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Ring of continuous functions

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Summary

Given any topological space X, we denote by $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{R})$ the set of all continuous functions from X with values in $\mathbb{R}$. The set $\mathcal{C}(X)$ has a ring structure: the sum and the product are defined pointwise, and the constant function equal to 1 plays the role of the unity. The main goal of this study is to show that the ring structure of $\mathcal{C}(X)$, on one side, reflects the topological properties of X and, on the other hand, for different topological spaces X it provides examples of rings which are useful for constructing counterexamples to certain statements in Commutative Ring Theory. For example, when X is a completely regular space ($T_{3\frac{1}{2}}$ or Tychonoff), the behavior of prime and maximal ideals is quite interesting:

- The maximal ideals of $\mathcal{C}(X)$ are parameterized by the points of the Stone–Čech compactification of X.
- Every prime ideal of $\mathcal{C}(X)$ is contained in a single maximal ideal.
- The ideals of $\mathcal{C}(X)$ containing a prime ideal form a chain.
- Every sum of prime ideals in $\mathcal{C}(X)$ is again a prime ideal.

This is a sample of the results we intend to state; that is, that the topological properties of the space X can be studied through algebraic properties of the ring $\mathcal{C}(X)$.

In this project the ideals play a fundamental role, and in particular the prime and maximal ideals. In addition, the following types of ideals appear: convex and absolutely convex, z-ideals, d-ideals, pure ideals, minimal prime ideals, essential and uniform ideals, the socle of the ring $\mathcal{C}(X)$ and many other types of ideals that can be denoted by topological properties on X.
In the same vein we can recover classical results such as Gel'fand–Kolmogorov theorem which states that if X is a compact Hausdorff space, and we take as a base the field $\mathbb{C}$, we obtain an identification of X with the set of maximal ideals of the ring $\mathcal{C}(X)$. This result can be seen as the topological version of the well known Hilbert zeros theorem in Algebraic Geometry.

Introduction

This paper is mainly focused in the algebra field, although it contains several topological results and topological theory. Our main objective is to introduce the set of continuous functions on a topological space X, endowing it with a lattice ring structure and studying prime and maximal ideals in it. It turns out that those ideals and some topological objects called, z–filters, are highly related. Our job is to give the necessary tools in order to join both of those concepts.

An intensive study has been made on the topology in our space X. Thanks to the book "*Ring of continuous functions*" [3] we were able to study this subject deeper. We have concluded that the best spaces for our purpose are the completely regular and normal ones. We have also been able to apply both Urysohn's Theorems; one of them has been used to extend continuous functions to a bigger space and the other one states that every normal space is completely regular.

Also, we studied another type of ideals called d–ideals in reduced rings. They are based in the annihilator so it's a purely algebraic object. They give us information about zero–divisors and we can characterize them making use of the Jacobson radical.

Finally, we wanted to study ideals in the ring $\mathcal{C}(X)$ when X is compact, and we obtained several properties and characterizations. We were also able to construct the Stone–Čech compactification and we discovered that this new space brings us some grateful properties. Of course we took advantage of this new concept. A bunch of useful results were born and it allowed us to present, for instance, Tychonoff's product Theorem.

In the last chapter we concluded this paper by giving and proving the Gel'fand-Kolmogorov Theorem, which characterizes totally the maximal ideals in $\mathcal{C}(X)$.

Although there appear many types of ideals, there are a bunch more that we haven't talked about, such as convex ideals, pure ideals or essential and uniform ideals for instance. This only means the theory behind this paper is huge and the reader can sharpen its knowledge about this subject by reading articles or books pointed in the references.

1 Preliminary results

In this chapter we will recall concepts of Algebra and Topology that will be recurrent throughout the project.

1.1 Introduction to algebra

It is assumed that the reader knows how to work with these algebra terms, however, it is convenient to have them on hand.

Definition 1.1. Given a partial order in a set $\mathcal{S}$, we say that $\mathcal{S}$ is **partially ordered set**, and for all $a, b \in \mathcal{S}$, if $a \leq b$ and $a \geq b$, then $a = b$.

Definition 1.2. In a partially ordered set $\mathcal{S}$, when both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist then $\mathcal{S}$ is called a **lattice**. If we take a subset $\mathcal{A} \subseteq \mathcal{S}$ and for whatever $a, b \in \mathcal{A}$ its supremum and infimum stays in $\mathcal{A}$, we call it a **sublattice**.

Definition 1.3. In a lattice $\mathcal{S}$, we denote $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ for every $a, b \in \mathcal{S}$.

Lemma 1.4. (Zorn's lemma) *Every partially ordered set containing upper bounds for every chain (that is, every totally ordered subset) necessarily contains at least one maximal element.*

The notation to refer to convergence, closure of a set or any basic property will not be strange to the reader. Now we are going to introduce the concepts of rings, homomorphisms of rings and ideals in order to work freely with them later. We will also see some examples that will be familiar to us.

Definition 1.5. A **ring** is a set R on which three operations are defined,

$$
\begin{aligned}
&+ : \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R},\\
&\cdot : \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R} \text{ and}\\
&1 : \{1\} \longrightarrow R,
\end{aligned}
$$

known as **sum**, **product**, and unity element that satisfies the next properties:

(1) The sum is **associative**: $r + (s + t) = (r + s) + t$, for all $r, s, t \in \mathrm{R}$.

(2) The sum is **commutative**: $r + s = s + r$, for all $r, s \in \mathrm{R}$.

(3) Existence of **zero element**: There exists an element $0 \in \mathrm{R}$ such that $r + 0 = r$ for all $r \in \mathrm{R}$.

(4) Existence of **opposite element**; For all $r \in \mathrm{R}$, there exists $-r \in \mathrm{R}$ such that $r + (-r) = 0$.

(5) The product is **associative**: $r \cdot (s \cdot t) = (r \cdot s) \cdot t$, for all $r, s, t \in \mathrm{R}$.

(6) Existence of **unity element**: There exists an element $1 \in R$ such that $r \cdot 1 = r = 1 \cdot r$ for all $r \in \mathrm{R}$.

(7) The product is **distributive** with respect to the sum: $r(s+t) = rs+rt$ and $(r+s)t = rt+st$, for all $r, s, t \in \mathrm{R}$.

If it also satisfies the commutative property for the product, R is said to be a **commutative ring**.

Remark 1.6. It is easy to proof that both neutral elements for the sum and the product are unique, so does the opposite element for the sum.

Example 1.7. If we take one of the most common rings we already know (these are $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$) with the usual sum, product, and unity element, we get a commutative ring.

Example 1.8. However, given a field $\mathbb{K}$, if we establish $\mathcal{M}_2(\mathbb{K})$ as the square matrix set with 2 rows and 2 columns with its coefficients in $\mathbb{K}$ and we define the sum coordinate by coordinate and the product as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{pmatrix}$$

Then is easy to prove that $(\mathcal{M}_2(\mathbb{K}), +, \cdot, 1)$ is a ring, although it is not commutative.

In the following we will work mainly with commutative rings.

Definition 1.9. Let R and R′ be two rings and $f : \mathrm{R} \longrightarrow \mathrm{R}'$ a mapping. Then f is said to be a **ring homomorphism** if it satisfies:

(1) For all $r, s \in \mathrm{R}, f(r+s) = f(r) + f(s)$.

(2) For all $r, s \in \mathrm{R}, f(r \cdot s) = f(r) \cdot f(s)$.

(3) $f(1) = 1$, where we must distinguish that they are the corresponding ones of their ring.

The last thing we are going to recall is the definition of an ideal, but also prime and maximal ideals.

Definition 1.10. Given R a ring, $\mathfrak{a} \subseteq \mathrm{R}$ is said to be an **ideal** of R if it satisfies two properties:
(1) $r + s \in \mathfrak{a}$ for all $r, s \in \mathfrak{a}$,
(2) $r \cdot s \in \mathfrak{a}$ for all $s \in \mathrm{R}$ and $r \in \mathfrak{a}$.

We will notate ideals as $\mathfrak{a} \leq \mathrm{R}$

Theorem 1.11. (First isomorphism theorem) *Let R, S be rings and $f : R \longrightarrow S$ an homomorphism. Then these three states hold true:*

(a) Ker(f) is an ideal of R.

(b) Im(f) is a subring of S.

(c) Im(f) is isomorphic to R/Ker(f).

In particular, if f is surjective, then S is isomorphic to $R/Ker(f)$.

Proposition 1.12. *The intersection of ideals is again an ideal.*

Proposition 1.13. *Given a ring R and $B \subseteq R$ we can define*

$$I(B) = \{a \in R |\ ab = 0 \text{ for every } b \in B\}.$$

Then, $I(B)$ is an ideal in R

Definition 1.14. An ideal $\mathfrak{a} \subseteq \mathrm{R}$ is said to be **prime** if for every pair $a, b \in \mathrm{R}$ such that $a \cdot b \in \mathfrak{a}$, then $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$. The set of all prime ideals is called the **spectrum** of R and is notated Spec(R).

Definition 1.15. An ideal $\mathfrak{a} \subseteq \mathrm{R}$ is said to be **maximal** if there isn't another ideal $\mathfrak{b} \leq \mathrm{R}$ such that $\mathfrak{a} \subsetneq \mathfrak{b} \subsetneq \mathrm{R}$. The **maximal spectrum**, Max(R), of R is the set of all maximal ideals of R.

Definition 1.16. An ideal $\mathfrak{a} \subseteq \mathrm{R}$ is said to be **principal** if it is generated by just one element. This is:

$$\mathfrak{a} = \langle a \rangle.$$

Definition 1.17. The **minimal spectrum**, Min(R), of R is the set of all minimal prime ideals of R.

Definition 1.18. A ring R is named **von Neumann regular** whenever for every element $a \in \mathrm{R}$ there exists $x \in \mathrm{R}$ satisfying $a = axa$.

Definition 1.19. An ideal $\mathfrak{a} \subseteq \mathrm{R}$ is said to be **radical** whenever

$$\mathfrak{a} = \mathrm{rad}(\mathfrak{a}) = \{a \in \mathrm{R} \mid \text{ such that } a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}.$$

Definition 1.20. The radical of $\{0\} \subseteq \mathrm{R}$ is $\mathrm{rad}(0) = \mathrm{Nil(R)}$, and it is called the **nilradical** of R. The ring R is **reduced** or **semiprime** whenever $\mathrm{Nil(R)} = 0$.

Definition 1.21. The **Jacobson radical** of R is

$$\mathrm{Jac(R)} = \bigcap \{\mathfrak{m} |\ \mathfrak{m} \in \mathrm{Max(R)}\} = \{x \in \mathrm{R} |\ 1 + xy \text{ is invertible for any } y \in \mathrm{R}\}.$$

The ring R is called **primitive** whenever $\mathrm{Jac(R)} = 0$.

Definition 1.22. In a ring R, the **annihilator** of an element $a \in \mathrm{R}$ is

$$Ann(a) = \{x \in \mathrm{R} |\ ax = 0\}$$

and the **double annihilator**

$$Ann^2(a) = \{x \in \mathrm{R} |\ xy = 0 \text{ for every } y \in Ann(a)\}.$$

Observe that, for any $a \in R$, we have: $\mathrm{Ann}^3(a) = \mathrm{Ann}(a)$.

Remark 1.23. It is easy to prove that $Ann^i(a)$ is an ideal of R for $i = 1, 2$.

Given these basic algebra terms, we will go on with some other basic topological concepts.

1.2 Introduction to topology

As for notation, we will use bold letters for constant functions. For example, $\mathbf{r}(x) = r \in \mathbb{R}$, for every $x \in \mathbb{R}$. In addition we will call $\mathbf{i}$ the identity function in $\mathbb{R}$ or in any subset of it and $\mathbf{j}$ the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$.

Definition 1.24. A **topological space** is a set X endowed with a collection of open sets $\mathcal{T}$ that satisfies four conditions:
(1) $\emptyset \in \mathcal{T}$.
(2) $X \in \mathcal{T}$.
(3) The intersection of a finite number of open sets is again an open set.
(4) The union of an arbitrary number of open sets is again an open set.

With these conditions, $(X, \mathcal{T})$ is a topological space.

Remark 1.25. We can also define the topology making use of the closed sets with the formula $F = X \backslash U$ for every $U \in \mathcal{T}$ (the intersection becomes arbitrary and the union becomes finite in these conditions).

Definition 1.26. In a topological space $(X, \mathcal{T})$, a subset $\mathcal{B} \subseteq \mathcal{T}$ is a **base of open sets** whenever it generated all subsets of open sets. This is, if $U \in \mathcal{T}$ and $\{B|\ i \in I\}$ is a base of open sets, then we can write $U = \bigcup_{j\in J} B_j$ with $J \subseteq I$.

Remark 1.27. As we saw before, we can also obtain a base of closed sets by the formula $F = X \backslash U$.

Definition 1.28. Given a topological space $(X, \mathcal{T})$ and $x \in X$. We say $N \subseteq X$ is a **neighborhood** of x if there exists $U \in \mathcal{T}$ such that $x \in U \subseteq N$.

We notice there are a few different ways to define a topology, making use of open sets, closed sets, a base of open sets, a base of closed sets or a base of neighborhoods. As we know, the importance of studying the topology of some spaces relies on the study of its invariants. In this paper we will study continuous functions, so we need to connect algebraic with topological objects, and how the properties of one object gives some other properties on the second. One of these properties is the Hausdorff property, which will be recurrent throughout the paper.

Definition 1.29. A topological space $(X, \mathcal{T})$ is **Hausdorff** whenever for every $x, y \in X$ with $x \neq y$ we can find N_x and N_y neighborhoods for x and y such that $N_x \bigcap N_y = \emptyset$.

Due to our need of using continuous functions, we shall study the smallest topology on X such that all functions in $\mathcal{C}$ (or in a subspace of it) are continuous, this is, the **weak topology**.

Definition 1.30. Let $(X, \mathcal{T})$ be a topological space. We say a function $f : X \longrightarrow \mathbb{R}$ is **continuous** whenever for any open subset $V \subseteq \mathbb{R}$ we have $f^{-1}(V) \in \mathcal{T}$.

Definition 1.31. For any space X and any subset of the real-valued functions on X we denote $\mathcal{S}$ as the collection of all the preimages of those functions. This means that any subset U contained in X belongs to $\mathcal{S}$ if and only if there exists a function f and V an open set in $\mathbb{R}$ such that $U = f^{-1}(V)$. The weak topology generated by any subset of those functions is the smallest topology containing $\mathcal{S}$. Therefore, it is the topology for which $\mathcal{S}$ is a **subbasis**.

2 Functions in a topological space

In this chapter we introduce the protagonist of this paper as well as the necessary tools in order to work with it. If we call $\mathbb{R}^X$ to the set of all the real–valued functions with a topological space X as its domain, then we can take some suitable functions in that set: Those are the continuous ones.
For any topological space X we can define:

$$\mathcal{C}(X, \mathbb{R}) = \{f : X \longrightarrow \mathbb{R} |\ f \text{ is continuous}\}.$$

From now on, the arrival domain will be the same throughout the paper, so we will denote that set as $\mathcal{C}(X)$, and it is the set of all the continuous functions from X to $\mathbb{R}$.
We need to endow it first with a ring structure. We need two operations in it and we are introducing them as intuitively as possible. For every $f, g \in \mathcal{C}(X)$

$$(f + g)(x) = f(x) + g(x) \quad (fg)(x) = f(x)g(x), \text{ for all } x \in X.$$

Due to the properties of $\mathbb{R}$, it is easy to prove that $(\mathcal{C}(X), +, \cdot, \mathbf{1})$ is a **commutative** ring. If we recall those properties, we need a neutral element for the sum, an opposite for the sum and the neutral element for the product. There are no surprises at this point; the zero element is the constant zero function, that gives for all $f \in \mathcal{C}(X)$,

$$(f + \mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x), \text{ for all } x \in X.$$

Now given $f \in \mathcal{C}(X)$,

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0, \text{ for all } x \in X.$$

Hence, we can deduce $-f$ is the opposite element for the sum for $f \in \mathcal{C}(X)$. Of course, the neutral element for the product can't be any other than the constantly one function; for every $f \in \mathcal{C}(X)$:

$$(f \cdot \mathbf{1})(x) = f(x) \cdot \mathbf{1}(x) = \mathbf{1}(x) \cdot f(x) = f(x) \text{ for all } x \in X.$$

In case there exists the inverse of a given function, we will notate it as it follows:

$$f^{-1}(x) = \frac{1}{f(x)}, \text{ for all } x \in X.$$

Once we have proven that $\mathcal{C}(X, \mathbb{R})$ is a ring, we should look for an order relation, that will allow us to compare functions. We are used to work with totally ordered rings (or fields), but that will not be our case when X contains more than two elements. We have to sharpen a little more the chosen order in this ring.

2.1 Ordered rings

Remember our ring changes every time we change the topological space. However, the idea is to convince the reader that the structure becomes invariant, no matter what space we choose.

Definition 2.1. Let R be a ring and $\leq$ be a partial order relation. We say

- $\leq$ is **compatible with the sum** whenever, for any $a, b, c \in \mathrm{R}$, if $a \leq b$, then $a + c \leq b + c$.
- $\leq$ is **compatible with the product** whenever, for any $a, b, c \in \mathrm{R}$, if $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

The pair $(\mathrm{R}, \leq)$ is an **ordered ring** whenever the partial order $\leq$ is compatible with the sum and the product.

Definition 2.2. In an ordered ring we define:

- The **positive cone** as:
$$\mathrm{R}_0^+ = \{a \in \mathrm{R} |\ 0 \leq a\}.$$
- The **negative cone** as:
$$\mathrm{R}_0^- = \{a \in \mathrm{R} |\ a \leq 0\}.$$

Some properties are immediate:

Lemma 2.3. *(1) The intersection of both cones is the zero element. This is:*

$$R_0^+ \cap R_0^- = \{0\}.$$

(2) The positive cone is closed under the sum and the product.

The following characterization is also very intuitive:

Lemma 2.4. *For any ring R and partial order relation $\leq$ compatible with the sum, the following statements are equivalent:*

(a) $\leq$ is compatible with the product.

(b) R_0^+ is closed under the product.

Proof (a) $\Rightarrow$ (b).If $0 \leq a, b$, then $0 \leq ab$ for any $a, b \in \mathrm{R}$.
(b) $\Rightarrow$ (a).If $a \leq b$ and $0 \leq c$, then $0 \leq b - a$, and $0 \leq (b - a)c = bc - ac$. Hence, $ac \leq bc$. □

We would like now to give some kind of characterization for a partial order. This result gives us that desire given a subset with some properties.

Proposition 2.5. *For any ring R and any subset $S \subseteq R$ such that*

- $0 \in S$.
- *S is closed under the sum and the product.*
- $S \cap (-S) = \{0\}$.

we shall define a partial order relation as it follows:

$$a \leq b \text{ whenever } b - a \in S.$$

This way, we claim:

(1) $\leq$ *is a partial order relation in R.*

(2) R is an ordered ring.

PROOF Since $0 \in S$, then $\leq$ satisfies the reflexive property. It also satisfies the transitive property by being closed under the sum. The point $S \cap (-S) = \{0\}$ gives us the anti-symmetric property. Hence, $\leq$ is a partial order relation in R.
Now, for any $a, b, c \in \mathrm{R}$, if $a \leq b$, then $0 \leq b - a$, and we have:

$$0 \leq b - a = b - a - c + c = (b + c) - (a + c),$$

thus $a + c \leq b + c$.
Therefore, R is an ordered ring because S is closed under the product. $\square$

We have managed to determine an ordered ring by its positive cone called S. Of course the negative cone is $-S$.
In the following, we define $\mathrm{R}^+ = \mathrm{R}_0^+ \backslash \{0\}$, the strict positive cone and we can do the same for R^-.

Proposition 2.6. *Every ordered ring R with a nontrivial order relation is infinite.*

PROOF Since the order relation is nontrivial, there exist $a, b \in \mathrm{R}$ such that $a < b$, hence $0 \neq b - a \in \mathrm{R}^+$. For any $0 \neq x \in \mathrm{R}^+$ we have $0 \leq x \leq 2x$, and so on. If $n = \mathrm{car}(\mathrm{R})$, we have:

$$0 \leq x \leq (n-1)x \leq nx = 0$$

therefore $x = 0$, which is a contradiction. $\square$

In general, we have used that R_0^+ is closed under the product. If we restrict it to the domain R^+ we have a different notion, which coincides with the one we had whenever R is an integral domain. Let's study this with two examples.

Example 2.7. Let us consider the quotient ring

$$\mathrm{R} = \frac{\mathbb{R}[X]}{X^2},$$

and take

$$\mathrm{R}_0^+ = \{(a, b) |\ 0 < a \text{ or } a = 0 \text{ and } 0 \leq b\}.$$

We can say that it contains the zero element, it is closed under the sum and the product and of course $\mathrm{R}_0^+ \cap (-\mathrm{R}_0^+) = \{(0, 0)\}$. Thus, R is an ordered ring and it also satisfies $(0, 1) \in \mathrm{R}^+$ and $(0, 1)^2 = (0, 0) \notin \mathrm{R}^+$.

An ordered ring R is called **strictly ordered ring** whenever if $0 < a$ and $0 < b$, then $0 < ab$. Every ordered ring which is also an integral domain is an strictly ordered ring. Moreover, for any $a, b \in \mathrm{R}^+$, since $a, b \neq 0$, then $0 \neq ab \in \mathrm{R}_0^+$, hence $ab \in \mathrm{R}^+$. The converse doesn't necessarily holds.

2.1.1 Totally ordered sets

We now know the notion of an ordered ring (this is also called a partially ordered ring). Now we call a **totally ordered ring** an ordered ring $(\mathrm{R}, \leq)$ in which the order relation is total, this is, for any $a, b \in \mathrm{R}$ either $a \leq b$ or $b \leq a$.

Lemma 2.8. *A partial ordered ring R with positive cone R_0^+ is a totally ordered ring if and only if $R = R_0^+ \cup R_0^-$.*

Proof (⇒). Of course if we take $a \in \mathrm{R}$, either $0 \leq a$ or $a \leq 0$, hence $a \in \mathrm{R}_0^+$ or $a \in \mathrm{R}_0^-$.
(⇐). Similarly, if we take $a, b \in \mathrm{R}$, if $(b-a) \in \mathrm{R}_0^+$ then $a \leq b$. On the other hand, if $(b-a) \in \mathrm{R}_0^-$ then $(a-b) \in \mathrm{R}_0^+$, hence $b \leq a$. □

Remark 2.9. We can deduce from that result that in a totally ordered ring we have a partition

$$\mathrm{R} = \mathrm{R}^+ \cup \{0\} \cup \mathrm{R}^-.$$

Of course we call the element of R^+ the **positive** elements of R and the element of R^- the **negative** ones.

Lemma 2.10. *In a totally ordered ring R, for any $a \in R\backslash\{0\}$ we have $a^2 \in R_0^+$. In particular, $1 \in R^+$, hence $-1 \in R^-$.*

Proof Since $a \neq 0$, then $a \in \mathrm{R}^+ \cup \mathrm{R}^-$. If $a \in \mathrm{R}^+$ then $a^2 \in \mathrm{R}^+$. If not, $a \in \mathrm{R}^-$, then $-a \in \mathrm{R}^+$ and $a^2 = (-a)^2 \in \mathrm{R}^+$. □

Remark 2.11. Totally ordered rings non necessarily are integral domains.

However, we can give a condition so we have a totally ordered ring which is also an integral domain.

Proposition 2.12. *Given a totally ordered ring R, the following statements are equivalent:*

(a) R is an integral domain.

(b) R is an strictly ordered ring.

Proof (a)⇒(b). If $a, b \in \mathrm{R}^+$, since $a, b \neq 0$ then $ab \in \mathrm{R}^+$.
(b)⇒(a). Let $0 \neq a, b \in \mathrm{R}$. If $ab = 0$ we have three different cases:
(1) $a, b \in \mathrm{R}^+$, hence $ab \in \mathrm{R}^+$ and $ab \neq 0$.
(2) $a \in \mathrm{R}^+$ and $b \in \mathrm{R}^-$, then $-b \in \mathrm{R}^+$ and $-ab = a(-b) \in \mathrm{R}^+$, hence $ab \neq 0$.
(3) $a, b \in \mathrm{R}^-$, then $-a, -b \in \mathrm{R}^+$, hence $ab = (-a)(-b) \in \mathrm{R}^+$. Therefore, $ab \neq 0$.

□

Definition 2.13. In a totally ordered ring R for any element $a \in \mathrm{R}\backslash\{0\}$ we can define the **absolute value** of a as

$$|a| = \begin{cases} a, & \text{if } a \in \mathrm{R}_0^+, \\ -a, & \text{if } a \in \mathrm{R}_0^-. \end{cases}$$

As expected, the absolute value of $0 \in \mathrm{R}$ is 0.

Proposition 2.14. *In a totally ordered ring R, for any $a, b \in R$ we have*

$$|ab| = |a||b|.$$

2.2 Induced orders

Now we know several things about order relations we are able to recognize some structure in our ring of the continuous functions $\mathcal{C}(X, \mathbb{R})$ with X a topological space.
From now on we are calling R^X the ring of all functions from X into R.

Proposition 2.15. *Let R be an ordered ring and $X \neq \emptyset$. Consider the relation*

$$f \leq g \text{ whenever } f(x) \leq g(x) \text{ for every } x \in X$$

in R^X. Then we have that:

(1) $\leq$ is a partial order relation, though in general it is not a total order.

(2) R^X is an ordered ring.

(3) There is no monotone ring map $\varphi : R \longrightarrow \mathrm{R}^X$ defined as $\varphi(a)(x) = a$ for every $x \in X$.

(4) If R is a totally ordered ring, for every $f \in \mathrm{R}^X$ we define $|f| \in \mathrm{R}^X$ as $|f|(x) = |f(x)|$ the absolute value of f.

An ordered ring R is a **lattice ring** whenever R is a lattice set.

Remark 2.16. In a different way, we can also define the absolute value of any element a of a lattice ring as it follows:

$$|a| = a \vee (-a).$$

Lemma 2.17. *If R is a lattice ring, for any elements $a, b, c \in R$ we have*

$$(a \vee b) + c = (a + c) \vee (b + c) \text{ and } (a \wedge b) + c = (a + c) \wedge (b + c).$$

Theorem 2.18. *For any lattice ring R and any non–empty set X, the ordered ring R^X is a lattice ring.*

PROOF For any $f, g \in \mathrm{R}^X$ we define for every $x \in X$

$$(f \vee g)(x) = f(x) \vee g(x).$$

First we have $f, g \leq f \vee g$. On the other hand, for any $h \in \mathrm{R}^X$ such that $f, g \leq h$ and any $x \in X$ we have:

$$(f \vee g)(x) = f(x) \vee g(x) \leq h(x),$$

hence $f \vee g \leq h$, and $f \vee g$ is the supremum of f and g. Similarly, we have that $f \wedge g$, defined

$$(f \wedge g)(x) = f(x) \wedge g(x)$$

for every $x \in X$ is the infimum of f and g in R^X. □

Corollary 2.19. *For any topological space X the ring $\mathcal{C} = \mathcal{C}(X, \mathbb{R})$ is a lattice ring.*

Proof Just consider the positive cone

$$\mathcal{C}^+ = \{f \in \mathcal{C} |\ 0 \leq f\}$$

with the point–wise order relation. As in the theorem, the supremum and infimum are well defined. □

2.3 The lattice ring C

Once we have endowed our ring $\mathcal{C}(X)$ with a lattice ring structure we are able to use its order relation.
$\mathcal{C}(X, \mathbb{R})$ is not totally ordered when X contains more than two elements, so we have a partially ordered ring and it looks like this:

$$f, g \in \mathcal{C}(X), f \leq g \Longleftrightarrow f(x) \leq g(x), \text{ for all } x \in X.$$

In addition, if $f, g \geq 0$ then $(fg) \geq 0$. This definition didn't surprise anyone, since its the most intuitive one and for sure it is the one the reader had in his mind.

Definition 2.20. We say a function $f \in \mathcal{C}(X)$ is **bounded** whenever $f(X)$ is a bounded subset of $\mathbb{R}$. The set of all the bounded functions in X is represented by $\mathcal{C}^*(X)$ and it is clear that $\mathcal{C}^*(X) \subseteq \mathcal{C}(X)$.

Proposition 2.21. *Let X be a topological space. Then, $\mathcal{C}^*(X)$ is a subring of $\mathcal{C}(X)$*

Proof There are three points we must prove. Both neutral elements for the sum and product must be in $\mathcal{C}^*(X)$ and this is true because the constant zero function is bounded and so does the identity function. Now we must prove that given $f, g \in \mathcal{C}^*(X)$, $(f - g) \in \mathcal{C}^*(X)$.

$$f \in \mathcal{C}^*(X) \Rightarrow \text{ we can find } m_1 \in \mathbb{R}^+ \text{ such that } |f| \leq m_1.$$

Using the same argument:

$$g \in \mathcal{C}^*(X) \Rightarrow \text{ we can find } m_2 \in \mathbb{R}^+ \text{ such that } |g| \leq m_2.$$

Therefore, we can ensure that $(f - g) \leq m_3$ with $m_3 \in \mathbb{R}^+$. Hence, $(f - g) \in \mathcal{C}^*(X)$.
Finally we must see if it's closed for the product. It is the same idea as we did for the sum but way easier because the product of two constant functions is again a constant function, so $fg \in \mathcal{C}^*(X)$ trivially for every $f, g \in \mathcal{C}^*(X)$. Hence $\mathcal{C}^*(X)$ is a subring of $\mathcal{C}(X)$. □

It is remarkable to say that the order stated can be described by giving those functions $f \in \mathcal{C}(X)$ that fulfill the condition $f(x) \geq 0$ for every $x \in X$. Nevertheless, the prerequisite $f \geq 0$ only means that f is a square, this is, $f = g^2$ of some $g \in \mathcal{C}(X)$. Furthermore, we can easily determine a function $|f|$ by taking one of the roots of the function f^2, specifically the non–negative one. A result can be deduced from this argument:

Proposition 2.22. *Let $\tau : \mathcal{C}(X) \to \mathcal{C}(Y)$ be an isomorphism. Then τ preserves order. Moreover, an isomorphism between $\mathcal{C}^*(Y)$ and $\mathcal{C}(X)$ preserves order too (whenever f is bounded and $f = g^2$, then g is also bounded).*

We can generalize this result:

Theorem 2.23. *Let $\varphi : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$ be a ring homomorphism. Then φ is a lattice homomorphism (we can change $\mathcal{C}(Y)$ by $\mathcal{C}^*(Y)$).*

Proof φ sends non–negative functions into non–negative functions since $f = g^2$ implies $\varphi f = (\varphi g)^2$, hence φ is order–preserving. Now,

$$(\varphi|f|)^2 = \varphi(|f|^2) = \varphi(f^2) = (\varphi f)^2,$$

and since $\varphi|f| \geq \mathbf{0}$, we have $\varphi|f| = |\varphi f|$. If we combine this with the formula

$$(f \vee g) + (f \vee g) = f + g + |f - g|,$$

we get

$$\varphi(f \vee g) + \varphi(f \vee g) = \varphi f + \varphi g + |\varphi f - \varphi g| = (\varphi f \vee \varphi g) + (\varphi f \vee \varphi g).$$

But $\varphi(f \vee g)$ and $\varphi f \vee \varphi g$ are both functions defined on X that take values in $\mathbb{R}$, hence

$$\varphi(f \vee g) = \varphi f \vee \varphi g$$

□

Another intuitive result comes immediately to our minds. We can also determine boundedness of those functions through the algebraic structure of $\mathcal{C}(X)$. It yields:

Theorem 2.24. *Every ring map $\varphi : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$ preserves the boundedness of functions (we can also change $\mathcal{C}(Y)$ by $\mathcal{C}^*(Y)$).*

Proof We have that $\varphi\mathbf{1}$ is an idempotent since

$$\varphi\mathbf{1} = \varphi(\mathbf{1} \cdot \mathbf{1}) = (\varphi\mathbf{1})(\varphi\mathbf{1}).$$

Therefore it only takes the value 0 or 1 in X. Therefore for every $n \in \mathbb{N}$, the function

$$\varphi\mathbf{n} = \varphi\mathbf{1} + \cdots + \varphi\mathbf{1}$$

assumes only the values 0 and n. Considering then an arbitrary function $f \in \mathcal{C}^*(Y)$; since $|f| \leq n$, for an appropriate $n \in \mathbb{N}$, we have $|\varphi f| \leq \varphi\mathbf{n} \leq \mathbf{n}$. □

As a consequence, we have the next corollary:

Corollary 2.25. *Let $\varphi : \mathcal{C}(Y) \to \mathcal{C}(X)$ be a ring homomorphism such that $\mathcal{C}^*(X) \subset Im(\varphi)$. Then $\varphi(\mathcal{C}^*(Y)) = \mathcal{C}^*(X)$.*

PROOF We need to prove first that $\varphi\mathbf{1} = \mathbf{1}$. Take $k \in \mathcal{C}(Y)$ such that $\varphi k = \mathbf{1}$. Then:

$$\varphi\mathbf{1} = (\varphi k)(\varphi\mathbf{1}) = \varphi(k \cdot \mathbf{1}) = \varphi k = \mathbf{1}.$$

This means that $\varphi\mathbf{n} = \mathbf{n}$ for each $n \in \mathbb{N}$.
Now, for a function $f \in \mathcal{C}^*(X)$, take $h \in \mathcal{C}(Y)$ such that $\varphi h = f$, and also choose $n \in \mathbb{N}$ satisfying $|f| \leq \mathbf{n}$. Last, we shall define $g = (-\mathbf{n} \vee h) \wedge \mathbf{n}$, so $g \in \mathcal{C}^*(Y)$ and by Theorem (2.23.),

$$\varphi g = (-\mathbf{n} \vee f) \wedge \mathbf{n} = f.$$

□

Definition 2.26. If $\mathcal{C}(X) = \mathcal{C}^*(X)$ for any topological space X, we will call it a **pseudo–compact** space.

Once we know the meaning for a space to be pseudo–compact, we get an important corollary from Corollary (2.25.):

Corollary 2.27. *If a topological space X is not pseudo–compact, there is no space Y that yields that the ring $\mathcal{C}(X)$ is a homomorphic image of $\mathcal{C}^*(Y)$.*

Specifically, $\mathcal{C}(X)$ and $\mathcal{C}^*(X)$ can only be isomorphic if, and only if, they are essentially the same.

Proposition 2.28. *These statements hold:*

(i) Every compact space X is pseudo–compact.

(ii) Every countable compact space X is pseudo–compact.

PROOF Let's prove the first result. X is a compact space, therefore it exists a closed set J such that $X \subseteq J$. In that case every continuous function will be bounded. In the second case we can take the subsets $J_n = \{x \in X \text{ such that } |f(x)| < n\}$. They constitute a countable open covering of X, thus we can find $n \in \mathbb{N}$ such that $J_n = X$ and we use the previous result. It doesn't mind if it's closed or not, the point is that both of those sets are bounded. □

2.4 Zero–sets

Until now We have a lattice ring structure on $\mathcal{C}(X, \mathbb{R})$. Now we shall to introduce the zero sets. They will be useful and recurrent throughout this paper.

Definition 2.29. Given a topological space X, for every $f \in \mathcal{C}(X)$ we define the **zero–set** of f as:

$$Z(f) = \{x \in X | \; f(x) = 0\}$$

Remark 2.30. The first thing we can deduce from this definition is that $Z(f) = Z(|f|) = Z(f^2)$.

Remark 2.31. Furthermore, we also have $Z(fg) = Z(f) \cup Z(g)$ for all $f, g \in \mathcal{C}(X)$.

Proposition 2.32. *The zero sets of $\mathcal{C}(X)$ and $\mathcal{C}^*(X)$ coincide.*

Proof Take $f \in \mathcal{C}(X)$, then we can define $g(x) = f(x) \wedge \mathbf{1}$ for each $x \in X$. Our function $g \in \mathcal{C}^*(X)$ and $Z(f) = Z(g)$. □

This definition is quite useful as we said. Using this notation we can say that f^{-1} exists if and only if $Z(f) = \emptyset$ for any function $f \in \mathcal{C}(X)$, this is, f does not vanish on X. We can state then:

$$f \text{ belongs to the units of } \mathcal{C}(X) \Longleftrightarrow Z(f) = \emptyset$$

Let's give an example which contains all of the concepts seen previously so we can assume them easier.

Example 2.33. Let X be $\mathbb{N}$, so $\mathcal{C}(\mathbb{N})$ is the set of the sequences of real numbers and $\mathcal{C}^*(\mathbb{N})$ is the set of the bounded sequences of real numbers. We take $\mathbf{i}$ the identity sequence as we defined it in the introduction. Without a doubt this function is unbounded, but $Z(\mathbf{i}) = \emptyset$ so that $\mathbf{i^{-1}}$ exists. This is the sequence $\mathbf{j}(n) = 1/n$ for all $n \in \mathbb{N}$.
Nevertheless, $\mathbf{j}$ is bounded and $Z(\mathbf{j}) = \emptyset$; in addition, $\mathbf{i} = \mathbf{j^{-1}} \notin \mathcal{C}^*(\mathbb{N})$, so we have found an example of one function in $\mathcal{C}^*(\mathbb{N})$ with empty zero–set which is not an unit, in $\mathcal{C}^*(\mathbb{N})$.

In order to work with ideals, we want to introduce a new point of view, this is, giving a more general meaning to our zero–sets.

Definition 2.34. Given X a topological space and $\mathcal{D} \subseteq \mathcal{C}(X)$, we define:

(i) $\mathcal{Z}(\mathcal{D}) = \{Z(f) \text{ such that } f \in \mathcal{D}\}$

(ii) $\mathcal{Z}(X) = \{Z(f) \text{ such that } f \in \mathcal{C}(X)\}$

Proposition 2.35. *The set $\mathcal{Z}(X)$ is closed under countable intersections.*

Proof Given $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}(X)$, define $g_n = |f_n| \wedge 2^{-n}$ and $g(x) = \sum_{n\in\mathbb{N}} g_n(x)$ for all $x \in X$. The series converges uniformly because $|g_n| \leq 2^{-n}$. Therefore, g is continuous and:

$$Z(g) = \bigcap\{Z(g_n)|\ n \in \mathbb{N}\} = \bigcap\{Z(f_n)|\ n \in \mathbb{N}\}.$$

□

However, this result does not hold true for unions.
Following these steps, we can go on studying the correspondences between algebraic properties of the ring $\mathcal{C}(X)$ and topological properties of the space X. If we take an ideal of functions we will get some substantial characteristics of the family of zero-sets and those properties will be analogous to those of a filter and this is playing a central role in this paper.
From now on, we will notate $\mathcal{C}(X)$ as $\mathcal{C}$ just for abbreviation. Let $\mathcal{S} \subseteq \mathcal{C}$ be a subset and we define:

$$\mathcal{Z}(\mathcal{S}) = \{Z(f)|\ f \in \mathcal{S}\}$$

and for any $Y \subseteq X$, we define:

$$I(Y) = \{f \in \mathcal{C}|\ f(y) = 0 \text{ for any y} \in Y\}$$

Proposition 2.36. *In this situation we have the following relationships:*

(1) For any $Y \subseteq X$ we have that $I(Y) \subseteq \mathcal{C}$ is an ideal.

(2) For any $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{D} \subseteq \mathcal{C}$ we have $\mathcal{Z}(\mathcal{S}_1) \subseteq \mathcal{Z}(\mathcal{S}_2)$.

(3) For any $\mathcal{S} \subseteq \mathcal{D} \subseteq \mathcal{C}$ we have $\mathcal{S} \subseteq I\mathcal{Z}(\mathcal{S})$.

(4) For any $Y_1 \subseteq Y_2 \subseteq X$ we have $I(Y_2) \subseteq I(Y_1)$.

(5) For any $Y \subseteq X$ we have $Y \subseteq \mathcal{Z}I(Y)$.

Proof Let's prove this result by result.
(1) Given $f, g \in I(Y) \implies (f+g)(y) = f(y) + g(y) = 0$ for all $y \in Y \implies (f+g) \in I(Y)$.
Now we take $f \in I(Y), g \in \mathcal{C} \implies (fg)(y) = f(y)g(y) = 0$ for all $y \in Ia \implies (fg) \in I(Y)$, thus $I(Y)$ is an ideal of $\mathcal{C}(X)$.
(2) Given $s \in \mathcal{Z}(S_1) \implies f(s) = 0$, but $f \in S_1 \subseteq S_2$ and f is arbitrary, so $f(s) = 0$ for every $s \in \mathcal{Z}(S_1)$, for all $f \in S_2$. This means that $\mathcal{Z}(S_1) \subseteq \mathcal{Z}(S_2)$.
(3) Given $f \in S \implies f(s) = 0$ for all $s \in \mathcal{Z}(S) \implies f \in I(\mathcal{Z}(S))$.
(4) Take $f \in I(Y_2) \implies f(s) = 0$ for every $s \in Y_2$. $Y_1 \subseteq Y_2$ so $f(s) = 0$ for all $s \in Y_1 \implies f \in I(Y_1)$.
(5) Given $y \in Y \implies f(y) = 0$ for all $y \in I(Y) \implies y \in \mathcal{Z}(I(Y)) \implies Y \subseteq \mathcal{Z}(I(Y))$. □

2.5 C–embedding and C^*–embedding

Before studying the protagonists of this paper, we need to introduce a new concept: $\mathcal{C}$–embedding.

Definition 2.37. Let S be a subspace of X. S is said to be **$\mathcal{C}$–embedded** in X whenever we can extend every function in $\mathcal{C}(S)$ to another function in $\mathcal{C}(X)$. In the case of $\mathcal{C}^*$, we say S is $\mathcal{C}^*$–embedded in X whenever each function in $f \in \mathcal{C}^*(S)$ can be extended to a function in $\mathcal{C}^*(X)$.

Remark 2.38. We can see immediately that if $S \subset X \subset Y$, and X is $\mathcal{C}$–embedded in Y, then S is $\mathcal{C}$–embedded in Y if and only if it is $\mathcal{C}$–embedded in X. This result holds true for $\mathcal{C}^*$

However, it is not common to find a $\mathcal{C}^*$–embedded subspace. Let's give an example:

Example 2.39. Take $\mathbb{R} - \{0\}$ and the function

$$f(x) = \begin{cases} -1 & if \quad x < 0 \\ \\ 1 & if \quad x > 0 \end{cases}$$

Then it has no continuous extension, giving us an example of a function that is not $\mathcal{C}^*$–embedded in $\mathbb{R}$.

Our main purpose in this section is to give the *Urysohn's extension theorem*, but we need to define first the concept of separated sets.

Definition 2.40. Let A, B be subsets in X. Then we say A is **completely separated** from B in X whenever we can find a function $f \in \mathcal{C}^*(X)$ such that

$$f(A) = \{0\}, f(B) = \{1\}, \text{ and } 0 \leq f \leq 1.$$

Theorem 2.41. *Two sets will be completely separated if, and only if, they are contained in disjoint zero-sets. Furthermore, completely separated sets have zero-set-neighborhoods which are disjoint.*

Proof Starting with sufficiency, suppose $Z(f) \cap Z(g) = \emptyset$, then $|f| + |g|$ has no zeros so we can define:

$$h(x) = \frac{|f(x)|}{|f(x)| + |g(x)|} \qquad (x \in X).$$

Then $h \in \mathcal{C}(X), h(Z(f)) = \{0\}$, and we have $h(Z(g)) = \{1\}$
On the other hand, if we have two subsets A and A' such that they are completely separated in X, then there exists $f \in (\mathcal{C}(X))$ such that $f(A) = \{0\}$ and $f(A') = \{1\}$. The sets

$$F = \{x |\ f(x) \leq 1/3\},\ F' = \{x |\ f(x) \geq 2/3\}$$

are both disjoint and zero–set–neighborhoods of A and A', respectively. □

Theorem 2.42. (Urysohn's extension theorem) *Any function of $\mathcal{C}^*(S)$, where $S \subset X$, can be extended in X if and only if for any two subsets $S_1, S_2 \subseteq S$ completely separated in S are also completely separated in X.*

Proof (⇒) We have that S is $\mathcal{C}^*$–embedded. Take A and B in S two completely separated sets in S, then there is $f \in \mathcal{C}^*(S)$ such that is $f(A) = 0$ and $f(B) = 1$. We know that S is $\mathcal{C}^*$–embedded, so we can extend f to a function $g \in \mathcal{C}^*(X)$. These sets will be then completely separated in X because we have found $g \in \mathcal{C}^*(X)$ such that $g[A] = 0$ and $g[B] = 1$.
(⇐) Take $f_1 \in \mathcal{C}^*(S)$. Then $|f_1| \leq m$ for some $m \in \mathbb{N}$. We also define:

$$\mathbf{r}_n = \frac{m}{2}\left(\frac{2}{3}\right)^n \qquad (n \in \mathbb{N}).$$

Then $|f_1| \leq m = \mathbf{3r_1}$. By induction, given $f_n \in \mathcal{C}^*(S)$, with $|f_n| \leq \mathbf{3r_n}$, define:

$$A_n = \{s \in S|\ f_n(s) \leq -r_n\},\ \text{and } B_n = \{s \in S|\ f_n(s) \geq r_n\}$$

Since the subsets A_n and B_n are completely separated sets in S, so they are also completely separated in X. Hence a function $g_n \in \mathcal{C}^*(X)$ exists such that it is equal to $-\mathbf{r}_n$ on A_n and to $\mathbf{r}_n$ on B_n with $|g| \leq \mathbf{r}_n$. The values of f_n and g_n on A_n lie between $-\mathbf{3r}_n$ and $-\mathbf{r}_n$; on B_n they lie between $\mathbf{r}_n$ and $\mathbf{3r}_n$; and, elsewhere on S, they are between $-\mathbf{r}_n$ and $\mathbf{r}_n$. Define now:

$$f_{n+1} = f_n - g_{n_{\upharpoonright S}},$$

and we have $|f_{n+1}| \leq \mathbf{2r}_n$, i.e,

$$|f_{n+1}| \leq \mathbf{3r}_{n+1}.$$

Now write

$$g(x) = \sum_{n\in\mathbb{N}} g_n(x) \qquad\qquad (x \in X).$$

Since the series converges uniformly, then g is a continuous function on X. The last thing we shall realize is that

$$(g_1 + \ldots + g_n)_{\restriction S} = (f_1 - f_2) + \cdots + (f_n - f_{n+1}) = f_1 - f_{n+1}.$$

Because the sequence $(f_{n+1}(s))$ approaches 0 at every point $s \in S$; this shows that $g(s) = f_1(s)$. Hence, g is an extension of f_1. □

2.6 z–filters

In order to study the zero-sets $\mathcal{Z}(\mathcal{S})$ we will define a z–filter as it follows:

Definition 2.43. A family $F \subseteq \mathcal{Z}(X)$ is a z–**filter** if it satisfies:

(i) $\emptyset \notin F$.

(ii) If $Z_1, Z_2 \in F$, then $Z_1 \cap Z_2 \in F$.

(iii) If $Z \in F$, for any $Z' \in \mathcal{Z}(X)$ such that $Z \subseteq Z'$ we have $Z' \in F$.

However, we can give an equivalent condition that contains (ii) and (iii), this is:

(iv) For any $Z_1, Z_2 \in F$ there exists $Z \in F$ such that $Z \subseteq Z_1 \cap Z_2$.

This definition is not the same as the given for a filter. They are different concepts, this is, a z–filter will be a topological object, and a filter is a purely set-theoretic one. We must give importance to the fact that in a discrete space, every set is a zero–set, thus filters and z–filters become the same concept in that case. Now we are able to study the relations between ideals and z–filters.

Theorem 2.44. *Using the usual concepts, we have:*
(1) For any ideal $\mathfrak{a} \subseteq \mathcal{C}$ the family $\mathcal{Z}[\mathfrak{a}] = \{Z(f)$ such that $f \in \mathfrak{a}\}$ is a z–filter.
(2) For any z–filter $F \subseteq \mathcal{Z}(X)$ the set $I(F) = \{f \in \mathcal{C}$ such that $Z(f) \in F\}$ is an ideal of $\mathcal{C}$.

For the second statement, we already knew that for any arbitrary set F, $I(F)$ is an ideal of the chosen ring R. However, what we are really proving is the form that these ideals take when you take as the ring $\mathrm{R} = \mathcal{C}(X)$ and why does it also work.

Proof (1). We have to prove that $\mathcal{Z}[\mathfrak{a}]$ is a z–filter.
If $\emptyset \in \mathcal{Z}[\mathfrak{a}]$, then it would be the zero-set of a function $f \in \mathfrak{a}$, hence f is a unit in $\mathfrak{a} \Longrightarrow \mathfrak{a} = \mathcal{C}$ which is not possible because $\mathfrak{a}$ is a proper ideal.
Now let's take $Z_1, Z_2 \in \mathcal{Z}[\mathfrak{a}] \Longrightarrow Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$ with $f_1, f_2 \in \mathfrak{a}$. We can define now $Z = Z(f_1^2 + f_2^2) \subseteq Z(f_1) \cap Z(f_2)$. Therefore, $\mathcal{Z}[\mathfrak{a}]$ is a z–filter.
(2). Now we must prove that $I(F)$ is an ideal of $\mathcal{C}$.

Given $f_1, f_2 \in I(F)$ we have $Z(f_1^2 + f_2^2) = Z(f_1) \cap Z(f_2) \subseteq Z(f_1 + f_2) \in I(F)$ because the intersection of elements of a z–filter stays in the z–filter. Take $f \in I(F)$ and $g \in \mathcal{C} \Longrightarrow Z(f) \subseteq Z(f) \cup Z(g) = Z(fg) \in I(F)$ where we are using the third property of the z–filters. Then, $I(F)$ is an ideal of $\mathcal{C}$. □

Let's give an example of z–filters and some relations with ideals so we can have an application to our paper.

Example 2.45. In our usual topological space $\mathcal{C}(\mathbb{R})$ we consider the principal ideal $\mathfrak{a} = \langle \mathbf{i} \rangle$ (remember that $\mathbf{i}$ is the identity function on $\mathbb{R}$). This ideal contains basically every continuous real-valued function f satisfying $f(x) = xg(x)$ with $g \in \mathcal{C}(\mathbb{R})$. Thus, every function belonging to $\mathfrak{a}$ vanishes at 0, which means 0 is contained in every zero-set in $\mathcal{Z}(\mathfrak{a})$. Then $\mathcal{Z}(\mathfrak{a})$ is the family of all zero-sets that contains 0 because it is a z–filter that includes the set $\{0\}$.

Definition 2.46. A **z–ultrafilter** in X is a maximal z–filter. Hence we have that every z–filter is contained in a z–ultrafilter.

Theorem 2.47. *Relation between maximal ideals and z–ultrafilters:*

(1) For any maximal ideal $\mathfrak{m} \subseteq \mathcal{C}$ we have that $\mathcal{Z}[\mathfrak{m}]$ is a z–ultrafilter. Moreover, if $Z(f)$ meets each element in $\mathcal{Z}[\mathfrak{m}]$, then $f \in \mathfrak{m}$.

(2) For any z-ultrafilter $F \subseteq \mathcal{Z}(X)$ we have $I(F)$ is a maximal ideal. Furthermore, if $U \in \mathcal{Z}(X)$ meets each element on Z, then $U \in Z$.

2.7 Abstract z–ideals

Once we have studied the z–filters which are topological objects as we mentioned before, we can define some kind of equivalence in the algebraic field. We are giving some notation first: Given a ring R, an element $a \in$ R, and a subset $D \subseteq$ R, we define

$$\mathcal{M}(a) = \{M \in \text{Max(R)} \mid a \in M\}$$
$$\mathcal{M}(D) = \{M \in \text{Max(R)} \mid D \subseteq M\}$$

Where Max(R) is the set of all maximal ideals of R. So basically these sets, $\mathcal{M}(a)$ and $\mathcal{M}(D)$, contain the maximal ideals in R that contains an element $a \in$ R, or a subset of $D \subseteq$ R.

Definition 2.48. An ideal $\mathfrak{a} \leq$ R is a **z–ideal** if for all $a, b \in$ R such that $\mathcal{M}(a) = \mathcal{M}(b)$ and $a \in \mathfrak{a}$, then $b \in \mathfrak{a}$.

Example 2.49. We can see easily some examples of these z–ideals.

(1) Every maximal ideal M is a z–ideal. Given $a, b \in R$ such that $\mathcal{M}(a) = \mathcal{M}(b)$ and $a \in M$, then $M \in \mathcal{M}(a) = \mathcal{M}(b)$, so $b \in M$.

(2) The intersection of z–ideals is again a z–ideal. Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ be z–ideals in R and $a, b \in$ R such that $\mathcal{M}(a) = \mathcal{M}(b)$ with a in the intersection of all the z–ideals. In particular, $a \in \mathfrak{a}_i$ for any arbitrary index i, so b must be in $\mathfrak{a}_i$ too because it is a z–ideal. Since we took i any index between 1 and n so it's true for every index, hence b is in the intersection too.

We can also see z–ideal from different points of view which are equivalent to the first definition. We need a previous lemma that will help us proving those results.

Lemma 2.50. *Given a ring R and elements $a, b \in R$, we have $\mathcal{M}(a) \supseteq \mathcal{M}(b)$ if, and only if, $\mathcal{M}(a) = \mathcal{M}(ab)$.*

Proof ($\Rightarrow$). We have that $\mathcal{M}(a) \supseteq \mathcal{M}(b)$.
First we show that $\mathcal{M}(a) \supseteq \mathcal{M}(ab)$. Suppose $M \in \mathcal{M}(ab)$, then $ab \in M$. M is a maximal ideal, so it is prime, thus $a \in M$ or $b \in M$. If the first case takes place, then trivially $M \in \mathcal{M}(a)$. If not, then $b \in M$ so $M \in \mathcal{M}(b)$, and since $\mathcal{M}(a) \supseteq \mathcal{M}(b)$ it follows that $M \in \mathcal{M}(a)$.
Second we show that $\mathcal{M}(a) \subseteq \mathcal{M}(ab)$. Now let $M \in \mathcal{M}(a)$. Then $a \in M$, and hence $ab \in M$ since M is an ideal. That gives us $M \in \mathcal{M}(ab)$ so.
Therefore, $\mathcal{M}(a) = \mathcal{M}(ab)$.
($\Leftarrow$). Now we assume $\mathcal{M}(a) = \mathcal{M}(ab)$ and we take $M \in \mathcal{M}(b)$. $ab \in M$ since M is an ideal, then $M \in \mathcal{M}(ab)$. But we knew that $\mathcal{M}(ab) = \mathcal{M}(a)$, so $M \in \mathcal{M}(a)$, proving that $\mathcal{M}(b) \subseteq \mathcal{M}(a)$.
□

Proposition 2.51. *In a ring R, an ideal $\mathfrak{a} \leq R$ is a z–ideal if, and only if, for any $a, b \in R$ such that $\mathcal{M}(b) \subseteq \mathcal{M}(a)$ and $b \in \mathfrak{a}$ then $a \in \mathfrak{a}$.*

Proof ($\Rightarrow$). Let $\mathfrak{a}$ be a z–ideal in R. If we consider any $a, b \in$ R with $b \in \mathfrak{a}$ and $\mathcal{M}(b) \subseteq \mathcal{M}(a)$, by the previous lemma we have that $\mathcal{M}(b) \subseteq \mathcal{M}(a)$ if, and only if, $\mathcal{M}(a) = \mathcal{M}(ab)$. Therefore, $ab \in \mathfrak{a}$ because $\mathfrak{a}$ is a z–ideal, implying $a \in \mathfrak{a}$.
($\Leftarrow$). Now we want to prove that $\mathfrak{a}$ is a z–ideal given those conditions, this is, we consider any $a, b \in$ R with $\mathcal{M}(b) = \mathcal{M}(a)$ and $b \in \mathfrak{a}$. But that equal implies $\mathcal{M}(b) \subseteq \mathcal{M}(a)$ and $b \in \mathfrak{a}$, so $a \in \mathfrak{a}$, hence $\mathfrak{a}$ is a z–ideal. □

For the next proposition we are calling, for any $a \in$ R,

$$M(a) = \bigcap \mathcal{M}(a)$$

the intersection of every maximal ideal that contains a which is again an ideal as we know.

Proposition 2.52. *In a ring R, an ideal $\mathfrak{a} \leq R$ is a z–ideal if, and only if, for any $a \in R$ we have $M(a) \subseteq \mathfrak{a}$.*

Proof ($\Rightarrow$). We have $\mathfrak{a}$ a z-ideal in R. Let $x \in M(a)$. Then, x is contained in every maximal ideal containing a. Therefore $\mathcal{M}(x) \subseteq \mathcal{M}(a)$ and $a \in \mathfrak{a}$ and that implies $x \in \mathfrak{a}$ because $\mathfrak{a}$ is a z-ideal. Our element x is arbitrary, so we have $M(a) \subseteq \mathfrak{a}$.
($\Leftarrow$). Now we assume $M(a) \subseteq \mathfrak{a}$ for every $a \in \mathfrak{a}$. As we have done in many occasions, we consider $x, y \in$ R such that $\mathcal{M}(x) = \mathcal{M}(y)$ and $y \in \mathfrak{a}$. We now have:

$$x \in M(x) = \bigcap \mathcal{M}(x) = \bigcap \mathcal{M}(y) = M(y)$$

and since $y \in \mathfrak{a}$, we have by hypothesis that $M(y) \subseteq \mathfrak{a}$. So $x \in \mathfrak{a}$ an $\mathfrak{a}$ is a z-ideal. □

Once we have our idea of what a z–ideal is, later we will notice that a z–ideal in our ring ($\mathcal{C}(X)$) is easier to define and we will see some results that we can find really useful and interesting and somehow we can relate them with the topological concept of z–filter.
We wrote before two examples of z–ideals. If we take both of them at the same time, this is, the intersections of maximal ideals, we will call them **strong z–ideals**. We can give a really strong theorem that shows us the relevance for a ring to be von Neumann regular, but first we need two lemmas.

Lemma 2.53. *Given a ring R and $\mathfrak{a} \leq R$ a z–ideal, then $\mathfrak{a}$ is radical. This is, $\mathfrak{a} = rad(\mathfrak{a})$.*

PROOF We know $\mathfrak{a} \subseteq \text{rad}(\mathfrak{a})$, so we only need to prove the other inclusion.
Take $a \in \text{rad}(\mathfrak{a}) \implies a^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. Since $\mathfrak{a}$ is a z–ideal so for any $a, b \in R$ with $\mathcal{M}(a) = \mathcal{M}(b)$ and $a \in \mathfrak{a}$ we have $b \in \mathfrak{a}$. If we prove that $\mathcal{M}(a) = \mathcal{M}(a^n)$, then $a \in \mathfrak{a}$.

$$\mathcal{M}(a) = \{M \in \text{Max}(R)/a \in M\}$$

Every maximal ideal is prime so if $a^n \in M$ then $a \in M$, hence $\mathcal{M}(a^n) \subseteq \mathcal{M}(a)$. On the other hand, if $a \in M$, then $a \cdot a \in M$, hence $a^n \in M$ and $\mathcal{M}(a) \subseteq \mathcal{M}(a^n)$, thus $a \in \mathfrak{a}$. □

Corollary 2.54. *If $\mathfrak{a}$ is a z–ideal in R, then $\mathfrak{a}$ is the intersection of all the prime ideals of R that contain $\mathfrak{a}$.*

Theorem 2.55. *The following are equivalent for a ring R.*

(a) Every ideal is a strong z–ideal.

(b) Every ideal is a z–ideal.

(c) Every principal ideal is a z–ideal.

(d) R is a von Neumann regular ring.

PROOF We will use the fact that for a principal ideal $\mathfrak{a} = \langle a \rangle$, then $\langle a \rangle^2 = \langle a^2 \rangle$
(a)⇒(b) If every ideal is a strong z–ideal then every ideal is the intersection of some maximal ideals, which is again a z–ideal.
(b)⇒(c) A principal ideal is a ideal, so by hyphotesis it is a z–ideal.
(c)⇒(d) Now we have that every principal ideal $\mathfrak{a} = \langle a \rangle$ is a z–ideal. Then $\langle a^2 \rangle = \langle a \rangle$ because every z–ideal is radical. Hence there exists $b \in R$ such that $a = a^2 b$ which means being a von Neumann regular ring.
(d)⇒(a) Every ideal in a von Neumann regular ring is the intersection of the maximal ideals containing it, then every ideal is a strong z–ideal. □

2.8 z–ideals in *C(X)*

As we mentioned before, the definition of z–ideal applied to $\mathcal{C}(X)$ will be easier to understand and we will work better due to the knowledge achieved in the previous section which of course applies to this case in particular.
When we replace the general ring R with our ring of the continuous functions it turns out that an ideal $\mathfrak{a} \leq \mathcal{C}$ is a z–ideal whenever if $Z(f) \in \mathcal{Z}(\mathfrak{a})$, then $f \in \mathfrak{a}$. This happens to be true because there exists a bijection between z–ideals and z–filters, therefore studying one of them gives us information about both of them. We are proving that result now so we can use it freely.
Now we can give some important results that connect all the concepts we have seen in this chapter.

Proposition 2.56. *These three results hold true:*

(1) For any z–filter F we have that $I(F)$ is a z–ideal.

(2) For any z–ideal $\mathfrak{a} \leq \mathcal{C}$ we have that $\mathfrak{a} = I(\mathcal{Z}(\mathfrak{a}))$.

(3) There is a bijection between z–ideals and z–filters.

Proof Let's prove them one by one:
(1). We have a z-filter F and $I(F) = \{f \in \mathcal{C}/f(x) = 0 \text{ for all } x \in F\} = \{f \in \mathcal{C}/Z(f) \in F\}$ and we did this proof in the Theorem (2.44.).
(2). We had $\mathfrak{a} \subseteq I(\mathcal{Z}(\mathfrak{a}))$ from Proposition (2.36.). Now we have to prove the inclusion $\supseteq$. Take $f \in \ I(\mathcal{Z}(\mathfrak{a}))$. We know that

$$I(\mathcal{Z}(\mathfrak{a})) = \{f \in \mathcal{C}(X)|\ f(x) = 0 \quad \forall x \in Z(\mathfrak{a})\}$$

so trivially $Z(f) \in Z(\mathfrak{a})$ and $\mathfrak{a}$ is a z–ideal, so by definition $f \in \mathfrak{a}$.
(3) If we have a look at Theorem (2.47.) and the two previous statements in this result we can deduce it easily. □

By now we know several things. One of them is that every z–ideal has to be an intersection of prime ideals, because they are radical ideals. Likewise, we also know that every intersection of prime ideals is an z–ideal. However, the converse is not true. Let us see an example of this.

Remark 2.57. Consider the set $\mathcal{O}_0$ of all real valued continuous functions f for which $Z(f)$ is a neighborhood of 0. It is not necessary to show that $\mathcal{O}_0$ is a z–ideal, and of course $\mathcal{O}_0 \subseteq \mathcal{M}_0$, i.e., it is contained in $\mathcal{M}_0$. This set is no more than the maximal ideal that contains all those functions that vanish at point 0. On the other hand, let us consider $\mathfrak{a}$ an ideal in which is contained $\mathcal{O}_0$, then $\mathcal{Z}(\mathcal{O}_0) \subseteq \mathcal{Z}(\mathfrak{a})$, and therefore the intersection of any element of $\mathcal{Z}(\mathfrak{a})$ with every neighborhood of 0 is nonempty; thus, it contains 0, and $\mathfrak{a} \subseteq \mathcal{M}_0$. Consequently, $\mathcal{M}_0$ is the unique maximal ideal in which $\mathcal{O}_0$ is contained. This means that $\mathcal{O}_0$ cannot be intersection of maximal ideals.

Theorem 2.58. *For any z–ideal $\mathfrak{a} \leq \mathcal{C}$ the following statements are equivalent:*

(a) $\mathfrak{a}$ is prime.

(b) $\mathfrak{a}$ *contains a prime ideal.*

(c) *For any* $f, g \in \mathcal{C}$ *if* $f \cdot g = 0$, *then either* $f \in \mathfrak{a}$ *or* $g \in \mathfrak{a}$.

(d) *For any* $f \in \mathcal{C}$ *there exists a zero-subset in* $\mathcal{Z}(\mathfrak{a})$ *in which* f *maintains the sign.*

Proof
(a)$\Longrightarrow$(b) If $\mathfrak{a}$ is prime then it contains a prime ideal which is itself.
(b)$\Longrightarrow$(c) If $\mathfrak{a}$ contains a prime ideal P and $fg = 0$ then $fg \in P$ (see 2.57.), whence either f or g is in $P \subset \mathfrak{a}$.
(c)$\Longrightarrow$(d) We just have to realize that $(f \vee 0)(f \wedge 0) = 0$ for every $f \in \mathcal{C}$.
(d)$\Longrightarrow$(a) Take $fg \in \mathfrak{a}$ and consider $|f| - |g|$, then there exist a zero-set $\mathcal{Z} \in \mathfrak{a}$ such that $|f| - |g|$ is nonnegative. Then every zero of f is a zero of g. Hence:

$$Z \supset Z \cap Z(g) = Z \cap Z(fg) \in \mathcal{Z}[\mathfrak{a}]$$

so that $Z(g) \in \mathcal{Z}[\mathfrak{a}]$. Since $\mathfrak{a}$ is a z-ideal, $g \in \mathfrak{a}$. Hence, $\mathfrak{a}$ is prime. ☐

Corollary 2.59. *If we have two ideals* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *such that* $\mathfrak{a} \not\subseteq \mathfrak{a}'$ *and* $\mathfrak{a}' \not\subseteq \mathfrak{a}$, *then* $\mathfrak{a} \cap \mathfrak{a}'$ *is not prime.*

Proof Take $a \in \mathfrak{a} \backslash \mathfrak{a}'$ and $a' \in \mathfrak{a} \backslash \mathfrak{a}'$, then neither a nor a' belongs to $\mathfrak{a} \cap \mathfrak{a}'$, but $aa' \in \mathfrak{a} \cap \mathfrak{a}'$. ☐

Definition 2.60. A z–filter F is **prime** whenever for any $Z_1, Z_2 \in \mathcal{Z}(X)$ such that $Z_1 \cup Z_2 \in F$ we have either $Z_1 \in F$ or $Z_2 \in F$.

Theorem 2.61. *The following statements holds true.*

(1) *Let* $\mathfrak{p} \leq \mathcal{C}$ *be a prime ideal. Therefore* $\mathcal{Z}(\mathfrak{p})$ *is a prime* z*–filter.*

(2) *Let* F *be a prime* z*–filter. Therefore* $I(F)$ *is a prime* z*–ideal.*

Proof
(1) We know from Theorem (2.44.) that $\mathfrak{p}$ is a z–filter. Now we have to prove that it is prime. Take $Z_1, Z_2 \in \mathcal{Z}(X)$ with $Z_1 \cup Z_2 \in \mathcal{Z}(\mathfrak{p})$. Then $Z_1 = Z(f)$ and $Z_2 = Z(g)$ with $f, g \in \mathcal{C}$. We know that

$$Z_1 \cup Z_2 = \overbrace{Z(f) \cup Z(g)}^{\in \mathcal{Z}(\mathfrak{p})} = Z(fg) \in \mathcal{C}.$$

Hence $fg \in \mathfrak{p}$ and using that it is a prime ideal, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, which means that $Z(f) = Z_1 \in \mathcal{Z}(\mathfrak{p})$ or $Z(g) = Z_2 \in \mathcal{Z}(\mathfrak{p})$. Thus, $\mathcal{Z}(\mathfrak{p})$ is a prime z–filter.
(2) We also know from Theorem (2.44.) that $I(F)$ is an ideal. Now we have to prove that it is prime. Take $fg \in I(F)$, then we need to prove that $f \in I(F)$ or $g \in I(F)$. If $fg \in I(F)$, then $(fg)(x) = 0$ for all $x \in F$. This means that $Z(f) \cup Z(g) = Z(fg) \in F$ and since F is a prime z–filter, $Z(f) \in F$ or $Z(g) \in F$. In any case, for instance, we can say that $f(x) = 0$ for every $x \in F$. Hence $f \in I(F)$ and $I(F)$ is a prime z–ideal. ☐

Remark 2.62. One important result that comes from the whole chapter is that any prime ideal in $\mathcal{C}(X)$ can be inserted in an unique maximal ideal. This can be easily explained. Since every ideal contains a maximal ideal, whenever $\mathfrak{m}$ and $\mathfrak{m}'$ are two different maximal ideals such that they contain the ideal, their intersection is a z–ideal, but it is not prime (look Corollary (2.59.)), so by Theorem (2.58.), $\mathfrak{m} \cap \mathfrak{m}'$ does not contain prime ideals.

In this chapter we have established some connections between z–ideals and z–filters that are very useful for the study of $\mathcal{C}(X)$ for any topological space X. Nevertheless, the space of the continuous functions has taken all the leading role. We can give some analogous results for the space of the bounded continuous functions $\mathcal{C}^*(X)$ but that theory is more difficult and it's not the point of the paper, but I recommend the reader to look for books or articles in the references in order to find some information about it if he's interested.

3 d–ideals

For this chapter we are assuming that every ring R is reduced, this is, it has no nonzero nilpotent elements. We have studied z–ideals in the previous section. d–ideals (also known as z^0–ideals) are similar to those z–ideals. However, the main object here is the annihilator, which is defined in the first chapter (1.1).

Definition 3.1. We say an ideal $\mathfrak{a}$ of a ring R is a d–**ideal** whenever for any $a, b \in \mathrm{R}, \mathrm{Ann}(a) = \mathrm{Ann}(b)$ and $a \in \mathfrak{a}$, it implies $b \in \mathfrak{a}$.

We are now introducing some notation that will be useful in this section.

Definition 3.2. If B is a subset of a ring R and $x \in B$, we can define

$$V(B) = \{P \in \mathrm{Min}(\mathrm{R}) |\ P \subseteq B\}, \qquad V(x) = \{P \in \mathrm{Min}(\mathrm{R}) |\ x \in P\}$$

and

$$D(B) = \mathrm{Min}(\mathrm{R}) - V(B).$$

Example 3.3. Assume R is a reduced ring.

(1) If $\mathfrak{a}$ is an ideal in a ring R, then $\mathrm{Ann}(\mathfrak{a})$ is a d–ideal because we can easily observe that $\mathrm{Ann}(\mathfrak{a}) = \bigcap\{P \in \mathrm{Min}(\mathrm{R}) |\ P \in D(\mathfrak{a})\}$.

(2) If S is a multiplicatively closed set in a ring R, then

$$O_s = \{a \in \mathrm{R} |\ as = 0 \text{ for some } s \in S\} = \bigcap_{s \in S} \mathrm{Ann}(s) = \sum_{s \in S} \mathrm{Ann}(s)$$

is a d–ideal.

(3) In a ring R, each minimal ideal and the socle of R, which is the sum of all minimal ideals, are d–ideals.

In the next result we are showing several ways of characterising these ideals. First, we need to introduce some notation. For any $a \in \mathrm{R}$ zero-divisor, let

$$P(a) = \bigcap\{Q \in \mathrm{Min}(\mathrm{R}) |\ a \in Q\}$$

and we will call it a basic d–ideal. In fact, it is easy to prove that $P(a) = \mathrm{Ann}^2(a)$.

Lemma 3.4. *These statements hold:*

(1) Every minimal prime ideal is a d–ideal.

(2) The intersection of d–ideals is a d–ideal.

(3) The nilradical Nil(R) is the smallest prime d–ideal.

The proof of this result is quite simple, we can easily figure it out from the definition of d–ideal.

Lemma 3.5. *For every ideal* $\mathfrak{a}$ *of a ring R and any element* $a \in \mathfrak{a}$ *we have* $Ann^2(x) \subseteq Ann^2(a)$ *for any element* $x \in P(a)$*.*

Proof Instead of $\mathrm{Ann}^2(x) \subseteq \mathrm{Ann}^2(a)$, we can prove the equivalent statement:

$$P(x) \subseteq P(a) \text{ for any } x \in P(a).$$

Take an arbitrary $x \in P(a)$. By definition, $P(a)$ is the intersection of all the minimal ideals that contain the element a. Call $\mathcal{Q}'$ the family of minimal ideals that contain a. Therefore, x is also contained in $\mathcal{Q}'$. However, there may exist more minimal ideals such that x is contained in them, this is, $\mathcal{Q}' \subseteq \mathcal{Q}$ where $\mathcal{Q}$ is the family of minimal ideals containing x. This means:

$$P(x) = \bigcap \mathcal{Q} \subseteq \bigcap \mathcal{Q}' = P(a)$$

since $P(x)$ contains at least the same minimal ideals than $P(a)$. Also, x was an arbitrary element in $P(a)$ so it yields for any element in $P(a)$. □

Proposition 3.6. *Let* $\mathfrak{a}$ *be an ideal of a ring R. Then the following are equivalent:*

(a) $\mathfrak{a}$ *is a d–ideal in R.*

(b) $P(a) \subseteq \mathfrak{a}$ *for every* $a \in \mathfrak{a}$*.*

(c) For $a, b \in R, P(a) = P(b)$ *and* $b \in \mathfrak{a}$ *imply* $a \in \mathfrak{a}$*.*

(d) For $a, b \in R, V(a) = V(b)$ *and* $a \in \mathfrak{a}$ *imply* $b \in \mathfrak{a}$*.*

(e) $a \in \mathfrak{a}$ *implies that* $Ann^2(a) \subseteq \mathfrak{a}$*.*

Proof (a)⇒(b) Assume that $\mathfrak{a}$ is a d–ideal and let $a \in \mathfrak{a}$. We need to show that

$$P(a) = \mathrm{Ann}^2(a) \subseteq \mathfrak{a}.$$

Let $x \in P(a)$. By Lemma (3.5.), $\mathrm{Ann}^2(x) \subseteq \mathrm{Ann}^2(a)$ and $a \in \mathfrak{a}$ imply $x \in \mathfrak{a}$ since $\mathfrak{a}$ is a d–ideal. Since x is an arbitrary element of $P(a)$, it follows that $P(a) \subseteq \mathfrak{a}$.
(b)⇒(c) Let $P(a) \subseteq \mathfrak{a}$ for every $a \in \mathfrak{a}$. Consider any x and y in R with $P(x) = P(y)$ and $y \in \mathfrak{a}$. We have

$$x \in P(x) = P(y)$$

but by hypothesis $P(y) \subseteq \mathfrak{a}$. Therefore $x \in \mathfrak{a}$.
(c)⇒(d) Assume that (3) holds. Take $x, y \in$ R with $V(x) = V(y)$ and $y \in \mathfrak{a}$. We have then

$$x \in P(x) = \bigcap V(x) = \bigcap V(y) = P(y)$$

and $y \in \mathfrak{a}$, so by hypothesis $x \in \mathfrak{a}$.

(d)⇒(e) Let $x \in \mathfrak{a}$. We must show that $\text{Ann}^2(x) \subseteq \mathfrak{a}$. It is shown in Lemma (3.5.) that $V(x) = V(\text{Ann}^2(x))$. We therefore have

$$V(x) = V(\text{Ann}^2(x)) = V(\text{Ann}^2(y)) = V(y)$$

and $x \in \mathfrak{a}$ imply by the stated condition that $\text{Ann}^2(x) \subseteq \mathfrak{a}$.
(e)⇒(a) We have $\text{Ann}^2(a) \subseteq \mathfrak{a}$ for every $a \in \mathfrak{a}$. To show that $\mathfrak{a}$ is a d–ideal, consider any $x, y \in \text{R}$ with $\text{Ann}^2(x) = \text{Ann}^2(y)$ and $y \in \mathfrak{a}$. But now we have

$$x \in \text{Ann}^2(x) = \text{Ann}^2(y).$$

Now, by hypothesis, $\text{Ann}^2(y) \subseteq \mathfrak{a}$ for every $y \in \mathfrak{a}$. So $x \in \mathfrak{a}$. Hence $\mathfrak{a}$ is a d–ideal. □

Now we are giving some results that the reader will find familiar because they are similar to the ones that we gave for z–ideals.

Proposition 3.7. *For any ring R and any zero-divisor $a \in R$ we have*

$$P(a) = \{x \in R |\ Ann(a) \subseteq Ann(x)\}.$$

PROOF Observe that $\text{Ann}^3(a) = \text{Ann}(a)$ for any $a \in \text{R}$. Therefore, if $x \in P(a)$ then $\text{Ann}^2(x) \subseteq P(a) = \text{Ann}^2(a)$, and $\text{Ann}(a) \subseteq \text{Ann}(x)$. On the other hand, if $\text{Ann}(a) \subseteq \text{Ann}(x)$, then $P(x) = \text{Ann}^2(x) \subseteq \text{Ann}^2(a) = P(a)$. Hence $x \in P(a)$. □

Proposition 3.8. *In a von Neumann regular ring, we can write any ideal as the intersection of some minimal prime ideals. This means that every ideal in a ring with that property is a d–ideal.*

Proposition 3.9. *Each d–ideal in a reduced ring is a radical ideal.*

Proposition 3.10. *A principal ideal is a d–ideal if and only if it is a basic d–ideal.*

Lemma 3.11. *We can only find zero-divisors in d–ideals.*

Theorem 3.12. *The Jacobson radical of a ring R is zero if and only if every d–ideal is a z–ideal.*

PROOF We need to prove both implications.
(⇒) Let $\mathfrak{a}$ be a d–ideal and suppose $\mathcal{M}(x) = \mathcal{M}(y)$ with $y \in \mathfrak{a}$. We claim that $x \in \text{Ann}^2(y) \subseteq \mathfrak{a}$. If $s \in \text{Ann}(y)$ and $xs \neq 0$, since $\text{Jac}(\mathfrak{a}) = 0$ we can find M a maximal ideal fulfilling that $xs \notin M$, that is $x \notin M$ and $s \notin M$. But $sy = 0$ and $s \notin M$ implies $y \in M$, so $x \in M$ which is a contradiction.
(⇐) Suppose $a \neq 0$ belongs to Jac(R) and take a minimal prime ideal P of R not containing a. Then P is a d–ideal and it is not z–ideal for $\mathcal{M}(a) = \mathcal{M}(0)$ and a does not belong to P. □

We have found then a type of ring where we can talk about z–ideals and d–ideals without the need to distinguish them because they are the same. However, we shall give a result in order to prove that not every z–ideal is a d–ideal.

Theorem 3.13. *Let R be a ring and M a maximal ideal in the polynomial ring $\mathrm{R}[x]$. Then M cannot consist entirely of zero-divisors.*

Proof Let's prove this by contradiction. Suppose M consist entirely of zero-divisors. Then $x \notin M$ since x is not a zero-divisor. Hence $\langle x, M\rangle = \mathrm{R}[x]$ and we can write $1 = xf + g$ with $f \in \mathrm{R}[x]$ and $g \in M$. But clearly $g = 1 - xf$ cannot be a zero-divisor. □

Definition 3.14. A d–ideal $\mathfrak{a}$ in a ring R is a **maximal d–ideal** if $\mathfrak{a}$ is a proper d–ideal and there is no other proper d–ideal $\mathfrak{a}'$ with $\mathfrak{a} \subseteq \mathfrak{a}' \subseteq \mathrm{R}$.

Given this definition, we can give a result in order to show that prime d–ideals are key elements in the concepts of d–ideals and we are getting several corollaries from it. Let's start with a lemma.

Lemma 3.15. *Let R be a ring and $Ann(S_i) \subseteq Ann(T_i), i = 1, 2, \ldots, n$, where S_i, T_i are subsets of R for every index i. Then $Ann(S_1S_2 \ldots S_n) \subseteq Ann(T_1T_2 \ldots T_n)$.*

Theorem 3.16. *Let R be a ring and $\mathfrak{a}$ be a d–ideal. Then, every prime ideal, minimal over $\mathfrak{a}$ is a prime d–ideal.*

Proof Take P a prime ideal, minimal over $\mathfrak{a}$ and $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ where $a \in P$ and $b \in \mathrm{R}$. Now since $P/\mathfrak{a}$ is a minimal prime ideal in $\mathrm{R}/\mathfrak{a}$ is a reduced ring, there exists $0 \neq c + \mathfrak{a}$ in $\mathrm{R}/\mathfrak{a}$ with $c \notin P$ and $ac \in \mathfrak{a}$. Now by the previous lemma, $\mathrm{Ann}(ac) = \mathrm{Ann}(bc)$. Since $\mathfrak{a}$ is a d–ideal and $ac \in \mathfrak{a}$, we have $bc \in \mathfrak{a} \subseteq P$. But $c \notin P$, that is $b \in P$. □

Corollary 3.17. *An ideal $\mathfrak{a}$ in a ring R is a d–ideal if, and only if, it can be written as the intersection of some prime d–ideals.*

Corollary 3.18. *If R is a ring, then every maximal d–ideal is a prime d–ideal.*

Corollary 3.19. *Let R be a ring and P a prime ideal in R. Then either P is a d–ideal or contains a maximal d–ideal which is a prime d–ideal.*

Proof Define

$$\mathcal{K} = \{\mathfrak{a} \subseteq \mathrm{R} |\ \mathfrak{a} \text{ is a } d\text{–ideal and } \mathfrak{a} \subseteq P\}.$$

Then $\{0\} \in \mathcal{K}$ and $\mathcal{K}$ is inductive so by Zorn's Lemma (1.4.), $\mathcal{K}$ has a maximal element. Let $\mathfrak{a}$ be one, then $\mathfrak{a} = P$ if and only if P is a prime d–ideal. If $\mathfrak{a} \subset P$ thus we can find a prime ideal Q minimal with respect to containing $\mathfrak{a}$ and contained in P (By Theorem (3.16.) $Q \neq P$). Moreover, either $Q = \mathfrak{a}$ in which case $\mathfrak{a}$ is prime, or $\mathfrak{a} \subset Q$, which contradicts the maximality of $\mathfrak{a}$. □

4 Tychonoff spaces

Now we are taking a topological space X and we are mostly using the continuous functions in it.

Definition 4.1. We say a space X is **completely regular** whenever for any F closed set and $x \in X \backslash F$, there exists a function $f \in \mathcal{C}(X)$ such that $f(x) = 1$ and $f(F) = 0$.

This means that F is completely separated from $\{x\}$. However, from now on we are going to deal mainly with the so-called Tychonoff spaces (or $T_{3\frac{1}{2}}$).

Definition 4.2. A completely regular space X which is also provided with the Hausdorff property is called a **Tychonoff** space.

From now on, we are treating every topological space as a Hausdorff space. We can see some immediate properties of Tychonoff spaces.

Remark 4.3. If $S \subseteq X$ and X is a Tychonoff space, then S is also Tychonoff.

Remark 4.4. Let X be a Tychonoff space. Then if $f(x) = f(y)$ for all $f \in \mathcal{C}$, then $x = y$.

Some examples of Tychonoff spaces are metric spaces. In particular, $\mathbb{R}$ and all its subspaces are Tychonoff. We actually find reasonable that this definition involves continuous functions because it's our final objective. One thing that may cause confusion is the fact that for this paper we are considering topological spaces in terms of its closed sets, instead of open sets as usual. This result shows why.

Theorem 4.5. *Let X be a Hausdorff space. Therefore, X is a Tychonoff space if, and only if, one of its base for the closed sets is the family $\mathcal{Z}(\mathcal{C})$ of all zero-sets.*

Remark 4.6. Remember for $f \in \mathcal{C}$ we had $Z(f) = \{x \in X \text{ such that } f(x) = 0\}$. We can also see the zero-set as $f^{-1}(0)$, which means that it is a closed set since f is continuous and $\{0\}$ is a closed set.

Let's prove the theorem then.

PROOF ($\Longrightarrow$). Let us assume X is a Tychonoff space. Hence whenever we have F a closed set in X and $x \in X \backslash F$, we can find a continuous function f such that $f(x) = 1$ and $f(F) = 0$. This means that $F \subsetneq Z(f)$ and $x \notin Z(f)$, hence $\mathcal{Z}(\mathcal{C})$ is a base.
($\Longleftarrow$). Assume that $\mathcal{Z}(\mathcal{C})$ is a base. Therefore, for every closed set F and $x \in X \backslash F$, we can find a function $g \in \mathcal{C}$ and a zero-set $Z(g)$ that fulfills the conditions $F \subsetneq Z(g)$ and $x \notin Z(g)$. Then $g(x) \neq 0$ and we can define $f = g \cdot g^{-1}(x) = 1 \in \mathcal{C}$. Evidently, $f(F) = 0$ and this means X is Tychonoff. $\square$

We can now use the fact that we can write any closed set as the intersection of some zero-sets whenever we work in a Tychonoff space.
It is convenient introducing the concept of weak topology because it will help us getting the environment we need to work with our functions.

Theorem 4.7. *In an arbitrary topological space X, the weak topology on X that is induced by the families $\mathcal{C}(X)$ and $\mathcal{C}^*(X)$ is the same. Moreover, the family $\mathcal{Z}(X)$ (containing every zero-set) constitutes a base for its closed sets and a basic neighborhood system for a point $x \in X$ is given by the collection of all sets*

$$\{y \in X | \, |f(x) - f(y)| < \epsilon\}, \quad f \in \mathcal{C}^*, \epsilon > 0.$$

Corollary 4.8. *For any Hausdorff space X, it is Tychonoff if and only the weak topology induced by $\mathcal{C}(X)$ and $\mathcal{C}^*(X)$ coincides with its topology.*

With this conclusion, we can state that our ring $\mathcal{C}(X)$ determines the topology of a space that is Tychonoff. Moreover, we can also figure out easily the continuous mappings into our space.

Theorem 4.9. *Let $\mathcal{C}' \subseteq \mathcal{C}(Y)$ be a subfamily such that the topology in Y is determined. Then a mapping $\sigma : S \longrightarrow Y$ is continuous if and only if the function obtained by the composition $g \circ \sigma$ belongs to $\mathcal{C}(Y)$ for all $g \in \mathcal{C}$'.*

Proof Trivially, if σ is continuous then for every $g \in \mathcal{C}'$ the composition is continuous in $\mathcal{C}(S)$. On the other hand, in order to prove that σ is continuous we will see the form of those closed subbasic sets F in Y when we apply σ^{-1}. By hypothesis, we can write them as $g^{-1}(F)$, where $g \in \mathcal{C}'$ and F is a closed set in $\mathbb{R}$. Finally,

$$\sigma^{-1}(g^{-1}(F)) = (g \circ \sigma)^{-1}(F)$$

and since $g \circ \sigma$ is continuous and F is a closed set, then this set is closed in S. Hence, σ is continuous. □

The next theorem will unify all of our theory in this chapter. Our objective is to work with rings $\mathcal{C}(X)$ where X is a topological space. Let us see that we can always assume that X is a Tychonoff space.

Theorem 4.10. *Let X be any topological space. Therefore there is always a Tychonoff space Y and a mapping $\tau \in \mathcal{C}(X)$ such that the mapping $g \longrightarrow g \circ \tau$ is a ring isomorphism between $\mathcal{C}(Y)$ and $\mathcal{C}(X)$.*

Proof Let us first define an equivalence relation in $\mathcal{C}(X)$. We define the equivalence class of x if given $x, x' \in X$, it happens that $f(x) = f(x')$ for every $f \in \mathcal{C}(X)$. The set of all those equivalence classes will be named Y. Now we notate $\tau : X \longrightarrow Y$ as follows:

τx is the equivalence class that contains x.

Then for every $f \in \mathcal{C}(X)$, we can connect a function $g \in \mathbb{R}^Y$ (remember $\mathbb{R}^Y$ contains every function $g : Y \longrightarrow \mathbb{R}$) in the following way: $g(y)$ is the common value of $f(x)$ at every point $x \in Y$. This way we have $f = g \circ \tau$. If we denote as $\mathcal{C}'$ the family of all such functions g. If we now see what happens when we look at the weak topology induced by $\mathcal{C}'$ in Y, every function in $\mathcal{C}'$ is continuous on Y. Due to the previous theorem (4.9.), τ is continuous.

No need to prove the fact that whenever we have two distinct points in Y, we can then find a function in $\mathcal{C}'$ such that the image is also different for both points. Thus, Y is a Hausdorff space and by Theorem (4.7.) it is also Tychonoff.
At last, if we consider any function $h \in \mathcal{C}(Y)$, $h \circ \tau$ is continuous on X because τ is also continuous. Hence $h \in \mathcal{C}'$ which means that $\mathcal{C}' \supset \mathcal{C}(Y)$. This would mean that $\mathcal{C}' = \mathcal{C}(Y)$, so $g \longrightarrow g \circ \tau$ is an isomorphism. □

We now present three significant findings about completely separated and compact sets. In order to understand this, it is necessary to be reminded that a Hausdorff space is considered to be compact if and only if every family of closed sets with the property of finite intersection has nonempty intersection. Let us study how can we take advantage of the separation properties whenever our space is Tychonoff.

Proposition 4.11. *These three states hold true:*

(a) Let $A, A' \subset X$ be two subsets such that $A \cap A' = \emptyset$ and one of them is compact. If X is a Tychonoff space, then they are completely separated.

(b) Let X be Tychonoff. Whenever we have $S \subseteq G_\delta \subset X$ where S is compact, then G_δ contains a zero-set containing S.

(c) Let X be a Tychonoff space and $S \subset X$ a compact subset. Hence S is $\mathcal{C}$–embedded.

Proof (a) Take A, A' with the given conditions. Suppose A is compact. Given $x \in A$, take disjoint zero-sets Z_x and Z'_x, with Z_x a neighborhood of x, and $A' \subset Z'_x$. Then we can find a finite subcover of the cover $\{Z_x\}_{x \in A}$ of the compact set A, this is

$$\{Z_{x_1}, \ldots, Z_{x_n}\}$$

so our sets A, A' belong to disjoint zero-sets

$$Z_{x_1} \cup \ldots \cup Z_{x_n} \text{ and } \mathcal{Z}'_{x_1} \cup \ldots \cup \mathcal{Z}'_{x_n},$$

respectively.
(b) A G_δ has the shape $\bigcap_{n \in \mathbb{N}} U_n$, being each U_n is an open set. If $S \subset A$, therefore S is completely separated from $X - U_n$, and there is a zero-set satisfying $S \subset F_n \subset U_n$. Then

$$S \subset \bigcap_{n \in \mathbb{N}} F_n \subset A$$

and $\bigcap_{n \in \mathbb{N}} F_n$ is also a zero-set since it is a countable intersection of those as well. Specifically, we have shown that in an arbitrary Tychonoff space, every compact G_δ must be a zero-set.
(c) Compact property does not depend on any embedding, thus keep in mind that compact is an absolute topological notion. Now consider $S \subset X$ a compact subspace (X is Tychonoff). If we take two arbitrary subsets in S such that they are completely separated in S, then their closures, since they are also compact, they will be again completely separated and by Proposition (a), they

are completely separated in X. We can apply then Urysohn's extension Theorem (2.42.) and get what we needed to prove. □

As the reader has imagined, we are looking for Tychonoff spaces. This is because with this structure, the behaviour of prime and maximal ideals is quite interesting. In order to create more of them, our next purpose is to prove Urysohn's Lemma, that will grant us much more Tychonoff spaces coming from normal spaces.

Definition 4.12. A topological space X is a **normal** space if, given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F that are also disjoint. More intuitively, this condition says that E and F can be separated by neighbourhoods.

We will deal in this paper with T_4 spaces, those are, Hausdorff spaces that are also normal.

Lemma 4.13. *For any X topological space, take $R_0 \subset \mathbb{R}$ any dense subset. We define open sets $U_r \subset X$ for all $r \in R_0$ such that*

$$\bigcup_{r \in R_0} U_r = X \qquad \bigcap_{r \in R_0} U_r = \emptyset$$

and

$$cl_X(U_r) \subset U_s \text{ whenever } r < s.$$

Then the formula

$$f(x) = \inf\{r \in R_0 | \; x \in U_r\} \qquad (x \in X)$$

defines a continuous function f on X.

Theorem 4.14. (Urysohn's Lemma) *In a normal space X, any two disjoint closed sets completely separated. Therefore, every normal space is Tychonoff.*

Proof We want to take advantage of the previous lemma. Let X be a normal space and $A, B \subset X$ two disjoint closed sets. Now we must define our open sets U_r (for every $r \in \mathbb{Q}$):

$$U_r = \emptyset \text{ for every } r < 0$$
$$U_r = X \text{ for every } r > 1$$

If we take the set $U_1 = X \backslash B$ we then have a neighborhood of A. Therefore it must contain a closed neighborhood of A because X is normal. If we notate is as U_0 then we have an open set that yields

$$A \subset U_0 \text{ and } \mathrm{cl}_X(U_0) \subset U_1.$$

Take then a rational sequence $\{r_n\}_{n \in \mathbb{N}}$ in $[0, 1] \cap \mathbb{Q}$ such that $r_1 = 1$ and $r_2 = 0$. The idae is to choose, for every $n > 2$, open sets U_{r_n} that meet the conditions

$$\mathrm{cl}_X(U_{r_k}) \subset U_{r_n} \text{ and } \mathrm{cl}_X(U_{r_n}) \subset U_{r_l}.$$

whenever $r_k < r_n < r_l$, with $k, l < n$. We are now in the hypothesis of Lemma (4.13.) and this gives us a continuous function that is equal to 0 on A, and equal to 1 on B. □

Corollary 4.15. *Let X be a compact space and $S \subset X$ any subset. Then S is Tychonoff.*

PROOF Since X is compact, then it is also normal. If we apply Urysohn's Lemma (4.14.) we find out X is also a Tychonoff space, so any subspace of itself is again Tychonoff. □

This ends this section where we defined a lot of topological concepts. Now, in the next one, we will study how they are related to the topological objects we defined in the first chapter, z–filters.

5 Fixed ideals and compact spaces

Finally Tychonoff spaces are defined, so we do not need to deal with other kind of spaces anymore. We will assume from now on that every given space is Tychonoff. Of course this condition must be checked every time we construct a space.

5.1 Fixed and free ideals

Definition 5.1. Take $\mathfrak{a} \leq \mathcal{C}(X)$ an ideal (we can replace $\mathcal{C}(X)$ with $\mathcal{C}^*(X)$). We say $\mathfrak{a}$ is a **fixed ideal** whenever

$$\bigcap \mathcal{Z}[\mathfrak{a}] \neq \emptyset.$$

If this condition does not hold, we will call $\mathfrak{a}$ a **free ideal**.

Remark 5.2. We can see easily that $\mathfrak{a}$ is free if and only if, for every point $x \in X$ we can find a function $f \in \mathfrak{a}$ such that $f(x) \neq 0$.

Remark 5.3. There are many examples of spaces that does not admit free ideals. These are the compact spaces and we shall prove this result in Theorem (5.11.).

Example 5.4. Given S any nonempty set in X, we know that the set

$$\{f \in \mathcal{C} |\ f(S) = \{0\}\}$$

is not only an ideal in $\mathcal{C}$, but also a z–ideal. It's not difficult to prove that this ideal is fixed and when we intersect it with $\mathcal{C}^*$ we obtain again a fixed ideal in $\mathcal{C}^*$. Moreover, whenever our set S is not dense in X, we can say those ideals are nonzero.

Example 5.5. Let's take an ideal in $\mathcal{C}^*(\mathbb{N})$ such that $\mathbf{j}$ is contained in that ideal. We have found then a free ideal. Specifically, there is no fixed maximal ideal containing $\mathbf{j}$.

Remark 5.6. As a trivial remark, a free ideal cannot be contained in any fixed ideal.

Example 5.7. Let's introduce some notation. We will name $C_k(\mathbb{N})$ the set of all functions on $\mathbb{N}$ that only vanishes at most in a finite number of points. In this way we have discovered another free ideal both in $\mathcal{C}(\mathbb{N})$ and $\mathcal{C}^*(\mathbb{N})$. In addition, $C_k(\mathbb{N})$ is the intersection of every free ideal, not only in $\mathcal{C}$, but in $\mathcal{C}^*$ too.

Take now both $\mathcal{C}(X)$ and $\mathcal{C}^*(X)$ rings. Let us talk about fixed maximal ideals there. Given any fixed ideal $\mathfrak{a}$ in $\mathcal{C}$, then the set

$$S = \mathcal{Z}(\mathfrak{a})$$

is nonempty, and the set

$$\mathfrak{a}' = \{f \in \mathcal{C} |\ f(S) = \{0\}\}$$

contains $\mathfrak{a}$ and is a fixed ideal. Thus if we have a fixed maximal ideal, it's going to have this form. Furthermore, we can only find as fixed maximal ideals those ideals $\mathfrak{a}$' for which S only contains one single point. This happens because $\mathfrak{a}$' can be enlarged by making our set S smaller.

Theorem 5.8. *In the ring $\mathcal{C}(X)$, where X is a Tychonoff space, the fixed maximal ideals have the following form:*

$$M_p = \{f \in \mathcal{C} |\ f(p) = 0\} \qquad\qquad (p \in X).$$

Those ideals are different whenever we change the point p. In any case, $\mathcal{C}/M_p$ is isomorphic with $\mathbb{R}$. Indeed, the mapping $M_p(f) \longrightarrow f(p)$ is not only an isomorphism, but the unique one of $\mathcal{C}/M_p$ onto $\mathbb{R}$.

Proof The set just defined can be seen as the kernel of the homomorphism $f \longrightarrow f(p)$ that goes from $\mathcal{C}$ to $\mathbb{R}$. Since that homomorphism stays in $\mathbb{R}$ because $\mathbf{r}(p) = r$ for every real r, then its kernel M_p is thus a maximal ideal. This happens because our space X is Tychonoff. Moreover, our point p is unique. If we reason along this path, when we have a fixed ideal in $\mathcal{C}$, then it is necessary that p exists. Of course $M \subseteq M_p$, which has just been shown to be a proper ideal. Thus, we can finish the proof by concluding that whenever we have M maximal, $M = M_p$ and since M_p is the kernel of a homomorphism over $\mathbb{R}$, $\mathcal{C}/M_p$ must be isomorphic (by Theorem (1.11.) with $\mathbb{R}$. The uniqueness comes from the fact that the identity is only automorphism possible of $\mathbb{R}$. □

Remark 5.9. We can substitute $\mathcal{C}$ by $\mathcal{C}^*$ and the result still holds true. The proof is identical except for notation.

This theorem grants us some kind of bridge between that theory about fixed maximal ideals in $\mathcal{C}$ and those in $\mathcal{C}^*$. Moreover, we get as an corollary an easy way in order to obtain this correspondence, this is:

$$M_p \longrightarrow M_p^* = M_p \cap \mathcal{C}^*$$

An easy way in order to obtain a prime ideal in $\mathcal{C}^*$ is to intersect any maximal ideal in $\mathcal{C}^*$ with the ring itself. This statement, however, does not imply that this ideal must be maximal nor that the free maximal ideals in $\mathcal{C}^*$ have this form. For example, take a look at the function $\mathbf{j}$ in $\mathcal{C}^*(\mathbb{N})$. This is known to be a unit of $\mathcal{C}(\mathbb{N})$, so it cannot be in any ideal of $\mathcal{C}(\mathbb{N})$. Soon we will show that in fact, this function is in every free maximal ideal of $\mathcal{C}^*(\mathbb{N})$. By now, let us just assume this to be true. Consider any maximal free ideal $M \in \mathcal{C}(\mathbb{N})$. Then $M \cap \mathcal{C}^*(\mathbb{N})$ will not be a fixed maximal ideal in $\mathcal{C}^*(\mathbb{N})$. It's impossible for it to be a free maximal ideal since it does not contain $\mathbf{j}$.

Proposition 5.10. *Let M^* be any fixed maximal ideal in $\mathcal{C}^*(\mathbb{N})$. Then, the function $\mathbf{j}$ is contained in M^*.*

Proof The proof will be quite simple. Given our fixed maximal ideal M^*, let's pretend that $\mathbf{j} \notin M^*$. Then we can reproduce the whole space by means of our ideal and $\mathbf{j}$, this is, $\langle M^*, \mathbf{j}\rangle = \mathcal{C}^*(\mathbb{N})$, which means that it will exist $f \in \mathcal{C}^*(\mathbb{N})$ such that $f\mathbf{j} - 1 \in M^*$. Remember that f is bounded, so the following set

$$A = \{n \in \mathbb{N} |\ f(n) > n/2\}$$

must be finite. By the Example (5.5.), we can find $g \in M^*$ that does not vanish in A (without going any further, the characteristic function of A can be used). But then $g^2 + (\mathbf{1} - f\mathbf{j})^2$ is bounded away from zero, so it is contained in M^* which is not true by hypothesis. □

5.2 Continuous functions on compact spaces

In the study of this new ideals, they seem to behave very well when the topological space X is compact. We are stating now some results in order to show that.

Theorem 5.11. *Every ideal belonging to any compact space X is fixed.*

PROOF The proof is quite simple. We have to realize that the family of closed sets $\mathcal{Z}(\mathfrak{a})$ has the finite intersection property, which gives us the proof. □

In the light of Theorem (5.8.), we now have:

Lemma 5.12. *Let X be a compact space and we shall remind the correspondence $p \longrightarrow M_p$. Then it is a one-one correspondence, this is, every point in X is attached to one maximal ideal in $\mathcal{C}(X)$.*

As far as we know, maximal ideals are algebraic invariants, so we can recover any point in a compact space from the algebraic structure of the ring. Remember that the zero-sets form a base for the closed sets and we can still use the relation already seen between these zero-sets and our ideals defined as M_p, that is, $p \in \mathcal{Z}(f)$ is equivalent to $f \in M_p$. We are thus relating topological concepts to algebraic objects. That is, we can recover the topology of X from $\mathcal{C}(X)$.

Theorem 5.13. *For any X and Y compact spaces, they are homeomorphic if and only if the corresponding rings $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are isomorphic.*

PROOF No need to prove necessity. We have also given a way to recover a space from its respective function ring, so the condition is also sufficient. □

Not only we have determined an equivalence between a compact space X and its respective ring of continuous functions $\mathcal{C}(X)$, but also we given a method on to recover that space from its ring. However, we aim to describe this action deeper.

We will denote $\mathcal{M} = \mathcal{M}(X)$ as the collection of every maximal ideal in $\mathcal{C}(X)$. The idea is giving $\mathcal{M}$ a topological structure by giving its base for the closed sets. They will have the following form:

$$\mathcal{L} = \{M \in \mathcal{M} |\ f \in M\} \qquad\qquad (f \in \mathcal{C}(X)).$$

Let us see that this space is indeed well defined. Let us take a function f. An ideal M_p for a certain $p \in X$ will belong to this set if and only if $f(p) = 0$. This means that the correspondence $p \longrightarrow M_p$ will uniquely take the zero sets in X to the family of all sets $\mathcal{L}$. Then, it is well defined, has topological space structure and is homeomorphic to X.

We have then presented the topology that is going to help us from now on. We can name it the **Stone topology** on $\mathcal{M}$. Whenever we endow any set $\mathcal{M}$ with this specific topology, it will be called the structure space of the ring $\mathcal{C}$.

5.3 Free and fixed z–filters

We will now look at the other side, the topological objects we know as z–filters. We will try to define topological objects using the previous algebraic concepts as sharpen as we can.

Definition 5.14. Let $\mathcal{F}$ be a z–filter. Then, in the same way as we did before, it will be a free z–filter if

$$\bigcap_{F\in\mathcal{F}} F = \emptyset.$$

If this does not yields, then it will be called a fixed z–filter.

By the characterization given in the second chapter (more specifically in the results Theorem (2.44.) and Proposition (2.56.)) we can easily give the following result:

Proposition 5.15. *Let $\mathfrak{a}$ be an ideal in $\mathcal{C}$. We will say that $\mathfrak{a}$ is fixed if and only if $\mathcal{Z}(\mathfrak{a})$ is fixed.*

In other words, we can deduce that every z–filter in X will be fixed if and only if every ideal belonging to $\mathcal{C}(X)$ is also fixed. For sure this was the obvious result that everyone expected at this point, z–filters and ideals are still sharing properties.

Lemma 5.16. *Let Z be a zero-set Z. Then it will be compact if and only if there is no free z–filter in which Z is contained.*

PROOF ($\Rightarrow$) No need to prove this implication.
($\Leftarrow$) If we take an arbitrary family $\mathcal{B}$ of closed subsets of $\mathcal{Z}$ with the finite intersection property, it is not difficult to see that the elements of $\mathcal{B}$ are closed in X. Name $\mathcal{F}$ as the set of every zero-sets in X containing a finite intersection of elements of $\mathcal{B}$. This will be a filter that satisfies that $Z \in \mathcal{F}$. Since X is Tychonoff, the zero sets form a base for the closed ones, which means that $\bigcap \mathcal{B} = \bigcap \mathcal{F}$. However, by hypothesis, this last intersection is non-empty, which means that $\bigcap \mathcal{B}$ is non-empty. Therefore, Z is compact. □

Remark 5.17. It is common to erroneously deduce that whenever we have that every element of a z–filter (or ultrafilter) is non compact, this means that is free. This result is, however, not true.

Now we are giving the strongest result about z–filters, fixed ideals, compact spaces and $\mathcal{C}(X)$ as well as $\mathcal{C}^*(X)$.

Theorem 5.18. *The next assertions are equivalent.*

(1) X is a compact space.

(2) Let $\mathfrak{a}$ be any ideal in $\mathcal{C}(X)$. Then $\mathfrak{a}$ is fixed (this also means that every z–filter is fixed).

(3) Let $\mathfrak{a}$ be any ideal in $\mathcal{C}(X)^*$. Then $\mathfrak{a}$ is fixed.

(4) Let $\mathfrak{m}$ be any ideal in $\mathcal{C}(X)$. Then $\mathfrak{m}$ is fixed (this also means that every z–ultrafilter is fixed).

(5) Let $\mathfrak{m}$ be any ideal in $\mathcal{C}(X)^*$. Then $\mathfrak{m}$ is fixed.

Proof This proof is quite simple because we have studied all these relations previously. The assertion (1) implies (2) is a particular case where $\mathcal{Z} = X$ of the Lemma (5.16.). In the same way, the equivalence (1) with (3), comes from the fact that $\mathcal{C} = \mathcal{C}^*$ when X is compact. Moreover, whenever an ideal $\mathfrak{a}$ is free in $\mathcal{C}$, then the ideal $\mathfrak{a} \cap \mathcal{C}^*$ is also free in $\mathcal{C}^*$ and that gives us another equivalence, (3) implies (2). Lastly, (2) implies (4), and (3) is equivalent with (5) because we can insert any free ideal into a maximal free ideal. □

6 The Stone–Čech compactification

We have seen some characterisations of maximal ideals in compact spaces. To do this, to each of these ideals we have added a point in that space. Our aim now is to characterise them for our known spaces, that is, in $\mathcal{C}(X)$ and in $\mathcal{C}(X)^*$. From these conclusions, the idea is to extend them to the general case, i.e. when we have a Tychonoff topological space. Whenever our space is not pseudo compact we'll face different problems: the $\mathcal{C}(X)$ one and the $\mathcal{C}^*(X)$ one.
Let us now summarise what we will do in both cases. While we are in $\mathcal{C}^*(X)$ we will look for a space which is not compact W as well as its compactification W^*, in which W is $\mathcal{C}^*$—embedded. This will mean that we can find an isomorphism between $\mathcal{C}^*(W)$ and $\mathcal{C}^*(W^*)$. Finally, using this mapping we can characterise the maximal ideals in any space finding a one-one relation between ideals and points of that compactification.
We will also use the results already seen about z–ultrafilters on a space X in the previous chapter. They will serve as a path that joins z–ultrafilters and maximal ideals in $\mathcal{C}(X)$. In an appropriate space we can associate a point to each z–ultrafilter. Somehow, this points will be the points of X itself, and we are trying to associate them with fixed z–ultrafilters in X. Only then we will look for a compactification of X.
Given these problems, one would normally think that we would have to find a different solution to each of them. However, this is not the case. We will be able to find a mutual solution to both problems. This happens thanks to the compactification βX. This object will grants us the holy grail of this theory, this is, characterizing maximal ideals in $\mathcal{C}(X)$. Moreover, it happens to be a space in which X is $\mathcal{C}^*$—embedded. This is known as βX, we will call it the Stone–Čech compactification of X and, basically, it is unique.

6.1 Accumulation points

First we want to deal with the case of $\mathcal{C}(X)$. So before constructing our compact space, I will give a general idea of what is to be done. To our Tychonoff space X we will attach a new point for each z–ultrafilter as a 'limit'. Of course, since we know that zero–sets form a base for the closed sets, they will be key in this process and we will use them repeatedly.

Definition 6.1. Let T be any space and X a dense subspace in T. Take then a z–filter $\mathcal{F}$ on X. We define a **accumulation point** $p \in T$ of $\mathcal{F}$ if every neighborhood in T of p meets every member of $\mathcal{F}$. This is,

$$p \in \bigcap_{Z \in \mathcal{F}} \mathrm{cl}_T(Z)$$

Remark 6.2. As a consequence of the definition itself, we know that any family of closed sets $\mathrm{cl}_T(Z)$ has the finite intersection property. We can then give a necessary condition for the space T to be compact. This is that every z–filter on X has an accumulation point in T. Moreover, the converse is also true. Moreover, it is enough that every z–ultrafilter on X has an accumulation point in T.

One of the meanings of the above observation is that whenever we have different free z–ultrafilters on X, then they must converge to distinct points in their compactification βX. We can draw an immediate consequence from this. This means that the one-point compactification $\mathbb{N}^*$ of $\mathbb{N}$ is not the best candidate for $\beta\mathbb{N}$. Of course this concludes that $\mathbb{N}^*$ does not pass the test of $\mathcal{C}^*$ because $\mathbb{N}$ is not even $\mathcal{C}^*$-embedded.
We can then ask when a dense subspace X of T will be $\mathcal{C}^*$–embedded in T. It is more worth asking the following, even more general, question: for any compact space Y, when can we extend a continuous mapping $\tau : X \longrightarrow Y$ to another continuous mapping $\overline{\tau} : T \longrightarrow Y$?
We will see that the condition that $\overline{\tau}$ carries closures into closures can be translated to an accumulation point language, this is, it takes accumulation points to accumulation points (or in the language of convergence, it takes limits to limits). The next result helps us solving this problem.

Theorem 6.3. *Given T a topological space and $X \subset T$ dense in it. Then, the next assertions are equivalent:*
(1) For any Y compact space, let $\tau : X \to Y$ be a continuous mapping. Then we can extend it to another continuous mapping $\overline{\tau} : T \longrightarrow Y$.
(2) X is $\mathcal{C}^*$–embedded in T.
(3) Let Z_1 and Z_2 be two zero-sets such that $Z_1 \cap Z_2 = \emptyset$ in X. Then $cl_T(Z_1) \cap cl_T(Z_2) = \emptyset$.
(4) For two arbitrary zero-sets Z_1 and Z_2 in X it yields that

$$cl_T(Z_1 \cap Z_2) = cl_T(Z_1) \cap cl_T(Z_2).$$

(5) For each point $p \in T$, it must be the limit of an unique z–ultrafilter on X.

Proof (1) ⇒ (2). If we take the set $\mathrm{cl}_{\mathbb{R}}(f(X)) \subseteq \mathbb{R}$ for any function $f \in \mathcal{C}^*$, then it is obviously a continuous mapping into that set, so it turns out that (2) is a special case of (1).
(2) ⇒ (3). Urysohn's extension Theorem (2.42.) grants us this implication.
(3) ⇒ (4). Let's suppose $p \in \mathrm{cl}_T(Z_1) \cap \mathrm{cl}_T(Z_2)$. Then if we look at each zero-set-neighborhood V (in T) of p, it happens that

$$p \in \mathrm{cl}_T(V \cap Z_1) \text{ and } p \in \mathrm{cl}_T(V \cap Z_2).$$

Thus (3) implies that $V \cap Z_1$ meets $V \cap Z_2$, this is, V meets $Z_1 \cap Z_2$. In that case:

$$p \in \mathrm{cl}(Z_1 \cap Z_2).$$

Hence, $\mathrm{cl}_T(Z_1) \cap \mathrm{cl}_T(Z_2)$ must be contained in $\mathrm{cl}_T(Z_1 \cap Z_2)$. The other inclusion is not difficult to prove.
(4) ⇒ (5). Given that the space X is dense in T, it implies the existence of a limit in the form of a z–ultrafilter for every point in T. Additionally, distinct z–ultrafilters encompass disjoint zero-sets. According to the hypothesis, a point p cannot simultaneously pertain to the closures of both zero-sets. Consequently, the two z–ultrafilters are unable to converge to p.
(5) ⇒ (1). For a given $p \in T$, let $\mathcal{A}$ represent the unique z–ultrafilter on X with a limit at p. We can then define

$$\tau^\sharp \mathcal{A} = \{E \in \mathcal{Z}(Y) | \ \tau^{-1}(E) \in \mathcal{A}\}.$$

This forms a z–filter on the compact space Y, which includes an accumulation point. Furthermore, as $\mathcal{A}$ is a prime z–filter, $\tau^\sharp\mathcal{A}$ is also prime. Hence, $\tau^\sharp\mathcal{A}$ possesses a limit in Y. We will call this limit $\overline{\tau}p$:

$$\bigcap \tau^\sharp\mathcal{A} = \{\overline{\tau}p\}.$$

This leads to the definition of a mapping $\overline{\tau} : T \longrightarrow Y$. If it happens that $p \in X$, we would have $p \in \bigcap \mathcal{A}$ which means that $\tau p \in \bigcap \tau^\sharp\mathcal{A}$. Then, $\overline{\tau}$ is nothing else than τ in X. In the case of $F, F' \in \mathcal{Z}(Y)$, we can write $Z = \tau^{-1}(F)$ and $Z' = \tau^{-1}(F')$. If $p \in \mathrm{cl}_T(Z)$ is given, then Z has to be in $\mathcal{A}$, and hence $F \in \tau^\sharp\mathcal{A}$. This means that $p \in \mathrm{cl}_T(Z)$ implies $\overline{\tau}p \in F$.
For the continuity of $\overline{\tau}$ at p, we must consider two neighborhood. Let's take any zero-set neighborhood F of $\overline{\tau}p$ and also another neighborhood of the point p that is carried by $\overline{\tau}$ in F. We can deduce that

$$\mathrm{cl}_T(Z) \cup \mathrm{cl}_T(Z') = T$$

since $F \cup F' = Y$ and $Z \cup Z' = X$. We have also $p \notin \mathrm{cl}_T(Z')$ because we know that $\overline{\tau}p \notin F'$. Next, it is a fact that $T \backslash \mathrm{cl}_T(Z')$ serves as a neighborhood of p, so every point q in this neighborhood is in $\mathrm{cl}_T(Z)$, i.e., $\overline{\tau}q \in F$. This concludes the theorem. □

6.2 z–filters on a Tychonoff space

Once we have defined key objects in our study we will move on to give it a more specific context. This section will be devoted to a brief study of the convergence of z–filters in a Tychonoff space. Likewise, we could do this study for filters in any Hausdorff space.
In fact, in the previous section we have dropped the concept of convergence of z–filters to a dense space somewhat intuitively. This time we will formalise that concept for our preferred spaces and see how all the objects studied so far relate to each other.
We are treating this short part as an introduction to notions about the convergence of z–filters in order to be able to take advantage of them when dealing with Tychonoff spaces. However, it is not our main aim to deal with this topic in depth, so many demonstrations will be left undone. If the reader is interested in this topic, he can study it in depth in the book "*Ring of continuous functions* [4]" and find answers to his doubts.
The definition of an accumulation point remains the same. A point $p \in X$ is an accumulation point of a z–filter whenever the intersection of every neighborhood of p with every member of that filter is non-empty. The first difference with respect to the previous case is to be found now. It turns out that each member of a z–filter is closed, so we can give a characterisation of the accumulation points.

Proposition 6.4. *Let X be a Tychonoff space along a z–filter $\mathcal{F}$ in X. Then p is an accumulation point of $\mathcal{F}$ if and only if $p \in \bigcap \mathcal{F}$.*

It follows that for a non-empty subset S of X, the set $\mathrm{cl}_X(S)$ precisely contains all the accumulation points of the z–filter $\mathcal{F}$, where $\mathcal{F}$ comprises all the zero–sets containing S. This is a consequence of the fact that in a Tychonoff space, the zero–sets serve as a base for the closed sets.
The definition of convergence for z–filters follows a familiar pattern. We assert that a z–filter converges to the limit p if every neighborhood of p contains at least one member of the filter. It

is straightforward to observe that under this condition, p must be an accumulation point of the z–filter.
About Tychonoff spaces, it yields that every neighborhood of a point p in that space contains at least one zero-set-neighborhood of p. In other words,

Proposition 6.5. *A z–filter $\mathcal{F}$ converges to a point $p \in X$ if and only if $\mathcal{F}$ contains the z–filter of all zero-set-neighborhoods of p.*

Let's give an example of this last proposition.

Example 6.6. Let's look for z–filters converging to 0 on $\mathbb{R}$. These are going to be provided by families of zero-sets Z in $\mathbb{R}$ meeting some conditions:

(i) Z must be a neighborhood of 0.

(ii) We can find $\epsilon > 0$ and $\delta > 0$ such that the intervals $[0, \epsilon]$ and $[-\delta, 0]$ are contained in Z.

(iii) The element $\frac{1}{n} \in Z$ for almost any $n \in \mathbb{N}$ (there may exist a set containing a finite number of elements in $\mathbb{N}$ such that this assertion will not yield).

(iv) Of course, $0 \in Z$.

We have then given a family of z–filters converging to 0 according to the notions already studied. Moreover, the last of these z–filters is a z–ultrafilter and it is the only one converging to 0.

From this example we can deduce the next proposition:

Proposition 6.7. *Let $\mathcal{F}$ be a family of z–filters and $p \in \mathcal{F}$ an accumulation point of that family. Then there is one or more z–ultrafilters converging to p and contains $\mathcal{F}$. Specifically, a z–ultrafilter converges to any accumulation point.*

Whenever X is a Tychonoff space, if $\mathcal{F}$ converges to a point $p \in X$, then it yields that

$$\bigcap \mathcal{F} = \{p\}.$$

Since our main objective is dealing with maximal ideals and they happen to be also prime ideals, it makes sense for us to give a result that relates accumulation points to these ideals.

Theorem 6.8. *Let $p \in X$ be a point of a Tychonoff space and $\mathcal{F}$ an arbitrary prime z–filter on the same space. Then these three are equivalent.*

(1) p is an accumulation point of $\mathcal{F}$.

(2) p is a convergence point of $\mathcal{F}$.

(3) $\bigcap \mathcal{F} = \{p\}$.

Proof It is enough to prove that (1) implies (2). We want to show that being an accumulation point implies convergence for a z–filter. Let us take any zero-set-neighborhood of p and call it V. We know V must contain a neighborhood of p of the form $X\backslash Z$ with Z a zero-set because X is Tychonoff. Since we have $V \bigcup Z = X$, then either V is in the prime z–filter $\mathcal{F}$ or Z is. However, the last one cannot be true since $p \notin Z$, which means that p is a convergence point of $V \in \mathcal{F}$ by Proposition (6.5.). $\square$

We can then deduce from this section that in a Tychonoff space, a z–ultrafilter (it yields for any prime z–filter in general) either it has an accumulation point or none at all.
Let's denote the family of all zero-sets containing a point p by A_p. This set is not only a z–filter, but a maximal one, this is, a z–ultrafilter. This is because if we take any zero-set that does not contain p, then our set is completely separated from $\{p\}$. In accordance with the Theorem (6.8.), the z–ultrafilters A_p are those that converge on X.

Proposition 6.9. *A point p in any Tychonoff space X is an accumulation point of a z–filter $\mathcal{F}$ if and only if $\mathcal{F} \subset A_p$.*

In other words, for p to be an accumulation point of $\mathcal{F}$ it is necessary and sufficient that p pertains to every element of $\mathcal{F}$. We can deduce several corollaries:

Corollary 6.10. *For any point $p \in X$ (where X is a Tychonoff space), there is only one z–ultrafilter that converges to that point, and that is A_p.*

Corollary 6.11. *It is not possible for different z–ultrafilters to have a common accumulation point.*

Corollary 6.12. *Let $\mathcal{F}$ be a z–filter on a Tychonoff space X and suppose it converges to p. Then the only z–ultrafilter that contains $\mathcal{F}$ is A_p.*

6.3 Construction of the Stone-Čech compactification

Now we are ready to solve the problem that we faced in this chapter, giving a method to construct βX.

Theorem 6.13. (Compactification theorem) *We can find the Stone-Čech compactification βX for any Tychonoff space X, along with the properties listed below, which are equivalent:*

*(1) **(Stone)** For any compact space Y, we can find a continuous extension $\overline{\tau} : \beta X \to Y$ for any continuous mapping $\tau : X \to Y$.*
*(2) **(Stone-Čech)** We can extend any function $f \in \mathcal{C}^*(X)$ into a function f^β in $\mathcal{C}(\beta X)$.*
*(3) **(Čech)** If Z_1 and Z_2 are zero-sets such that $Z_1 \cap Z_2 = \emptyset$ in X, then $cl_{\beta X}(Z_1) \cap cl_{\beta X}(Z_2) = \emptyset$.*
(4) Take two arbitrary zero-sets $Z_1, Z_2 \subset X$, then

$$cl_{\beta X}(Z_1 \cap Z_2) = cl_{\beta X}(Z_1) \cap cl_{\beta X}(Z_2).$$

(5) Whenever we choose distinct z–ultrafilters on X, their limits are different in its compactification.

Moreover, if a compactification T of X satisfies any of the enumerated specifications, then there exists a homeomorphism of βX on T that leaves X pointwise fixed, which means that βX is essentially unique.

Whenever βX satisfies (1) we will say that it meet the Stone's theorem, we will call by the Stone extension of τ into Y to the mapping $\overline{\tau}$.

PROOF Let's start by dealing with uniqueness. The Theorem (6.3.), tells us that if T satisfies one of the statements, it satisfies all of them. Then, by Theorem (1), we can extend the identity mapping that has as its domain X to go from βX to the continuous space T since we are talking about continuous mappings. Similarly, you have a Stone extension from T to βX. We can deduce that these extensions are homeomorphisms. □

Finally we will give a method to construct the Stone-Čech compactification. As we mentioned before, we must relate one by one the points of βX with the z–ultrafilters in X. They are going to be actually the convergence points of these z–ultrafilters. We have given at the moment this relation in Corollary (6.10.), so our Tychonoff space X will constitute a ready-made index set for the fixed z–ultrafilters. Dealing with our notation:

The members of this enlarged index set are actually the points of βX. For notation, we write the family of all z–ultrafilters on X as

$$(A^p)_{p\in\beta X}$$

Whenever we look for emphasis, we will rewrite A^p by A_p, for $p \in X$. Hence, A_p is clearly the same as $\mathcal{Z}(M_p)$. From now on and in order to define correctly the topology on βX, p will not only be the limit of the z–ultrafilter A^p for every $p \in X$, but also for $p \in \beta X$. Now we are writing

$$\overline{Z} = \{p \in \beta X |\ Z \in A^p\},$$

this means, $p \in \overline{Z}$ if and only if $Z \in A^p$. Specifically, it yields that $\overline{X} = \beta X$ because X itself belongs to every z–ultrafilter. We know that $Z_1 \cup Z_2 \in A^p$ if and only if $Z_1 \in A^p$ or $Z_2 \in A^p$. Thus

$$\overline{Z}_1 \cup \overline{Z}_2 = \overline{Z_1 \cup Z_2}.$$

And in fact, since $\emptyset$ is not in any z–ultrafilter, it follows that $\emptyset = \overline{\emptyset}$. We have thus shown mainly two things:

$$\bigcup_{i\in I} \overline{Z}_i = \overline{\bigcup_{i\in I} Z_i}$$

where I is a finite index set, and

$$\emptyset \in \overline{Z}.$$

Having this in mind, nothing can delay the appearance of the topology in βX.

A topology can be defined in βX by taking as a base for the closed sets the family of all sets $\overline{Z}$.

Let us first check that indeed our space X is inside its own Stone–Čech compactification. We have the relation $p \in \overline{Z} \cap X$ if and only if $Z \in A_p$, so $p \in Z$. This means that $\overline{Z} \cap X = Z$, and, therefore, the identity on X carries the elements of the base of closed sets in the relative topology to another family of basic closed sets in the original topology. Therefore, it is effectively a homeomorphism.
We shall prove next that X is dense in βX. We will prove a more general case instead, this is:

$$\mathrm{cl}_{\beta X}(Z) = \overline{Z}.$$

We can then deduce that $\mathrm{cl}_{\beta X}(X) = \overline{X} = \beta X$. It's already known that $Z \subset \overline{Z}$. Besides that, whenever we have $\overline{Z}'$ a basic closed set that contains Z, then,

$$Z' = \overline{Z}' \cap X \supset Z,$$

so that $\overline{Z}' \supset \overline{Z}$. Therefore, $\mathrm{cl}_{\beta X}(Z) \supset \overline{Z}$. Then we have:

$$p \in cl_{\beta X}(Z) \textit{ if and only if } Z \in A^p.$$

$Z_1 \cap Z_2 \in A^p$ is equivalent to $Z_1 \in A^p$ and $Z_2 \in A^p$, and that gives us (4). As a consequence, by the time we know that βX is compact, this proof will be finished.
In order to prove that βX is Hausdorff, take $p \neq p'$. Take zero-sets $A \in A^p$ and $A' \in A^{p'}$ such that $A \cap A' = \emptyset$ and therefore there exists zero-sets Z, Z' disjoint from A and A' respectively such that $Z \cup Z' = X$. This means that Z does not belong to A^p and neither do Z' in $A^{p'}$. In other words, $p \notin \mathrm{cl}_{\beta X}(Z)$ and $p' \notin \mathrm{cl}_{\beta X}(Z')$. Since

$$\mathrm{cl}_{\beta X}(Z) \cup \mathrm{cl}_{\beta X}(Z') = \beta X,$$

the neighborhoods $\beta X \backslash \mathrm{cl}_{\beta X}(Z)$ of p, and $\beta X \backslash \mathrm{cl}_{\beta X}(Z')$ of p', are disjoint.
In order to finish the proof, we need a family of basic closed sets $\mathrm{cl}_{\beta X}(Z)$ such that Z is in some auxiliary family $\mathcal{B}$. As a consequence, $\mathcal{B}$ also has this property and we can then deduce that there exists always a manner to embed $\mathcal{B}$ in a z–ultrafilter with the form A^p, hence

$$p \in \bigcap_{Z \in A^p} \mathrm{cl}_{\beta X}(Z) \subset \bigcap_{Z \in \mathcal{B}} \mathrm{cl}_{\beta X}(Z),$$

and effectively, there is at least one element in that last intersection, thus it is not empty. Hence βX is compact.
Indeed, the fact that (2) implies (3) is a trivial result of Urysohn's extension Theorem (2.42.), because disjoint closed sets are completely separated in βX.

Theorem 6.14. *The next statements are equivalent with the results* (1) *to* (5) *of Theorem* (6.3.), *with X being dense in T.*
(6) $X \subset T \subset \beta X$.
(7) $\beta T = \beta X$.

Proof The idea is to use the second condition (2), this is, that X is $\mathcal{C}^*$–embedded in T.
(2) ⇒ (7). We have just said that X is $\mathcal{C}^*$–embedded plus it is dense in T, thus $\beta X = \beta T$.
(7) ⇒ (6). $X \subset T \subset \beta T = \beta X$.
(6) ⇒ (2). X is clearly also embedded in T since it is $\mathcal{C}^*$–embedded in βX. □

Remark 6.15. What the above result gives us those spaces in which we can $\mathcal{C}^*$–embed X and it is also dense in them. Those are just the subspaces of its Stone–Čech compactification which contain X (clearly the only compact space contained in βX and containing X is βX itself).

6.4 Examples of the compactification and applications

We are now giving away some results that are mainly consequences of the last theorem, so we are not proving them at the moment.

Theorem 6.16. (Tychonoff product theorem) *Let $X_{i\in I}$ be any compact spaces for each $i \in I$. Then,*

$$X = \bigtimes_{i\in I} X_i$$

is a compact space.

Let us now take $S \subseteq X$. At this point we know that this subspace is $\mathcal{C}^*$-embedded in X if and only if it is $\mathcal{C}^*$-embedded in βX. Equivalently, given that the compact set $\mathrm{cl}_{\beta X}(S)$ is $\mathcal{C}^*$—embedded in βX, we can say that S will be $\mathcal{C}^*$—embedded in X if and only if S is $\mathcal{C}^*$—embedded in $\mathrm{cl}_{\beta X}(S)$. We can deduce from all this that $\mathrm{cl}_{\beta X}(S)$ clearly meets the well-known conditions of βS: thus we have found a compactification of S such that S is $\mathcal{C}^*$—embedded. Then:

Corollary 6.17. *S is $\mathcal{C}^*$–embedded in X if and only if $cl_{\beta X}(S) = \beta S$.*

Corollary 6.18. *For every $S \subset X$ compact subset, it is $\mathcal{C}^*$–embedded in X.*

There is no need to explain this. We can deduce it easily from the first corollary: whenever we have S a compact set, then $\mathrm{cl}_{\beta X}(S) = S = \beta S$, whence S is $\mathcal{C}^*$–embedded in X.

Corollary 6.19. *If S is both an open and closed set at the same time in X, then it yields that $cl_{\beta X}(S)$ and $cl_{\beta X}(X\backslash S)$ are open sets in βX and*

$$cl_{\beta X}(S) \cup cl_{\beta X}(X\backslash S) = \beta X$$
$$cl_{\beta X}(S) \cap cl_{\beta X}(X\backslash S) = \emptyset.$$

Corollary 6.20. *Isolated points in X are also isolated in βX. We can then deduce that X is open in βX if and only if X is locally compact.*

Now let's look for examples and we are taking the most known fields which are also metric spaces, hence they are Tychonoff spaces. These are, $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$.

Example 6.21. (The space $\beta\mathbb{N}$.) By Corollary (6.20.), $\mathbb{N}$ is open in $\beta\mathbb{N}$. Moreover, the same corollary tells us that isolated points in $\mathbb{N}$ are also isolated in $\beta\mathbb{N}$. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, these will be the only isolated points. By Corollary (6.19.), if we compute the closure in $\beta\mathbb{N}$ of any subset in $\mathbb{N}$, it will be open in $\beta\mathbb{N}$. Returning to the relation calculated above, we can ensure that those points $p \in \beta\mathbb{N}\backslash\mathbb{N}$ are in one-to-one correspondence with free ultrafilters of the form A^p in $\mathbb{N}$, with A^p converging to p. This means that for any neighbourhood of p, it intersects the set A^p. Conversely, if we have $Z \in A^p$, then $\mathrm{cl}_{\beta\mathbb{N}}(Z)$ is an open neighbourhood of p. Since $\beta\mathbb{N}$ is totally disconnected: if we take points $p \neq q$, we can take $Z \in A^p\backslash A^q$, so that $\mathrm{cl}_{\beta\mathbb{N}}(Z)$ is at the same time open and closed set in $\beta\mathbb{N}$ that contains p but not q.
For instance, the subset N_1 containing every odd integer is $\mathcal{C}^*$–embedded in $\mathbb{N}$. Hence $\mathrm{cl}_{\beta\mathbb{N}}(N_1) = \beta N_1$ by Corollary (6.17.), and therefore $\mathrm{cl}_{\beta\mathbb{N}}(N_1)$ is homeomorphic with $\beta\mathbb{N}$. It happens the same for N_2, defined as the subset of even integers in $\mathbb{N}$. Hence, we can write $\beta\mathbb{N}$ as:

$$\mathrm{cl}_{\beta\mathbb{N}}(N_1) \cup \mathrm{cl}_{\beta\mathbb{N}}(N_2) = \beta\mathbb{N}.$$

Example 6.22. (The space $\beta\mathbb{Q}$) Let us proceed similarly with this space, since it is again totally disconnected. Let $p \neq q$ and closed neighbourhoods $p \in U$ and $q \in V$ be such that $U \cap V = \emptyset$. Then,

$$(U \cap \mathbb{Q}) \cap (V \cap \mathbb{Q}) = \emptyset$$

and both $(U \cap \mathbb{Q})$ and $(V \cap \mathbb{Q})$ are again closed in $\mathbb{Q}$. Then we can find an open and closed set L in $\mathbb{Q}$ that contains $U \cap \mathbb{Q}$ and is also disjoint from $V \cap \mathbb{Q}$. By Corollary (6.19.), the open and closed set $\mathrm{cl}_{\beta\mathbb{Q}}(L)$ in $\beta\mathbb{Q}$ contains p but not q.
If we take any mapping $\tau : \mathbb{N} \to \mathbb{Q}$, it will be continuous in the compact space $\beta\mathbb{Q}$. In such a way, it has a Stone extension $\overline{\tau} : \beta\mathbb{N} \to \beta\mathbb{Q}$. As this mapping arrives at a compact set in $\beta\mathbb{Q}$ and contains $\mathbb{Q}$, which is dense, then it must be the whole $\beta\mathbb{Q}$. This means that $\beta\mathbb{Q}$ is a continuous image of $\beta\mathbb{N}$. What is more, Corollary (6.17.) implies that

$$\mathrm{cl}_{\beta\mathbb{Q}}\mathbb{N} = \beta\mathbb{N}.$$

Therefore $\beta\mathbb{Q}$ is equipotent with $\beta\mathbb{N}$.

Example 6.23. (The space $\beta\mathbb{R}$.) Proceeding in a similar manner, $\beta\mathbb{R}$ is also a continuous image of $\beta\mathbb{N}$, and

$$\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{N}) = \beta\mathbb{N}.$$

Again, this means that $\beta\mathbb{R}$ is equipotent with $\beta\mathbb{N}$.
We know that whenever we have X a connected set, then the closure of itself is again connected. Then we deduce that $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)$ and $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^-)$, as well as $\beta\mathbb{R} = \mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R})$, are connected. Clearly, $\mathbb{R}^+$ is $\mathcal{C}^*$–embedded in $\mathbb{R}$, hence $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+) = \beta\mathbb{R}^+$. The fact that $\mathbb{R}^+$ is homeomorphic with $\mathbb{R}^-$ implies that $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)$ is homeomorphic with $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^-)$, and $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)\backslash(\mathbb{R}^+)$ with $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^-)\backslash(\mathbb{R}^-)$. If we take any neighborhood of a point in $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)\backslash(\mathbb{R}^+)$, then it does intersect $\mathbb{R}^+$ in a set that is not bounded. $\mathbb{R}^+$ is open in $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)$ because it is locally compact, so we can assure $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)\backslash(\mathbb{R}^+)$ is compact.

Obviously, $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+) \cup \mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^-) = \beta\mathbb{R}$. We can take for instance the function $f(x) = arctg(x)$, which is a continuous real-valued function. As well, it can be extended to a continuous function in $\beta\mathbb{R}$. This extension only takes the value $\pi/2$ on $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)\backslash(\mathbb{R}^+)$, and the value $-\pi/2$ on $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^-)\backslash(\mathbb{R}^-)$. This means that $\beta\mathbb{R}\backslash\mathbb{R}$ is not connected: $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^+)\backslash(\mathbb{R}^+)$ and $\mathrm{cl}_{\beta\mathbb{R}}(\mathbb{R}^-)\backslash(\mathbb{R}^-)$ do not intersect, and their union is the whole set $\beta\mathbb{R}\backslash\mathbb{R}$.

7 Maximal ideals in C and C^*

One of our objectives in this paper is to show the reader what form do maximal ideals have both in $\mathcal{C}(X)$ and $\mathcal{C}^*(X)$ for a topological space X. The intermediate tool has been already given: the Stone–Čech compactification. It gives an isomorphism between $\mathcal{C}^*(X)$ and $\mathcal{C}(\beta X)$, and then we can associate every maximal ideal in $\mathcal{C}(\beta X)$ with one unique point of βX.
Nevertheless, whenever $\mathcal{C}(X) \neq \mathcal{C}^*(X)$, by Corollary (2.27.), $\mathcal{C}(X)$ and $\mathcal{C}(\beta X)$ cannot be isomorphic. In this case, we do not want to change any plan. In fact, maximal ideals in $\mathcal{C}(X)$ will still be in that one-one correspondence with points of βX. There are several ways to define βX, however, we are going to choose one based on z–ultrafilters, which will make the construction even more intuitive since we have a lot of knowledge about that subject. z–ultrafilters will form the connection between maximal ideals in $\mathcal{C}(X)$ and points of βX.

Theorem 7.1. *In $\mathcal{C}^*(X)$, maximal ideals take the following form:*

$$M^{*p} = \{f \in \mathcal{C}^*(X) |\ f^{\beta}(p) = 0\}$$

with $p \in \beta X$. If $p, p' \in \beta X$ are two points such that $p \neq p'$, then $M^{*p} \neq M^{*p'}$

Proof We know that βX is compact, and the mapping $f \longrightarrow f^{\beta}$ is an isomorphism from $\mathcal{C}^*(X)$ onto $\mathcal{C}(\beta X)$. This means that maximal ideals will be the fixed ideals, that is,

$$\{f^{\beta} \in \mathcal{C}(\beta X) |\ f^{\beta}(p) = 0\}.$$

□

Trivially, M^{*p} is free or fixed according as $p \in \beta X \backslash X$ or $p \in X$.
Finally we are able to state the theorem that characterizes maximal ideals in $\mathcal{C}(X)$:

Theorem 7.2. (Gel'fand-Kolmogorov Theorem) *A subset M of $\mathcal{C}(X)$ is a maximal ideal of $\mathcal{C}(X)$ if and only if there is a unique point $p \in \beta X$ such that M coincides with the set*

$$M^p = \{f |\ f \in \mathcal{C}(X), p \in cl_{\beta X}(Z(f))\}$$

Proof ($\Leftarrow$). First we need to prove that M^p is a maximal ideal of $\mathcal{C}(X)$. It is just enough to prove that the family $\mathcal{Z}(M^p)$ is a maximal subfamily of $\mathcal{C}$ having the finite intersection property and not containing the empty set. It is obvious that the empty set is not contained in it, so there is one more property left to prove.
Since any two disjoint zero–sets are completely separated and any two completely separated subsets of X have disjoint closures in βX, it follows that $\mathcal{Z}(M^p)$ has the finite intersection property.
In order to establish maximality with respect to this property, consider any $g \in \mathcal{C}$ for which $Z(g)$ meets every member of $\mathcal{Z}(M^p)$; we are to prove that $g \in M^p$. Take two arbitrary neighborhoods of p in βX, for instance Ω, Σ, such that $\mathrm{cl}_{\beta X}(\Sigma) \subset \Omega$. By normality βX, there is an $f \in \mathcal{C}^*(X)$ of p such that $\overline{f}(\mathrm{cl}_{\beta X}(\Sigma)) = 0, \overline{f}(\beta X \backslash \Omega) = 1$. Since X is dense in βX, the set $\Sigma \cap X$ is nonempty, and its closure contains p, and we also have $f(\Sigma \cap X) = 0$. Hence $p \in \mathrm{cl}_{\beta X}(Z(f))$, so $f \in M^p$.

Therefore $Z(g)$ meets $Z(f)$. Since $Z(f) \subset \Omega$ and Ω was an arbitrary neighborhood of p, it follows that $Z(g)$ meets every neighborhood of p. Thus $p \in \mathrm{cl}_{\beta X}(Z(g))$, this is, $g \in M^p$.
($\Rightarrow$). Now take M any maximal ideal in $\mathcal{C}(X)$. Then the family $\mathcal{Z}(M)$ has the finite intersection property and, since M is a proper ideal, the empty set is not contained in M. We know βX is compact, so the intersection

$$\Delta(M) = \bigcap_{f \in M} \mathrm{cl}_{\beta X}(Z(f)) \neq \emptyset.$$

For each $p \in \Delta(M)$ it is obvious that $M \subset M^p$. Since M^p is an ideal, and M is a maximal ideal, we have $M = M^p$. The uniqueness of p comes from the complete regularity of the space. □

Finally we have achieved our objective, the one-one correspondence that connects maximal ideals and points of βX has been given. Let's give an application of this theorem that at first they seem to be unrelated, however, Gel'fand-Kolmogorov Theorem (7.2.) will manage to join them into the same subject.

Remark 7.3. Consider the purely algebraic proposition

$$f^2 + g^2 \in M^p \text{ if and only if } f \in M^p \text{ and } g \in M^p,$$

and the purely topological statement

$$p \in \mathrm{cl}_{\beta X}(Z \cap Z') \text{ if and only if } p \in \mathrm{cl}_{\beta X}(Z) \text{ and } p \in \mathrm{cl}_{\beta X}(Z').$$

By Gel'fand-Kolmogorov Theorem, both of them are equal since

$$Z(f^2 + g^2) = Z(f) \cap Z(g).$$

Remark 7.4. Since $\mathcal{C}^*(X)$ and $\mathcal{C}(\beta X)$ are isomorphic, every maximal ideal M^* of $\mathcal{C}^*(X)$ assumes the form $M^* = M^{*p} = \{f \mid f \in \mathcal{C}^*(X), \overline{f}(p) = 0\}$ for some $p \in \beta X$. Of course, M^{*p} is fixed or free according as $p \in X$ or $p \in \beta X \backslash X$.

Definition 7.5. For every point $p \in \beta X$ we define N^p as the ideal in $\mathcal{C}(X)$ consisting of every continuous function in X so that $Z(f)$ contains an X-neighborhood of p. Whenever $p \in X$, we will write N_p instead of N^p.

Notice that whenever $p \in \beta X \backslash X$, then N^p is a free ideal.

Lemma 7.6. *If $\mathfrak{p}$ is any prime ideal in $\mathcal{C}(X)$ such that the maximal ideal M^p contains $\mathfrak{p}$, then $N^p \subset \mathfrak{p}$.*

Proof Take $f \in N^p$. By definition, there exists an open subset $\Omega \subseteq \beta X$ such that $p \in U = \Omega \cap X$, and $f(U) = 0$. Since βX is Tychonoff, we can find $g \in \mathcal{C}^*(X)$ with $\overline{g}(p) = 1$ and $\overline{g}(\beta X \backslash \Omega) = 0$. Clearly $fg = 0$, and since $\mathfrak{p}$ is prime and $g \notin M^p$, it yields that $f \in \mathfrak{p}$. Thus $N^p \subset \mathfrak{p}$. □

Theorem 7.7. *Every prime ideal $\mathfrak{p} \leq \mathcal{C}(X)$ with X a Tychonoff space belongs to an unique maximal ideal on $\mathcal{C}(X)$. The result holds true if we change $\mathcal{C}(X)$ for $\mathcal{C}^*(X)$.*

Proof At least $\mathfrak{p}$ is contained in one maximal ideal, say M^p. Let q be any point of βX distinct from p and the idea is to prove that $\mathfrak{p}$ is not contained in the maximal ideal M^q. If we take Ω_p and Ω_q disjoint neighborhoods in βX of p and q respectively, by the Tychonoff property there exists $f \in \mathcal{C}^*(X)$ such that $\overline{f}(q) = 1$ and $\overline{f}(\beta X \backslash \Omega_q) = 0$. Since $\Omega_p \subset \beta X \backslash \Omega_q$, f is contained in N^p. But by Lemma (7.6.), $N^p \subset \mathfrak{p}$. Hence $\mathfrak{p}$ is not contained in M^q. This proves the result for $\mathcal{C}(X)$. If we apply it to $\mathcal{C}(\beta X)$, it yields the theorem for $\mathcal{C}^*(X)$. □

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