# Critical point theory for the Lorentz force equation


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**Abstract**

In this paper we study the existence and multiplicity of solutions of the Lorentz force equation

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = E(t,q) + q' \times B(t,q)$$

with periodic or Dirichlet boundary conditions. In Special Relativity, it models the motion of a slowly accelerated electron under the influence of an electric field $E$ and a magnetic field $B$. We provide a rigourous critical point theory by showing that the solutions are the critical points in the Szulkin's sense of the corresponding Poincaré non-smooth Lagrangian action. By using a novel minimax principle, we prove a variety of existence and multiplicity results. Based on the associated Planck relativistic Hamiltonian, an alternative result is proved for the periodic case by means of a minimax theorem for strongly indefinite functionals due to Felmer.



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# 1 Introduction

In the relativistic regime with the velocity of light in the vacuum and the charge-to-mass ratio normalized to one by simplicity, the motion of a slowly accelerated charged particle under the influence of an electromagnetic field is modeled by the Lorentz force equation (LFE)

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' + (W(t,q))' = \mathcal{E}(t,q,q') - \nabla_q V(t,q), \tag{1}$$

where $V : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are two $C^1$-functions and $\mathcal{E} : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathcal{E}(t,q,p) = (p \cdot D_{q_1} W(t,q), p \cdot D_{q_2} W(t,q), p \cdot D_{q_3} W(t,q)).$$

By a solution of the above equation we mean a function $q = (q_1, q_2, q_3)$ of class $C^2$ such that $|q'(t)| < 1$ for all $t$, and which verifies the equation. In what follow $\mathbb{R}^3$ is endowed with the Euclidean scalar product " $\cdot$ " and the Euclidean norm "$| \quad |$". Denoting by

$$E = -\nabla_q V - \frac{\partial W}{\partial t}, \qquad B = \operatorname{curl}_q W,$$

the electric and magnetic fields respectively, the equation above is written as

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = E(t,q) + q' \times B(t,q),$$

which is the classical form of the Lorentz force equation that can be found in many textbooks and monographies on Classical Mechanics and Electrodynamics, see for instance [16, Chapter 12] or [18, Chapter 3]. Historically, the Lorentz force equation dates back to Poincaré [26] and Planck [24].

A deeper understanding of the dynamics of charged particles induced by external electromagnetic fields is of primary importance, not only from the theoretical point of view but also for applications like particle accelerators, where the consideration of relativistic effects becomes essential. In spite of being one of the fundamental equations of Mathematical Physics, most of the studies on the dynamics of Lorentz force equation are limited to the identification of exact solutions for particular cases of simple electromagnetic fields (uniform and static fields [18], circular, linear or elliptically polarized electromagnetic waves [1, 3, 29] and other variants). One of the main reasons of the lack of qualitative results is that a neat and rigorous variational approach is not available up to the date. It is well known that (1) is formally the Euler-Lagrange equation of the relativistic Lagrangian

$$\mathcal{L}(t,q,p) = 1 - \sqrt{1-|p|^2} + p \cdot W(t,q) - V(t,q).$$

See [18] or Feynmann 19th lesson [15]. More precisely, observe that the second part

$$L(t,q,p) = p \cdot W(t,q) - V(t,q), \quad (t,q,p) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$$

is a smooth function that accounts for the effect of the fields on the particle, while the term $1-\sqrt{1-|p|^2}$ is only defined for $|p| \leq 1$ and it is not differentiable. This means that the corresponding action functional

$$\mathcal{I}(q) := \int_0^T \mathcal{L}(t,q,q')dt$$

is not of class $C^1$ and the usual critical point theory [4, 28] is not applicable. Historically, the relativistic Lagragian $1-\sqrt{1-|p|^2}$ dates back to Poincaré and the pioneering papers [25, 26] (see [11] for a detailed discussion). More than one century after this crucial contribution, a detailed study of the related literature reveals that the mentioned lack of regularity has supposed a serious obstacle for an adequate development of a proper critical point theory for the LFE.

This lack of regularity of the action functional is a typical situation inherent to problems involving the relativistic acceleration. It was solved for the first time by Brezis and Mawhin in [10] for the case of a forced relativistic pendulum with periodic boundary conditions. In this paper, the authors suceeded to prove that the global minimizer of the action functional is in fact a solution by means of a suitable variational inequality. The problem considered in [10] is scalar and, in addition, $W \equiv 0$ in it. Later, in [9] the action functional is identified for the first time as the sum of a proper convex lower semicontinuous functional and a $C^1$-functional in the space of the continuous functions, then Szulkin's critical point theory from [30] is applicable. Indeed, when $W = 0$, the smooth part $\int_0^T L(t,q,q')dt = -\int_0^T V(t,q)dt$ of the action functional can be defined in the space of continuous functions and then it is not difficult to prove that every Palais-Smale sequence (in a suitable sense) admits a subsequence converging in that space; i.e., the Palais-Smale condition required in [30] holds. In our case, the presence of a magnetic potential requires a complete reformulation from the very beginning, since now the functional is not properly defined in the continuous functions space (as in [9]). We are forced to define the action functional in the Sobolev space $W_0^{1,\infty}$ (resp. $W_T^{1,\infty}$), the set of functions $q = (q_1, q_2, q_3)$ such that the components $q_i$ are Lipschitz functions with Dirichlet (resp. periodic) boundary conditions. By the definition in $W_0^{1,\infty}$, we cannot expect the convergence in this space of a subsequence of every Palais-Smale sequence. As a consequence the Szulkin's critical point theory is not directly applicable. To overcome this technical difficulty, observing additionally that the action functional is continuous in its domain, we have developed in Section 2 some new results that do not need any compactness assumption and rely essentially on the geometry of the functional, in the spirit of [5, 20]. From a theoretical point of view, such results may be interesting by themselves. Once the Palais-Smale sequence is obtained, we discuss (see Lemma 5) what convergence is satisfied by a subsequence of it and we show that *"its limit"* is a critical point of the action functional.

In Section 3, we state the precise functional framework for the Dirichlet problem, and we prove rigourously that a critical point of the action functional in the sense of Szulkin is a solution of the Dirichlet problem. Once it is proved that the Lagrangian action is bounded below and attains its infimum, we can state a principle of least action for the Lorentz force equation with Dirichlet conditions. In this way, we give a positive answer to Hilbert's 20th problem for the Lorentz force equation with Dirichlet boundary conditions (see Theorem 3). This is a type of "universal" existence result in the spirit of the known result by Bartnik-Simon [6]. As a matter of fact, the mere existence result for the Dirichlet problem can be deduced from the main result in [21], or more simply from a basic application of Schauder's fixed point theorem [8, Section 9]. The advantage of our approach is that, once the critical point theory is settled on a solid ground, one can look for critical points that are not the global minimum by a suitable minimax theorem, giving rise to multiplicity results.

The consideration of the periodic problem is very natural as well, once it is observed that a constant uniform magnetic field supports circular trajectories in the plane perpendicular to the field (see for instance Chapter 2.21 in [18]). Section 4 is dedicated to the periodic problem. Based on the functional setting given in [9, 19] for $W = 0$, we show in Section 4.1 that $q$ is a $T$-periodic solution of (1) if and only if it is a critical point of the relativistic Lagrangian action $\mathcal{I}$. By means of such characterization of the $T$-periodic solution set, we give a principle of the least action for $\mathcal{I}$ in Section 4.2. The very special case $W = 0$ was proved in [9, 19]. Using this principle we provide sufficient conditions on $V$ and $W$ for the existence of $T$-periodic solutions. Our second objective for the periodic problem has been to develop a minimax principle for the Lagrangian action, by using again the Minimax Principle for non-smooth functionals developed in Section 2. Section 4.3 is devoted to this purpose.

As it is explained in many texbooks (see for instance [22, Chapter 1.2, Example 5]), from the Lagrangian of the problem we can easily find the corresponding Hamiltonian, that turns out to be

$$\mathcal{H}(t, p, q) = \sqrt{1 + |p - W(t, q)|^2} - 1 + V(t, q).$$

Historically, it seems that this hamiltonian was formulated for the first time by Planck in [24]. Since (1) is the Hamilton-Jacobi equation of the above relativistic Hamiltonian, an alternative method consists in studying the critical points of the associated Hamiltonian action. The difficulty in this case is that the action functional is strongly indefinite. Section 5 is dedicated to develop this Hamiltonian approach for the periodic problem. We use a minimax theorem for strongly indefinite functionals due to Felmer [14] for the Hamiltonian action associated to a modification of $\mathcal{H}$ in order to prove the existence of a nonconstant $T$-periodic solution. Felmer's result is a generalization of the minimax theorem given in [7] which is applied to study superquadratic Hamiltonian systems. The main result of Section 5, Theorem 5.1, is new even for the case $W = 0$. The conditions upon the electric potential $V$ were introduced by Rabinowitz in [27] to deal with the existence of $T$-periodic solutions of second order Hamiltonian systems.

**Notation** The letter $C$ denotes a positive constant which will not necessarily be the same at different places in the paper and which may sometimes change from line to line.

## 2 Non-smooth functionals without compactness

Szulkin proved in [30] suitable versions of some well known minimax theorems by a combination of geometrical and compactness conditions, to handle convex lower semicontinuous perturbations of $C^1$ functionals in Banach spaces. Here, following [5, 20], we emphasize the role of the geometrical assumptions of those minimax theorems to deduce the existence of a suitable Palais-Smale sequence without assuming any compactness hypothesis.

**Theorem 1** *Assume that $E$ is a Banach space and that the functional $\mathcal{I} : E \to (-\infty, +\infty]$ is the sum of two functionals $\mathcal{I} = \Psi + \mathcal{F}$ where*

- *(i) $\Psi : E \to (-\infty, +\infty]$ is a convex and proper functional with a closed domain $Dom\,\Psi := \{v \in E : \Psi(v) < \infty\}$ in $E$ and $\Psi$ is continuous in $Dom\,\Psi$.*
- *(ii) $\mathcal{F} : E \to \mathbb{R}$ is a $C^1$-functional.*

*Let also $K$ be a compact metric space, $K_0 \subset K$ a closed subset and $\gamma_0 : K_0 \to E$ a continuous map. Consider the set*

$$\Gamma = \{\gamma : K \to E \ : \gamma \text{ is continuous and } \gamma|_{K_0} = \gamma_0\}.$$

*If*

$$c_1 := \sup_{t \in K_0} \mathcal{I}(\gamma_0(t)) < c := \inf_{\gamma \in \Gamma} \sup_{t \in K} \mathcal{I}(\gamma(t)) < \infty, \tag{2}$$

*then, for every $\varepsilon > 0$ and $\gamma \in \Gamma$ such that*

$$c \le \max_{t \in K} \mathcal{I}(\gamma(t)) \le c + \frac{\varepsilon}{2}, \tag{3}$$

*there exist $\overline{\gamma}_\varepsilon \in \Gamma$ and $q_\varepsilon \in \overline{\gamma}_\varepsilon(K) \subset E$ satisfying*

$$c \le \max_{t \in K} \mathcal{I}(\overline{\gamma}_\varepsilon(t)) \le \max_{t \in K} \mathcal{I}(\gamma(t)) \le c + \frac{\varepsilon}{2},$$

$$\max_{t \in K} \|\overline{\gamma}_\varepsilon(t) - \gamma(t)\| \le \sqrt{\varepsilon},$$

$$c - \varepsilon \le \mathcal{I}(q_\varepsilon) \le c + \frac{\varepsilon}{2},$$

*and*

$$\Psi(\varphi) - \Psi(q_\varepsilon) + \mathcal{F}'(q_\varepsilon)[\varphi - q_\varepsilon] \ge -\sqrt{\varepsilon}\|\varphi - q_\varepsilon\| \quad \text{for all } \varphi \in E.$$

**Remark 1** Notice that the continuity of $\Psi$ in its closed domain implies that $\Psi$ is lower semicontinuous in $E$. A similar theorem of mountain-pass type is proved in the very recent paper [2] without the continuity condition (see Theorem 3.1 therein). We are grateful to an anonymous referee for pointing out this analogy. Since our aim is to apply the result to the LFE, we prefer to keep the continuity condition and to provide an independent proof.

**Remark 2** Observe that every $\gamma \in \Gamma$ such that $\sup_{t\in K} \mathcal{I}(\gamma(t)) < \infty$ satisfies that $\gamma(t) \in \operatorname{Dom}\Psi$ for every $t \in K$. Hence, the continuity of $\Psi$ implies that $\mathcal{I}\circ\gamma$ is continuous in the compact $K$ for every $\gamma \in \Gamma$ such that $\sup_{t\in K} \mathcal{I}(\gamma(t)) < \infty$ (and then the previous supremum of $\mathcal{I}\circ\gamma$ is attained). Similarly, since we are assuming that $\sup_{t\in K_0} \mathcal{I}(\gamma_0(t)) < \infty$, we also have $c_1 = \max_{t\in K_0} \mathcal{I}(\gamma_0(t))$. It should be observed that if the convex functional $\Psi$ is only lower semicontinuous in $E$ (instead of continuous in $\operatorname{Dom}\Psi$), then it is not true in general that $\mathcal{I}\circ\gamma$ be continuous (see [23, Example 3]).

*Proof.* Clearly, $\Gamma$ is a complete metric space endowed with the uniform distance

$$\mathrm{d}_\Gamma(\gamma_1, \gamma_2) = \max_{t\in K} \|\gamma_1(t) - \gamma_2(t)\|, \qquad (\gamma_1, \gamma_2 \in \Gamma).$$

Consider the functional $\Upsilon : \Gamma \to (-\infty, +\infty]$ given by

$$\Upsilon(\gamma) = \sup_{t\in K} \mathcal{I}(\gamma(t)), \qquad (\gamma \in \Gamma),$$

which is lower semicontinuous by [30, Lemma 3.1]. By Remark 2, for every $\gamma$ in the domain of $\Upsilon$, that is $\Upsilon(\gamma) < +\infty$, we have

$$\Upsilon(\gamma) = \max_{t\in K} \mathcal{I}(\gamma(t)).$$

By (2) the functional $\Upsilon$ is proper and bounded from below by $c_1$. Fix $\varepsilon > 0$ that, without loss of generality, can be assumed less than $c - c_1$. Applying the Ekeland variational principle [13], we deduce that for every $\gamma \in \Gamma$ verifying (3), there exists $\overline{\gamma}_\varepsilon \in \Gamma$ satisfying

$$c \le \Upsilon(\overline{\gamma}_\varepsilon) \le \Upsilon(\gamma) \le c + \frac{\varepsilon}{2},$$

$$\mathrm{d}_\Gamma(\overline{\gamma}_\varepsilon, \gamma) = \max_{t\in K} \|\overline{\gamma}_\varepsilon(t) - \gamma(t)\| \le \sqrt{\varepsilon},$$

and

$$\Upsilon(\overline{\gamma}_\varepsilon) < \Upsilon(\vartheta) + \sqrt{\varepsilon}\, \mathrm{d}_\Gamma(\overline{\gamma}_\varepsilon, \vartheta) \quad \text{for all } \vartheta \in \Gamma. \tag{4}$$

Consider

$$\mathcal{T} := \{t \in K : \mathcal{I}(\overline{\gamma}_\varepsilon(t)) \geq c - \varepsilon\}.$$

Notice that $\mathcal{T}$ is compact and nonempty. To conclude the proof it suffices to show the existence of $t_\varepsilon \in \mathcal{T}$ such that, if $q_\varepsilon = \overline{\gamma}_\varepsilon(t_\varepsilon)$, then

$$\Psi(\varphi) - \Psi(q_\epsilon) + \mathcal{F}'(q_\varepsilon)[\varphi - q_\varepsilon] \geq -\sqrt{\varepsilon}\|\varphi - q_\varepsilon\| \quad \text{for all } \varphi \in E.$$

Indeed, assume by contradiction that for every $t \in \mathcal{T}$ there exists $\varphi_t \in E \backslash \{\overline{\gamma}_\varepsilon(t)\}$ such that

$$\Psi(\varphi_t) - \Psi(\overline{\gamma}_\epsilon(t)) + \mathcal{F}'(\overline{\gamma}_\epsilon(t))[\varphi_t - \overline{\gamma}_\epsilon(t)] < -\sqrt{\varepsilon}\|\varphi_t - \overline{\gamma}_\epsilon(t)\|.$$

Since $\Psi, \mathcal{F}'$ and $\overline{\gamma}_\varepsilon$ are continuous, for each $t \in \mathcal{T}$ there exist $\delta_t > 0$ and an open ball $B_t$ in $K$ with $t \in B_t$ such that

$$\varphi_t \neq \overline{\gamma}_\varepsilon(s) \quad \text{for all } s \in \overline{B}_t,$$

and

$$\Psi(\varphi_t) - \Psi(\overline{\gamma}_\epsilon(s)) + \mathcal{F}'(\overline{\gamma}_\varepsilon(s) + q)[\varphi_t - \overline{\gamma}_\epsilon(s)] < -\sqrt{\varepsilon}\|\varphi_t - \overline{\gamma}_\epsilon(s)\| \tag{5}$$

for every $s \in B_t$ and $q \in E$ such that $\|q\| < \delta_t$. Since the set $\mathcal{T}$ is compact, there exist $B_{t_1},\ B_{t_2}, \cdots, B_{t_k}$ such that $\mathcal{T} \subset \cup_{j=1}^k B_{t_j}$. Take also $0 < \delta \leq \min\{\delta_{t_1}, \delta_{t_2}, \cdots, \delta_{t_k}\}$ and consider functions $\eta, \eta_j \in C(K, [0,1])$ satisfying

$$\eta(s) = \begin{cases} 1, & \text{if } c \leq \mathcal{I}(\overline{\gamma}_\varepsilon(s)) \\ \\ 0, & \text{if } \mathcal{I}(\overline{\gamma}_\varepsilon(s)) \leq c - \varepsilon \end{cases}$$

and

$$\eta_j(s) = \begin{cases} \dfrac{\text{dist } (s, K \setminus B_{t_j})}{\displaystyle\sum_{i=1}^k \text{dist}(s, K \setminus B_{t_i})}, & \text{if } s \in B_{t_j} \\ \\ 0, & \text{if } s \in K \setminus B_{t_j}. \end{cases}$$

Define $\gamma^* : K \to E$ by

$$\gamma^*(t) = \overline{\gamma}_\varepsilon(t) + \delta\eta(t) \sum_{j=1}^k \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|}(\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)) \quad \forall t \in K.$$

Taking into account that $\varepsilon < c - c_1$, we deduce that $\eta(t) = 0$ for every $t \in K_0$ and thus $\gamma^*(t) = \overline{\gamma}_\varepsilon(t) = \gamma_0(t)$. Hence $\gamma^* \in \Gamma$. Observe that if $t \in K$, then $\gamma^*(t)$ can be written as a linear combination of the points $\overline{\gamma}_\varepsilon(t)$, $\varphi_{t_1}$, $\ldots$, $\varphi_{t_k}$. Indeed, for every $t \in K$,

$$\gamma^*(t) = \left[1 - \delta\eta(t) \sum_{j=1}^k \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|}\right] \overline{\gamma}_\varepsilon(t) + \delta\eta(t) \sum_{j=1}^k \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|}\varphi_{t_j}.$$

In addition, using that $\eta_j(t) = 0$ for every $t \notin B_{t_j}$ we deduce that there exists $M > 0$ such that

$$\delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} \leq \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\min\limits_{s \in \overline{B}_{t_j}} \|\varphi_{t_j} - \overline{\gamma}_\varepsilon(s)\|} \leq \delta M \quad \forall t \in K.$$

Thus, if $\delta < 1/M$, the above linear combination is also a convex combination. Therefore, by the convexity of $\Psi$, we obtain

$$\begin{aligned} \Psi(\gamma^*(t)) \leq & \left[1 - \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|}\right] \Psi(\overline{\gamma}_\varepsilon(t)) \\ & + \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} \Psi(\varphi_{t_j}) \quad \forall t \in K. \end{aligned} \tag{6}$$

On the other hand, by the mean value theorem, for every $t \in K$ there exists $\tau \in (0, 1)$ such that

$$w_\tau := \overline{\gamma}_\varepsilon(t) + \tau\delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} [\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)],$$

satisfies that

$$\mathcal{F}(\gamma^*(t)) - \mathcal{F}(\overline{\gamma}_\varepsilon(t)) = \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} \mathcal{F}'(w_\tau) [\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)]$$

and, by (6),

$$\begin{aligned} \mathcal{I}(\gamma^*(t)) =& \Psi(\gamma^*(t)) + \mathcal{F}(\gamma^*(t)) \\ \leq & \left[1 - \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|}\right] \Psi(\overline{\gamma}_\varepsilon(t)) + \mathcal{F}(\overline{\gamma}_\varepsilon(t)) \\ & + \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} \Psi(\varphi_{t_j}) \\ & + \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} \mathcal{F}'(w_\tau) [\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)] \\ =& \mathcal{I}(\overline{\gamma}_\varepsilon(t)) \\ & + \delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} \left[\Psi(\varphi_{t_j}) - \Psi(\overline{\gamma}_\varepsilon(t)) + \mathcal{F}'(w_\tau) [\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)]\right]. \end{aligned}$$

Using that

$$\left\| \tau\delta\eta(t) \sum_{j=1}^{k} \frac{\eta_j(t)}{\|\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)\|} [\varphi_{t_j} - \overline{\gamma}_\varepsilon(t)] \right\| \leq \delta$$

and (5) we obtain

$$\mathcal{I}(\gamma^*(t)) < \mathcal{I}(\overline{\gamma}_\varepsilon(t)) + \delta\eta(t)\sum_{j=1}^{k}\frac{\eta_j(t)}{\|\varphi_{t_j}-\overline{\gamma}_\varepsilon(t)\|}(-\sqrt{\varepsilon}\|\varphi_{t_j}-\overline{\gamma}_\varepsilon(t)\|)$$

i.e.

$$\mathcal{I}(\gamma^*(t)) < \mathcal{I}(\overline{\gamma}_\varepsilon(t)) - \delta\eta(t)\sum_{j=1}^{k}\eta_j(t)\sqrt{\varepsilon} \quad \forall t\in K.$$

In particular,

$$\mathcal{I}(\gamma^*(t)) < \mathcal{I}(\overline{\gamma}_\varepsilon(t)) - \delta\eta(t)\sqrt{\varepsilon}, \text{ for all } t\in\mathcal{T}(\subset \cup_{j=1}^{k} B_{t_j}).$$

Observing also that $\gamma^*(t)=\overline{\gamma}_\varepsilon(t)$ for every $t\in K\setminus\mathcal{T}$ (since $\eta(t)=0$), we get $\mathcal{I}(\gamma^*(t))=\mathcal{I}(\overline{\gamma}_\varepsilon(t))<c-\varepsilon$ for every $t\in K\setminus\mathcal{T}$. Thus, the maximum of $\mathcal{I}\circ\gamma^*$ is attained at some $s_0\in\mathcal{T}$. Since $\gamma^*\in\Gamma$, we have

$$c\le\Upsilon(\gamma^*)=\mathcal{I}(\gamma^*(s_0))<\mathcal{I}(\overline{\gamma}_\varepsilon(s_0))-\delta\eta(s_0)\sqrt{\varepsilon}<\mathcal{I}(\overline{\gamma}_\varepsilon(s_0))$$

which implies that $\eta(s_0)=1$ and thus

$$\Upsilon(\gamma^*)=\mathcal{I}(\gamma^*(s_0))<\mathcal{I}(\overline{\gamma}_\varepsilon(s_0))-\delta\sqrt{\varepsilon}\le\Upsilon(\overline{\gamma}_\varepsilon)-\delta\sqrt{\varepsilon}.$$

Taking into account that

$$\mathrm{d}_\Gamma(\gamma^*,\overline{\gamma}_\varepsilon)=\max_{t\in K}\ \delta\eta(t)\left\|\sum_{j=1}^{k}\frac{\eta_j(t)}{\|\varphi_{t_j}-\overline{\gamma}_\varepsilon(t)\|}(\varphi_{t_j}-\overline{\gamma}_\varepsilon(t))\right\|\le\delta$$

we deduce that

$$\Upsilon(\gamma^*)<\Upsilon(\overline{\gamma}_\varepsilon)-\sqrt{\varepsilon}\mathrm{d}_\Gamma(\gamma^*,\overline{\gamma}_\varepsilon).$$

This contradicts (4) and completes the proof. ■

**Remark 3** Choosing $\varepsilon=\varepsilon_n\to 0$ as $n$ tends to infinity, the previous theorem implies the existence of a suitable Palais-Smale sequence, i.e. a sequence $q_n = q_{\varepsilon_n}$ in $E$ satisfying

$$\lim_{n\to\infty}\mathcal{I}(q_n)=c$$

and

$$\Psi(\varphi)-\Psi(q_n)+\mathcal{F}'(q_n)[\varphi-q_n]\ge-\sqrt{\varepsilon_n}\|\varphi-q_n\|,$$

for all $\varphi\in E$, or equivalently for all $\varphi\in\mathrm{Dom}\,\Psi$.

Some particular hypotheses in which the *"linking"* condition (2) holds correspond to the geometrical assumptions of either the mountain pass theorem of Ambrosetti and Rabinowitz [4, Theorem 2.1], or the classical saddle point theorem of Rabinowitz [28, Theorem 4.6].

Specifically, we can derive the non-smooth version of the mountain pass theorem (without requiring any compactness condition) by applying Theorem 1 (with $K=[0,1]$, $K_0=\{0,1\}$, $\gamma_0(0)=0$ and $\gamma(1)=e$), as follows.

**Corollary 1** *If there exists* $e \in E \setminus \{0\}$ *such that*

$$\max\{\mathcal{I}(0), \mathcal{I}(e)\} < c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(\gamma(t)),$$

*where* $\Gamma = \{\gamma : [0,1] \to E \, : \, \gamma$ *is continuous and* $\gamma(0) = 0, \; \gamma(1) = e\}$*, then there exists a sequence* $(q_n) \subset E$ *such that* $\lim_{n\to\infty} \mathcal{I}(q_n) = c$ *and there exists* $0 < \epsilon_n \to 0$ *such that*

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\epsilon_n \|\varphi - q_n\| \tag{7}$$

*for all positive integer* $n$ *and for all* $\varphi \in Dom\,\Psi$*.* ■

Similarly, if the Banach space $E$ is split into $E = \bar{E} \oplus \widetilde{E}$ with $\dim \bar{E} < \infty$, then applying Theorem 1 with $K$ the closed ball $B_\rho$ in $\bar{E}$ of center zero and radius $\rho$, $K_0$ the boundary in $\bar{E}$ of this ball and $\gamma_0$ the identity function in $K_0$, we obtain the following corollary.

**Corollary 2** *Assume that* $E = \bar{E} \oplus \widetilde{E}$ *with* $dim\,\bar{E} < \infty$*. If there exists* $\rho > 0$ *such that the boundary* $\partial B_\rho$ *of the ball* $B_\rho$ *in* $\bar{E}$ *of center zero and radius* $\rho$ *satisfies that*

$$\sup_{\partial B_\rho} \mathcal{I} < \inf_{\widetilde{E}} \mathcal{I},$$

*and* $\Gamma = \{\gamma \in C(\overline{B}_\rho, E) \, : \, \gamma(x) = x, \; \forall x \in \partial B_\rho\}$*, then there exists a sequence* $(q_n) \subset E$ *such that*

$$\lim_{n\to\infty} \mathcal{I}(q_n) = c := \inf_{\gamma \in \Gamma} \sup_{x \in \overline{B}_\rho} \mathcal{I}(\gamma(x)), \tag{8}$$

*and there exists* $0 < \epsilon_n \to 0$ *such that* (7) *holds for all positive integer* $n$ *and for all* $\varphi \in Dom\,\Psi$*.* ■

**Remark 4** Both corollaries state the existence of sequences $(q_n) \subset E$ and $0 < \epsilon_n \to 0$ such that (7) holds for all positive integer $n$ and for all $\varphi \in \mathrm{Dom}\,\Psi$. We remark explicitly that this inequality is trivially verified for all $\varphi \notin \mathrm{Dom}\,\Psi$ as well.

# 3 Relativistic Lagrangians and Dirichlet problems

## 3.1 The functional framework

Let $T > 0$ be fixed. If $W^{1,\infty}(0,T)$ denotes the space of all Lipschitz functions in $[0,T]$ (or equivalently the absolutely continuous functions in $[0,T]$ with bounded derivatives), we consider the Banach space

$$W^{1,\infty} = [W^{1,\infty}(0,T)]^3$$

endowed with the usual norm $\|\cdot\|_{1,\infty}$ given by

$$\|q\|_{1,\infty} = \|q\|_\infty + \|q'\|_\infty \quad (q \in W^{1,\infty}),$$

where $\|q\|_\infty = \max_{t\in[0,T]} |q(t)|$ and $\|q'\|_\infty = \max_{t\in[0,T]} |q'(t)|$. Consider also two $C^1$-functions $V : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ which give the "smooth" part $L$ of the *relativistic Lagrangian* $\mathcal{L}$, that is

$$L(t,q,p) = p\cdot W(t,q) - V(t,q) \qquad ((t,q,p)\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3).$$

One has that

$$D_{q_i}L(t,q,p) = p\cdot D_{q_i}W(t,q) - D_{q_i}V(t,q),\ i=1,2,3$$

and

$$\nabla_p L(t,q,p) = W(t,q),$$

for every $(t,q,p)\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$. Consider also $\mathcal{E} : [0,T]\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathcal{E}(t,q,p) = (p\cdot D_{q_1}W(t,q), p\cdot D_{q_2}W(t,q), p\cdot D_{q_3}W(t,q)). \tag{9}$$

The action functional associated to $L$ is $\mathcal{F} : W^{1,\infty}\to\mathbb{R}$ given by

$$\mathcal{F}(q) := \int_0^T L(t,q,q')dt = \int_0^T [q'\cdot W(t,q) - V(t,q)]dt, \qquad \forall q\in W^{1,\infty}.$$

It is standard (see for instance [20]) to prove the differentiability of $\mathcal{F}$.

**Lemma 1** *The functional $\mathcal{F}$ is of class $C^1$ in $W^{1,\infty}$, i.e. $\mathcal{F}\in C^1(W^{1,\infty},\mathbb{R})$, with*

$$\begin{aligned}\mathcal{F}'(q)[\varphi] &= \int_0^T [\nabla_q L(t,q,q')\cdot\varphi + \nabla_p L(t,q,q')\cdot\varphi']dt \\ &= \int_0^T (\mathcal{E}(t,q,q') - \nabla_q V(t,q))\cdot\varphi dt + \int_0^T W(t,q)\cdot\varphi' dt,\end{aligned}$$

*for every $q,\varphi\in W^{1,\infty}$.*

Next, we deal with the "nonsmooth" part of the relativistic Lagrangian in the subspace $W_0^{1,\infty}$ of all functions $q\in W^{1,\infty}$ such that $q(0) = 0 = q(T)$. Consider

$$\mathcal{K}_0 = \{q\in W_0^{1,\infty} : \|q'\|_\infty \le 1\},$$
$$\Phi(s) = 1-\sqrt{1-s^2} \qquad (s\in[-1,1]),$$

and the action functional corresponding to the "nonsmooth" part of the relativistic Lagrangian, i.e., the functional $\Psi_0 : W_0^{1,\infty}\to(-\infty,+\infty]$ defined by

$$\Psi_0(q) = \begin{cases} \displaystyle\int_0^T \Phi(q')dt = \int_0^T [1-\sqrt{1-|q'|^2}]dt, & \text{if } q\in\mathcal{K}_0, \\ +\infty, & \text{if } q\notin W_0^{1,\infty}\setminus\mathcal{K}_0. \end{cases}$$

**Lemma 2** *The restriction of the functional $\Psi_0$ to its domain $\mathcal{K}_0$ is continuous.*

*Proof.* Let $(q_n) \subset \mathcal{K}_0$ a sequence converging in $W^{1,\infty}$ to $q$. Necessarily, $q \in \mathcal{K}_0$ ($\mathcal{K}_0$ is closed in $W^{1,\infty}$). Then, using the convergence of $(q_n'(t))$ to $q'(t)$ for a.e. $t \in [0,T]$ and the continuity of the function $\Phi$ we deduce the pointwise convergence

$$\Phi(q_n'(t)) = 1 - \sqrt{1 - |q_n'(t)|^2} \longrightarrow \Phi(q'(t)) = 1 - \sqrt{1 - |q'(t)|^2}, \quad \text{a.e. } t \in [0,T].$$

Taking into account that $0 \leq \Phi \leq 1$, the sequence $(\Phi(q_n))$ is dominated by the constant function 1. Therefore, the Lebesgue dominated convergence theorem implies that $(\Psi_0(q_n))$ converges to $\Psi_0(q)$, i.e., $\Psi_0$ is continuous in $\mathcal{K}_0$ ■

Following the ideas of [10, Lemma 1 and the proof of Theorem 1] (see also [9, equation (7)] or [19, Lemma 12]) it is not difficult to prove the following properties of $\mathcal{K}_0$ and $\Psi_0$.

**Lemma 3** *(i) The set $\mathcal{K}_0$ is convex and closed in $C([0,T],\mathbb{R}^3)$ and thus in $W_0^{1,\infty}$. Moreover, if $(q_n)$ is a sequence in $\mathcal{K}_0$ converging pointwise in $[0,T]$ to a continuous function $q : [0,T] \to \mathbb{R}^3$, then $q \in \mathcal{K}_0$ and $q_n' \to q'$ in the $w^*$-topology $\sigma(L^\infty, L^1)$.*

*(ii) If $(q_n)$ is a sequence in $\mathcal{K}_0$ converging in $C([0,T],\mathbb{R}^3)$ to $q$, then*

$$\Psi_0(q) \leq \liminf_{n\to\infty} \Psi_0(q_n).$$

*In particular, the functional $\Psi_0$ is weakly lower semicontinuous and convex in $W_0^{1,\infty}$.* ■

Next, consider the Euler-Lagrange action functional associated to the relativistic Lagrangian $\mathcal{L}$ with zero Dirichlet boundary conditions, i.e.,

$$\mathcal{I}_0 : W_0^{1,\infty} \to (-\infty, +\infty], \quad \mathcal{I}_0 = \Psi_0 + \mathcal{F}_0,$$

where $\mathcal{F}_0$ is the restriction of $\mathcal{F}$ to $W_0^{1,\infty}$. Since $\mathcal{I}_0$ is the sum of the proper convex lower semicontinuous functional $\Psi_0$ and of the $C^1$-functional $\mathcal{F}_0$, Szulkin's critical point theory from [30] is applicable for $\mathcal{I}_0$. According to this theory, we have the following definition.

**Definition 1** *A function $q \in W_0^{1,\infty}$ is a* **critical point** *of $\mathcal{I}_0$ if $q \in \mathcal{K}_0$ and*

$$\Psi_0(\varphi) - \Psi_0(q) + \mathcal{F}_0'(q)[\varphi - q] \geq 0 \quad \textit{for all } \varphi \in W_0^{1,\infty},$$

*or, equivalently*

$$\int_0^T (\Phi(\varphi) - \Phi(q))dt + \mathcal{F}_0'(q)[\varphi - q] \geq 0 \quad \textit{for all } \varphi \in \mathcal{K}_0,$$

*that is*

$$\int_0^T [\sqrt{1-|q'|^2} - \sqrt{1-|\varphi'|^2}]dt + \int_0^T [\mathcal{E}(t,q,q') - \nabla_q V(t,q)] \cdot (\varphi - q)dt$$
$$+ \int_0^T W(t,q) \cdot (\varphi' - q')dt \geq 0, \quad \textit{for all } \varphi \in \mathcal{K}_0.$$

The following lemma, which is essentially contained in [9, Lemma 1], is the main tool to prove that the critical points of $\mathcal{I}_0$ coincide with solutions in $W_0^{1,\infty}$ of the Euler-Lagrange equation associated to the relativistic Lagrangian $\mathcal{L}$.

**Lemma 4** *For every $f \in L^1 = [L^1(0,T)]^3$, the Dirichlet problem*

$$\left( \frac{q'}{\sqrt{1-|q'|^2}} \right)' = f, \qquad q(0) = 0 = q(T),$$

*has a unique solution, which is also the unique solution of the variational inequality*

$$\Psi_0(\varphi) - \Psi_0(q) + \int_0^T f \cdot (\varphi - q)dt \geq 0, \qquad \textit{for all } \varphi \in \mathcal{K}_0.$$

■

**Theorem 2** *If $V : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are two $C^1$-functions, then a function $q \in W_0^{1,\infty}$ is a critical point of $\mathcal{I}_0$ if and only if $q$ is a solution of the Lorentz force equation with zero Dirichlet boundary conditions on $[0,T]$, i.e.*

$$\left( \frac{q'}{\sqrt{1-|q'|^2}} \right)' + (W(t,q))' = \mathcal{E}(t,q,q') - \nabla_q V(t,q), \qquad q(0) = 0 = q(T). \quad (10)$$

*Proof.* Let $q \in W_0^{1,\infty}$ be a critical point of $\mathcal{I}_0$, i.e. $q \in \mathcal{K}_0$ satisfying

$$\Psi_0(\varphi) - \Psi_0(q) + \int_0^T [\mathcal{E}(t,q,q') - \nabla_q V(t,q)] \cdot (\varphi - q)dt + \int_0^T W(t,q) \cdot (\varphi' - q')dt \geq 0,$$

for every $\varphi \in \mathcal{K}_0$. Integrating by parts one has that

$$\int_0^T W(t,q) \cdot (\varphi' - q')dt = -\int_0^T (W(t,q))' \cdot (\varphi - q)dt$$

and thus, if we consider $f_q := \mathcal{E}(t,q,q') - \nabla_q V(t,q) - (W(t,q))' \in L^\infty$, we have

$$\Psi_0(\varphi) - \Psi_0(q) + \int_0^T f_q \cdot (\varphi - q)dt \geq 0, \qquad \text{for all } \varphi \in \mathcal{K}_0.$$

By Lemma 4, this means that $q$ is the solution of

$$\left( \frac{q'}{\sqrt{1-|q'|^2}} \right)' = f_q, \qquad q(0) = q(T) = 0$$

and therefore $q$ solves (10). The reversed implication follows similarly. ■

### 3.2 Principle of least action for Dirichlet problems

By Proposition 1.1 in [30], every local minimum of $\mathcal{I}_0$ is a critical point of $\mathcal{I}_0$, hence Theorem 2 provides the following direct consequence.

**Proposition 1** *Each local minimum of $\mathcal{I}_0$ is a solution of the Dirichlet boundary value problem* (10) *associated to the Lorentz equation.*

The following result gives a positive answer to Hilbert's 20th problem for the Lorentz force equation with zero Dirichlet boundary conditions.

**Theorem 3 (Principle of the least action for the Lorentz equation with Dirichlet boundary conditions.)** *The Lagrangian action $\mathcal{I}_0$ associated to Lorentz equation with Dirichlet boundary conditions is bounded from below and attains its infimum at some $q \in \mathcal{K}_0$, which is a solution of* (10)*.*

*Proof.* Let $(q_n)$ be a minimizing sequence of $\mathcal{I}_0$, that is

$$(q_n) \subset \mathcal{K}_0, \quad \mathcal{I}_0(q_n) \to \inf_{W_0^{1,\infty}} \mathcal{I}_0 = \inf_{\mathcal{K}_0} \mathcal{I}_0 \text{ as } n \to \infty.$$

We split the proof in three steps:

**Step 1:** The sequence $(q_n)$ is bounded in $W_0^{1,\infty}$.

**Step 2:** Up to a subsequence, $(q_n)$ is convergent in $L^\infty$ to some $q \in \mathcal{K}_0$ with

$$\Psi_0(q) \le \liminf_{n\to\infty} \Psi_0(q_n), \quad \lim_{n\to\infty} \int_0^T V(t,q_n)dt = \int_0^T V(t,q)dt$$

and

$$\lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q_n)dt = \int_0^T q' \cdot W(t,q)dt.$$

**Step 3:** $\mathcal{I}_0(q) = \min_{W_0^{1,\infty}} \mathcal{I}_0$ and $q$ is a solution of (10).

Indeed, to prove Step 1, from $q(t) = \int_0^t q'(s)ds$ and $|q'(s)| \le 1$, we deduce that

$$\|q\|_\infty \le T, \quad \forall q \in \mathcal{K}_0. \tag{11}$$

and thus $\|q_n\|_{1,\infty} \le T + 1$ and the proof of the Step 1 is concluded.

To prove the second step, we observe that, by the compact embedding of $W_0^{1,\infty}$ into $C([0,T],\mathbb{R}^3)$, we can assume, passing to a subsequence if necessary, that there exists $q \in C([0,T],\mathbb{R}^3)$ such that $\|q_n - q\|_\infty \to 0$. By Lemma 3 we deduce that

$$q \in \mathcal{K}_0, \quad \Psi_0(q) \le \liminf_{n\to\infty} \Psi_0(q_n), \quad \lim_{n\to\infty} \int_0^T V(t,q_n)dt = \int_0^T V(t,q)dt.$$

and also that $q_n' \to q'$ in the w*-topology $\sigma(L^\infty, L^1)$, that is

$$\lim_{n\to\infty} \int_0^T (q_n' - q') \cdot \varphi dt = 0 \quad \text{for all } \varphi \in L^1.$$

Since $W(t,q) \in C([0,T], \mathbb{R}^3) \subset L^1$, it follows then that

$$\lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q) dt = \int_0^T q' \cdot W(t,q) dt.$$

On the other hand, using that $\|q_n - q\|_\infty \to 0$ and $\|q_n'\|_\infty \leq 1$ it follows also that

$$\left| \int_0^T q_n' \cdot (W(t,q_n) - W(t,q)) dt \right| \leq \|q_n'\|_\infty \int_0^T |W(t,q_n) - W(t,q)| dt \to 0$$

as $n \to \infty$. Consequently

$$\begin{aligned} \lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q_n) dt &= \lim_{n\to\infty} \int_0^T q_n' \cdot (W(t,q_n) - W(t,q)) dt \\ &\quad + \lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q) dt. \\ &= \int_0^T q' \cdot W(t,q) dt \end{aligned}$$

proving the Step 2.

Finally, Step 3 is easily deduced because Step 2 implies that

$$\mathcal{I}_0(q) = \inf_{\mathcal{K}_0} \mathcal{I}_0,$$

i.e., $\mathcal{I}_0$ attains its infimum at $q \in \mathcal{K}_0$ which is also a solution of (10) by Proposition 1. ■

As a simple application of the above principle we have the following result of existence of a nonzero solution.

**Theorem 4** *If there exist* $1 \leq \mu < \min\{\nu, 2\}$ *and* $d > 0$ *such that*

$$V(t,q) \geq d|q|^\mu + V(t,0) \ \textit{and} \ |W(t,q)| \leq d|q|^\nu, \tag{12}$$

*for every* $(t,q) \in [0,T] \times \mathbb{R}^3$ *with* $|q| \leq T$, *then,* (10) *has at least one nonzero solution which is a minimizer of the Lagrangian action* $\mathcal{I}_0$.

*Proof.* Since $W(t,0) = 0$ for every $t \in [0,T]$, we observe that

$$\mathcal{I}_0(0) = -\int_0^T V(t,0) dt.$$

If $q_0 \in \mathcal{K}_0 \setminus \{0\}$, then $\epsilon q_0 \in \mathcal{K}_0$ for all $\epsilon \in (0,1]$. In addition,

$$\int_0^T \left[1 - \sqrt{1 - |\epsilon q_0'|^2}\right] dt \leq \int_0^T |\epsilon q_0'|^2 dt.$$

By (11), we deduce from hypothesis (12) the existence of positive constants $C_1, C_2 > 0$ such that

$$\mathcal{I}_0(\epsilon q_0) \leq C_1(\epsilon^2 + \epsilon^\nu) - C_2 \epsilon^\mu - \int_0^T V(t,0)dt.$$

In particular, for $\epsilon$ small enough we have $\mathcal{I}_0(\epsilon q_0) < \mathcal{I}_0(0)$, which implies that

$$\inf_{W_0^{1,\infty}} \mathcal{I}_0 < \mathcal{I}_0(0).$$

Using the principle of least action for Dirichlet problem (Theorem 3), $\mathcal{I}_0$ attains its infimum at some $q \in \mathcal{K}_0 \setminus \{0\}$, which is a nonzero solution of (10) by Proposition 1. ■

### 3.3 Solutions via Minimax critical point theory

We prove in this subsection the existence of solution of (10) via the non-smooth mountain pass theorem (see Corollary 1). We begin by establishing the suitable Palais-Smale condition for the functional $\mathcal{I}_0$.

**Lemma 5** *If $(q_n)$ is a sequence in $W_0^{1,\infty}$ satisfying that*

$$\lim_{n\to\infty} \mathcal{I}_0(q_n) = c \in \mathbb{R}$$

*and that there is a sequence $(\varepsilon_n)$ of positive numbers converging to zero such that*

$$\Psi_0(\varphi) - \Psi_0(q_n) + \mathcal{F}_0'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\|_{1,\infty}, \quad \forall \varphi \in \mathcal{K}_0, \tag{13}$$

*then there exists a subsequence $(q_{n_k})$ of $(q_n)$ converging in $C([0,T], \mathbb{R}^3)$ to a critical point $q \in \mathcal{K}_0$ of $\mathcal{I}_0$ with level $\mathcal{I}_0(q) = c$.*

*Proof.* Repeating the argument of Steps 1 and 2 in the proof of Theorem 3, we obtain, up to a subsequence, the convergence in $C([0,T], \mathbb{R}^3)$ of $(q_n)$ to $q$ with

$$q \in \mathcal{K}_0, \quad \Psi_0(q) \leq \liminf_{n\to\infty} \Psi_0(q_n), \quad q_n' \to q' \text{ in w*-topology } \sigma(L^\infty, L^1),$$

and

$$\lim_{n\to\infty} \int_0^T V(t,q_n)dt = \int_0^T V(t,q)dt, \ \lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q_n)dt = \int_0^T q' \cdot W(t,q)dt.$$

Similarly, using, as in the cited Step 2, the convergence in the w*-topology $\sigma(L^\infty, L^1)$ of $(q_n')$ to $q'$, we also deduce that

$$\int_0^T (\mathcal{E}(t,q_n,q_n')-\nabla_q V(t,q_n))\cdot(\varphi-q_n)dt \to \int_0^T (\mathcal{E}(t,q,q')-\nabla_q V(t,q))\cdot(\varphi-q)dt,$$

as $n$ tends to $\infty$. Consequently,

$$\lim_{n\to\infty} \mathcal{F}_0(q_n) = \mathcal{F}_0(q)$$

and

$$\lim_{n\to\infty} \mathcal{F}_0'(q_n)[\varphi - q_n] = \mathcal{F}_0'(q)[\varphi - q].$$

By taking lim inf as $n \to \infty$ in (13) it follows that

$$\Psi_0(\varphi) - \Psi_0(q) + \mathcal{F}_0'(q)[\varphi - q] \geq 0, \quad \forall \varphi \in \mathcal{K}_0,$$

which implies that $q$ is a critical point of $\mathcal{I}_0$.

In addition, by choosing $\varphi = q$ in (13), we get

$$\Psi_0(q_n) \leq \Psi_0(q) + \mathcal{F}_0'(q_n)[q - q_n] + \varepsilon_n \|q - q_n\|_{1,\infty}, \quad \forall n \in \mathbb{N},$$

and taking limits we deduce that

$$\Psi_0(q) \leq \liminf_{n\to\infty} \Psi_0(q_n) \leq \limsup_{n\to\infty} \Psi_0(q_n) \leq \Psi_0(q);$$

i.e.,

$$\Psi_0(q) = \lim_{n\to\infty} \Psi(q_n).$$

Therefore,

$$c = \lim_{n\to\infty} \mathcal{I}_0(q_n) = \lim_{n\to\infty} \Psi_0(q_n) + \lim_{n\to\infty} \mathcal{F}_0(q_n) = \Psi_0(q) + \mathcal{F}_0(q) = \mathcal{I}_0(q).$$

■

In order to prove the existence of multiple solutions of (10), we introduce a parameter $\lambda > 0$ in the electric potential $V$; namely, we consider the problem

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' + (W(t,q))' = \mathcal{E}(t,q,q') - \lambda \nabla_q V(t,q), \qquad q(0) = 0 = q(T). \tag{14}$$

**Theorem 5** *Assume that there exist constants $\mu, \nu > 2$ and $d > 0$ such that*

$$V(t,q) \leq d|q|^\mu \text{ and } |W(t,q)| \leq d|q|^\nu, \tag{15}$$

*for every $(t,q) \in [0,T] \times \mathbb{R}^3$ with $|q| \leq T$. If there is a function $q_0 \in \mathcal{K}_0 \setminus \{0\}$ satisfying $\int_0^T V(t,q_0)dt > 0$, then there exists $\lambda^* > 0$ such that for every $\lambda \geq \lambda^*$, problem (14) possesses at least one nonzero solution.*

*If in addition $q = 0$ is solution of (14), i.e., if $V(t,0) = 0$ for every $t \in [0,T]$, then there exists also a second nonzero solution.*

**Remark 5** It will be proved that first nonzero solution of (14) corresponds to a global minimum of the functional $\mathcal{I}_0$, while the other one is obtained by the non-smooth version of the mountain pass theorem given in Corollary 1.

*Proof.* We have

$$\mathcal{I}_0(q_0) \leq \Psi_0(q_0) + \int_0^T q_0' \cdot W(t, q_0)dt - \lambda \int_0^T V(t, q_0)dt,$$

and thus, since $\int_0^T V(t, q_0)dt > 0$, there exists $\lambda^* > 0$ such that $\mathcal{I}_0(q_0) < 0$ for every $\lambda \geq \lambda^*$. In consequence,

$$\inf_{q \in \mathcal{K}_0} \mathcal{I}_0(q) \leq \mathcal{I}_0(q_0) < 0, \quad \forall \lambda \geq \lambda^*.$$

Taking into account that (15) implies that $V(t, 0) \leq 0$ for every $t \in [0, T]$, we deduce that $\mathcal{I}_0(0) \geq 0$ and using the principle of least action for the Dirichlet problem (Theorem 3), that $\mathcal{I}_0$ attains its infimum at some $q_* \in \mathcal{K}_0 \setminus \{0\}$, which is a nonzero solution of (14) by Proposition 1.

To prove the existence of a second nonzero solution for $\lambda \geq \lambda^*$, we observe that necessarily $\mathcal{I}_0(0) = 0$ provided that it is additionally assumed that $V(t, 0) = 0$ for every $t \in [0, T]$. Moreover, the inequality $1 - \sqrt{1 - s^2} \geq s^2/2$ for every $s \in [-1, 1]$ implies that

$$\Psi_0(q) \geq \frac{1}{2} \int_0^T |q'|^2 dt = \frac{1}{2} \|q\|_{H_0^1}^2, \quad \forall q \in \mathcal{K}_0.$$

On the other hand, using that $\|q'\|_\infty \leq 1$ and (15) we also have

$$\left| \int_0^T q' \cdot W(t, q)dt \right| \leq \int_0^T |W(t, q)|dt \leq C \int_0^T |q|^\nu dt = C\|q\|_{L^\nu}^\nu,$$

and

$$\mathcal{I}_0(q) \geq \frac{1}{2} \|q\|_{H_0^1}^2 - C\|q\|_{L^\nu}^\nu - \lambda C\|q\|_{L^\mu}^\mu, \quad \forall q \in \mathcal{K}_0.$$

Taking into account that $H_0^1$ is embedded in $L^\mu$ and in $L^\nu$ (since $\mu, \nu > 2$), we deduce

$$\mathcal{I}_0(q) \geq \frac{1}{2} \|q\|_{H_0^1}^2 - C\|q\|_{H_0^1}^\nu - \lambda C\|q\|_{H_0^1}^\mu, \quad \forall q \in \mathcal{K}_0,$$

which implies the existence of $r \in (0, \|q_*\|_{H_0^1})$ and $\alpha > 0$ such that

$$\mathcal{I}_0(q) \geq \alpha, \quad \forall q \in \mathcal{K}_0 \text{ with } \|q\|_{H_0^1} = r.$$

Let $\Gamma$ be the family of all continuous paths $\gamma : [0, 1] \to W_0^{1,\infty}$ joining 0 with $q_*$, i.e. satisfying $\gamma(0) = 0$ and $\gamma(1) = q_*$. If $\gamma \in \Gamma$, then, by the embedding of $W_0^{1,\infty}$ into $H_0^1$ and the connectedness of $\gamma([0, 1])$, there exists $t_0 \in [0, 1]$ such that $\|\gamma(t_0)\|_{H_0^1} = r$. Hence, the above inequality implies that

$$\max_{t \in [0,1]} \mathcal{I}_0(\gamma(t)) \geq \alpha > \mathcal{I}_0(0) = 0 > \mathcal{I}_0(q_*), \quad \forall \gamma \in \Gamma,$$

for every $\lambda \geq \lambda^*$. Applying the non-smooth version of the mountain pass theorem given in Corollary 1, we obtain the existence of a sequence $(q_n) \subset W_0^{1,\infty}$ such that

$$\lim_{n\to\infty} \mathcal{I}_0(q_n) = c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_0(\gamma(t)) \geq \alpha$$

and there exists $0 < \epsilon_n \to 0$ such that (13) holds. By using Lemma 5 we deduce the existence of a critical point $q^* \in \mathcal{K}_0$ of $\mathcal{I}_0$ with level $\mathcal{I}_0(q^*) = c \geq \alpha > 0$, which implies that $q^*$ is different from 0 and $q_*$. Therefore, by Proposition 1, $q^*$ is a second nonzero solution of (14). ■

**Remark 6** Other variants of the mountain pass theorem for non-smooth functionals different from the Szulkin's one are considered in the literature. To apply the metric mountain pass theorem discovered independently by Degiovanni - Marzocchi [12] and Katriel [17] the functional $\mathcal{I}_0$ should be continuous on the entire space $W_0^{1,\infty}$ and to verify a Palais-Smale type condition. However, our functional is continuous only on its domain of definition. On the other hand, if $q$ is not in the interior of the domain $\mathcal{K}_0$ of $\mathcal{I}_0$, we observe that $\mathcal{I}_0$ is not differentiable at $q$ through any direction $\varphi \in W_0^{1,\infty}$, and the main abstract result from [5] is neither applicable.

# 4 Relativistic Lagrangians and periodic boundary value problems

## 4.1 The functional framework

Now for two $C^1$-functions $V : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$, we study the boundary value problem associated to the Lorentz force equation with periodic boundary conditions

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' + (W(t,q))' = \mathcal{E}(t,q,q') - \nabla_q V(t,q),$$
$$q(0) = q(T), \quad q'(0) = q'(T). \tag{16}$$

In this case, we consider the subspace $W_T^{1,\infty}$ of all $T$-periodic functions $q \in W^{1,\infty}$ (i.e. $q \in W^{1,\infty}$ such that $q(0) = q(T)$) and the convex and closed set $\mathcal{K}_*$ given by

$$\mathcal{K}_* = \{q \in W_T^{1,\infty} : \|q'\|_\infty \leq 1\}.$$

The Lagrangian action $\mathcal{I}_* : W_T^{1,\infty} \to (-\infty, +\infty]$ associated to the problem (16) is given by

$$\mathcal{I}_* = \Psi_* + \mathcal{F}_*.$$

where $\Psi_*$ is defined by

$$\Psi_*(q) = \begin{cases} \displaystyle\int_0^T [1 - \sqrt{1 - |q'|^2}]dt, & \text{if } q \in \mathcal{K}_*, \\ +\infty, & \text{if } \ q \notin W_T^{1,\infty} \setminus \mathcal{K}_*, \end{cases}$$

and $\mathcal{F}_*$ is the restriction of $\mathcal{F}$ to $W_T^{1,\infty}$. Since $\Psi_*$ is a proper convex function which is continuous in its closed domain $\mathcal{K}_*$ (similar proof to that of Lemma 2) and $\mathcal{F}_*$ is a function of class $C^1$, Szulkin's critical point theory from [30] is applicable for $\mathcal{I}_*$. According to this theory, we have the following definition.

**Definition 2** *A function $q \in W_T^{1,\infty}$ is a* **critical point** *of $\mathcal{I}_*$ if $q \in \mathcal{K}_*$ and*

$$\Psi_*(\varphi) - \Psi_*(q) + \mathcal{F}_*'(q)[\varphi - q] \geq 0 \quad \textit{for all } \varphi \in W_T^{1,\infty},$$

*or, equivalently*

$$\int_0^T [\sqrt{1 - |q'|^2} - \sqrt{1 - |\varphi'|^2}]dt + \int_0^T [\mathcal{E}(t, q, q') - \nabla_q V(t, q)] \cdot (\varphi - q)dt \\ + \int_0^T W(t, q) \cdot (\varphi' - q')dt \geq 0, \quad \textit{for all } \varphi \in \mathcal{K}_*.$$

Given $f \in L^1$, as in the Dirichlet case, a function $q \in W_T^{1,\infty}$ is a solution of the periodic problem

$$\left( \frac{q'}{\sqrt{1 - |q'|^2}} \right)' = f, \qquad q(0) = q(T), \ q'(0) = q'(T),$$

if and only if it satisfies the variational inequality

$$\Psi_*(\varphi) - \Psi_*(q) + \int_0^T f \cdot (\varphi - q)dt \geq 0, \qquad \text{for all } \varphi \in \mathcal{K}_*.$$

This allows to prove the analogous result to Theorem 2.

**Theorem 6** *A function $q \in W_T^{1,\infty}$ is a critical point of $\mathcal{I}_*$ if and only if $q$ is a solution of the Lorentz force equation with periodic boundary conditions* (16). ■

**Remark 7** In addition, similarly to the Proposition 1, it is easy to show that each local minimum of $\mathcal{I}_*$ is a solution of the Lorentz force equation with periodic boundary conditions (16).

## 4.2 The principle of the least action for periodic problems

For every $q \in W_T^{1,\infty}$ we denote by

$$\bar{q} = \frac{1}{T}\int_0^T q dt, \qquad \tilde{q} = q - \bar{q}.$$

Thus $q = \bar{q} + \tilde{q}$ with

$$\int_0^T \tilde{q} dt = 0.$$

Then, by using the mean value theorem in each component of $q$, it is easy to find

$$\|\tilde{q}\|_\infty \le T \qquad \text{for all } q \in \mathcal{K}_*. \tag{17}$$

We also denote

$$\mathcal{K}_\rho = \{q \in \mathcal{K}_* : |\bar{q}| \le \rho\}, \quad \forall \rho > 0.$$

**Theorem 7** *If there exists $\rho > 0$ such that*

$$\inf_{\mathcal{K}_\rho} \mathcal{I}_* = \inf_{\mathcal{K}_*} \mathcal{I}_*, \tag{18}$$

*then $\mathcal{I}_*$ is bounded from below on $W_T^{1,\infty}$ and attains its infimum at some $q \in \mathcal{K}_\rho$ which is a solution of* (16)*.*

*Proof.* It suffices to show that for every $\rho > 0$, the restriction $\mathcal{I}_*\big|_{\mathcal{K}_\rho}$ attains its infimum at some $q \in \mathcal{K}_\rho$. Indeed, if this is proved, then the hypothesis (18) implies that $q$ is a minimum of $\mathcal{I}_*$ and, by Remark 7, a solution of (16). Hence, let $(q_n)$ be a minimizing sequence of $\mathcal{I}_*\big|_{\mathcal{K}_\rho}$, that is

$$(q_n) \subset \mathcal{K}_\rho, \quad \mathcal{I}_*(q_n) \to \inf_{\mathcal{K}_\rho} \mathcal{I}_* \text{ as } n \to \infty.$$

By (17) we deduce that

$$\|q_n\|_\infty \le |\bar{q}_n| + \|\tilde{q}_n\|_\infty \le \rho + T, \quad \forall n \in \mathbb{N},$$

and, using that $\|q_n'\|_\infty \le 1$, also that

$$\|q_n\|_{1,\infty} \le 1 + \rho + T, \quad \forall n \in \mathbb{N},$$

that is, $(q_n)$ is bounded in $W_T^{1,\infty}$. Since $W^{1,\infty}$ is compactly embedded in $C([0,T],\mathbb{R}^3)$, passing to a subsequence if necessary, there exists $q \in C([0,T],\mathbb{R}^3)$ such that $\|q_n - q\|_\infty \to 0$.

By a similar result to Lemma 3, we can repeat the argument of Step 2 in the proof of Theorem 3 to deduce that (notice that $\bar{q}_n \to \bar{q}$ and $|\bar{q}| \le \rho$)

$$q \in \mathcal{K}_\rho, \quad \Psi_*(q) \le \liminf_{n\to\infty} \Psi_*(q_n), \quad q_n' \to q' \text{ in w}^*\text{-topology } \sigma(L^\infty, L^1),$$

and

$$\lim_{n\to\infty}\int_0^T V(t,q_n)dt = \int_0^T V(t,q)dt,$$

$$\lim_{n\to\infty}\int_0^T q_n'\cdot W(t,q_n)dt = \int_0^T q'\cdot W(t,q)dt,$$

which implies that

$$q\in\mathcal{K}_\rho,\qquad \mathcal{I}_*(q)=\inf_{\mathcal{K}_\rho}\mathcal{I}_*,$$

i.e., $\mathcal{I}_*\big|_{\mathcal{K}_\rho}$ attains its infimum at $q\in\mathcal{K}_\rho$. ■

As an application of the above principle we show that if $V$ and $W$ hold convenient a relation, then problem (16) has a solution.

**Theorem 8** *If there exist $\mu>\nu\geq 1$ and $d,r>0$ such that*

$$V(t,q)\leq -d|q|^\mu \text{ and } |W(t,q)|\leq d|q|^\nu,$$

*for every $(t,q)\in[0,T]\times\mathbb{R}^3$ with $|q|\geq r$, then (16) has at least one solution, which is a minimizer of $\mathcal{I}_*$.*

*Proof.* Let $C>0$ be such that

$$V(t,q)\leq -d|q|^\mu + C$$

and

$$|W(t,q)|\leq d|q|^\nu + C,$$

for all $(t,q)\in[0,T]\times\mathbb{R}^3$. For any $q\in\mathcal{K}_*$, using that $\|q'\|_\infty\leq 1$ and (17) we deduce that

$$\begin{aligned}\left|\int_0^T q'\cdot W(t,q)dt\right| &\leq \int_0^T |W(t,q)|dt \leq d\int_0^T |q|^\nu dt + TC\\ &\leq 2^{\nu-1}d\int_0^T [|\tilde{q}|^\nu + |\bar{q}|^\nu]dt + TC\\ &\leq 2^{\nu-1}d[T^{\nu+1}+T|\bar{q}|^\nu] + TC.\end{aligned}$$

On the other hand one has that

$$\int_0^T V(t,q)dt \leq -d\int_0^T |q|^\mu dt + TC,$$

and using again (17) that

$$T|\bar{q}|^\mu \leq \int_0^T (|q|+|\tilde{q}|)^\mu dt \leq 2^{\mu-1}\int_0^T (|q|^\mu + T^\mu)dt.$$

It follows that there exist a constants $C_1, C_2 > 0$ such that

$$\int_0^T |q|^\mu dt \geq C_1 |\bar{q}|^\mu - C_2.$$

We deduce that

$$\mathcal{I}_*(q) \geq C_1|\bar{q}|^\mu - C_2 - 2^{\nu-1} d[T^{\nu+1} + T|\bar{q}|^\nu] - TC,$$

which together with $\nu < \mu$ imply that there exists $\rho > 0$ for which the hypothesis (18) is satisfied. The conclusion follows from the principle of least action given above. ■

Another application of the principle of least action is the following one.

**Proposition 2** *Assume that the function $W$ is independent of time variable, that is, $W(t,q) = W(q)$ and that there exists $a_1, a_2 > 0$ such that $|\nabla_q V| \leq a_1$ and $|\nabla W| \leq a_2$. If*

$$\limsup_{|q|\to\infty} \int_0^T V(t,q)dt < -T^2(a_1 + a_2) - \int_0^T V(t,0)dt, \tag{19}$$

*then* (16) *has at least one solution which is a minimizer of $\mathcal{I}_*$.*

*Proof.* For any $q \in \mathcal{K}_*$, using that $\Psi_*(q) \geq 0$ and

$$\int_0^T q' \cdot W(\bar{q}) = 0,$$

we deduce that

$$\mathcal{I}_*(q) \geq \int_0^T q' \cdot (W(q) - W(\bar{q}))dt + \int_0^T (V(t,\bar{q}) - V(t,q))dt - \int_0^T V(t,\bar{q})dt.$$

Reminding that $q = (q_1, q_2, q_3)$ and $W = (W_1, W_2, W_3)$, by the mean value theorem and the Cauchy-Schwarzt inequality, we obtain

$$\begin{aligned}
\mathcal{I}_*(q) &\geq \int_0^T \sum_{i=1}^3 q_i' \int_0^1 \nabla W_i(\bar{q} + s\tilde{q}) \cdot \tilde{q} ds dt - \int_0^T \int_0^1 \nabla_q V(t, \bar{q} + s\tilde{q}) \cdot \tilde{q} ds dt - \int_0^T V(t,\bar{q})dt \\
&\geq -\int_0^T |q'| \sqrt{\sum_{i=1}^3 \left[\int_0^1 |\nabla W_i(\bar{q} + s\tilde{q})||\tilde{q}| ds\right]^2} dt - \\
&\quad - \int_0^T \int_0^1 \nabla_q V(t, \bar{q} + s\tilde{q}) \cdot \tilde{q} ds dt - \int_0^T V(t,\bar{q})dt
\end{aligned}$$

which, by using $|q'| \leq 1$, (17) and the boundedness of $\nabla_q V$ and $\nabla W$, implies that

$$\mathcal{I}_*(q) \geq -T^2(a_1 + a_2) - \int_0^T V(t,\bar{q})dt, \qquad \forall q \in W_T^{1,\infty}. \tag{20}$$

In consequence, the hypothesis (19) implies that there exists $\rho > 0$ such that

$$\mathcal{I}_*(q) > \mathcal{I}_*(0) = \int_0^T V(t,0)dt, \qquad \forall q \in W_T^{1,\infty} \setminus \mathcal{K}_\rho,$$

and

$$\inf_{\mathcal{K}_\rho} \mathcal{I}_* = \inf_{W_T^{1,\infty}} \mathcal{I}_*.$$

The result follows now from Theorem 7. ■

### 4.3 Solutions via Minimax critical point theory

In this subsection, we will show the existence of a solution of (16) via the non-smooth saddle point theorem of Rabinowitz (see Corollary 2). We begin by establishing the following compactness condition for bounded Palais-Smale sequences of the functional $\mathcal{I}_*$.

**Lemma 6** *If $(q_n)$ is a bounded sequence in $W_T^{1,\infty}$ satisfying that*

$$\lim_{n\to\infty} \mathcal{I}_*(q_n) = c \in \mathbb{R}$$

*and that there is a sequence $(\varepsilon_n)$ of positive numbers converging to zero such that*

$$\Psi_*(\varphi) - \Psi_*(q_n) + \mathcal{F}_*'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\|_{1,\infty}, \quad \forall \varphi \in \mathcal{K}_*, \tag{21}$$

*then there exists a subsequence $(q_{n_k})$ of $(q_n)$ which is converging in $C([0,T],\mathbb{R}^3)$ to a critical point $q \in \mathcal{K}_*$ of $\mathcal{I}_*$ with level $\mathcal{I}_*(q) = c$.*

*Proof.* As in the proof of Theorem 7, repeating the argument of Step 2 of Theorem 3 (using the similar result to Lemma 3), we obtain, up to a subsequence, the convergence in $C([0,T],\mathbb{R}^3)$ of $(q_n)$ to $q$ with

$$q \in \mathcal{K}_*, \quad \Psi_*(q) \leq \liminf_{n\to\infty} \Psi_*(q_n), \quad q_n' \to q' \text{ in w}^*\text{-topology } \sigma(L^\infty, L^1),$$

and

$$\lim_{n\to\infty} \int_0^T V(t,q_n)dt = \int_0^T V(t,q)dt, \ \lim_{n\to\infty} \int_0^T q_n' \cdot W(t,q_n)dt = \int_0^T q' \cdot W(t,q)dt.$$

The rest of the proof follows similarly to the proof of Lemma 5. ■

In the following result we show again that a good connection between the growths of $V$ and $W$ implies existence of solutions corresponding to critical points obtained by min-max methods.

**Theorem 9** *If there exist $\mu > \nu \geq 1$ and $d, r > 0$ such that*

$$V(t,q) \geq d|q|^\mu \text{ and } |W(t,q)| \leq d|q|^\nu,$$

*for all $(t,q) \in \mathbb{R}^3$ with $|q| \geq r$, then (16) has at least one solution, which corresponds to a saddle point of $\mathcal{I}_*$.*

*Proof.* Consider $\widetilde{W}_T^{1,\infty} = \{q \in W_T^{1,\infty} : \bar{q} = 0\}$, which gives the decomposition

$$W_T^{1,\infty} = \mathbb{R}^3 \oplus \widetilde{W}_T^{1,\infty},$$

that is, every $q \in W_T^{1,\infty}$ can be written as

$$q = \bar{q} + \tilde{q} \text{ with } \bar{q} \in \mathbb{R}^3, \tilde{q} \in \widetilde{W}_T^{1,\infty}.$$

Let $\tilde{q} \in \mathcal{K}_* \cap \widetilde{W}_T^{1,\infty}$. Using $\|\tilde{q}'\|_\infty \leq 1$ and (17) one has that

$$\begin{aligned}\mathcal{I}_*(\tilde{q}) &= \Psi_*(\tilde{q}) + \int_0^T \tilde{q}' \cdot W(t,\tilde{q})dt - \int_0^T V(t,\tilde{q})dt \\ &\geq -T\big(\max_{[0,T]\times[-T,T]^3} |W| + \max_{[0,T]\times[-T,T]^3} |V|\big),\end{aligned}$$

which implies that

$$\inf_{\widetilde{W}_T^{1,\infty}} \mathcal{I}_* \geq -T\big(\max_{[0,T]\times[-T,T]^3} |W| + \max_{[0,T]\times[-T,T]^3} |V|\big). \tag{22}$$

On the other hand, if $C > 0$ is such that

$$V(t,q) \geq C|q|^\mu - C \text{ and } |W(t,q)| \leq C|q|^\nu + C, \quad \forall (t,q) \in [0,T] \times \mathbb{R}^3, \tag{23}$$

one has that

$$\mathcal{I}_*(\bar{q}) = -\int_0^T V(t,\bar{q})dt \leq T(-C|\bar{q}|^\mu + C), \qquad \forall \bar{q} \in \mathbb{R}^3;$$

which implies that

$$\mathcal{I}_*(\bar{q}) \to -\infty \text{ as } |\bar{q}| \to \infty,\ \bar{q} \in \mathbb{R}^3.$$

It follows that there exists $\rho > 0$ such that

$$\sup_{\partial B_\rho} \mathcal{I}_* < \inf_{\widetilde{W}_T^{1,\infty}} \mathcal{I}_*.$$

Then, it follows from Corollary 2 (with $\bar{E} = \mathbb{R}^3$ and $\widetilde{E} = \widetilde{W}_T^{1,\infty}$) that there exist $0 < \epsilon_n \to 0$ and a sequence $(q_n) \in \mathcal{K}_*$ such that $\mathcal{I}_*(q_n) \to c$ (with $c$ given by (8)) and (21) holds true.

Using (23) and similar arguments like in the proof of Theorem 8 we deduce that the functions $q_n = \bar{q}_n + \tilde{q}_n$ satisfies for all positive integers $n$ that

$$\mathcal{I}_*(q_n) \leq C(|\bar{q}_n|^\nu - |\bar{q}_n|^\mu + 1),$$

which together with $\mu > \nu \geq 1$ and $\mathcal{I}_*(q_n) \to c$ implies that the sequence $(|\bar{q}_n|)$ is bounded. Hence, by (17), it follows that $\|q_n\|_{1,\infty}$ is bounded.

By Lemma 6, we can assume, up to a subsequence, that $(q_n)$ is converging in $C([0,T],\mathbb{R}^3)$ to a critical point $q \in \mathcal{K}_*$ of $\mathcal{I}_*$ with level $\mathcal{I}_*(q) = c$. Then, using Theorem 6, we deduce that q is a solution of (16). ∎

A different application of the non-smooth saddle point theorem of Rabinowitz (Corollary 2) is the following one.

**Proposition 3** *Assume that the function $W$ is independent of time variable, that is, $W(t,q) = W(q)$ and that there exists $a > 0$ such that $|\nabla_q V| \le a$ and $|\nabla W| \le a$. If*

$$\lim_{|q|\to\infty} \int_0^T V(t,q)dt = +\infty, \tag{24}$$

*then* (16) *has at least one solution which is a saddle point of $\mathcal{I}_*$.*

*Proof.* As in the proof of Theorem 9 we consider again the decomposition and we use the inequality (22). Indeed, since

$$\mathcal{I}_*(\bar{q}) = -\int_0^T V(t,\bar{q})dt \qquad \text{for all } \bar{q} \in \mathbb{R}^3,$$

we deduce from (24) and (22) that

$$\sup_{\partial B_\rho} \mathcal{I}_* < \inf_{\widetilde{W}_T^{1,\infty}} \mathcal{I}_*,$$

for $\rho$ large enough. Then, it follows from Corollary 2 (with $\bar{E} = \mathbb{R}^3$ and $\widetilde{E} = \widetilde{W}_T^{1,\infty}$) that there exist $0 < \epsilon_n \to 0$ and $(q_n) \in \mathcal{K}_*$ such that $\mathcal{I}_*(q_n) \to c$ and

$$\Psi_*(\varphi) - \Psi_*(q_n) + \mathcal{F}'_*(q_n)[\varphi - q_n] \ge -\epsilon_n \|\varphi - q_n\|_{1,\infty}$$

for all positive integer $n$ and for all $\varphi \in \mathcal{K}_*$. Using that

$$\Psi_*(q_n) \le T \qquad (n \ge 1)$$

and a similar argument to that used to prove inequality (20) in the proof of Proposition 2, one has that for all $n \ge 1$,

$$\mathcal{I}_*(q_n) \le T + CT^2 - \int_0^T V(t,\bar{q}_n), dt$$

and we deduce by (24) that the sequence $(|\bar{q}_n|)$ is bounded. It follows by (17) that $\|q_n\|_{1,\infty}$ is bounded. By Lemma 6, we can assume, up to a subsequence, that $(q_n)$ is converging in $C([0,T],\mathbb{R}^3)$ to a critical point $q \in \mathcal{K}_*$ of $\mathcal{I}_*$ with level $\mathcal{I}_*(q) = c$. Theorem 6 implies then that q is a solution of (16). ■

# 5 Relativistic Hamiltonians for periodic problems

## 5.1 Main result

In what follows we consider a function $V : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfying the following hypotheses introduced in [27]:

($V_0$) $V \in C^1(\mathbb{R} \times \mathbb{R}^3, \mathbb{R})$ and $V$ is $2\pi$-periodic in time variable; i.e., $V(t, q) = V(t + 2\pi, q)$ for all $(t, q) \in \mathbb{R} \times \mathbb{R}^3$,

($V_1$) $V(t, q) \geq 0$ for all $(t, q) \in \mathbb{R} \times \mathbb{R}^3$,

($V_2$) there exist constants $\beta > 2$ and $d, \eta > 0$ such that

$$V(t, q) \leq d|q|^\beta,$$

for all $(t, q) \in \mathbb{R} \times \mathbb{R}^3$ with $|q| \leq \eta$,

($V_3$) there exist constants $\mu > 2$ and $r > 0$ such that

$$0 < \mu V(t, q) \leq q \cdot \nabla_q V(t, q),$$

for all $(t, q) \in \mathbb{R} \times \mathbb{R}^3$ with $|q| \geq r$.

On the other hand, we consider a function $W : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfying the following hypotheses

($W_0$) $W \in C^1(\mathbb{R} \times \mathbb{R}^3, \mathbb{R}^3)$, and $W(t, q) = W(t + 2\pi, q)$ for all $(t, q) \in \mathbb{R} \times \mathbb{R}^3$,

($W_1$) there exist constants $\alpha > 1$ and $d, \gamma > 0$ such that

$$|W(t, q)| \leq d|q|^\alpha,$$

for all $(t, q) \in \mathbb{R} \times \mathbb{R}^3$ with $|q| \leq \gamma$.

($W_2$) there exist constants $d, \hat{\gamma} > 0$ and $1 \leq \theta < \min\{\mu, 2\alpha, \beta\}$ such that

$$|W(t, q)| \leq d|q|^\theta,$$

for all $(t, q) \in \mathbb{R} \times \mathbb{R}^3$ with $|q| \geq \hat{\gamma}$.

The relativistic Hamiltonian $\mathcal{H} : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\mathcal{H}(t, p, q) = \sqrt{1 + |p - W(t, q)|^2} - 1 + V(t, q).$$

If $J$ denotes the standard symplectic matrix and $z = (p, q)$, the Hamiltonian system is

$$z' = J\nabla_z \mathcal{H}(t, z),$$

i.e.,

$$\begin{aligned} p' &= \nabla_q W(t, q) \cdot \frac{p - W(t, q)}{\sqrt{1 + |p - W(t, q)|^2}} - \nabla_q V(t, q), \\ q' &= \frac{p - W(t, q)}{\sqrt{1 + |p - W(t, q)|^2}}. \end{aligned} \tag{25}$$

Observe that if $(p, q)$ is a solution of (25), then $q$ satisfies the Lorentz force equation (1) with $\mathcal{E}$ given by (9).

**Theorem 10** *If $V$ satisfies $(V_0)-(V_3)$ and $W$ satisfies $(W_0)-(W_2)$, then the Hamiltonian system* (25) *possesses a nonconstant $2\pi$-periodic solution $(p,q)$, such that $q \in C^2(\mathbb{R},\mathbb{R}^3)$, $|q'(t)| < 1$ for every $t \in \mathbb{R}$ and $q$ verifies the Lorentz force equation* (1).

The idea of the proof is to apply a suitable version proved in [14] for functionals of class $C^1$ in a Hilbert space $E$ of the well known generalized mountain pass theorem of Benci and Rabinowitz [28, Theorem 5.29] to obtain a $2\pi$-periodic solution $z_K = (p_K, q_K)$ of the modified Hamiltonian system $z' = J\nabla_z H_K(t,z)$, where for every $K > 0$, $H_K$ is a suitable truncation of the Hamiltonian function. An a priori estimate of $\|q_K\|_\infty$ will imply that $H_K(t,z_K) = \mathcal{H}(t,z_K)$ and then for large $K$, $z_K$ is a solution of (25). Finally, we will also show that this solution cannot be constant.

## 5.2 A modified Hamiltonian system

Assume that $V$ satisfies $(V_0)-(V_3)$ and let us fix $\nu \in (\max\{2,\theta\}, \min\{\mu, 2\alpha, \beta\})$. For any $K > r + \eta + \gamma$, consider a function $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ such that

- $\chi(t) = 1$ if $t \leq K$,
- $\chi(t) = 0$ if $t \geq K+1$,
- and $\chi'(t) < 0$ if $t \in (K, K+1)$.

If

$$R \geq \max\left\{\frac{V(t,q)}{|q|^\nu} : (t,q) \in \mathbb{R}\times\mathbb{R}^3, K \leq |q| \leq K+1\right\},$$

then we set

$$V_K(t,q) = \chi(|q|)V(t,q) + (1-\chi(|q|))R|q|^\nu, \quad W_K(t,q) = \chi(|q|)W(t,q),$$

for every $(t,q) \in \mathbb{R}\times\mathbb{R}^3$. Define also the associated Hamiltonian function

$$H_K(t,p,q) = \sqrt{1+|p-W_K(t,q)|^2} - 1 + V_K(t,q),$$

for all $(t,p,q) \in \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$. Recalling that $J$ denotes the standard symplectic matrix, we consider the Hamiltonian system

$$z' = J\nabla_z H_K(t,z). \tag{26}$$

Notice that $V_K$ satisfies $(V_0)-(V_2)$ and $(V_3)$ with $\mu$ replaced by $\nu$. This implies that there is a constant $C > 0$ which is independent of $K$ such that

$$V_K(t,q) \geq C(|q|^\nu - 1) \qquad \text{for all } (t,q) \in \mathbb{R}\times\mathbb{R}^3. \tag{27}$$

Moreover, there is a constant $C > 0$, which depends on $K$, such that

$$|V_K(t,q)| \leq C(1+|q|^\nu), \qquad |\nabla_q V_K(t,q)| \leq C(1+|q|^{\nu-1}), \tag{28}$$

for all $(t,q) \in \mathbb{R} \times \mathbb{R}^3$.

On the other hand, the function $W_K$ is $C^1$ and satisfies $(W_1)$ with $\nu/2$ instead of $\alpha$. Moreover, there is a constant $C > 0$ depending on $K$ such that

$$|W_K(t,q)| + |\nabla_q W_K(t,q)| \le C, \qquad ((t,q) \in \mathbb{R} \times \mathbb{R}^3). \tag{29}$$

As a consequence of (28) and (29), there exists $C > 0$ depending on $K$ such that

$$|H_K(t,z)| \le C(1 + |z|^\nu) \text{ for all } (t,z) \in \mathbb{R} \times \mathbb{R}^6. \tag{30}$$

### 5.3 Existence of solution $z_k$ of (26)

In this subsection, following [28, 7, 14] we consider the Sobolev space $H^{1/2}_{2\pi}$ of the functions $u$ in the space $L^2 = L^2((0,2\pi),\mathbb{R}^3)$ whose coefficients $\alpha_0, \alpha_k, \beta_k \in \mathbb{R}^3$ of its associated Fourier series

$$u(t) \approx \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos kt + \beta_k \sin kt),$$

satisfy that

$$\sum_{k=1}^{\infty} k(|\alpha_k|^2 + |\beta_k|^2) < \infty.$$

If the Fourier series of $u \in H^{1/2}_{2\pi}$ is given above and $v \in H^{1/2}_{2\pi}$ has the corresponding Fourier series

$$v(t) \approx \tilde{\alpha}_0 + \sum_{k=1}^{\infty} (\tilde{\alpha}_k \cos kt + \tilde{\beta}_k \sin kt),$$

then

$$(u|v) := 2\pi \alpha_0 \cdot \tilde{\alpha}_0 + \pi \sum_{k=1}^{\infty} k(\alpha_k \cdot \tilde{\alpha}_k + \beta_k \cdot \tilde{b}_k),$$

defines a scalar product and $H^{1/2}_{2\pi}$ is a Hilbert space. It deserves to be mentioned that the Sobolev space $H^1_{2\pi} := W^{1,2}_{2\pi}$ is contained in $H^{1/2}_{2\pi}$ and also that for each $s \in [1,\infty)$, the space $H^{1/2}_{2\pi}$ is compactly embedded in $L^s$; i.e., there is an $\alpha_s > 0$ such that

$$\|u\|_{L^s} \le \alpha_s \|u\|, \qquad \forall u \in H^{1/2}_{2\pi}. \tag{31}$$

In the sequel we will use the Hilbert space $E = H^{1/2}_{2\pi} \times H^{1/2}_{2\pi}$ endowed with the natural scalar product $(\cdot|\cdot)$.

Observe that the associated functional $I_K : E \to \mathbb{R}$ to the Hamiltonian system (26) is given by

$$I_K(z) := A(z) - \int_0^{2\pi} H_K(t,z)dt,$$

where $A : E \to \mathbb{R}$ is given for every $z(t) \approx a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$ in $E$, by

$$A(z) := \pi \sum_{k=1}^{\infty} kJa_k \cdot b_k.$$

(Notice that

$$A(z) = \int_0^{2\pi} pq' dt, \quad \forall z = (p, q) \in H^1_{2\pi} \times H^1_{2\pi}.)$$

Indeed, if $B : E \times E \to \mathbb{R}$ is the symmetric continuous bilinear form defined for every $z(t) \approx a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$ and $w(t) \approx \tilde{a}_0 + \sum_{k=1}^{\infty}(\tilde{a}_k \cos kt + \tilde{b}_k \sin kt)$ in $E$ by

$$B(z, w) := \pi \sum_{k=1}^{\infty} k(Ja_k \cdot \tilde{b}_k - Jb_k \cdot \tilde{a}_k)$$

then $A$ is its associated quadratic form ($A(z) = \frac{1}{2}B(z, z)$ for $z \in E$) and $A \in C^1(E, \mathbb{R})$ with

$$A'(z)[w] = B(z, w), \qquad \forall z, w \in E.$$

On the other hand, the growth condition (30) implies that $I_K \in C^1(E, \mathbb{R})$ with

$$I'_K(z)[w] = B(z, w) - \int_0^{2\pi} \nabla_z H_K(t, z) \cdot w \, dt, \qquad \forall z, w \in E.$$

that is, if $z = (p, q)$ and $w = (\varphi, \psi)$ in $E$,

$$\begin{aligned} I'_K(z)[w] =& B(z, w) - \int_0^{2\pi} \frac{p - W_K(t, q)}{\sqrt{1 + |p - W_K(t, q)|^2}} \cdot \varphi dt \\ &+ \int_0^{2\pi} \left( \nabla_q W_K(t, q) \cdot \frac{p - W_K(t, q)}{\sqrt{1 + |p - W_K(t, q)|^2}} - \nabla_q V_K(t, q) \right) \cdot \psi dt. \end{aligned}$$

Hence $z \in C^1(\mathbb{R}, \mathbb{R}^6)$ is a $2\pi$-periodic solution of the modified Hamiltonian system (26) if and only if $z \in E$ is a critical point of the $C^1$-functional $I_K$.

As it has been previously mentioned, the existence of a critical point of $I_K$ will be deduced by applying to this functional the following theorem.

**Theorem 11** ([14, Theorem 3.1]) *Assume that the Hilbert space $E$ has a splitting $E = \bar{E} \oplus \tilde{E}$ and let $\mathcal{I} : E \to \mathbb{R}$ a functional given by*

$$\mathcal{I}(z) = \langle Lz, z \rangle + b(z),$$

*where*

($\mathcal{I}_1$) *$L : E \to E$ is a linear, bounded, selfadjoint operator,*

($\mathcal{I}_2$) *$b : E \to \mathbb{R}$ is of class $C^1$ in $E$ with compact derivative $b'$,*

($\mathcal{I}_3$) *there exist two linear bounded invertible operators $B_1, B_2 : E \to E$ such that if $P : E \to E$ is the projection of $E$ onto $\bar{E}$, for every $\tau \geq 0$ the linear operator*

$$\hat{B}(\tau) := PB_1^{-1}\, exp\,(\tau L) B_2 : \bar{E} \to \bar{E}$$

*is invertible.*

*Assume also that*

(PS) *any sequence $(z_m) \subset E$ for which $(\mathcal{I}(z_m))$ is bounded and $\mathcal{I}'(z_m) \to 0$ as $m \to \infty$, possesses a convergent subsequence.*

*Let $\rho > 0$ and $e \in \tilde{E}$ with $\|e\| = 1$ be fixed and for $r_1 \geq \rho/\|B_1^{-1}B_2 e\|$, $r_2 > \rho$, we define*

$$S = \{B_1 z : \|z\| = \rho,\ z \in \tilde{E}\},$$

$$Q = B_2(\{re + z : 0 \leq r \leq r_1,\ z \in \bar{E}, \|z\| \leq r_2\}),$$

*and denotes by $\partial Q$ the boundary of $Q$ relative to the subspace $B_2(span\,\{e\} \oplus \bar{E})$. Let also $\Gamma$ be the family of all functions $\gamma \in C(E \times [0,1], E)$ satisfying*

($\Gamma_1$) *there exist $\tau \in C(E \times [0,1], [0,\infty))$ which maps bounded sets into bounded sets, and a compact operator $T : E \times [0,1] \to E$ such that*

$$\gamma(z,t) = \ exp\,(\tau(z,t)L)x + T(z,t), \quad \forall z \in E,\ \forall t \in [0,1],$$

($\Gamma_2$) *$\gamma(z,t) = 0$, for every $z \in \partial Q$,*

($\Gamma_3$) *$\gamma(z,0) = z$, for every $z \in Q$.*

*If there is a constant $\delta > 0$ such that*

*(i) $\mathcal{I}(z) \geq \delta$ for all $z \in S$,*

*(ii) $\mathcal{I}(z) \leq 0$ for all $z \in Q$,*

*then there exists $z \in E$ such that*

$$\mathcal{I}(z) = c := \inf_{\gamma \in \Gamma} \sup_{z \in Q} \mathcal{I}(\gamma(z)) \text{ and } \mathcal{I}'(z) = 0.$$

■

In order to apply the above theorem to the functional $I_K$ we need to split the space $E$. We consider the orthogonal decomposition of $E$ given by

$$E = E^- \oplus E^0 \oplus E^+, \quad z = z^- + z^0 + z^+, \quad \forall z \in E,$$

where, if $a_k, b_k \in \mathbb{R}^6$ are the Fourier coefficients of $z \in E$,

$$E^0 = \{z^0 = a_0 : a_0 \in \mathbb{R}^6\},$$

and

$$E^{\pm} = \{z^{\pm} = \sum_{k=1}^{\infty}(a_k \cos kt \pm Ja_k \sin kt) \,:\, a_1, a_2, \ldots \in \mathbb{R}^6\}.$$

Taking into account that if $a_k, b_k \in \mathbb{R}^6$ (respectively, $\tilde{a}_k, \tilde{b}_k \in \mathbb{R}^6$) are the Fourier coefficients of $z \in E$ (respectively, $w \in E$), then the bilinear form $B : E \times E \to \mathbb{R}$ satisfies

$$B(z,w) = \pi \sum_{k=1}^{\infty} k(Ja_k \cdot \tilde{b}_k - Jb_k \cdot \tilde{a}_k), \quad \forall z, w \in E,$$

we deduce for every $z = z^- + z^0 + z^+ \in E$ that

$$A(z) = \frac{1}{2}B(z,z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2).$$

Moreover, if $z = (p,q) \in E^{\pm}$, then for some $a_k = (a_k^1, a_k^2) \in \mathbb{R}^3 \times \mathbb{R}^3$ $(k \geq 1)$, we have

$$z = \sum_{k=1}^{\infty}(a_k \cos kt \pm Ja_k \sin kt),$$

and

$$p = \sum_{k=1}^{\infty}(a_k^1 \cos kt \mp a_k^2 \sin kt), \qquad q = \sum_{k=1}^{\infty}(a_k^2 \cos kt \pm a_k^1 \sin kt).$$

Hence,

$$\|p\| = \|q\| = \pi \sum_{k=1}^{\infty} k|a_k|^2, \qquad \|z\|^2 = 2\|p\|^2. \tag{32}$$

In order to apply Theorem 11, take

$$\bar{E} = E^- \oplus E^0, \quad \tilde{E} = E^+, \quad \mathcal{I} = I_K, \quad b(z) = \int_0^{2\pi} H_K(t,z)dt$$

and $L : E \to E$ the bounded, symmetric operator given by $B(z,w) = (Lz|w)$ for any $z, w \in E$. Choosing $u, v > 1$ such that

$$\frac{1}{2} < \frac{u}{u+v}, \quad \frac{1}{\nu} < \frac{v}{u+v}, \tag{33}$$

(since $1/2 + 1/\nu < 1$) we consider the linear, bounded, invertible operators $B_1, B_2 : E \to E$ given by

$$B_1(p,q) = (\rho^{u-1}p, \rho^{v-1}q), \quad \forall (p,q) \in E, \quad B_2 = \text{Identity}.$$

Let $P : E \to E$ be the projection of $E$ onto $E^- \oplus E^0$ and for $\tau \geq 0$ consider

$$B(\tau) = PB_1^{-1}\exp(\tau L) : E^- \oplus E^0 \to E^- \oplus E^0.$$

Observe that for every $z \in E^- \oplus E^0$ with $z = (p^-, q^-) + (p^0, q^0)$ one has that

$$B(\tau)z = (\rho^{1-u} + \rho^{v-1}) \exp(-\tau)(p^-, q^-) + (\rho^{1-u}p^0, \rho^{1-v}q^0),$$

which implies that $B(\tau)$ is invertible for any $\tau \geq 0$. Consequently, the hypotheses $(\mathcal{I}_1) - (\mathcal{I}_3)$ hold true in this case. In addition, (PS) is satisfied.

**Lemma 7** *$I_K$ satisfies $(PS)$ condition.*

*Proof.* Consider $(z_m) \subset E$ such that $(I_K(z_m))$ is bounded and $I'_K(z_m) \to 0$ as $m \to \infty$. We write $z_m$ as $z_m = (p_m, q_m)$ for simplicity. Using that

$$B((p_m, q_m), (p_m, 0)) = A(z_m) = B((p_m, q_m), (0, q_m)),$$

it follows that

$$\begin{aligned}
I'_K(z_m)[(p_m, 0)] - I_K(z_m) &= \int_0^{2\pi} (\sqrt{1 + |p_m - W_K(t, q_m)|^2} - 1 + V_K(t, q_m))dt \\
&- \int_0^{2\pi} \frac{p_m - W_K(t, q_m)}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} \cdot (p_m - W_K(t, q_m) + W_K(t, q_m))dt \\
&= \int_0^{2\pi} \frac{1}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} dt + \int_0^{2\pi} V_K(t, q)dt - 2\pi \\
&- \int_0^{2\pi} \frac{p_m - W_K(t, q_m)}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} \cdot W_K(t, q_m)dt.
\end{aligned}$$

This together with (27) and (29) imply that

$$I'_K(z_m)[(p_m, 0)] - I_K(z_m) \geq C(\|q_m\|_\nu^\nu - 1). \tag{34}$$

On the other hand, one has that

$$\begin{aligned}
I'_K(z_m)[(0, q_m)] - I_K(z_m) &= \int_0^{2\pi} (\sqrt{1 + |p_m - W_K(t, q_m)|^2} - 1 + V_K(t, q_m))dt \\
+ \int_0^{2\pi} &\left( \nabla_q W_K(t, q_m) \cdot \frac{p - W_K(t, q_m)}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} - \nabla_q V_K(t, q_m) \right) \cdot q_m dt
\end{aligned}$$

This together with (28), (29) and the inequality $\sqrt{1 + s^2} - 1 \geq s - 2$ for every $s \geq 0$ imply that

$$\|p_m\|_1 \leq I'_K(z_m)[(0, q_m)] - I_K(z_m) + C(\|q_m\|_\nu^\nu + 1).$$

Using this inequality and (34), we deduce that for $m$ large enough,

$$\|p_m\|_1 + \|q_m\|_\nu^\nu \leq \frac{1}{2}\|z_m\| + C. \tag{35}$$

Next, writing $z_m = z_m^- + z_m^0 + z_m^+ = (p_m^-, q_m^-) + (p_m^0, q_m^0) + (p_m^+, q_m^+)$, one has that

$$I_K'(z_m)[z_m^+] = \|z_m^+\|^2 - \int_0^{2\pi} \frac{p_m - W_K(t, q_m)}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} \cdot p_m^+ dt$$
$$+ \int_0^{2\pi} \left( \nabla_q W_K(t, q_m) \cdot \frac{p_m - W_K(t, q_m)}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} - \nabla_q V_K(t, q_m) \right) \cdot q_m^+ dt.$$

Using (31) it follows that

$$\left| \int_0^{2\pi} \frac{p_m - W_K(t, q_m)}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} \cdot p_m^+ dt \right| \le \|p_m^+\|_1 \le C\|p_m^+\|,$$

and

$$\left| \int_0^{2\pi} \left( \nabla_q W_K(t, q_m) \cdot \frac{p_m - W_K(t, q_m)}{\sqrt{1 + |p_m - W_K(t, q_m)|^2}} \right) \cdot q_m^+ dt \right| \le C\|q_m^+\|_1 \le C\|q_m^+\|.$$

Using again (31), (28) and Holder inequality, we deduce that

$$\begin{aligned} \left| \int_0^{2\pi} \nabla_q V_K(t, q_m) \cdot q_m^+ \right| dt &\le \int_0^{2\pi} |\nabla_q V_K(t, q_m)||q_m^+| dt \\ &\le C\|q_m^+\|_1 + C\||q_m|^{\nu-1}|q_m^+|\|_1 \\ &\le C\|q_m^+\|_1 + C\|q_m\|_\nu^{\nu-1}\|q_m^+\|_\nu \\ &\le C\|q_m^+\| + C\|q_m\|_\nu^{\nu-1}\|q_m^+\|. \end{aligned}$$

Then, using that $I_K'(z_m) \to 0$ as $m \to \infty$, it follows that

$$|I_K'(z_m)[z_m^+]| \le \|z_m^+\|$$

for large $m$, and then

$$\|z_m^+\|^2 \le C(\|z_m^+\| + \|p_m^+\| + \|q_m^+\| + \|q_m\|_\nu^{\nu-1}\|q_m^+\|) \le C\|z_m^+\|(1 + \|q_m\|_\nu^{\nu-1}),$$

which implies that

$$\|z_m^+\| \le C(1 + \|q_m\|_\nu^{\nu-1}). \tag{36}$$

Similarly,

$$\|z_m^-\| \le C(1 + \|q_m\|_\nu^{\nu-1}). \tag{37}$$

Next, taking projection onto the space of constants and using (34), it follows that

$$|q_m^0|^\nu \le C\|q_m\|_\nu^\nu \le C(1 + \|p_m\|),$$

that is

$$|q_m^0| \leq C(1 + \|p_m\|^{1/\nu}). \tag{38}$$

On the other hand, using (31), (36) and (37) it follows that

$$\begin{aligned} 2\pi|p_m^0| &\leq \|p_m\|_1 + \|p_m^+\|_1 + \|p_m^-\|_1 \\ &\leq \|p_m\|_1 + C(\|z_m^+\| + \|z_m^-\|), \end{aligned}$$

and

$$|p_m^0| \leq \|p_m\|_1 + C(1 + \|q_m\|_\nu^{\nu-1}). \tag{39}$$

Finally, using (35), (36), (37), (38) and (39) it follows that

$$\|z_m\| \leq C(\|z_m\|^{1/\nu} + \|z_m\|^{\frac{\nu-1}{\nu}} + 1) + \frac{1}{2}\|z_m\|,$$

from where

$$\|z_m\| \leq C(\|z_m\|^{1/\nu} + \|z_m\|^{\frac{\nu-1}{\nu}} + 1),$$

and then $(z_m)$ is bounded in $E$. This implies, using a standard argument, that $(z_m)$ has a convergent subsequence. ■

In order to show that hypotheses *(i)* and *(ii)* of Theorem 11 hold true, define for $\rho > 0$

$$S = \{B_1 z : \|z\| = \rho,\ z \in E^+\},$$

and for fixed $e \in E^+$ with $\|e\| = 1$, $r_1 \geq \rho/\|B_1^{-1}e\|$ and $r_2 > \rho$, we also define

$$Q = \{re : 0 \leq r \leq r_1\} \oplus \{z \in E^- \oplus E^0\ :\ \|z\| \leq r_2\},$$

and consider $\partial Q$ as the boundary of $Q$ relative to the subspace

$$\hat{E} = \text{span}\ \{e\} \oplus E^- \oplus E^0.$$

**Lemma 8** *There exist $0 < \rho < 1$ and $\delta > 0$, which may depend on $K$, such that $I_K(z) \geq \delta$ for all $z \in S$.*

*Proof.* Let $(p, q) \in E^+$ and take $z = (\rho^{u-1}p, \rho^{v-1}q)$ for some $\rho > 0$ that will be chosen later. One has that

$$A(z) = \rho^{u+v-2}A(p, q) = \frac{1}{2}\rho^{u+v-2}\|(p, q)\|^2,$$

and by (32)

$$\|(p, q)\| = \sqrt{2}\|p\| = \sqrt{2}\|q\|.$$

Using $(V_2)$ and $(W_1)$ it follows that there exists $C > 0$ which may depend on $K$ such that

$$|W_K(t, q)|^2 + V_K(t, q) \leq C|q|^\nu \text{ for all } (t, q) \in \mathbb{R} \times \mathbb{R}^3.$$

This inequality together with

$$\sqrt{1+s^2}-1 \le \frac{1}{2}s^2 \text{ for all } s \in \mathbb{R},$$

implies that there exists $C > 0$ depending on $K$ such that

$$H_K(t,p,q) \le C(|p|^2 + |q|^\nu) \text{ for all } (t,p,q) \in \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3.$$

Then, using this inequality we can follow the arguments in Lemma 2.3 from [14]. Indeed, the cited inequality together with (31) implies

$$\begin{aligned} I_K(z) &\ge \frac{1}{2}\rho^{u+v-2}\|(p,q)\|^2 - C(\rho^{2(u-1)}\|p\|_2^2 + \rho^{\nu(v-1)}\|q\|_\nu^\nu) \\ &\ge \frac{1}{2}\rho^{u+v-2}\|(p,q)\|^2 - C(\rho^{2(u-1)}\|p\|^2 + \rho^{\nu(v-1)}\|q\|^\nu). \end{aligned}$$

Hence, if we consider $\|(p,q)\| = \rho$, then by (33) one has that

$$I_K(z) \quad \ge \quad \frac{1}{2}\rho^{u+v} - C(\rho^{2u} + \rho^{\nu v}) \ge \delta$$

for some $0 < \rho < 1$ and $\delta > 0$. ■

**Lemma 9** *If $z + re \in \hat{E}$, then $I_K(z+re) \to -\infty$, as $\|z+re\| \to \infty$.*

*Proof.* Given $z + re \in \hat{E}$, reminding that $\|e\| = 1$, one has that

$$\|z+re\|^2 = r^2 + \|z^-\|^2 + 2\pi|z^0|^2.$$

It follows that if $\|z+re\| \to \infty$, then either one has that $|r| \to \infty$ or one has that $|r| \le M$ for some $M$ and $\|z^-\| + |z^0| \to \infty$.

In the case that $|r| \to \infty$, if $z+re = (p,q)$, using that $\sqrt{1+|p-W_K(t,q)|^2} - 1 \ge |p - W_K(t,q)| - 2$ and (29) it follows that

$$\sqrt{1+|p-W_K(t,q)|^2} - 1 \ge |p| - c_4 - 2.$$

This together with (27) imply that there exists $c > 0$, which may depend on $K$, such that

$$I_K(z+re) \le \frac{1}{2}(r^2 - \|z^-\|^2) - C(\|p\|_1 + \|q\|_\nu^\nu + 1) \le \frac{1}{2}r^2 - C\|q\|_\nu^\nu,$$

Observing now that there exists $c_\nu > 0$ such that

$$\|q^+\|_\nu \le \|q^+\|_\nu + \|q^0\|_\nu + \|q^- + q^0\|_\nu \le c_\nu\|q\|_\nu, \ (\forall q \in H^{1/2}_{2\pi}),$$

we deduce that there exists $d_\nu > 0$ such that if we denote $e = (e_1, e_2)$, then

$$\|q\|_\nu^\nu \ge d_\nu |r|^\nu \|e_2\|_\nu^\nu, \tag{40}$$

and hence

$$I_K(z+re) \le \frac{1}{2}r^2 - Cd_\nu \|e_2\|_\nu^\nu |r|^\nu$$

and, using that $\nu > 2$ implies that $\frac{1}{2}r^2 - d_\nu \|e_2\|_\nu^\nu |r|^\nu \to -\infty$ as $|r| \to \infty$ we get the conclusion of the lemma in this case.

Assume now that $|r| \le M$ for some $M$ and $\|z^-\| + |z^0| \to \infty$. Then, using that an analogous inequality to (40) holds true for $p$ instead of $q$, it follows that

$$|p^0| \le C\|p\|_1, \qquad |q^0| \le C\|q\|_\nu,$$

which implies that $I_K(z+re) \to -\infty$ also in the second case. ■

**Lemma 10** *For any $K$ there exists $r_K > 0$ such that if $r_1, r_2 \ge r_K$ then $I_K(z) \le 0$ for all $z \in \partial Q$.*

*Proof.* It follows from $(V_1)$ that

$$I_K(z) \le 0 \text{ for all } z \in E^- \oplus E^0.$$

The proof is then concluded by Lemma 9. ■

By Lemmas 7, 8 and 10 it follows that $I_K$ satisfies all hypotheses of Theorem 11. Therefore, we conclude the following existence result:

**Corollary 3** *The modified Hamiltonian system* (26) *possesses a $2\pi$-periodic solution $z_K = (p_K, q_K) \in E$ such that $I_K(z_K) > 0$.*

## 5.4 Proof of Theorem 10

Consider the $2\pi$-periodic solution of the modified Hamiltonian system (26) given by Corollary 3, i.e. $z_K = (p_K, q_K) \in E$ such that $I_K(z_K) > 0$ and

$$\begin{aligned} p_K' =& \nabla_q W_K(t, q_K) \cdot \frac{p_K - W_K(t, q_K)}{\sqrt{1 + |p_K - W_K(t, q_K)|^2}} - \nabla_q V_K(t, q_K), \\ q_K' =& \frac{p_K - W_K(t, q_K)}{\sqrt{1 + |p_K - W_K(t, q_K)|^2}}. \end{aligned}$$

We claim that there exists a positive constant $C$ independent of $K$ such that $\|q_K\|_\infty \le C$ for every $K$. Indeed, by (27) and (29) and a similar argument to the one used in (34) we obtain

$$
\begin{aligned}
I_K'(z_K)[(p_K,0)] - I_K(z_K) =& \int_0^{2\pi} \frac{1}{\sqrt{1+|p_K - W_K(t,q_K)|^2}} dt \\
&+ \int_0^{2\pi} V_K(t,q_K)dt - 2\pi \\
&- \int_0^{2\pi} \frac{p_K - W_K(t,q_K)}{\sqrt{1+|p_K - W_K(t,q_K)|^2}} \cdot W_K(t,q_K)dt \\
\geq& C\|q\|_\nu^\nu - 2\pi(1+C) \\
&- \int_0^{2\pi} \frac{p_K - W_K(t,q_K)}{\sqrt{1+|p_K - W_K(t,q_K)|^2}} \cdot W_K(t,q_K)dt.
\end{aligned}
$$

From the definition of $W_K$ and $(W_2)$ it follows that there exists $C>0$ *independent* of $K$ such that

$$|W_K(t,q)| \leq C(|q|^\theta + 1),$$

for all $(t,q) \in \mathbb{R} \times \mathbb{R}^3$. This implies that

$$\left| \int_0^{2\pi} \frac{p_K - W_K(t,q_K)}{\sqrt{1+|p_K - W_K(t,q_K)|^2}} \cdot W_K(t,q_K)dt \right| \leq C\|q_K\|_\theta^\theta + 2\pi C.$$

Then, it follows that

$$0 = I_K'(z_K)[(p_K,0)] \geq C\|q_K\|_\nu^\nu - 2\pi(1+C) + I_K(z_K) - C\|q_K\|_\theta^\theta + 2\pi C,$$

which together with $I_K(z_K) > 0$ imply that there exists $C>0$ *independent* of $K$ such that (here it is important to note that $\theta < \nu$),

$$\|q_K\|_\nu^\nu \leq C(1 + \|q_K\|_\theta^\theta) \leq C(1 + \|q_K\|_\nu^\theta),$$

and hence there exists $C>0$ *independent* of $K$ such that

$$\|q_K\|_\nu \leq C. \tag{41}$$

Now, for every $i = 1,2,3$, choose $t_i \in \mathbb{R}$ such that the i-th component $q_K^i$ of $q_K$ satisfies that

$$q_K^i(t_i) = \frac{1}{2\pi} \int_0^{2\pi} q_K^i dt.$$

Using then that

$$q_K^i(\tau) = \int_{t_i}^\tau (q_K^i)'dt + q_K^i(t_i) = \int_{t_i}^\tau (q_K^i)'dt + \frac{1}{2\pi}\int_0^{2\pi} q_K^i dt,$$

together with (41) and $\|(q_K^i)'\|_\infty < 1$ we obtain the existence of a positive constant $C$ *independent* of $K$ such that

$$|q_K^i(\tau)| = \left| \int_{t_i}^\tau (q_K^i)'dt + q_K^i(t_i) \right| \leq C, \quad \forall \tau \in \mathbb{R}.$$

As a consequence, there exists a positive constant $C$ *independent* of $K$ such that

$$\|q_K\|_\infty \leq C,$$

proving the claim. By this, for every $K \geq C$, we have

$$V_K(t, q_K(t)) = V(t, q_K(t)), \quad W_K(t, q_K(t)) = W(t, q_K(t)), \quad \forall t \in \mathbb{R}$$

and thus, $z_K$ is a solution of (25).

Moreover, if $z_K$ were constant, then using $(V_1)$ it would be deduced that

$$I_K(z_k) = -\int_0^{2\pi} H_K(z_K)dt \leq 0,$$

contradicting that $I_K(z_K) > 0$. Therefore, $z_K$ is a $2\pi$-periodic nonconstant solution of (25) and the proof of Theorem 10 is complete. ■

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