EFFECT OF THE IMPLICIT COMBINATORIAL MODEL ON COMBINATORIAL REASONING IN SECONDARY SCHOOL PUPILS

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ABSTRACT

Elementary combinatorial problems may be classified into three different combinatorial models (selection, partition and distribution). The main goal of this research was to determine the effect of the implicit combinatorial model on pupils' combinatorial reasoning before and after instruction. When building the questionnaire, we also considered the combinatorial operation and the nature of elements as task variables. The analysis of variance of the answers from 720 14-15 year-old pupils showed the influence of the implicit combinatorial model on problem difficulty and the interaction of all the factors with instruction. Qualitative analysis also revealed the dependence of error types on task variables. Consequently, the implicit combinatorial model should be considered as a didactic variable in organising elementary combinatorics teaching.

1. INTRODUCTION

Combinatorics is an essential component of discrete mathematics and, as such, it has an important role to play in school mathematics. In 1970, Kapur presented the following reasons, which could still be valid, to justify the teaching of elementary Combinatorics at school:

- Since it does not depend on calculus, it has suitable problems for different grades; usually very challenging problems can be discussed with pupils, so that they discover the need for more mathematics to be created;
- It can be used to train pupils in enumeration, making conjectures, generalization and systematic thinking; it can help the development of many concepts, such as equivalence and order relations, function, sample, etc.
- Many applications in different fields can be presented.

As regards Probability, and according to Piaget and Inhelder (1951), if the subject does not possess combinatorial capacity, he/she is not able to use the idea of probability, except with very elementary random experiments. Moreover, these authors related the emergence of the chance concept to understanding the idea of permutation, and the correct estimates of probabilities to the development of the concept of combination.

All these reasons justify the interest in improving the teaching of the topic. Nevertheless, Combinatorics is a field that most pupils find very difficult. Two fundamental steps for making the learning of this subject easier are understanding the nature of pupils' mistakes when solving combinatorial problems and identifying the variables that might influence this difficulty.

In this paper we analyze the effect of the combinatorial model implicit in the statement of simple combinatorial problems (ICM variable) on pupils' solutions. The possible influence of this variable was suggested by Dubois (1984), although it has not been assessed in experimental research work until now. As dependent variables, we have considered the percentage of correct solutions to the problems and the specific type of error, in case of incorrect solutions. We have also controlled whether the pupils had received instruction or not. The analysis of our data shows the effect of the ICM variable on the problem difficulty, as well as the interaction of ICM with the following variables studied by Fischbein and Gazit (1988): combinatorial operation, type of elements and teaching. As an additional result we present a systematic description of the pupils' errors when solving combinatorial problems.

Research by Piaget and Fischbein

Besides its importance in developing the idea of Probability, combinatorial capacity is a fundamental component of formal thinking. This capacity can be related to the stages described in Piaget's theory. Children at Stage I use random listing procedures, without trying to find a systematic strategy. At Stage II, they use trial and error, discovering some empirical procedures with a few elements. After the period of formal operations, adolescents discover systematic procedures of combinatorial construction, although for permutations, it is necessary to wait until children are 15 years old. According to Piaget and Inhelder (1951), combinations involve the coordination of seriation and correspondence, permutations imply an arrangement according to a mobile and reversible system of reference; therefore they are operations on operations, characteristics of the formal thought level.

However, more recent results showed, such as Fischbein (1975), that the combinatorial problem-solving capacity is not always reached, not even at the formal operations level, without specific teaching. On the other hand, Fischbein, Pampu and Minzat (1970) and Fischbein and Gazit (1988) studied the effect of specific instruction on the combinatorial capacity, discovering that even 10 year-old pupils can learn some combinatorial ideas with the help of the tree diagram. In 1988, Fischbein and Gazit also analyzed the relative difficulty of combinatorial problems in terms of the nature and the number of elements, identifying some typical errors when solving combinatorial problems with one operation.

Before teaching, they found the following order in the difficulty of combinatorial operations: permutations, arrangements with repetition, arrangements without repetition, and combinations, confirming Piaget's findings. After teaching this order changed, and combinations provided the lowest frequency of correct solutions. On the contrary, permutations, which was to be the most difficult task before teaching, became easier after the pupils had learnt the use of the tree diagram. Fischbein and Gazit argued that the formula and tree diagram for permutations are simpler than for combinations. They also pointed out that the teaching of the formula for combinations, seems to disturb the intuitive empirical strategies for this type of problems.

As regards the nature of the elements to be combined, Fischbein and Gazit found that pupils obtained a higher proportion of correct responses when using digits than with committees or colored flags, because pupils are more used to operating mentally with digits than with other elements. They were also interested in the specific mistakes in the pupils' solutions. The following types of errors were identified:

- Using a formula that corresponds to a different type of combinatorial operations.
- Multiplying the numbers representing the data from the problem.
- One of the numbers contained in the problem is presented as the solution.
- The subject gives a number apparently not clearly related to the data in the problem.

2. IMPLICIT COMBINATORIAL MODEL IN COMBINATORIAL PROBLEMS WITH ONE OPERATION

According to Dubois (1984), simple combinatorial configurations may be classified into three models: *selections*, which emphasize the concept of sampling; *distributions*, related to the concept of mapping, and *partitions* or divisions of a set into subsets.

In the model of *selection* a set of *m* (usually distinct) objects is considered, from which a sample of *n* elements must be drawn, for example, in Item 11. If we substitute the marbles with people, we could interpret Items 8 and 13 in a similar way. In selecting a sample, sometimes it is permitted to repeat one or more elements in the sample, as in Item 11, and other times it is not possible, as in Item 5. According to this possibility and whether the order in which the sample is taken is relevant or not, we obtain the four basic combinatorial operations shown in Table 1: $AR_{m,n}$ (arrangements with repetition of m elements, taken n at a time), $A_{m,n}$ (arrangements of m elements, taken n at a time), $CR_{m,n}$ (combinations with repetition of m elements, taken n at a time) and $C_{m,n}$ (combinations of m elements, taken n at a time). We should also note that permutation is a particular case of arrangement.

Another type of problem refers to the **distribution** of a set of *n* objects into *m* cells, such as Item 3, in which each of three identical cards must be placed into one of four different envelopes. The solution to this problem is $C_{4,3}$, but there are many different possibilities in this model, depending on the following features:

- Whether the objects to be distributed are identical or not.
- Whether the containers are identical or not.
- Whether we must order the objects placed into the containers.
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Table I. Different possibilities in the selection model

	Ordered sample	Non ordered sample
Replacement	$AR_{m, n}$	$CR_{m, n}$
No replacement	$A_{m, n}$	$C_{m,n}$

Dubois (1984) differentiated six basic types in the combinatorial model of distribution:

1. Ordered distributions of different objects in different containers.
2. Ordered distributions of different objects in identical containers.
3. Non-ordered distribution of different objects in different containers.
4. Non-ordered distributions of different objects in identical containers.
5. Distributions of identical objects in different containers (because the objects are identical, the order is irrelevant).
6. Distributions of identical objects in identical containers (order is irrelevant).

In addition, other conditions, such as the maximum number of objects in each cell or the possibility of having empty cells are basic to finding the solution to the problem. There is not a different combinatorial operation for each different type of distribution mentioned, and, moreover, the same combinatorial operation may be obtained with two different distribution problems.

For example, we could define the arrangements as the number of possible distributions of *n* different objects into *m* different cells with, at most, one object in each cell (whether the distribution is ordered or not is irrelevant). When considering indistinguishable objects, we obtain the combinations. However, we might also consider some distributions that could not be expressed by a basic combinatorial operation. For example, if we consider the non-ordered distribution of *n* different objects into *m* identical cells, we obtain the Stirling numbers $S_{n,\,m}$ of the second kind. Consequently, it is not possible to translate each different distribution problem into a different sampling problem.

Assigning the *n* objects to the *m* cells is, from a mathematical point of view, equivalent to establishing a mapping from the set of the *n* objects into the set of the *m* cells. For injective mappings we obtain the arrangements; in case of a bijection we obtain the permutations. Nevertheless, there is no direct definition for the combinations, using the idea of mapping. Furthermore,, if we consider a non-injective mapping, we could obtain a problem for which the solution is not a basic combinatorial operation.

Finally, we might also be interested in splitting a set of *n* objects into *m* subsets, that is, in performing a **partition** of the set, as in Item 10. We could visualize the distribution of *n* objects into *m* cells as the partition of a set of *n* elements into *m* subsets (the cells). Therefore, there is a bijective correspondence between the models of partition and distribution considered by Dubois, although for the pupils this might be not evident.

Therefore, we cannot assume that the three types of problems described (selections, distributions and partition) are equivalent in difficulty, though they may correspond to the same combinatorial operation. This hypothesis was suggested in Dubois (1984) although, until now, there has been no experimental confirmation thereof. Moreover, we have analyzed Spanish textbooks, and we have found that combinatorial operations are usually defined using the idea of sampling. As regards the exercises in these textbooks, most of them refer either to sampling or to distribution problems. Situations of partition of a set into subsets are hardly employed in these exercises at all. Due to these reasons, we have taken the implicit model in the problems as a fundamental task variable in assessing pupils' combinatorial capacity.

3. DESCRIPTION OF THE QUESTIONNAIRE AND SUMMARY OF THE DATA

An initial item bank was set up to produce the questionnaire, with the Spanish translation of several items taken from different sources, such as those developed by Green (1981) and Fischbein and Gazit (1988). Some modifications were needed to obtain a more representative sample of problems and to homogenize the items. The suggestions of some teachers and pupils concerning the comprehension and difficulties of the problems were also taken into account. Two pilot samples of 108 and 56 pupils were used to estimate the time needed to complete the test and to revise the values of the parameters in some problems. Also generalizability indexes (Brennan, 1983) were obtained with values $G_I = 0.71$ for the generalizability to the item population, that is, the possibility of generalizing the results to other combinatorial problems with similar task variable values and $G_P=0.93$ for the generalizability to the pupil population. We considered the following task variables when choosing the problems:

a. Implicit combinatorial model: Selection, distribution and partition models were chosen as the background for the problems.
b. Type of combinatorial operation (permutations, combinations, arrangements).
c. Nature of elements to be combined: letters, numbers, people and objects.

d. Value given to the parameters *m* and *n*.

In Table II, we present the design used to allow a balance of different task variables in the questionnaire so that a representative sample of problems could be achieved. The order of the different items in the questionnaire was chosen so that different combinatorial models and combinatorial operations alternated. To neutralize as far as possible the effect of order, two different questionnaires (A and B) were used. Questionnaire A is included as an Appendix. Questionnaire B was obtained by reversing the order of the items in Questionnaire A, that is, beginning with Item 13 and finishing with Item 1. The two types of questionnaires were randomly distributed to pupils, so one half the pupils in each classroom received a different questionnaire.

Table II. Design of the questionnaire

Combinatorial operation	Mathematical model		
	Distribution	Selection	Partition
Combinations	Objects; $C_{4,3}$ Item 3	People; $C_{5,3}$ Item 8	Numbers; $C_{4,2}$ Item 10
Permutations with repetition	Letters; $PR_{5,1,1,3}$ Item 12	Objects; $PR_{4,1,1,2}$ Item 2	People; $PR_{4,2,2}$ Item 7
Arrangement with repetition	People; $AR_{2,4}$ Item 6	Numbers; $AR_{4,3}$ Item 11	Objects; $AR_{3,4}$ Item 4
Permutations	People; P_4 Item 1	Numbers; P_3 Item 5	
Arrangements	Objects; $A_{5,3}$ Item 9	People; $A_{4,3}$ Item 13	

The final sample included 720 pupils (14-15 years old) in 9 different secondary schools (24 groups of pupils). About half of the pupils (352) had been taught Combinatorics and the others (348) had not. In the first case the pupil, could identify the combinatorial operation, which is a major difficulty in solving combinatorial problems, according to Hadar and Hadass (1981). If he or she did not study the subject, he or she could find the solution by applying the three basic combinatorial rules of product, addition and quotient. Usually, solving the problems also requires recursive reasoning.

The completion of the questionnaire took place during their normal mathematics class. The time needed to complete the questionnaire varied between one hour and an hour and a half. In Table III we present the percentage of correct solutions in both groups of pupils. In this table, we can see that both groups of pupils had great difficulty in giving the correct answer, although the problems involved only one combinatorial operation. Even when the values of parameters were small, the total number of the combinatorial configurations increased quickly, as in Item 4, in which there is a total of 81 possible partitions. The pupils showed a lack of recursive reasoning required to either write down all the possible configurations or to compute the number without listing.

Before teaching, there was no great difference in the difficulty between the three types of models (distribution, selection and partition) except for problem 5, in which the pupils found the solution by trial and error, even with no systematic listing procedure.

Table III. Percentage of correct solutions in the two groups of pupils

Item	Operation	Model	Percent correct (Group with instruction)	Percent correct (Group with no instruction)
1	P_4	Distribution	71.0	23.9
2	$PR_{4,1,1,2}$	Selection	27.5	16.3
3	$C_{4,3}$	Distribution	26.7	26.9
4	$AR_{3,4}$	Partition	6.0	3.0
5	P_3	Selection	80.7	77.2
6	$AR_{2,4}$	Distribution	7.4	13.0
7	$PR_{4,2,2}$	Partition	39.2	32.3
8	$C_{5,3}$	Selection	46.0	22.5
9	$A_{5,3}$	Distribution	41.8	3.8
10	$C_{4,2}$	Partition	37.2	31.0
11	$AR_{4,3}$	Selection	59.1	12.5
12	$PR_{5,1,1,3}$	Distribution	29.5	10.6
13	$A_{4,3}$	Selection	59.6	9.5

After instruction, we found an improvement in a subset of the items. There was a general reduction in the difficulty for the selection problems and in the arrangements, permutations and permutations with repetition problems. In the distribution problems, the improvement was not general, and in the partition problems, there was no improvement at all. This could be explained by the definitions used to introduce combinatorial operations in the Spanish curriculum. These definitions are mainly based on the idea of sampling (selection model) to which, in some textbooks, the distribution model is added for the arrangements and permutations. Therefore, we should emphasize the need of considering the three types of models in future Combinatorics curricular developments.

4. EFFECT OF TASK VARIABLES ON ITEM DIFFICULTY

To test the statistical significance of our task variables on the items' difficulty, a multifactor analysis of variance was performed using the BMDP statistical package. The dependent variable was obtained scoring 1 for the correct solution in each item and 0 for any erroneous answer. We have considered the following within-subjects factors: Combinatorial model (3 levels), combinatorial operation (5 levels) and type of elements (3 levels). We have also controlled the following between-subjects factors: sex (2 levels), group of pupils (instruction and no instruction), and questionnaire (2 versions).The design of the questionnaire was not a complete factorial model, but allowed us to study the main effects of the different factors and the first order interaction between them. In Table IV, we present the mean percentage of success, standard error and sample size for each factor.

In relation to the between subjects factors, we found no differences regarding sex or the type of questionnaire. On the contrary, the F value was highly significant for instruction ($F=210.22$; $p<0.005$), showing the statistical significance of the differences found after instruction, which proved to be very effective for increasing the pupils' combinatorial capacity for solving the problems.

Table IV. Mean percentage and standard error of the success at different factor levels

Factor	Level	Mean percent	StandardError	Sample size
Sex	Female Male	0.3090 0.3092	0.0068 0.0067	352 368
Test version	Test A Test B	0.3122 0.3060	0.0068 0.0068	362 358
Instruction	Without With	0.2153 0.4075	0.0060 0.0073	352 368
Combinatorial Model	Distribution Selection Partition	0.2510 0.4073 0.2422	0.0072 0.0082 0.0091	3500 3500 2100
Combinatorial Operation	C PR AR P A	0.3157 0.2576 0.1599 0.6283 0.2810	0.0100 0.0094 0.0079 0.0128 0.0119	2100 2100 1400 1400 2100
Type of element	Objects People Letter/num	0.1643 0.3271 0.3687	0.0071 0.0085 0.0093	2800 2100 2800

All three within subjects factors were significant in the analysis of variance. The F values obtained for these factors were $F=197.97$; $p<0.001$ for the combinatorial model; $F=314.44$; $p<0.001$ for the combinatorial operation, and $F=304.90$; $p<0.001$ for the type of element. Consequently, we should add the effect of the combinatorial model to the results of Fischbein and Gazit (1988) concerning the effect of the combinatorial operation and the type of elements on the problem difficulty.

To analyze the effect of the combinatorial model on pupils' solutions in depth, clinical interviews were carried out on a total of 17 pupils, who were chosen because they showed typical errors in the partition problems or non-systematic listing procedures. As a result of these interviews, we noticed that many of these pupils did not consider two combinatorial problems with a different combinatorial model to be equivalent, even when the solution to both problems was the same combinatorial operation, e.g., Items 3 and 8.

Moreover, pupils' strategies were influenced by the combinatorial model and, though most pupils, after instruction, preferred employing a formula to solve the selection problems, many of them used listing with the partition or distribution problems. They were unable to transfer the definition of the combinatorial operations to this type of problem, or to translate the partition problem into an equivalent selection problem. As an example, we present one pupil's solution to Item 6 (distributing four children into two rooms):

"Reasoning Number of combinations

4, 0	*1*
3, 1	*4*
2, 2	*6*
1, 3	*4*
0, 4	*1*
Total	*16 possible combinations"*

Although this pupil found a correct solution, he obtained the number of combinations by considering the different ordered decompositions of the number 4 into two addends *m* and *n*; then he computed the number of combinations of the four children, taken *m* at a time and, finally, he added these numbers of combinations for the different decompositions of the number 4. Therefore, this pupil did not identify the combinatorial operation ($AR_{2,4}$) for this problem, but obtained his solution by applying the *addition rule* 16=1+4+6+4+1, because he had not translated the item into a selection problem.

It is worthwhile mentioning that our results concerning the relative order of difficulty before instruction, as a consequence of combinatorial operation, do not coincide with the order obtained in the Fischbein and Gazit (1988) experiment, although we obtained coincidence after instruction. We attribute the differences found between our research and Fischbein and Gazit's research to the fact that they did not control the implicit combinatorial model and, in particular, they did not include partition problems in their assessment.

This hypothesis could be supported by the fact that we obtained a significant effect of the interaction between the implicit combinatorial model and the combinatorial operation ($F=163.97$; $p<0.001$), which suggests that the order of relative difficulty for the different combinatorial operations might not be the same in the different combinatorial models. Nevertheless, since we did not include all the combinations of combinatorial operations and models in our questionnaire, this point needs further investigation, and now it is only held as a hypothesis for future research.

Our results concerning the type of elements showed a similarity to Fischbein and Gazit's research (1988). Finally, we highlight the interaction between instruction and combinatorial model ($F=49$; $p<0.01$) and instruction and combinatorial operation ($F=48.65$; $p<0.01$). Thus, the effect due to instruction was not homogeneous throughout the different combinatorial operations and combinatorial models. There was no interaction of instruction with the other factors. All these findings, as well as the implications between the success in the different items, were also studied using implicative analysis (Gras and Larher, 1993), and were presented in Batanero et al., (1995).

5. DESCRIPTION OF THE MAIN ERROR TYPES

In the previous section, we analysed the success rate in the different items before and after instruction. However, solving a combinatorial problem is not a simple process and so the differentiation between the possible incorrect solutions is essential to both the teacher and the pupil. Once the questionnaire was completed by the pupils, we analyzed their responses, classifying all the mistakes according to the categorization that we shall describe in the following paragraphs. (A code will be included to identify the error type in frequency Table V).

Common errors in the different models of selection, distribution and partition

1. (STATEMENT): Misinterpretation of the problem statement: In particular there were three typical misinterpretations of the verbal statement:

a. Changing the type of mathematical model in the statement of the problem.
b. Some pupils transformed a single problem into a compound combinatorial problem. For example, in Item 7, some pupils correctly found the number of ways in which it is possible to distribute the mathematics project, but, after this, they multiplied this number by 2, because of the two different projects considered.
c. Interpreting the verbs " distribute", "divide" or "share" used in the partition problems as requiring a division of the two data given in the problem. For example in Item 4 (distribution of four cars for three children):

a. "*4/3=1; one car for each brother and there is a car left"*

2. (ORDER): Error of order: This mistake consists of confusing the criteria of combinations and arrangements, that is, distinguishing the order of the elements when it is irrelevant or, on the contrary, not considering the order when it is essential. This is one example taken from the solution of a pupil to Item 8 (selecting three pupils to clean the blackboard):

" E= Elisabeth, F= Ferdinand, G= George, L= Lucy and M=Mary
E F G, E F L, E F M, E G F, E G L, E G M, E LM, E L F, E L G
E M F, E M G, E M L; 12 x5= 60; you have 60 different ways."

*3. (REPETITION): Error of repetition:*The pupil does not consider the possibility of repeating the elements when it is possible, or he/she repeats the elements when there is no possibility of doing so. This is an example from Item 5 (selecting three numbers without replacement):

> "*724-742-722-772-744-472-427-477-444-422-274-247-277-222-244; 15 different numbers."*

4.(OBJECTS): Confusing the type of object: Considering that identical objects are distinguishable or that different objects are undistinguishable. For example, in Item 2 (permutations with repetitions of colored marbles):

> *" White: W Blue: B Red: R Blue 2: Z*
>
> *WBRZ WBZR WRBZ WRZB WZBR WZRB,...*
>
> *For each different marble 6 options; 4 marbles, so 6x4=24 different options"*

5. (EXCLUSION): Excluding some elements to form the configurations: This error was typical in the permutation with repetition problems, in which some pupils had considered that the repeated elements did not intervene in the permutation. For example, in Item 2, in which two blue counters, a white counter and another red counter must be permuted:

> *"4 counters; 2 blue, 1 white, 1 red. Factorial of 3, because blue is repeated. 3!=3x2=6"*

6. (LISTING): Non-systematic listing: This type of error was described by Fischbein and Gazit and consists of trying to solve the problem by listing using trial and error, without a recursive procedure that leads to the formation of all the possibilities.

*7. (INTANSWER): Mistaken intuitive answer:*The pupils only give a mistaken numerical solution, without justifying the response.

8. (OPERATIONS): Incorrect arithmetic operations for finding the solutions: Some pupils used some combination of the basic rules of product, addition and quotient to solve the problem, instead of applying a known formula or listing all the possibilities. Sometimes, they failed to apply the appropriate operation. For example, in Item1 (permutation of four boys) some pupils fixed the first boy in the permutation and obtained the right number of permutations by trial and error (six permutations). Then, instead of multiplying by 4, they added 4 to obtain the permutation of the 4 boys.

9. (FORMULA): Not remembering the correct formula of a combinatorial operation that has been correctly identified. For example, giving "$C_{4,3}$ = 4x3=12" as a solution to Item 3.

10. (PARAMETERS): Not remembering the meaning of the values of parameters in the combinatorial formula, e.g., the following answer to Item 4 (distribution of four cars between three children):

> *" That is an arrangement with repetition, because the same person may have the four cars: $VR_{4,3}=4^3$ = 64 ways in which the boy could give the cars to his brothers".*

11. (TREEDIAG): Faulty interpretation of the tree diagram: In spite of its importance, as a tool to produce the solution, very few pupils used a tree diagram after teaching, preferring to look for a convenient formula. Moreover, some pupils who tried to build a tree diagram to solve the problem, either produced an inadequate diagram or incorrectly interpreted the diagram produced.

12. (PROPERTY): Failing to apply a property of the combinatorial number, when needed: In the bipartition items the pupils must remember that, given a set with *m* elements, for each subset with *n* elements there is another complementary subset with exactly *m-n* elements, and therefore, $C_{m,n} = C_{m, m-n}$. Nevertheless, some of them failed to apply this property, as in the following example from Item 7 (distributing two projects between four pupils):

> *" Mathematics: C4,2= V4,2/P2= 4x3/2! =6 ways;*
>
> *Language: C4,2= V4,2/P2= 4x3/2! =6 ways;*
>
> *In total 36 ways to do the projects. The order is irrelevant. They are combinations"*

Additional specific errors in distribution and partition problems

13. (CELLS): Confusing the type of cell (the type of subsets): That is to say, believing that we could distinguish identical (subsets) cells or that it is not possible to differentiate the distinguishable cells (subsets). For example, in Item 7 (assigning two different tasks to four pupils) some pupils did not differentiate which group was going to complete the mathematics project and which was going to undertake the language project.

14. (PARTITION): Error in the partition obtained. This can occur in the following two ways: a) The union of all the subsets in a partition does not contain all the elements in the total set. For example, in Item 4:

> *"black, orange, white, grey for Peggy;*
>
> *black, orange, white, grey for John;*
>
> *black, orange, white, grey for Linda;*
>
> *black for Peggy;*
>
> *black for Linda;... "*

b) Some possible types of partitions are forgotten: For example, in Item 6, we have to divide a group of four children into two subgroups. To solve the problem, we need to consider all the following decompositions of the number 4: 4= 4+0= 3+1= 2+2=1+3=0+4. Nevertheless, as in the following example, some pupils only consider a subset of all the possible partitions:

" *10 ways:*

A, B, C, D = First floor	*A, B. C. D= Upstairs*
A, B, C= First floor; D= Upstairs	*A, B, D= First floor; C= Upstairs*
A, D, C= First floor; D= Upstairs	*B, C, D,= First floor; A= Upstairs*
A, B, C= Upstairs; D= First floor	*A, B, D= Upstairs; C= First floor*
A, D, C= Upstairs; D= First floor	*B, C, D,= Upstairs; A= First floor"*

In Table V, we present, for each of these types of errors, the frequency and average number of errors per pupil in the whole questionnaire in both samples. We can see that, before instruction, the main type of difficulty was their lack of systematic listing capacity. We also note the confusion in the type of objects, type of partition and the mistaken intuitive answer. With respect to the group of pupils with instruction, the two main errors were the error of order and repetition and new errors such as those of the *formula* and *misinterpretation of the tree diagram* appear in some pupils' answers.

There was not a big difference regarding the mean number of problems for which the pupils provided no solution in both groups. Nevertheless, the average number of errors in the whole test was 10.59 for the pupils without instruction and 7.01 after teaching. This shows a positive effect of instruction, although it is obvious that many pupils have not grasped the meaning of the combinatorial operation, since new types of errors appeared after teaching.

Table V. Frequency and average number of errors per pupil in both groups

Errors	Instruction group		No instruction group		Items in which this error had special incidence
	N.of errors	Mean per pupil	N.of errors	Mean per pupil	
1: STATEMENT	145	0.41	36	0.1	4; 6; 10
2: ORDER	787	2.24	153	0.44	3; 6; 7; 8; 9; 10; 12
3: REPETITION	563	1.6	145	0.42	2, 4, 6, 12
4:EOBJECTS	26	0.07	241	0.69	2, 3, 4
5: EXCLUSION	20	0.06	3	0.01	2, 12
6: LISTING	50	0.14	1678	4.82	All, except 4, 5
7: INTANSWER	29	.08	220	0.63	1, 9, 13
8: OPERATIONS	24	0.06	19	0.05	No difference
9: FORMULA	156	0.44			No difference
10: PARAMETERS	458	1.3	30	0.08	4, 6
11: TREEDIAG	36	0.09			3, 4, 6
12: PROPERTY	103	0.29	213	0.08	6, 7, 10
13: CELLS	42	0.1	280	0.8	4, 6, 7, 10
14: PARTITION	36	0.1	272	0.78	4, 6
More than 2 errors	226	0.64	388	1.05	No difference
No solution	549	1.6	648	1.86	No difference

To study the structure of these errors, a correspondence analysis (Greenacre, 1984) of the cross tabulation of items according to the type of answer (correct and error type) was performed using the BMDP statistical package, the results of which are described in great detail in Batanero et al. (1995). The Chi-square statistics value (χ^2=7968; p=0.000) was highly significant and the result of the analysis showed a multidimensional structure. The following four factors were identified:

a) First factor: complexity of the verbal statement of the problem (44.1% of the total inertia): This factor separated conceptual errors, in which the pupils failed to discriminate the basic features of the combinatorial configurations they were asked to count and the procedural errors. Items were ordered according to the difficulty index on this factor showing that the problem's difficulty was linked to conceptual type errors. The lack of systematic listing capacity and other procedural errors did not have such a great influence on the difficulty as the interpretation of different data in the statement of the problem.

b) Second factor: the combinatorial model (21.7 % of the inertia). In this factor, partition problems were opposed to the other two models, showing the specificity of some errors related to the partition problems, which we have noticed in the description of these errors

*c)Third factor: specificity of permutation problems (*11.3 % of the inertia): In this factor we noticed specific behavior of permutation problems, in which the variety of errors increased when adding supplementary conditions (repeated elements).

d) Fourth factor: Similarities and discrimination between combinations and permutation with repetitions (10.2 % of the inertia): This factor reflected the duality and discrimination between these two combinatorial operations and between the errors linked to each of them.

Finally, we noticed that the supplementary variable instruction had no influence on any of the factors. So, although instruction improved the overall difficulty of the problems, it did not influence the differentiation between the specific types of error linked to the particular type of items (partition model; permutations; permutations with repetition and combinations problems).

6. SOME PRACTICAL AND THEORETICAL IMPLICATIONS

Some task variables described in this paper, especially the *implicit combinatorial model*, revealed themselves as *didactic variables*, showing their strong effect on both the problem difficulty and the type of error. In particular, we noticed that some pupils who could apply the definition of the combinatorial operation for the selection model were not able to transfer this definition, when changing the problem to a different combinatorial model.

These variables have to be considered when organizing the teaching, which should also emphasize the translation of combinatorial problems into the different models, recursive reasoning and systematic listing procedures, instead of the mere centering on algorithmic aspects and on definitions of combinatorial operations. A possible development of Combinatorics in which this approach has been adopted may be found in our text (Batanero, Godino and Navarro-Pelayo; 1994) that includes a curricular proposal for 10-18 year-old pupils.

Another theoretical consequence refers to the assessment of the subjects' combinatorial knowledge, in which these task variables have to be considered to obtain a more valid and comprehensive account of its genesis and evolution.

Finally, some more general theoretical reflexions concerning didactic research could be raised, because of our results. Understanding a concept (e.g. combinations) cannot be reduced to simply being able to reproduce its definition. Concepts emerge from the *system of practices* carried out to solve problem-situations. Nevertheless, problems do not appear isolated, but they can be grouped into a *problem field*, for which a similar mathematical solution is applicable, with the frontier between different problem fields being fuzzy. For mathematicians, it is straightforward to extend or modify the valid solution for a given class of situations to other related problems. However, this may not be an easy task for pupils, because understanding a concept is a continuously increasing process achieved through a variety of situations in which this concept is brought into play.

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APPENDIX: QUESTIONNAIRE ON ELEMENTARY COMBINATORIAL REASONING

1. Four boys are sent to the headmaster for cheating. They have to line up in a row outside the head's room and wait for their punishment. No one wants to be first of course! Suppose the boys are called Andrew, Burt, Charles and Dan (A, B, C, D, for short). We want to write down all the possible orders in which they could line up. For example : A (first), B (second), C (third), D(fourth), we write ABCD. How many ways can the boys be lined up in?

2. In a box there are four colored counters: two of them are blue, another is white and the last one is red. We take one of the counters at random and we note down its color. We take another counter at random from the box without replacing the first one. We continue this process until we have selected all four counters. In how many different ways is it possible to select the counters? For example we could select the counters in the following sequence: white, blue, red and blue.

3. Suppossing we have three identical letters, we want to place them into four different colored envelopes: yellow, blue, red and green. It is only possible to introduce one letter in each different envelope. How many ways can the three identical letters be placed into the four different envelopes? For example, we could introduce a letter into the yellow envelope, another into the blue envelope and the last one into the green envelope.

4. A boy has four different colored cars (black, orange, white and grey) and he decides to distribute the cars to his friends Peggy, John and Linda. In how many different ways can he distribute the cars? For example he could give all the cars to Linda.

5. In an urn there are three marbles numbered with the digits 2, 4 and 7. We extract a marble from the urn and note down its number. Without replacing the first marble, we extract another one and note down its number. Finally, we extract the last marble from the urn. How many three-digit numbers can we obtain with this method? For example, we could obtain the number 724.

6. Four children: Alice, Bert, Carol and Diana go to spend the night at their grandmother's home. She has two different rooms available (one on the ground floor and another upstairs) in which she could place all or some of the children to sleep. In how many different ways can the grandmother place the children in the two different rooms? (She could use only one room to place the children). For example, she could use only one room to place the children, or she could place Alice, Bert and Carol in the ground floor room and Diana in the upstairs room.

7. Four friends Ann, Beatrice, Cathy and David must complete two different projects: one in Mathematics and the other one in Language. They decide to split up into two

groups of two pupils, so that each group could perform one of the projects. In how many different ways can the group of four pupils be divided to perform these projects? For example, Ann and Cathy could complete the Mathematics project and Beatrice and David the Language project.

8. Five pupils Elisabeth, Ferdinand, George, Lucy and Mary have volunteered to help the teacher in rubbing out the blackboard. In how many different ways can the teacher select three of the five pupils? For example, he could select Elisabeth, Mary and George.

9. The garage in Angel's building has five numbered places. As the building is very new, at the moment there are only three residents, Angel, Beatrice and Carmen to park their cars in the garage. This is a plan of the garage:

1	2	3	4	5

For example, Angel could park his car in place number 1, Beatrice in place number 2 and Carmen in place number 4. In how many different ways could Angel, Beatrice and Carmen park their cars in the garage?

10. Mary and Cindy have four stamps numbered from 1 to 4. They decide to share out the stamps, two for each of them. In how many ways can they share out the stamps? For example, Mary could keep the stamps numbered 1 and 2 and Cindy the stamps numbered 3 and 4.

11. In a box there are four numbered marbles (with the digits 2, 4, 7, 9). We choose one of the marbles and note down its number. Then we put the marble back into the box. We repeat the process until we form a three -digit number. How many different three- digit numbers is it possible to obtain? For example, we could obtain the number 222.

12. Each one of five cards has a letter: A, B, C, C and C. In how many different ways can I form a row by placing the five cards on the table? For example I could place the cards in the following way: ACBCC.

13. Given a three member committee (president, cashier and secretary) and 4 candidates: (Arthur, Ben, Charles and David), how many different commitees could be selected?

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