2 \rho^2 > 0 \quad \text{for all } q \in \widetilde{W}_*^{1,\infty} \text{ with } \|q\|_\infty = \rho.$$

Hence, using Theorem 2 it follows that $\mathcal{I}_*^\lambda$ has at least $k - 3$ distinct pairs of nontrivial critical points with positive levels for any $\lambda \geq \Lambda_m$ and the proof is completed. ■

Remark 5 Let V be given by

$$V(t, q) = \beta(t) |q|^\mu \quad \text{for all } (t, q) \in [0, T] \times \mathbb{R}^3,$$

where $\mu > 2$ and $\beta : [0, T] \to \mathbb{R}$ is a positive, continuous function. Assume that the magnetic potential W is such that $W(t, \cdot)$ is odd for all $t \in [0, T]$, and

$$\lim_{|q| \to 0} \frac{|W(t, q)|}{|q|^2} = 0, \qquad \limsup_{|q| \to \infty} \frac{|W(t, q)|}{|q|^\mu} < \infty,$$

uniformly in $t \in [0, T]$. Clearly, since $\mu > 2$, hypotheses (H_1) and (H_5) are satisfied. In addition, since β is continuous and positive, for $\lambda > 0$ sufficiently large, condition (H_2') holds true and then, by the previous theorem, for any integer $m \geq 1$ there is $\Lambda_m > 0$ such that the Lorentz force equation (8) has at least $2m$ nontrivial T-periodic solutions for any $\lambda \geq \Lambda_m$.

Next, consider the Lorentz force equation given by

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' + (W(t,q))' = \mathcal{E}(t,q,q') - [\lambda q + \nabla_q V(t,q)], \tag{9}$$

where V, W and $\mathcal{E}$ are defined above. This means that in this case the electric potential is given by $\lambda\frac{|q|^2}{2} + V(t,q)$. We have the following result.

Theorem 9 *If V and W satisfy conditions $(H_{1,2})$ with $\mu > 2$ and*

(H_6) *$V(t,0) = 0$ for every $t \in [0,T]$ and*

$$\lim_{|q|\to 0} \frac{|W(t,q)| - V(t,q)}{|q|^2} = 0 \quad \textit{uniformly in } t \in [0,T],$$

then for any integer $m \geq 1$, the Lorentz force equation (9) has at least $6m$ pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the action functional) for any $\lambda > 8\left(\frac{\pi m}{T}\right)^2$.

Proof. Let us take

$$\mathcal{F}_\lambda(q) = \int_0^T \left[q' \cdot W(t,q) - \left(\lambda\frac{|q|^2}{2} + V(t,q)\right)\right] dt, \qquad \text{for all } q \in W_*^{1,\infty},$$

and the action

$$\mathcal{I}_*^\lambda = \Psi_* + \mathcal{F}_\lambda,$$

corresponding to the problem associated to the equation (9) with periodic boundary conditions. Observe that $\mathcal{F}_\lambda$ is even and $\mathcal{I}_*^\lambda(0) = 0$. From (H_2) and same arguments used in the proof of Theorem 5 it follows that there are positive constants C_i $(1 \leq i \leq 3)$ depending on λ such that

$$\mathcal{I}_*^\lambda(q) \geq C_1|\overline{q}|^\mu - C_2|\overline{q}|^2 - C_3 \quad \text{for all } q \in W_*^{1,\infty}.$$

This implies, as in the proof of Theorem 5, that $\mathcal{I}_*^\lambda$ is bounded from below and satisfies (wPS)-condition.

Now, let us fix an integer $m \geq 1$ and consider the $6m$-dimensional subspace X_m of $W_*^{1,\infty}$ given by

$$X_m = \{a \sin j\omega t + b \cos j\omega t : a, b \in \mathbb{R}^3, 1 \leq j \leq m\},$$

where $\omega = 2\pi/T$. Fix $\lambda > 8\left(\frac{\pi m}{T}\right)^2$. Consider $\varepsilon > 0$ such that $m^2\omega^2 - \frac{\lambda}{2} + \varepsilon < 0$ and, using (H_6), choose $r \in (0,1)$ such that

$$|W(t,q)| - V(t,q) \leq \varepsilon|q|^2 \quad \text{for all } |q| \leq r,\ t \in [0,T].$$

Take

$$K = \{q \in X_m : \|q\|_{1,\infty} = r\}.$$

One has that $K \subset \mathcal{K}_*$ and, using that

$$1 - \sqrt{1 - s^2} \le s^2 \quad \text{for all } s \in [0, 1],$$

and

$$m^2\omega^2 \int_0^T |q|^2 dt \ge \int_0^T |q'|^2 dt \quad \text{for all } q \in X_m,$$

it follows that

$$\begin{aligned} \mathcal{I}_*^\lambda(q) &= \int_0^T [1 - \sqrt{1 - |q'|^2}]dt + \int_0^T [q' \cdot W(t,q) - V(t,q)]dt - \frac{\lambda}{2}\int_0^T |q|^2 dt \\ &\le (m^2\omega^2 - \frac{\lambda}{2}) \int_0^T |q|^2 dt + \int_0^T [|W(t,q)| - V(t,q)]dt, \end{aligned}$$

for all $q \in K$. By the choice of r we deduce that

$$\mathcal{I}_*^\lambda(q) \le (m^2\omega^2 - \frac{\lambda}{2} + \varepsilon) \int_0^T |q|^2 dt \le r^2 T (m^2\omega^2 - \frac{\lambda}{2} + \varepsilon) < 0 = \mathcal{I}_*^\lambda(0),$$

for all $q \in K$. The proof is completed via Corollary 1. ■

Next, consider the Lorentz force equation given by

$$\left(\frac{q'}{\sqrt{1 - |q'|^2}}\right)' + (W(t,q))' = \mathcal{E}(t,q,q') - [\lambda\beta(t)|q|^{\mu-1}q - \alpha(t)|q|^{\nu-1}q], \quad (10)$$

where W and $\mathcal{E}$ are defined above and the electric potential V is given by

$$V(t,q) = \frac{1}{\mu}\lambda\beta(t)|q|^\mu - \frac{1}{\nu}\alpha(t)|q|^\nu \quad \text{for all } (t,q) \in [0,T] \times \mathbb{R}^3,$$

where $\mu, \nu \ge 1$, $\lambda > 0$ is a parameter, $\alpha, \beta : [0,T] \to \mathbb{R}$ are positive, continuous functions. We have the following result.

Theorem 10 *If $\nu < \mu$ and the magnetic potential W satisfies*

(H_7) *$W(t,\cdot)$ is odd for all $t \in [0,T]$, and*

$$\limsup_{|q|\to 0} \frac{\nu|W(t,q)|}{|q|^\nu} < \min_{[0,T]} \alpha, \qquad \limsup_{|q|\to\infty} \frac{|W(t,q)|}{|q|^\mu} < \infty,$$

uniformly in $t \in [0,T]$,

then for any integer $m \ge 1$, there is $\Lambda_m > 0$ such that the Lorentz force equation (10) has at least m pairs of nontrivial T-periodic solutions (corresponding to positive critical values of the action functional) for any $\lambda \ge \Lambda_m$. Moreover, if $m \ge 4$, then the Lorentz force equation (10) has at least $m-3$ pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the Poincaré action functional) for any $\lambda \ge \Lambda_m$.

Proof. From (H_7) it follows that there exist $C_1 > 0$ such that

$$|W(t,q)| \leq C_1(|q|^\mu + 1) \quad \text{for all } (t,q) \in [0,T] \times \mathbb{R}^3.$$

Hence, using same arguments as in the proof of Theorem 5 it follows that there exist $C_2, C_3, C_4, C_5 > 0$ depending only on W, α, β, T such that

$$\mathcal{I}_*^\lambda(q) \leq (C_2 - C_3\lambda)|\overline{q}|^\mu + C_4|\overline{q}|^\nu + C_5 \quad \text{for all } q \in \mathcal{K}_*.$$

Consequently, taking $\Lambda = C_2/C_3$ and $\lambda > \Lambda$, using $\mu > \nu$ it follows that $\mathcal{I}_*^\lambda(q) \to -\infty$ as $q \in \mathcal{K}_*$ and $|\overline{q}| \to \infty$. This together with Lemma 5 implies that $\mathcal{I}_*^\lambda$ satisfies (wPS)-conditon for all $\lambda > \Lambda$.

Next, let $m \geq 1$ be a fixed integer and X_m be a m-dimensional subspace of $W_*^{1,\infty}$ endowed with the norm $\|\cdot\|_\infty$. Consider $r > 0$ such that

$$Q := \{q \in X_m : \|q\|_\infty \leq r\} \subset \mathcal{K}_*.$$

Note that $\partial Q = \{q \in X_m : \|q\|_\infty = r\}$. It is clear that there exists $\Lambda_m \geq \Lambda$ such that

$$\mathcal{I}_*^\lambda(q) \; < 0 \quad \text{for all } q \in \partial Q,$$

for all $\lambda \geq \Lambda_m$.

Next, let us fix $\lambda \geq \Lambda_m$. Using the assumption (H_7) consider $\rho > 0$ and $0 < C_6 < \min_{[0,T]} \alpha$ such that

$$|W(t,q)| \leq C_6 \frac{|q|^\nu}{\nu}$$

for all $t \in [0,T]$ and $q \in \mathbb{R}^3$ with $|q| \leq \rho$. In particular, for $q \in \mathcal{K}_*$ with $\|q\|_\infty = \rho$ it is deduced that

$$\begin{aligned}
\mathcal{I}_*^\lambda(q) \;&\geq\; -\int_0^T |W(t,q)|dt + \int_0^T [\alpha(t)\frac{|q|^\nu}{\nu} - \lambda\beta(t)\frac{|q|^\mu}{\mu}]dt \\
&\geq\; \int_0^T \left[(\alpha(t) - C_6)\frac{|q|^\nu}{\nu} - \lambda\beta(t)\frac{|q|^\mu}{\mu}\right] dt.
\end{aligned}$$

Hence, there exist $C_7, C_8 > 0$ such that for every $q \in \mathcal{K}_*$ with $\|q\|_\infty = \rho$, one has

$$\begin{aligned}
\mathcal{I}_*^\lambda(q) \;&\geq\; C_7\int_0^T |q|^\nu dt - C_8\int_0^T |q|^\mu dt \\
&=\; C_7\int_0^T |q^\nu|(1 - C_7^{-1}C_8|q|^{\mu-\nu})dt \\
&\geq\; C_7\int_0^T |q^\nu|(1 - C_7^{-1}C_8\rho^{\mu-\nu})dt.
\end{aligned}$$

Therefore, for $0 < \rho < r$ (depending on λ) small enough, there exists $C_9 > 0$ which depends on ρ such that

$$\mathcal{I}_*^\lambda(q) \quad \geq \quad C_9 \int_0^T |q|^\nu dt \quad \text{for all } q \in \mathcal{K}_* \text{ with } \|q\|_\infty = \rho.$$

If we prove the claim

$$a := \inf \left\{ \int_0^T |q|^\nu dt : q \in \mathcal{K}_*,\ \|q\|_\infty = \rho \right\} > 0,$$

we obtain the existence of $b > 0$ such that $\mathcal{I}_*^\lambda(q) \geq b$ for all $q \in W_*^{1,\infty}$ with $\|q\|_\infty = \rho$ and the first assertion will be completed via Theorem 2 (with $\overline{k} = 0$, $\widetilde{X}_0 = \mathcal{X} = W_*^{1,\infty}$). Indeed, to show the claim, observe that if $(q_n) \subset \mathcal{K}_*$ with $\|q_n\|_\infty = \rho$ is a minimizing sequence; i.e., satisfying $\int_0^T |q_n|^\nu dt \to 0$ as $n \to \infty$, then the boundedness of (q_n) in $W^{1,\infty}$ and the Arzelà-Ascoli theorem imply that there exists $q \in C([0,T], \mathbb{R}^3)$ such that, up to a subsequence, $\|q_n - q\|_\infty \to 0$ as $n \to \infty$. It follows then that $\|q\|_\infty = \rho$ and $\int_0^T |q|^\nu dt = a$ and therefore $a > 0$.

Next, using that for each $\lambda \geq \Lambda_m$, $\mathcal{I}_*^\lambda$ is bounded from below on $\widetilde{W}_*^{1,\infty}$ and that $\mathcal{I}_*^\lambda < 0$ on ∂Q, the existence of at least $m - 3$ pairs of nontrivial T-periodic solutions (corresponding to negative critical values of the Poincaré action functional) follows in a clear way from Corollary 1, taking $k = m$, $l = 4$, $\widetilde{X}_3 = \widetilde{W}_*^{1,\infty}$ and $K = \partial Q$. ■

4 Remarks and extensions

4.1 The Lorentz force equation with zero Dirichlet boundary value conditions

In the latter section, we have focused on the periodic problem. However, the Lorentz force equation (1) with zero Dirichlet boundary value conditions

$$q(0) = 0 = q(T) \tag{11}$$

can be considered in a very similar way. In this case, we must consider the functional in $W_0^{1,\infty} := \{q \in W^{1,\infty} : q(0) = q(T) = 0\}$ given by $\mathcal{I}_0 = \Psi_0 + \mathcal{F}$, where Ψ_0 is defined as in (7) just changing the domain K_* by

$$\mathcal{K}_0 = \{q \in W_0^{1,\infty} : \|q'\|_\infty \leq 1\}.$$

Then, the notion of critical point of $\mathcal{I}_0$ is analogous to this one of $\mathcal{I}_*$ (again changing to the domain $\mathcal{K}_0$, see [2, Definition 1]). In addition, in a similar way to Theorem 4, these critical points of $\mathcal{I}_0$ are just the solutions of the Lorentz force equation (1) with zero Dirichlet boundary conditions (11), see [2, Theorem 2]. Using that $\mathcal{I}_0$ is bounded from below for any potentials V and W and that the functional $\mathcal{I}_0$ satisfies the weak Palais-Smale condition (wPS) with $Y = C([0,T],\mathbb{R}^3)$ (see [2, Lemma 5]), we deduce from Corollary 1 the following result.

Theorem 11 *If condition (H_1) holds true and there exist a subspace X_k of $W_0^{1,\infty}$ with $\dim X_k = k$ and $r > 0$ such that $\mathcal{I}_0(q) < \mathcal{I}_0(0)$ for all $q \in X_k$ with $\|q\|_\infty = r$, then $\mathcal{I}_0$ possesses at least k distinct pairs of nontrivial critical points which are solutions of the Lorentz force equation (1) with Dirichlet boundary conditions (11).*

By using Theorem 11 and the same strategy like in Theorems 5, 7, 9 respectively, the reader can deduce the following results.

Corollary 2 *If V and W satisfy $(H_{1,3})$, then the Lorentz force equation (1) has infinitely many pairs of nontrivial solutions satisfying the Dirichlet boundary conditions (11).*

Corollary 3 *If V and W satisfy $(H_{1,4})$, then for any integer $m \geq 1$, there is $\Lambda_m > 0$ such that the Lorentz force equation (8) has at least m pairs of nontrivial solutions satisfying the Dirichlet boundary conditions (11) for any $\lambda \geq \Lambda_m$.*

Corollary 4 *If V and W satisfy $(H_{1,5})$, then for any integer $m \geq 1$, the Lorentz force equation (9) has at least $3m$ pairs of nontrivial solutions satisfying the Dirichlet boundary conditions (11) for any $\lambda > 2\left(\frac{\pi m}{T}\right)^2$.*

On the other hand using Theorem 3 we have the following application.

Corollary 5 *If V and W satisfy $(H_{1,4})$ and*

(H_8) *there exist constants $\mu, \nu > 2$ and $d > 0$ such that*

$$V(t,q) \leq d|q|^\mu, \quad |W(t,q)| \leq d|q|^\nu,$$

for every $(t,q) \in [0,T] \times \mathbb{R}^3$ with $|q| \leq T$,

then for any integer $m \geq 1$, there is $\Lambda_m > 0$ such that the Lorentz force equation (8) has at least $2m$ pairs of nontrivial solutions satisfying the Dirichlet boundary conditions (11) for any $\lambda \geq \Lambda_m$.

Proof. For each $m \geq 1$, let X_m be a m-dimensional subspace of $W_0^{1,\infty}$. Choose $\alpha_m > 1$ such that

$$\|q\|_{1,\infty} \leq \alpha_m \|q\|_\infty \quad \text{for all } q \in X_m,$$

and $r > 0$ such that $\alpha_m r < r_1$ (where r_1 is given by (H_4)). For every $q \in X_m$ with $\|q\|_\infty = r$ we have

- $\|q'\|_\infty < r_1$ and, since $r_1 \leq 1$, $q \in \mathcal{K}_0$,

- and, by (H_4),

$$\inf\left\{\int_0^T V(t,q)dt : q \in X_m \text{ with } \|q\|_\infty = r\right\} > 0.$$

Consequently, there exists $\Lambda_m > 0$ such that

$$\mathcal{I}_0(q) < 0 = \mathcal{I}_0(0) \quad \text{for all } q \in X_m \text{ with } \|q\|_\infty = r,$$

for every $\lambda \geq \Lambda_m$. In particular, $\mathcal{I}_0$ satisfies $(\mathcal{I}_1)$ with k=m provided that $\lambda \geq \Lambda_m$.

Next, since

$$\int_0^T [1-\sqrt{1-|q'|^2}]dt \geq \frac{1}{2}\int_0^T |q'|^2 dt = \frac{1}{2}\|q\|^2_{H_0^1}, \quad \text{for all } q \in \mathcal{K}_0,$$

we deduce by using (H_8) that there exists $C_1 > 0$ such that

$$\mathcal{I}_0(q) \geq \frac{1}{2}\|q\|^2_{H_0^1} - C_1\|q\|^\nu_{L^\nu} - \lambda C_1\|q\|^\mu_{L^\mu}, \quad \text{for all } q \in \mathcal{K}_0.$$

Taking into account that H_0^1 is embedded into $C([0,T],\mathbb{R}^3)$ and $C([0,T],\mathbb{R}^3)$ is embedded into L^μ and L^ν, it follows that there exist constants $C_2, C_3 > 0$ such that

$$\mathcal{I}_0(q) \geq C_2\|q\|^2_\infty - C_3\|q\|^\nu_\infty - \lambda C_3\|q\|^\mu_\infty, \quad \text{for all } q \in \mathcal{K}_0.$$

Therefore, using that $\mu, \nu > 2$ we deduce that $\mathcal{I}_0$ satisfies $(\mathcal{I}_2)$ with $\overline{k} = 0$ and $\widetilde{X}_0 = W_0^{1,\infty}$ for some $\rho < r$. Now the proof follows from the Theorem 3. ■

4.2 The generalized Lorentz force equation in 4-vector form

Following for example [17, Lecture 26] or [18, Section 23], if q is a T-periodic solution (respectively a solution with zero boundary conditions) for the LFE in $[0,T]$, then the function $\alpha : [0,T] \to [0, T^* := \alpha(T)]$ given by $\alpha(t) = \int_0^t \sqrt{1-|q'(\tau)|^2}d\tau$ is a bijection. From the physical point of view, $\alpha(t)$ is the proper time of the particle. Let $(r,\delta) : [0,T^*] \to \mathbb{R}^4$ be given by $\delta = \alpha^{-1}$ and $r = q \circ \delta$. It follows that (r,δ) satisfies the Lorentz force equation in 4-dimensions (2) with boundary conditions on $[0,T^*]$,

$$r(0) = r(T^*),\ r'(0) = r'(T^*),\ \delta(0) = 0,\ \delta'(0) = \delta'(T^*). \tag{12}$$

(respectively,

$$.r(0) = r(T^*) = 0,\ \delta(0) = 0). \tag{13}$$

Moreover, (r, δ) satisfies a fundamental property of LFE, namely

$$|r'(s)|^2 - \delta'^2(s) = -1 \quad \text{for all } s \in [0, T^*]. \tag{14}$$

Conversely, let (r, δ) be a solution of the LFE in 4-dimensions (2) with boundary conditions (12) (respectively, (13)) on $[0, T^*]$, then there exists a constant $h \in \mathbb{R}$ such that

$$|r'(s)|^2 - \delta'^2(s) = h, \quad \text{for all } s \in [0, T^*]. \tag{15}$$

Assume that $h < 0$. In this case $\delta : [0, T^*] \to [0, T := \delta(T^*)]$ is a bijection and $q : [0, T] \to \mathbb{R}^3$ given by $q(t) = r(\delta^{-1}(t))$ is a T-periodic solution (respectively, a solution with zero Dirichlet boundary conditions) of the LFE with the rest mass

$$m_0 = \sqrt{-h}.$$

Thus, in this case the rest mass is not prescribed and changes with the solution of (2) considered. Consider the Minkowski space (M, g) where $M = M_0 \times \mathbb{R}$ with $M_0 = \mathbb{R}^3$ endowed with the Euclidean scalar product and the Minkowski semi-metric given by $g = dr^2 - d\delta^2$. A curve $(r, \delta) : [0, T^*] \to M$ that satisfies (14) is called a material particle in M, while if the curve satisfies (15) with $h < 0$, it is called timelike (see [23, Chapter 6]). The Lorentz force equation in 4-dimensions (2) has a natural generalization to a general Lorentz manifold $M = M_0 \times \mathbb{R}$ where M_0 is a Riemannian manifold (see for example [4] or [27, §3.8.]). Existence results for the generalized Lorentz force equation with periodic boundary conditions without the prescribed rest mass can be found in [3, 8]. To the best of our knowledge, the results in the mentioned papers do not contain any qualitative result about solutions of the original Lorentz force equation when the rest mass and the period are prescribed.

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