Some analytical results about periodic orbits in the restricted three body problem with dissipation

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Abstract

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We present some analytical results about the existence of periodic orbits for the planar restricted three body problem with dissipation considered recently by Celletti et al. We show that, under fairly general conditions on the dissipation term, the circular orbits cannot be continued to the dissipative framework. Moreover, we give general conditions for the occurrence or not of an Hopf bifurcation around the libration points L_4 and L_5. Our results are consistent with the numerical findings of Celletti et al.

Keywords: Restricted Problems, Dissipative Forces, Periodic Orbits, Fredholm Alternative, Hopf Bifurcation.

1 Introduction

The dynamics of circular planar restricted three body problem in a synodic frame of reference is determined by the differential equation

$$\ddot{z} + 2i\dot{z} - z = -(1-\mu)\frac{\mu + z}{|\mu + z|^3} + \mu\frac{1-\mu-z}{|1-\mu-z|^3} + \varepsilon\mathcal{D}(z, \dot{z}, \mu), \qquad (1)$$

where $\mu \in [0, 1/2]$ and $z = x + iy \in \mathbb{C} \setminus \{-\mu, 1-\mu\}$ and $\dot{z} = \dot{x} + i\dot{y}$. The parameter $\varepsilon \geq 0$ and the function $\mathcal{D}$ are derived from dissipative forces acting on the small body. The classical setting corresponds to $\varepsilon = 0$, $\mathcal{D} = 0$. Recently, the dissipative case has been considered in [2]. In that paper several choices for $\mathcal{D}$ are discussed,

- $\mathcal{D}(z, \dot{z}, \mu) = -\left(iz + \dot{z} + \alpha\frac{iz}{|z|^3}\right)$, $\quad \alpha \in [0, 1[$ (Stoke's drag);
- $\mathcal{D}(z, \dot{z}, \mu) = -\frac{1}{r_1^2}(iz + \dot{z})$ (Poynting - Robertson (PR) force)

where $r_1 := |\mu + z|$.

The particular case of the Stoke's drag which corresponds to $\alpha = 0$ is called the linear drag, and in this case

$$\mathcal{D}(z, \dot{z}, \mu) = -(iz + \dot{z}). \qquad (2)$$

It is interesting to notice that the incorporation of dissipative terms goes back to the origins of Celestial Mechanics. Already Jacobi considered Kepler's problem with resistance in his book in Mechanics [4]. He introduced dissipative forces of the type $-f|\dot{Z}|^{n-1}\dot{Z}$, where $Z = e^{it}z$ is the coordinate of the small body in the inertial frame. The linear drag corresponds to $n = 1$.

The study in [2] is based on numerical computations and covers many aspects of the dynamics of (1). We will concentrate on the existence of closed orbits using analytical tools. The simplest closed orbits of (1) for $\varepsilon = 0$ are obtained by local continuation in μ of the circular orbits $z(t) = \rho e^{i\omega t}$, $\mu = 0$ and $\rho^3(\omega + 1)^2 = 1$. Indeed some additional restrictions on the parameter ω are required and we refer to [6] for more details. A first attempt to produce closed orbits in the dissipative case could be to try to continue the circular orbits in the parameters μ and ε simultaneously. We will prove that such a continuation is not possible, at least if ε is not too small compared to μ. This seems to be consistent with the numerical findings of [2], as all the closed orbits drawn in that paper are far from being circular. The libration points $L_i = L_i(\mu)$ are the equilibria of system (1) for $\varepsilon = 0$ and it is well known that L_4 and L_5 are elliptic if $\mu < \mu^* = 0.03852...$ and hyperbolic if $\mu > \mu^*$, see [5]. This exchange of stability suggests an alternative method for the search of closed orbits. First we continue the libration point $L_4 = L_4(\mu, \varepsilon)$ to the dissipative case, then by analogy with the conservative case we expect the existence of $\mu^* = \mu^*(\varepsilon)$ such that $L_4(\mu, \varepsilon)$ is linearly asymptotically stable if $\mu < \mu^*(\varepsilon)$ and unstable (hyperbolic) for $\mu > \mu^*(\varepsilon)$. In such case we can look for a Hopf bifurcation and closed orbits when $\mu - \mu^*(\varepsilon)$ is positive and small. In this paper we will show that this program works for the linear drag and for the Stoke's drag with small positive values of α. In contrast, for the PR drag we will prove that the perturbed L_4 and L_5 are hyperbolic unstable equilibria for all the values of μ, a fact that rules out the possibility of a Hopf bifurcation around these points. These results about the occurrence or not of a Hopf bifurcation will follow from two general theorems, which are stated for general dissipative forces. Incidentally we notice that no closed orbits for the PR drag are found in [2]. Many questions on the existence or non-existence of closed orbits for the dissipative problem (1) seem to be open and interesting.

2 Non continuation of the circular orbits

We consider the equation (1) when ε depends on μ. In this section $\varepsilon = \varepsilon(\mu)$ is a function in $C^1[0, \delta]$, $\delta > 0$, satisfying

$$\varepsilon(0) = 0, \quad \varepsilon(\mu) > 0 \text{ if } \mu \in]0, \delta].$$

For $\mu = 0$ we obtain Kepler's problem in a synodic frame. This problem has the circular solutions $z(t) = \rho e^{i\omega t}$ where ω is a real parameter, $\omega \neq -1, 0$ and $\rho^3(\omega + 1)^2 = 1$. Let us select one of these circular orbits so that ρ and ω become fixed numbers. The dissipative term $\mathcal{D} : \Omega \subset \mathbb{C}^2 \times [0, \frac{1}{2}] \to \mathbb{C}$ will be a function defined on an open set Ω which is open relative to $\mathbb{C}^2 \times [0, \frac{1}{2}]$ and contains the circular orbit. More precisely,

$$\{(z, w, \mu) \in \mathbb{C}^2 \times [0, \frac{1}{2}] : \ |z| = \rho, \ w = i\omega z, \ \mu = 0 \} \subset \Omega.$$

The notation $\mathcal{D} = \mathcal{D}(z, \dot{z}, \mu)$ can be misleading because in many cases $\mathcal{D}$ will not be holomorphic in z, $\dot{z}$ and μ. Notice that for Stoke's drag there appear a term of the type $|z|^3$, while for the P-R force there is a term $|\mu + z|^2$. Typically the function $\mathcal{D}$ will be real analytic but for our purposes it will be sufficient to understand $\mathcal{D}$ as a function of real variables defined in $\Omega \subset \mathbb{R}^2 \times \mathbb{R}^2 \times [0, \frac{1}{2}]$ and belonging to $C^1(\Omega, \mathbb{R}^2)$.

We say that the circular orbit admits a smooth continuation if there exists a function $z : \mathbb{R} \times [0, \delta_1] \to \mathbb{C}$, $(t, \mu) \mapsto z(t, \mu)$, where $0 < \delta_1 \leq \delta$, satisfying

(i) For each $\mu \in [0, \delta_1]$, $z(\cdot, \mu)$ is a solution of (1) with $\varepsilon = \varepsilon(\mu)$.

(ii) There exists a function $T = T(\mu)$ in $C^1[0, \delta_1]$ with $T(0) = \frac{2\pi}{\omega}$ and such that

$$z(t + T(\mu), \mu) = z(t, \mu).$$

(iii) $z(t, 0) = \rho e^{i\omega t}$.

(iv) The function $(t, \mu) \in \mathbb{R} \times [0, \delta_1] \mapsto (z(t, \mu), \dot{z}(t, \mu)) \in \mathbb{C} \times \mathbb{C}$ is C^1.

We will prove the following result:

Theorem 2.1 *In the previous conditions assume that $\varepsilon'(0) > 0$ and*

$$\mathrm{Im}[\int_0^{\frac{2\pi}{\omega}} d(t) e^{-i\omega t} dt] \neq 0,$$

where $d(t) = \mathcal{D}(\rho e^{i\omega t}, i\omega\rho e^{i\omega t}, 0)$. Then the circular orbit does not admit a smooth continuation.

Remark. The condition $\varepsilon'(0) > 0$ is essential in the previous theorem. For $\varepsilon \equiv 0$ the circular solution admits a smooth continuation if $\omega \neq -2, 0$ and $\omega \neq \frac{1}{k}$, $k \in \mathbb{Z} \setminus \{0\}$. We refer to [6] for more details.

Proof. We will argue by contradiction. Assume that $z(t, \mu)$ is a smooth continuation of $\rho e^{i\omega t}$ and denote by $\eta(t)$ the derivative with respect to the parameter at $\mu = 0$. That is,

$$\eta(t) = \frac{\partial z}{\partial \mu}(t, 0).$$

From the differential equation (1) and the condition (**iv**) we deduce that $\frac{\partial \ddot{z}}{\partial \mu}(t,\mu)$ exists and it is continuous. Hence the derivatives in t and μ commute. Differentiating (1) with respect to μ and evaluating the result at $\mu = 0$ we are lead to the linear differential equation

$$L[\eta] = b(t),$$

where L is the differential operator

$$L[\xi] = \ddot{\xi} + 2i\dot{\xi} - (1 + \frac{1}{2\rho^3})\xi - \frac{3}{2\rho^3}e^{2i\omega t}\overline{\xi}$$

and

$$b(t) = \frac{1}{\rho^2}e^{i\omega t} + \frac{1-\rho e^{i\omega t}}{|1-\rho e^{i\omega t}|^3} + \frac{1}{2\rho^3} + \frac{3}{2\rho^3}e^{2i\omega t} + \varepsilon'(0)d(t).$$

The function $b(t)$ is periodic with period $\frac{2\pi}{\omega}$. Also $\eta(t)$ has this property but this is not so obvious. In general the derivative with respect to parameters of a periodic function is not periodic. This is easily illustrated by the function $F(t,\mu) = \sin(\mu t)$. With some work we will prove that η is $\frac{2\pi}{\omega}$-periodic. Let us differentiate with respect to μ the identity appearing in (**ii**). After evaluating the result at $\mu = 0$ we obtain

$$i\omega\rho e^{i\omega t}T'(0) + \eta(t + \frac{2\pi}{\omega}) = \eta(t).$$

We will prove that $T'(0) = 0$. To this end we consider the function

$$\varphi(t) = \eta(t) + \beta t e^{i\omega t}$$

with $\beta = i\omega^2\rho T'(0)/2\pi$. The number β has been adjusted so that φ is $\frac{2\pi}{\omega}$-periodic. After some computations we find that

$$L[te^{i\omega t}] = -\frac{3}{\rho^3}te^{i\omega t} + 2i(\omega+1)e^{i\omega t}.$$

From the identity $\beta L[te^{i\omega t}] = -b(t) + L[\varphi]$ we deduce that β vanishes, for otherwise an unbounded function should coincide with a continuous periodic function. We conclude that $T'(0) = 0$.

If we follow the steps used in [6] and set $\sigma = e^{-i\omega t}\eta$ and $\nu = e^{-i\omega t}\dot{\eta}$ we get the system

$$\begin{cases} \dot{\sigma} = -i\omega\sigma + \nu \\ \\ \dot{\nu} = (1 + \frac{1}{2\rho^3})\sigma + \frac{3}{2\rho^3}\overline{\sigma} - i(\omega+2)\nu + e^{-i\omega t}b(t). \end{cases}$$

Taking $\sigma = x_1 + ix_2$ and $\nu = x_3 + ix_4$ we obtain the following system in real coordinates

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + B(t)$$

where

$$A = \begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 1+2(\omega+1)^2 & 0 & 0 & \omega+2 \\ 0 & 1-(\omega+1)^2 & -(\omega+2) & 0 \end{bmatrix} \quad \text{and}$$

$$B(t) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{\rho^2} + \dfrac{\cos(\omega t) - \rho}{|1-\rho e^{i\omega t}|^3} + 2(\omega+1)^2\cos(\omega t) + \varepsilon'(0)\mathrm{Re}[d(t)e^{-i\omega t}] \\ -\dfrac{\sin(\omega t)}{|1-\rho e^{i\omega t}|^3} + (\omega+1)^2\sin(\omega t) + \varepsilon'(0)\mathrm{Im}[d(t)e^{-i\omega t}]. \end{bmatrix}$$

Now we note that this real system admits a periodic solution of period $2\pi/\omega$ which is obtained by considering $\eta(t)$ in the new coordinates. It follows from Fredholm alternative for periodic systems that $B(t)$ must be orthogonal to the $2\pi/\omega$ periodic solutions of the adjoint system $\dot{y} = -A^T y$. Here the notion of orthogonality refers to $L^2(0, 2\pi/\omega)$. In particular it must be orthogonal to

$$\psi = \begin{bmatrix} \omega+2 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

which is a constant solution of $\dot{y} = -A^T y$. Then, the orthogonality condition may be written as

$$\int_0^{2\pi/\omega} < B(t), \psi > dt = \int_0^{2\pi/\omega} B_4(t)dt = 0, \qquad (3)$$

where $B_4(t)$ is the fourth component of the vector $B(t)$. The function $-\frac{\sin(\omega t)}{|1-\rho e^{i\omega t}|^3} + (\omega+1)^2\sin(\omega t)$ is odd and so it has zero average. In consequence,

$$\int_0^{2\pi/\omega} B_4(t)dt = \varepsilon'(0)\int_0^{2\pi/\omega} \mathrm{Im}[d(t)e^{-i\omega t}]dt$$

and we arrive at a contradiction if this integral does not vanish. ∎

It is easy to check that the previous theorem is applicable to the dissipative terms presented in the introduction. For Stoke's drag

$$d(t) = -i(\rho(\omega+1) + \frac{\alpha}{\rho^2})e^{i\omega t}$$

and for P-R force

$$d(t) = -\frac{i}{\rho}(\omega+1)e^{i\omega t}.$$

3 Study of the Hopf bifurcation around the libration point L_4

In this section we study the occurrence of the Hopf bifurcation around L_4 and L_5 for the restricted three body problem with dissipation. Actually, the computations are similar for the two points, and we focus on L_4. We will perform our computations in real coordinates. Accordingly, we start this section by rewriting in real form the general equations of the restricted three body problem with dissipation. Denoting by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

the symplectic matrix in $\mathbb{R}^2$, we can write the equations of motion as follows:

$$\begin{cases} z' &= w \\ w' &= -2Jw + \nabla\phi(z,\mu) - \varepsilon F(z,w,\mu) \end{cases} \qquad (\mathcal{P}_{\mu,\varepsilon})$$

where

$$z = (x,y),\ w = (u,v) := (x',y'),\ \phi(z,\mu) = \frac{1-\mu}{|\mu+z|} + \frac{\mu}{|1-\mu-z|}$$

and the dissipative term $\mathcal{D}$ is expressed (up to a sign) in the form

$$F(z,w,\mu) = \begin{pmatrix} f(z,w,\mu) \\ g(z,w,\mu) \end{pmatrix}.$$

Again $\varepsilon > 0$ is a free parameter, ∇ denotes the gradient in z and $f,g : (\mathbb{R}^4 \times [0,1/2]) \setminus \Delta \to \mathbb{R}$ are two real analytic functions with a singular set Δ. We assume that Δ is compact and with positive distance from the curves described by the points $(L_4(\mu),0,\mu)$ and $(L_5(\mu),0,\mu)$ as μ varies in $[0,1/2]$.

Recall that $L_4(\mu) = (-\mu + 1/2, \sqrt{3}/2)$ and $L_5(\mu) = (-\mu + 1/2, -\sqrt{3}/2)$. In words, the dissipation terms are well defined in a neighbourhood of the libration points, and this neighbourhood can be chosen to be the same for all the values of μ. This property holds for all the drags considered in [2]. In fact, in the case of the linear drag Δ is empty, for the Stoke's drag with $\alpha > 0$ it is

$$\Delta = \{(z, w, \mu) \mid z = 0,\ \mu \in [0, 1/2]\},$$

and for the PR drag it is

$$\Delta = \{(z, w, \mu) \mid z = -\mu,\ \mu \in [0, 1/2]\}.$$

Notice that in our general setting we have not considered additional parameters in the perturbation terms, such as the parameter α of the Stoke's drag. The only reason we did this is to keep our notation simpler. In fact, the general results about the smooth dependence of suitable functions on the parameter μ which we use in our proofs hold for any number of parameters (as long as the perturbations depend smoothly on them). The extra parameter α will appear in Corollary 3.3.

The libration point $L_4(\mu)$ is a non-degenerate solution of the equation

$$\nabla\phi(z, \mu) = 0, \qquad \mu > 0.$$

Actually a well known computation shows that the Hessian matrix

$$D^2\phi(L_4(\mu), 0) = \begin{bmatrix} \frac{3}{4} & \frac{3\sqrt{3}}{4}(1-2\mu) \\ \frac{3\sqrt{3}}{4}(1-2\mu) & \frac{9}{4} \end{bmatrix},$$

and the determinant of this matrix does not vanish if $\mu \in]0, \frac{1}{2}]$. This implies that for any $\delta \in]0, \frac{1}{2}[$ there exists $\varepsilon_1 = \varepsilon_1(\delta) > 0$ such that the equation

$$\nabla\phi(z, \mu) - \varepsilon F(z, 0, \mu) = 0,$$

defines a smooth function $(\mu, \varepsilon) \to \tilde{L}_4(\mu, \varepsilon)$ on the set $[\delta, 1/2] \times [0, \varepsilon_1]$ which satisfies $\tilde{L}_4(\mu, 0) = L_4(\mu),\ \mu \in [\delta, 1/2]$. This shows that the libration point L_4 can be continued for small values of ε, uniformly with respect to $\mu \in [\delta, 1/2]$ giving rise to the corresponding perturbed equilibrium $(\tilde{L}_4(\mu, \varepsilon), 0)$ of system $(\mathcal{P}_{\mu,\varepsilon})$. To justify the previous assertions we first apply the implicit function theorem at each point $(L_4(\mu), \mu)$ and $\varepsilon = 0$, then a compactness argument on the interval $[\delta, \frac{1}{2}]$ together with the uniqueness of the implicit function allows to find a common continuation on $\mu \in [\delta, \frac{1}{2}]$ and $\varepsilon \in [0, \varepsilon_1(\delta)]$. If we denote the second member of system $(\mathcal{P}_{\mu,\varepsilon})$ by $\Psi(z, w, \mu, \varepsilon)$ we have that the

Jacobian matrix of Ψ with respect to (z, w) around the perturbed libration point is given by:

$$D\Psi(\tilde{L}_4(\mu,\varepsilon), 0, \mu, \varepsilon) = \begin{bmatrix} 0 & I_2 \\ D^2\phi - \varepsilon\partial_1 F & -2J - \varepsilon\partial_2 F \end{bmatrix},$$

where each component indicates a 2×2 block and $\partial_1 F = \frac{\partial(f,g)}{\partial(x,y)}$, $\partial_2 F = \frac{\partial(f,g)}{\partial(u,v)}$. All the derivatives of ϕ are computed at $(\tilde{L}_4(\mu,\varepsilon), \mu)$ and all the derivatives of f and g are computed at $(\tilde{L}_4(\mu,\varepsilon), 0, \mu)$. In what follows, to keep our notation simple, we will sometimes drop the dependence of such derivatives and of other quantities on (μ, ε). However, to avoid any confusion, we will leave explicit such dependence in the first members of the equalities. In particular

$$A(\mu,\varepsilon) = \frac{\partial^2\phi}{\partial^2 x}, \ B(\mu,\varepsilon) = \frac{\partial^2\phi}{\partial x\partial y}, \ C(\mu,\varepsilon) = \frac{\partial^2\phi}{\partial^2 y}.$$

To compute the characteristic polynomial $P_{\mu,\varepsilon}(\lambda)$ of $D\Psi$ it is convenient to employ the following observation: given a 4×4 matrix defined by two dimensional blocks

$$\begin{bmatrix} 0 & I_2 \\ M & N \end{bmatrix},$$

the characteristic polynomial is $-\det(M + \lambda N - \lambda^2 I_2)$. In our case $P_{\mu,\varepsilon}(\lambda) = -\det(D^2\phi - \varepsilon\partial_1 F - \lambda(2J + \varepsilon\partial_2 F) - \lambda^2 I_2) = \sum_{k=0}^{4} a_k(\mu,\varepsilon)\lambda^k$, where

$$a_4(\mu,\varepsilon) = 1, \ a_3(\mu,\varepsilon) = \varepsilon\tilde{a}_3(\mu,\varepsilon), \ a_2(\mu,\varepsilon) = 4 - A - C + O(\varepsilon),$$

$$a_1(\mu,\varepsilon) = \varepsilon\tilde{a}_1(\mu,\varepsilon), \ a_0(\mu,\varepsilon) = AC - B^2 + O(\varepsilon),$$

and

$$\tilde{a}_3(\mu,\varepsilon) := \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v};$$

$$\tilde{a}_1(\mu,\varepsilon) := -A\frac{\partial g}{\partial v} + B\left(\frac{\partial f}{\partial v} + \frac{\partial g}{\partial u}\right) - C\frac{\partial f}{\partial u} + 2\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) + O(\varepsilon).$$

Since we will obtain our results by perturbing from $\varepsilon = 0$, it is worth recalling the behavior of the roots of $P_{\mu,0}$ as μ varies in the interval $]0, 1/2]$. See also [5]. When $\varepsilon = 0$ we know that

$$A(\mu,0) = \frac{3}{4}, \ B(\mu,0) = \frac{3\sqrt{3}}{4}(1-2\mu), \ C(\mu,0) = \frac{9}{4},$$

so that the eigenvalues of $D\Psi(\mu, 0)$ satisfy the equation

$$\lambda^2 = -\frac{1}{2} \pm \frac{\sqrt{1 - 27\mu(1-\mu)}}{2}.$$

If we denote by μ^* the smallest root of the equation $1 - 27\mu(1 - \mu) = 0$, then we have three occurrences:

1) $0 < \mu < \mu^*$; the eigenvalues of $D\Psi(\mu, 0)$ are two pairs of distinct non zero and conjugate eigenvalues of the form $\pm\beta_k(\mu)i$, $\beta_k(\mu) > 0$, $k = 1, 2$.

2) $\mu = \mu^*$; the two pairs above merge, that is $\beta_1(\mu^*) = \beta_2(\mu^*) = \sqrt{2}/2$.

3) $\mu^* < \mu \leq 1/2$; there are four distinct eigenvalues of the form $\pm\alpha_k(\mu) \pm i\beta_k(\mu)$, with $\alpha_k(\mu) > 0$ and $\beta_k(\mu) > 0$, $k = 1, 2$.

Of course, in case 3) we have two complex conjugate eigenvalues with positive real part and two complex conjugate eigenvalues with negative real part. Then, by the continuous dependence of the eigenvalues on parameters (on which we will be more specific below) if we consider any $\mu_2 > \mu^*$, this configuration of the eigenvalues will persist *for any* f and g if ε is small enough. As a consequence, to have (or not) an Hopf bifurcation around the perturbed libration point will depend essentially on the position in the complex plane of the eigenvalues of $D\Psi$ for $\mu < \mu^*$. It turns out that this position will be determined by some inequalities which involve some first derivatives of f and g.

In our first result, we will give conditions on these functions which guarantee that, for a suitable $\mu_1 < \mu^*$ and for all sufficiently small positive ε, the four complex roots of $P_{\mu_1,\varepsilon}$ have negative real parts. Actually, these conditions will imply that there is a unique value $\mu(\varepsilon) \in [\mu_1, \mu_2]$ of μ such that exactly two complex conjugate eigenvalues of $D\Psi$ cross the imaginary axis transversally. As a consequence, a Hopf bifurcation occurs around the perturbed libration point. This result applies to the linear drag.

In the second result our assumptions on f and g will rule out the possibility of a crossing of the imaginary axis by the eigenvalues for small ε, so that a pair of complex conjugate eigenvalues will always have negative real part and a pair of complex conjugate eigenvalues will always have positive real part. In this case, for any small ε no Hopf bifurcation occurs around the libration point, that will always be unstable. This second result applies to the PR drag.

One of the tools we will use in the proof of our first result is the Routh-Hurwitz criterion (see [3]), which gives some necessary and sufficient conditions for all the roots of a polynomial to have negative real parts. This criterion involves some quantities which we introduce below before stating our result.

The quantities we need are the following:

$$H_3(\mu,\varepsilon) = a_3 a_2 a_1 - a_4 a_1^2 - a_3^2 a_0 = \varepsilon^2 \tilde{H}_3(\mu,\varepsilon),$$

where

$$\tilde{H}_3(\mu,\varepsilon) := \tilde{a}_3 a_2 \tilde{a}_1 - \tilde{a}_1^2 - \tilde{a}_3^2 a_0,$$

and

$$H_2(\mu,\varepsilon) := a_3 a_2 - a_4 a_1 = \varepsilon \tilde{H}_2(\mu,\varepsilon),$$

where

$$\tilde{H}_2(\mu,\varepsilon) := \tilde{a}_3 a_2 - \tilde{a}_1.$$

Now we can state our first result.

Theorem 3.1 *Assume that there exist* $\mu_1 \in [\delta, \mu^*[$ *and* $\mu_2 \in]\mu^*, 1/2]$ *such that the next inequalities hold:*

$$i) \;\; \tilde{H}_3(\mu_1, 0) > 0;$$

$$ii) \;\; \tilde{a}_3(\mu, 0) > 0, \qquad \frac{\partial \tilde{H}_3(\mu, 0)}{\partial \mu} < 0, \qquad \forall \mu \in [\mu_1, \mu_2].$$

Then, there exists $0 < \varepsilon^* \leq \varepsilon_1$ *such that for any* $\varepsilon \in]0, \varepsilon^*]$ *there is a unique* $\mu(\varepsilon) \in]\mu_1, \mu_2[$ *such that a Hopf bifurcation occurs around* $\tilde{L}_4(\mu(\varepsilon), \varepsilon)$.

The notion of Hopf bifurcation is understood in the sense of [1]. More precisely, we get that for any $\varepsilon \in]0, \varepsilon^*]$ there is a unique $\mu(\varepsilon) \in]\mu_1, \mu_2[$ such that the following holds: there exists a neighborhood J_ε of $s = 0^+$, functions $\omega(s), \mu(s) \in C^1(J_\varepsilon)$ and a family of non-constant periodic solutions (z_s, w_s) of system $(\mathcal{P}_{\mu,\varepsilon})$ such that:

i) $\omega(s) \to \operatorname{Im} \lambda_2(\mu(\varepsilon), \varepsilon)$ as $s \to 0^+$, where $\lambda_2(\mu(\varepsilon), \varepsilon) \neq 0$ is a suitable eigenvalue of $D\Psi$ on the imaginary axis (see also Lemma 3.2 below);

ii) (z_s, w_s) has period $T_s = \frac{2\pi}{\omega(s)}$;

iii) the amplitude of the orbits tends to zero as $s \to 0^+$.

To prepare the proof we need some properties on the behavior of roots for certain polynomials of degree 4 of the type

$$q(\lambda,\mu)=\lambda^4+\sum_{m=0}^{3}a_m(\mu)\lambda^m$$

with $a_m\in C^1[\mu_1,\mu_2]$, $0\le m\le 3$. The polynomial q satisfies the property of *transversal crossing* if, given any root $(\hat\lambda,\hat\mu)$ with $\mathrm{Re}\hat\lambda=0$, then $\hat\lambda$ is simple and

$$\frac{d}{d\mu}\mathrm{Re}\lambda(\mu)\mid_{\mu=\hat\mu}>0,$$

where $\lambda=\lambda(\mu)$ is the C^1 branch of roots satisfying $\lambda(\hat\mu)=\hat\lambda$.

The dependence of roots with respect to parameters is not always smooth. However, for simple roots the implicit function theorem guarantees the existence of a smooth branch passing through the given root. This observation has been used in the previous definition.

Lemma 3.2 *In the previous notations assume that the roots of q can be labelled as $\lambda_k=\lambda_k(\mu)$, $1\le k\le 4$, with*

$$\mathrm{Re}\lambda_1(\mu)\le\mathrm{Re}\lambda_2(\mu),\quad \lambda_3(\mu)=\overline{\lambda_1(\mu)},\ \lambda_4(\mu)=\overline{\lambda_2(\mu)}. \tag{4}$$

In addition the property of transversal crossing holds and

$$\mathrm{Re}\lambda_1(\mu_1)\le\mathrm{Re}\lambda_2(\mu_1)<0,\quad \mathrm{Re}\lambda_1(\mu_2)<0<\mathrm{Re}\lambda_2(\mu_2). \tag{5}$$

Then there exists a unique $\tilde\mu\in]\mu_1,\mu_2[$ such that $q(\lambda,\tilde\mu)$ has roots on the imaginary axis. More precisely,

$$\mathrm{Re}\lambda_1(\mu)<0\ \ for\ each\ \mu\in[\mu_1,\mu_2]\ \ and\ \ (\mu-\tilde\mu)\mathrm{Re}\lambda_2(\mu)>0\ if\ \mu\ne\tilde\mu.$$

Proof. Define the sets $N_0=\{\mu\in[\mu_1,\mu_2]:\ \mathrm{Re}\lambda_1(\mu)>0\}$, $N_1=\{\mu\in[\mu_1,\mu_2]:\ \mathrm{Re}\lambda_1(\mu)<0<\mathrm{Re}\lambda_2(\mu)\}$ and $N_2=\{\mu\in[\mu_1,\mu_2]:\ \mathrm{Re}\lambda_2(\mu)<0\}$. These sets are open relative to $[\mu_1,\mu_2]$, as can be shown using degree theory because roots of complex polynomials have always positive Brouwer index. Since N_0, N_1 and N_2 are pairwise disjoint and $\mu_1\in N_2$, $\mu_2\in N_1$, the complement $C=[\mu_1,\mu_2]\setminus(N_0\cup N_1\cup N_2)$ must be non-empty. For every number $\tilde\mu$ in C, the polynomial $q(\lambda,\tilde\mu)$ has roots on the imaginary axis. To prove the uniqueness of $\tilde\mu$ we notice that the property of transversal crossing has strong consequences on the structure of C. Indeed every point $\tilde\mu$ in C has to be isolated and, for some small $\delta>0$, one of the alternatives below holds,

$$\textbf{(i)}\quad [\tilde\mu-\delta,\tilde\mu[\subset N_2,\ \ ]\tilde\mu,\tilde\mu+\delta]\subset N_1$$

$$\textbf{(ii)}\quad [\tilde\mu-\delta,\tilde\mu[\subset N_2,\ \ ]\tilde\mu,\tilde\mu+\delta]\subset N_0$$
$$\textbf{(iii)}\quad [\tilde\mu-\delta,\tilde\mu[\subset N_1,\ \ ]\tilde\mu,\tilde\mu+\delta]\subset N_0.$$

In consequence C has at most two points. From the configuration $\mu_1 \in N_1$, $\mu_2 \in N_1$ we deduce that C is a singleton and (**i**) holds. This also implies that $[\mu_1, \tilde\mu[\subset N_2$ and $]\tilde\mu, \mu_2] \subset N_1$. Finally we observe that $\mathrm{Re}\lambda_1(\tilde\mu) < 0$, for otherwise $\mathrm{Re}\lambda_1(\tilde\mu) = \mathrm{Re}\lambda_2(\tilde\mu) = 0$ and the transversal crossing would imply that we are in case (**ii**). ■

Proof of theorem 3.1. We will show that the polynomial $P_{\mu,\varepsilon}(\lambda)$ is in the conditions of the previous lemma for small $\varepsilon > 0$. Then theorem 2.6 in [1] implies that there is a Hopf bifurcation. The set of roots of the polynomial $P_{\mu,\varepsilon}(\lambda)$ depends continuously on parameters. This is a well known property of polynomials depending on parameters and having a constant leading coefficient. For $\varepsilon = 0$ we know that the roots of $P_{\mu,0}(\lambda)$ are complex conjugate and so this property also holds for small ε and arbitrary $\mu \in [\delta, \frac{1}{2}]$. From now on we assume that ε is sufficiently small so that the roots of $P_{\mu,\varepsilon}(\lambda)$ are $\lambda_i(\mu,\varepsilon)$, $1 \le i \le 4$, with $\mathrm{Im}\lambda_i(\mu,\varepsilon) > 0$, $i = 1, 2$, and such that (4) holds. Since $\mu_2 > \mu^*$ we know that $\mathrm{Re}\lambda_1(\mu_2, 0) < 0 < \mathrm{Re}\lambda_2(\mu_2, 0)$ and this implies that the inequality for $\mu = \mu_2$ in (5) holds if ε is small. The inequality for $\mu = \mu_1$ is more delicate since $\mathrm{Re}\lambda_1(\mu_1, 0) = \mathrm{Re}\lambda_2(\mu_1, 0) = 0$. Since $a_4(\mu, \varepsilon) = 1$, for any $\mu \in [\delta, 1/2]$, by the Routh-Hurwitz criterion the characteristic polynomial $P_{\mu_1,\varepsilon}$ has four roots with negative real parts for a fixed $\varepsilon \in]0, \varepsilon^*]$ if and only if the following conditions hold:

a) $\quad a_k(\mu_1, \varepsilon) > 0, \quad k = 0, \cdots, 3;$

b) $\quad H_k(\mu_1, \varepsilon) > 0, \quad k = 2, 3.$

In order to prove that *a*) and *b*) hold for any sufficiently small and positive ε, we start by noticing that

$$a_2(\mu, 0) = 1 > 0, \quad \text{and} \quad a_0(\mu, 0) = \frac{27\mu(1-\mu)}{4} > 0, \quad \text{for any } \mu \in [\delta, 1/2].$$

Moreover, by *i*) and the first inequality of *ii*) we have

$$\frac{\tilde a_1(\mu_1, 0)}{\tilde a_3(\mu_1, 0)} - \left(\frac{\tilde a_1(\mu_1, 0)}{\tilde a_3(\mu_1, 0)}\right)^2 > \frac{27\mu_1(1-\mu_1)}{4} > 0,$$

so that $0 < \frac{\tilde a_1(\mu_1,0)}{\tilde a_3(\mu_1,0)} < 1$. We conclude that $\tilde a_1(\mu_1, 0) > 0$ and $\tilde H_2(\mu_1, 0) = \tilde a_3(\mu_1, 0) - \tilde a_1(\mu_1, 0) > 0$.

By the continuous dependence of the coefficients of $P_{\mu,\varepsilon}$ on the parameters, we conclude that there exists $\varepsilon^* = \varepsilon^*(\mu_1, \mu_2) \le \varepsilon_1$ such that *a*) and *b*) hold for any $\varepsilon \in]0, \varepsilon^*]$ and the roots of $P_{\mu_1,\varepsilon}$ have four complex roots with negative real parts.

Finally we check the condition of transversal crossing. For this aim we start by observing that, choosing if necessary a smaller ε^*, we may assume that

$$\tilde{a}_3(\mu, \varepsilon) > 0, \qquad \frac{\partial \tilde{H}_3(\mu, \varepsilon)}{\partial \mu} < 0, \qquad \forall \mu \in [\mu_1, \mu_2] \times [0, \varepsilon^*]. \tag{6}$$

Assume that $\hat{\mu}$ is such that $\mathrm{Re}\lambda_j(\hat{\mu}, \epsilon) = 0$ for some $j = 1, 2$. Let us denote by k the other index in the set $\{1, 2\}$. Then, by the first inequality of (6) we have

$$2\mathrm{Re}\lambda_k(\hat{\mu}, \varepsilon) = \sum_1^4 \lambda_i(\hat{\mu}, \varepsilon) = \mathrm{trace} D\Psi(\hat{\mu}, \varepsilon) = -\varepsilon \tilde{a}_3(\hat{\mu}, \varepsilon) < 0, \tag{7}$$

so that $\lambda_j(\hat{\mu}, \varepsilon)$ is simple. Indeed the four roots must be simple and we denote by $\lambda_i(\mu, \varepsilon)$ the associated smooth branch with $i = 1, 2, 3, 4$. Let us check that $\dfrac{\partial \mathrm{Re}\lambda_j(\hat{\mu}, \varepsilon)}{\partial \mu} \neq 0$. To show that this last condition holds, we will make use of Orlando's Formula ([3], page 25), which in our case takes the following form:

$$H_3(\mu, \varepsilon) = 4\mathrm{Re}\lambda_1(\mu, \varepsilon) \cdot \mathrm{Re}\lambda_2(\mu, \varepsilon) \cdot |\lambda_1(\mu, \varepsilon) + \lambda_2(\mu, \varepsilon)|^2 \cdot |\lambda_1(\mu, \varepsilon) + \lambda_4(\mu, \varepsilon)|^2. \tag{8}$$

Then, by differentiating with respect to μ both sides of equation (8) and using the second inequality of (6) we get:

$$\frac{\partial H_3(\hat{\mu}, \varepsilon)}{\partial \mu} = \varepsilon \frac{\partial \tilde{H}_3(\hat{\mu}, \varepsilon)}{\partial \mu} =$$

$$= 4 \frac{\partial \mathrm{Re}\lambda_j(\hat{\mu}, \varepsilon)}{\partial \mu} \left(-\frac{\varepsilon}{2} \tilde{a}_3(\hat{\mu}, \varepsilon) \right) |\lambda_1 + \lambda_2|^2_{(\hat{\mu}, \varepsilon)} |\lambda_1 + \lambda_4|^2_{(\hat{\mu}, \varepsilon)} < 0 \tag{9}$$

and we conclude that

$$\frac{\partial \mathrm{Re}\lambda_j(\hat{\mu}, \varepsilon)}{\partial \mu} > 0. \tag{10}$$

Our proof is concluded. ∎

As an immediate consequence of Theorem 3.1 we have the following:

Corollary 3.3 *If $F(z, w, \mu) = Jz + w$ (linear drag) then for any sufficiently small positive ε there exists a unique value $\mu(\varepsilon) \in]0, 1/2[$ such that a Hopf bifurcation occurs around $(\tilde{L}_4(\mu(\varepsilon), \varepsilon), 0, 0)$. An analogous conclusion holds for the Stoke's drag for α sufficiently small.*

Proof.

In the linear drag case it is :

$$\tilde{a}_3(\mu, 0) = 2,\ \tilde{H}_3(\mu, 0) = 1 - 27\mu(1-\mu),\ \frac{\partial \tilde{H}_3(\mu, 0)}{\partial \mu} = 27(-1+2\mu)$$

and the assumptions of Theorem 3.1 are satisfied by choosing any $\mu_1 \in [\delta, \mu^*[$ and any $\mu_2 \in]\mu^*, 1/2[$.

Notice now that the Stoke's drag depends linearly on α, so that the corresponding functions $\tilde{a}_3$ $\tilde{H}_3$ and $\frac{\partial \tilde{H}_3}{\partial \mu}$ will depend in a smooth (actually real analytic) way on (μ, α, ϵ). This implies that, if α is small, these functions still satisfy the assumptions of Theorem 3.1, and our claim follows.

■

To deal with the PR drag we will need the following general result:

Theorem 3.4 *Assume that there exists $\mu_1 \in [\delta, \mu^*[$ and $\mu_2 \in]\mu^*, 1/2]$ for which the following inequalities hold:*

$$\tilde{a}_3(\mu, 0) > 0, \quad \textit{and} \quad \tilde{a}_1(\mu, 0) < 0, \quad \forall \mu \in [\mu_1, \mu_2]. \tag{11}$$

Then, there exists $\varepsilon^* > 0$ such that

$$Re\lambda_1(\mu, \varepsilon) \cdot Re\lambda_2(\mu, \varepsilon) < 0, \quad \forall (\mu, \varepsilon) \in [\mu_1, \mu_2] \times]0, \varepsilon^*], \tag{12}$$

where λ_1, λ_2 and their conjugates are the eigenvalues of $D\Psi$.

Proof. Choose ε^* such that the following inequalities hold:

$$\tilde{a}_3(\mu, \varepsilon) > 0, \quad \text{and} \qquad \tilde{a}_1(\mu, \varepsilon) < 0, \quad \text{for any } (\mu, \varepsilon) \in [\mu_1, \mu_2] \times [0, \varepsilon^*]. \tag{13}$$

Our claim follows now immediately by considering the sign conditions we get from (13) together with the relationships of a_3 and a_1 with the roots of the characteristic polynomial, namely:

$$0 < a_3 = \varepsilon \tilde{a}_3 = -2(\text{Re}\lambda_1 + \text{Re}\lambda_2)$$

and

$$0 > a_1 = \varepsilon \tilde{a}_1 = -2\text{Re}\lambda_1 |\lambda_2|^2 - 2\text{Re}\lambda_2 |\lambda_1|^2$$

for any $(\mu, \varepsilon) \in [\mu_1, \mu_2] \times [0, \varepsilon^*]$.

In fact, these last inequalities imply that no eigenvalue of $D\Psi$ can have zero real part and that λ_1 and λ_2 have real part with opposite sign.

■

Corollary 3.5 *In the case of the PR force if ε is sufficiently small, then for any $\mu \in [\delta, 1/2]$ the libration point is always unstable and no Hopf bifurcation occurs around it.*

Proof. In the case of the PR dissipation it is

$$\tilde{a}_3(\mu, 0) = 2, \ \ \tilde{a}_1(\mu, 0) = -3 + 2\mu,$$

and the assumptions of the last theorem are fulfilled with $\mu_1 = \delta$ and $\mu_2 = 1/2$.

■

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