Compact complete proper minimal immersions in strictly convex bounded regular domains of $\mathbb{R}^3$

Antonio Alarcón

Departamento de Matemática Aplicada, Universidad de Murcia, E-30100 Espinardo, Murcia, Spain

Abstract. For any strictly convex bounded regular domain C of $\mathbb{R}^3$ we find a compact Riemann surface $\mathscr{M}$, an open domain $M \subset \mathscr{M}$ with arbitrary finite topological type, and a conformal complete proper minimal immersion $X : M \to C$ which can be extended to a continuous map $X : \overline{M} \to \overline{C}$.

Keywords: Complete minimal surface, Proper immersion, Plateau problem, Limit set
PACS: 02.40.-k

INTRODUCTION AND PRELIMINARIES

One of the central questions in the global theory of complete minimal surfaces in $\mathbb{R}^3$ has been the Calabi-Yau problem, which consists of determining the existence or not of a complete bounded minimal surface in $\mathbb{R}^3$. The most important result in this line is the construction of a complete minimal disk in a ball [14]. After this discovery, new questions related to the embeddedness, properness and topology of surfaces of this type were posed [17].

Concerning the embedded question, complete embedded minimal surfaces in $\mathbb{R}^3$ with either finite genus and countably many ends or positive injectivity radius are proper in $\mathbb{R}^3$ [4, 12, 13]. In particular, they must be unbounded.

Regarding the properness of the examples, a domain $\mathscr{D}$ in $\mathbb{R}^3$ which is either convex or smooth and bounded, admits a complete proper minimal immersion of any open surface [9, 10, 2, 6]. In contrast to this result, any Riemannian three-manifold contains many nonsmooth domains with compact closure which do not admit any complete properly immersed surfaces with at least one annular end and bounded mean curvature [7, 8].

The study of the Calabi-Yau problem gave rise to new lines of work and techniques. Among other things, these new ideas established a surprising relationship between the theory of complete minimal surfaces in $\mathbb{R}^3$ and the Plateau problem. This problem consists of finding a minimal surface spanning a given family of closed curves in $\mathbb{R}^3$, and it is solved for any Jordan curve [5, 16]. The link between complete minimal surfaces and the Plateau problem is the existence of compact complete minimal immersions in $\mathbb{R}^3$, according to the following definition [3, 1].

Definition 1 *By a compact minimal immersion we mean a minimal immersion $X : M \to \mathbb{R}^3$, where M is an open region of a compact Riemann surface $\mathscr{M}$, and such that X can be extended to a continuous map $X : \overline{M} \to \mathbb{R}^3$.*

Let $\mathbb{D}$ denote the unit disk in the complex plane. Compact complete conformal minimal immersions $X : \mathbb{D} \to \mathbb{R}^3$ such that $X_{|\partial\mathbb{D}}$ is an embedding and $X(\mathbb{S}^1)$ is a Jordan curve with Hausdorff dimension 1 exist [11]. Moreover, there exist compact complete conformal minimal immersions of Riemann surfaces of arbitrary finite topology [1]. In addition, the set of closed curves given by the limit sets of these immersions is dense in the space of finite families of closed curves in $\mathbb{R}^3$ which admit a solution to the Plateau problem. In spite of this density result, there are some requirements for the limit set of a compact complete minimal immersion. For instance, given $\mathscr{D} \subset \mathbb{R}^3$ a regular domain and $X : \mathbb{D} \to \mathscr{D}$ a compact complete conformal proper minimal immersion, then the second fundamental form of the surface $\partial\mathscr{D}$ at any point of the limit set of X must be non negatively definite [3, 15].

The aim of the present paper is to join the techniques used in the construction of complete proper minimal surfaces in convex domains of $\mathbb{R}^3$, and those used to construct compact complete minimal immersions, in order to prove the following result.

Theorem 2 *For any C strictly convex bounded regular domain of $\mathbb{R}^3$, there exist compact complete proper minimal immersions $X : M \to C$ with arbitrary finite topology.*

Moreover, for any finite family Σ of closed curves in ∂C which admits a solution to the Plateau problem, and for any $\xi > 0$, there exists a minimal immersion $X : M \to C$ in the above conditions and such that $\delta^H(\Sigma, X(\partial M)) < \xi$, where δ^H means the Hausdorff distance.

Convex domains and Hausdorff distance

Given E a bounded regular convex domain of $\mathbb{R}^3$, and $p \in \partial E$, we let $\kappa_2(p) \geq \kappa_1(p) \geq 0$ denote the principal curvatures of ∂E at p associated to the inward pointing unit normal. Moreover, we write $\kappa_1(\partial E) := \min\{\kappa_1(p) \mid p \in \partial E\} \geq 0$. If E is in addition strictly convex, then $\kappa_1(\partial E) > 0$. If we consider $\mathscr{N} : \partial E \to \mathbb{S}^2$ the outward pointing unit normal or Gauss map of ∂E, then there exists a constant $a > 0$ (depending on E) such that $\partial E_t = \{p + t \cdot \mathscr{N}(p) \mid p \in \partial E\}$ is a regular (convex) surface $\forall t \in [-a, +\infty[$. We label E_t as the convex domain bounded by ∂E_t.

The set of convex bodies of $\mathbb{R}^3$, i.e. convex compact sets of $\mathbb{R}^3$ with nonempty interior, can be made into a metric space by using the Hausdorff distance. Recall that given two compact subsets C, $D \subset \mathbb{R}^3$, the Hausdorff distance between C and D is defined by $\delta^H(C,D) = \max\left\{\sup_{x\in C}\inf_{y\in D}\|x-y\| \, , \, \sup_{y\in D}\inf_{x\in C}\|x-y\|\right\}$.

Riemann surfaces

Throughout the paper we consider M' a fixed but arbitrary compact Riemann surface of genus $\sigma \in \mathbb{N} \cup \{0\}$, and ds^2 a Riemannian metric in M'.

Consider a subset $W \subset M'$, and a Riemannian metric $d\tau^2$ in W. Given $U, V \subset W$, we denote by $\operatorname{dist}_{(W,d\tau)}(U,V)$ the distance in W between U and V with the metric $d\tau^2$. Given a conformal minimal immersion $Y : \overline{W} \to \mathbb{R}^3$, by ds_Y^2 we mean the Riemannian metric induced by Y in $\overline{W}$. Moreover, we write $\operatorname{dist}_{(\overline{W},Y)}$ instead of $\operatorname{dist}_{(\overline{W},ds_Y)}$.

Let $\mathrm{E} \in \mathbb{N}$, and let $\mathbb{D}_1, \ldots, \mathbb{D}_{\mathrm{E}} \subset M'$ be open disks such that $\{\gamma_i := \partial \mathbb{D}_i\}_{i=1}^{\mathrm{E}}$ are analytic Jordan curves and $\overline{\mathbb{D}}_i \cap \overline{\mathbb{D}}_j = \emptyset$ for all $i \neq j$.

Definition 3 *Each curve γ_i is called a cycle on M' and the family $\mathscr{J} = \{\gamma_1, \ldots, \gamma_{\mathrm{E}}\}$ is called a multicycle on M'. We denote by $Int(\gamma_i)$ the disk $\mathbb{D}_i$, for $i = 1, \ldots, \mathrm{E}$. We also define $M(\mathscr{J}) = M' \setminus (\cup_{i=1}^{\mathrm{E}} \overline{Int(\gamma_i)})$. Notice that $M(\mathscr{J})$ is a hyperbolic Riemann surface with genus σ and E ends.*

Given $\mathscr{J} = \{\gamma_1, \ldots, \gamma_{\mathrm{E}}\}$ and $\mathscr{J}' = \{\gamma_1', \ldots, \gamma_{\mathrm{E}}'\}$ two multicycles on M' we write $\mathscr{J}' < \mathscr{J}$ if $\overline{\mathrm{Int}(\gamma_i)} \subset \mathrm{Int}(\gamma_i')$ for $i = 1, \ldots, \mathrm{E}$. Notice that $\mathscr{J}' < \mathscr{J}$ implies $\overline{M(\mathscr{J}')} \subset M(\mathscr{J})$.

Let $\mathscr{J} = \{\gamma_1, \ldots, \gamma_{\mathrm{E}}\}$ be a multicycle on M'. If $\varepsilon > 0$ is small enough, we can consider the multicycle $\mathscr{J}^{\varepsilon} = \{\gamma_1^{\varepsilon}, \ldots, \gamma_{\mathrm{E}}^{\varepsilon}\}$, where by γ_i^{ε} we mean the cycle satisfying $\overline{\mathrm{Int}(\gamma_i)} \subset \mathrm{Int}(\gamma_i^{\varepsilon})$ and $\mathrm{dist}_{(M',ds)}(q, \gamma_i) = \varepsilon$ for all $q \in \gamma_i^{\varepsilon}$. Notice that $\mathscr{J}^{\varepsilon} < \mathscr{J}$.

A Preliminary Lemma

Although next lemma can be found in [2], its usefulness in the construction of compact complete minimal immersions has been entirely exploited in this paper.

Lemma 4 *Let $\mathscr{J}$ be a multicycle on M', $X : \overline{M(\mathscr{J})} \to \mathbb{R}^3$ a conformal minimal immersion, and $p_0 \in M(\mathscr{J})$ with $X(p_0) = 0$. Consider E a strictly convex bounded regular domain, and E' a convex bounded regular domain, with $0 \in E \subset \overline{E} \subset E'$. Let a and ε be positive constants satisfying that $p_0 \in M(\mathscr{J}^{\varepsilon})$, E_{-a} makes sense and*

$$X(\overline{M(\mathscr{J})} \setminus M(\mathscr{J}^{\varepsilon})) \subset E \setminus \overline{E_{-a}}. \tag{1}$$

Consider also $b > 0$ such that E_{-2b-a} and E'_{-b} make sense.

Then there exist a multicycle $\widehat{\mathscr{J}}$ and a conformal minimal immersion $\widehat{X} : \overline{M(\widehat{\mathscr{J}})} \to \mathbb{R}^3$ with the following properties:

(L1) $\widehat{X}(p_0) = 0$.

(L2) $\mathscr{J}^{\varepsilon} < \widehat{\mathscr{J}} < \mathscr{J}$.

(L3) $1/\varepsilon < dist_{(\overline{M(\widehat{\mathscr{J}})}, \widehat{X})}(p, \mathscr{J}^{\varepsilon})$, $\forall p \in \widehat{\mathscr{J}}$.

(L4) $\widehat{X}(\widehat{\mathscr{J}}) \subset E' \setminus \overline{E'_{-b}}$.

(L5) $\widehat{X}(\overline{M(\widehat{\mathscr{J}})} \setminus M(\mathscr{J}^{\varepsilon})) \subset \mathbb{R}^3 \setminus E_{-2b-a}$.

(L6) $\|\widehat{X} - X\| < \varepsilon$ *in* $\overline{M(\mathscr{J}^{\varepsilon})}$.

(L7) $\|\widehat{X} - X\| < \mathtt{m}(a, b, \varepsilon, E, E')$ *in* $\overline{M(\widehat{\mathscr{J}})}$, *where*

$$\mathtt{m}(a, b, \varepsilon, E, E') := \varepsilon + \sqrt{\frac{2(\delta^H(E, E') + a + 2b)}{\kappa_1(\partial E)} + (\delta^H(E, E') + a)^2}.$$

For a detailed proof of this lemma we refer the reader to [2, Lemma 5]. We have stated it here just to make this paper self-contained.

THE THEOREM

The Theorem stated in the introduction trivially follows from the following one.

Theorem 5 *Let C be a strictly convex bounded regular domain of $\mathbb{R}^3$. Consider $\mathscr{J}$ a multicycle on the Riemann surface M' and $\phi : \overline{M(\mathscr{J})} \to \overline{C}$ a conformal minimal immersion satisfying $\phi(\mathscr{J}) \subset \partial C$.*

Then, for any $\mu > 0$ there exist a domain M_μ and a complete proper conformal minimal immersion $\phi_\mu : M_\mu \to C$ such that:

(i) $\overline{M(\mathscr{J}^\mu)} \subset M_\mu \subset \overline{M_\mu} \subset M(\mathscr{J})$, *and M_μ has the topological type of $M(\mathscr{J})$.*

(ii) *ϕ_μ admits a continuous extension $\Phi_\mu : \overline{M_\mu} \to \overline{C}$ and $\Phi_\mu(\partial M_\mu) \subset \partial C$.*

(iii) $\|\phi - \Phi_\mu\| < \mu$ *in $\overline{M_\mu}$.*

(iv) $\delta^H(\phi(\overline{M(\mathscr{J})}), \Phi_\mu(\overline{M_\mu})) < \mu$.

(v) $\delta^H(\phi(\mathscr{J}), \Phi_\mu(\partial M_\mu)) < \mu$.

The proof of the above Theorem consists roughly of the following. First we look for an exhaustion sequence $\{E^n\}_{n\in\mathbb{N}}$ of strictly convex bounded regular domains covering C. Then we use Lemma 4 in a recursive way in order to construct a sequence of minimal immersions $\{X_n\}_{n\in\mathbb{N}}$, starting at $X_1 = \phi$. To construct the immersion X_{n+1} we apply the lemma to the data $X = X_n$, $E = E^n$, $E' = E^{n+1}$ and constants $a = b_n$, $b = b_{n+1}$ and $\varepsilon = \varepsilon_{n+1}$. These constants and the convex domains $\{E^n\}_{n\in\mathbb{N}}$ are suitably chosen so that the sequence $\{X_n\}_{n\in\mathbb{N}}$ has a limit immersion ϕ_μ which satisfies the conclusion of Theorem 5.

Proof. Assume $\mathscr{J} = \{\gamma_1, \ldots, \gamma_{\mathrm{E}}\}$. First of all, we define a positive constant $\varepsilon < \mu/2$. In order to do it, consider $\mathscr{T}(\gamma_i)$ a tubular neighborhood of γ_i in $\overline{M(\mathscr{J})}$, and denote by $\mathrm{P}_i : \mathscr{T}(\gamma_i) \to \gamma_i$ the natural projection, $i = 1, \ldots, \mathrm{E}$. Choose $\varepsilon > 0$ small enough so that $\overline{M(\mathscr{J})} \setminus M(\mathscr{J}^\varepsilon) \subset \cup_{i=1}^{\mathrm{E}} \mathscr{T}(\gamma_i)$, and

$$\|\phi(p) - \phi(\mathrm{P}_i(p))\| < \frac{\mu}{2}, \quad \text{for any } p \in (\overline{M(\mathscr{J})} \setminus M(\mathscr{J}^\varepsilon)) \cap \mathscr{T}(\gamma_i),\ i = 1, \ldots, \mathrm{E}. \tag{2}$$

This choice is possible since the uniform continuity of ϕ. The definition of ε is nothing but a trick to obtain statements (iv) and (v) from statement (iii).

Let us describe how to define the family $\{E^n\}_{n\in\mathbb{N}}$ of convex sets. Consider $t_0 > 0$ small enough so that, for any $t \in]0, t_0[$, C_{-t} is a well defined strictly convex bounded regular domain, and $\Gamma_{-t} := \phi^{-1}((\partial C_{-t}) \cap \phi(M(\mathscr{J})))$ is a multicycle on M'.

Let c_1 be a positive constant (which will be specified later) small enough so that $c_1^2 \cdot \sum_{k\geq 1} \frac{1}{k^4} < \min\{t_0, \varepsilon\}$, and define, for any natural n, $t_n := c_1^2 \cdot \sum_{k\geq n} \frac{1}{k^4}$. Then, $\forall n \in \mathbb{N}$, we consider the strictly convex bounded regular domain $E^n := C_{-t_n}$.Notice that $\cup_{n\in\mathbb{N}} E^n = C$, $E^n \subset E^{n+1}$, and

$$\delta^H(E^{n-1}, E^n) = \frac{c_1^2}{n^4}, \quad \forall n \in \mathbb{N}. \tag{3}$$

Finally, consider a decreasing sequence of positives $\{b_n\}_{n\in\mathbb{N}}$ satisfying

$$b_1 < 2(t_0 - t_1), \quad \text{and} \quad b_n < \frac{c_1^2}{n^4}. \tag{4}$$

These numbers will take the role of the constants a and b of Lemma 4 in the recursive process. Now we use Lemma 4 to construct, for any $n \in \mathbb{N}$, a family $\chi_n = \{\mathscr{J}_n, X_n, \varepsilon_n, \xi_n\}$, where $\mathscr{J}_n$ is a multicycle on M', $X_n : \overline{M(\mathscr{J}_n)} \to C$ is a conformal minimal immersion and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are decreasing sequences of positive real numbers with

$$\xi_n < \varepsilon_n < \frac{c_1}{n^2}. \tag{5}$$

Moreover, the sequence $\{\chi_n\}_{n\in\mathbb{N}}$ must satisfy the following list of properties:

(A$_n$) $\mathscr{J}^{\varepsilon} < \mathscr{J}_{n-1}^{\xi_{n-1}} < \mathscr{J}_{n-1}^{\varepsilon_n} < \mathscr{J}_n^{\xi_n} < \mathscr{J}_n < \mathscr{J}_{n-1}$.

(B$_n$) $1/\varepsilon_n < \operatorname{dist}_{(\overline{M(\mathscr{J}_n^{\xi_n})}, X_n)}(\mathscr{J}_{n-1}^{\xi_{n-1}}, \mathscr{J}_n^{\xi_n})$.

(C$_n$) $\|X_n - X_{n-1}\| < \varepsilon_n$ in $\overline{M(\mathscr{J}_{n-1}^{\varepsilon_n})}$.

(D$_n$) $ds_{X_n} \geq \alpha_n \cdot ds_{X_{n-1}}$ in $\overline{M(\mathscr{J}_{n-1}^{\xi_{n-1}})}$, where the sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ is given by $\alpha_1 := \frac{1}{2}e^{1/2}$, $\alpha_k := e^{-1/2^k}$ for $k > 1$. Notice that $0 < \alpha_k < 1$ and $\{\prod_{m=1}^{k} \alpha_m\}_{k\in\mathbb{N}} \to 1/2$.

(E$_n$) $X_n(p) \in E^n \setminus \overline{(E^n)_{-b_n}}$, for any $p \in \mathscr{J}_n$.

(F$_n$) $X_n(p) \in \mathbb{R}^3 \setminus (E^{n-1})_{-b_{n-1}-2b_n}$, for any $p \in \overline{M(\mathscr{J}_n)} \setminus M(\mathscr{J}_{n-1}^{\varepsilon_n})$.

(G$_n$) $\|X_n - X_{n-1}\| < \mathtt{m}(b_{n-1}, b_n, \varepsilon_n, E^{n-1}, E^n)$ in $\overline{M(\mathscr{J}_n)}$, where $\mathtt{m}$ is defined in Lemma 4.

To define χ_1, we choose $X_1 = \phi$ and $\mathscr{J}_1 = \Gamma_{-t_1-b_1/2}$. The first inequality of (4) guarantees that $\mathscr{J}_1$ is well defined. From this choice we conclude that $X_1(\mathscr{J}_1) \subset \partial C_{-t_1-b_1/2} \subset E^1 \setminus \overline{(E^1)_{-b_1}}$, and so property (E$_1$) holds. Then, we take ε_1 and ξ_1 satisfying (5) and being ξ_1 small enough so that $\mathscr{J}^{\varepsilon} < \mathscr{J}_1^{\xi_1}$. The remainder properties of the family χ_1 do not make sense. The definition of χ_1 is done.

Now, assume that we have constructed the families $\chi_1, \ldots, \chi_n$ satisfying the desired properties. Let us show how to construct χ_{n+1}. First of all, notice that property (E$_n$) guarantees the existence of a positive constant λ such that $X_n(\overline{M(\mathscr{J}_n)} \setminus M(\mathscr{J}_n^{\lambda})) \subset E^n \setminus \overline{(E^n)_{-b_n}}$. Then, Lemma 4 can be applied to the data

$$\mathscr{J} = \mathscr{J}_n, \quad X = X_n, \quad E = E^n, \quad E' = E^{n+1}, \quad a = b_n, \quad \varepsilon, \quad b = b_{n+1},$$

for any $0 < \varepsilon < \lambda$. Now, consider a sequence of positives $\{\widehat{\varepsilon}_m\}_{m\in\mathbb{N}}$ decreasing to zero and such that

$$\widehat{\varepsilon}_m < \min\{\lambda, \xi_n, c_1/(n+1)^2\}, \quad \text{for any } m \in \mathbb{N}. \tag{6}$$

Let $\mathscr{I}_m$ and $Y_m : \overline{M(\mathscr{I}_m)} \to \mathbb{R}^3$ be the multicycle and the conformal minimal immersion given by Lemma 4 for the above data and $\varepsilon = \widehat{\varepsilon}_m$. Statement (L2) and (6) imply that

$$\mathscr{J}_n^{\xi_n} < \mathscr{J}_n^{\widehat{\varepsilon}_m} < \mathscr{I}_m, \quad \text{for any } m \in \mathbb{N}. \tag{7}$$

Taking (7) into account, (L6) guarantees that $\{Y_m\}_{m\in\mathbb{N}}$ converges to X_n uniformly in $\overline{M(\mathscr{J}_n^{\xi_n})}$. In particular, $\{ds_{Y_m}\}_{m\in\mathbb{N}}$ converges to ds_{X_n} uniformly in $\overline{M(\mathscr{J}_n^{\xi_n})}$. Therefore, there exists $m_0 \in \mathbb{N}$ large enough so that

$$ds_{Y_{m_0}} \geq \alpha_{n+1} \cdot ds_{X_n} \quad \text{in } \overline{M(\mathscr{J}_n^{\xi_n})}. \tag{8}$$

Define $\mathscr{J}_{n+1} := \mathscr{I}_{m_0}$, $X_{n+1} := Y_{m_0}$, and $\varepsilon_{n+1} := \widehat{\varepsilon}_{m_0}$. From (7) and statement (L3) in Lemma 4 we deduce that $1/\varepsilon_{n+1} < \operatorname{dist}_{(\overline{M(\mathscr{J}_{n+1})},X_{n+1})}(\mathscr{J}_{n+1}, \mathscr{J}_n^{\xi_n})$. Then, taking into account (7), the existence of a positive ξ_{n+1} small enough so that (5), (A_{n+1}) and (B_{n+1}) hold is guaranteed. Properties (C_{n+1}), (D_{n+1}), (E_{n+1}), (F_{n+1}) and (G_{n+1}) follow from (L6), (8), (L4), (L5) and (L7), respectively. The definition of χ_{n+1} is done.

Define

$$M_\mu := \bigcup_{n\in\mathbb{N}} M(\mathscr{J}_n^{\varepsilon_{n+1}}) = \bigcup_{n\in\mathbb{N}} M(\mathscr{J}_n^{\xi_n}).$$

Since (A_n) holds, $n \in \mathbb{N}$, the set M_μ is an expansive union of domains with the same topological type as $M(\mathscr{J})$. Therefore, elementary topological arguments give that M_μ is a domain with the same topological type as $M(\mathscr{J})$. Furthermore, (A_n), $n \in \mathbb{N}$, also imply that

$$\overline{M_\mu} = \bigcap_{n\in\mathbb{N}} \overline{M(\mathscr{J}_n)}. \tag{9}$$

This is the moment of specifying c_1. Take it small enough so that

$$\sum_{n=2}^{\infty} \mathtt{m}(b_{n-1}, b_n, \varepsilon_n, E^{n-1}, E^n) < \varepsilon. \tag{10}$$

Taking into account (9), (10) and properties (G_n), $n \in \mathbb{N}$, we infer that $\{X_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence uniformly in $\overline{M_\mu}$ of continuous maps. Hence, it converges to a continuous map $\Phi_\mu : \overline{M_\mu} \to \mathbb{R}^3$. Define $\phi_\mu := (\Phi_\mu)_{|M_\mu} : M_\mu \to \mathbb{R}^3$. Let us check that ϕ_μ satisfies the conclusion of the theorem.

- Properties (D_n), $n \in \mathbb{N}$, guarantee that ϕ_μ is a conformal minimal immersion.
- The completeness of ϕ_μ follows from properties (B_n), (D_n), $n \in \mathbb{N}$, and the fact that the sequence $\{1/\varepsilon_n\}_{n\in\mathbb{N}}$ diverges.
- The properness of ϕ_μ in C is equivalent to the fact that $\Phi_\mu(\partial M_\mu) \subset \partial C$. Let us check it. Consider $p \in \partial M_\mu$. For any $n \in \mathbb{N}$, let p_n be a point in $M(\mathscr{J}_n^{\xi_n})$ such that the sequence $\{p_n\}_{n\in\mathbb{N}}$ converges to p. Fix $k \in \mathbb{N}$. The convex hull property for minimal surfaces and (E_n) imply that $X_n(p_k) \in E^n$, for any $n \geq k$. Taking limits as $n \to \infty$ we obtain that $\Phi_\mu(p_k) \in \overline{C}$. Now, taking limits as $k \to \infty$, we get that $\Phi_\mu(p) \in \overline{C}$. On the other hand, $p \in \partial M_\mu \subset \overline{M(\mathscr{J}_n)} \setminus M(\mathscr{J}_{n-1}^{\varepsilon_n})$, $\forall n \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Properties (F_n), $n \in \mathbb{N}$, imply that $X_n(p) \in \mathbb{R}^3 \setminus (E^{k-1})_{-b_{k-1}-2b_k}$, for any $n > k$. Taking limits as $n \to \infty$ we have that $\Phi_\mu(p) \in \overline{C} \setminus (E^{k-1})_{-b_{k-1}-2b_k}$. Hence, $\Phi_\mu(p) \in \overline{C} \setminus (\cup_{k\in\mathbb{N}} (E^{k-1})_{-b_{k-1}-2b_k}) = \overline{C} \setminus C = \partial C$.
- Statement (i) follows from (A_n), $n \in \mathbb{N}$.
- Statement (ii) trivially holds.

• Taking into account (9), (10) and properties (G_n), $n \in \mathbb{N}$, we conclude that

$$\|\phi - \Phi_\mu\| < \varepsilon \quad \text{in } \overline{M_\mu}. \tag{11}$$

This inequality implies statement (iii).
• From (2) follows $\delta^H(\phi(M(\mathscr{J}^\varepsilon)), \phi(\overline{M(\mathscr{J})})) < \mu/2$. Then, to prove (iv) we use (11), the fact that $M(\mathscr{J}^\varepsilon) \subset \overline{M_\mu} \subset \overline{M(\mathscr{J})}$ and the above inequality in the following way: $\delta^H\big(\Phi_\mu(\overline{M_\mu}), \phi(\overline{M(\mathscr{J})})\big) < \delta^H\big(\Phi_\mu(\overline{M_\mu}), \phi(\overline{M_\mu})\big) + \delta^H\big(\phi(\overline{M_\mu}), \phi(\overline{M(\mathscr{J})})\big) < \varepsilon + \delta^H\big(\phi(M(\mathscr{J}^\varepsilon)), \phi(\overline{M(\mathscr{J})})\big) < \varepsilon + \frac{\mu}{2} < \mu$.
• Finally, let us check statement (v). Consider $p \in \partial M_\mu$. Let $i \in \{1, \ldots, \mathrm{E}\}$ such that $p \in \mathscr{T}(\gamma_i)$ and label $q = \mathrm{P}_i(p) \in \mathscr{J}$. Then $\|\Phi_\mu(p) - \phi(q)\| < \|\Phi_\mu(p) - \phi(p)\| + \|\phi(p) - \phi(q)\| < \varepsilon + \frac{\mu}{2} < \mu$, where we have used (11) and (2). On the other hand, given $q \in \mathscr{J}$ we can find a point $p \in \partial M_\mu$ such that $q = \mathrm{P}_i(p)$ for some $i \in \{1, \ldots, \mathrm{E}\}$. The above computation gives $\|\Phi_\mu(p) - \phi(q)\| < \mu$. In this way we have proved (v).

The proof is done. □

REFERENCES

1. A. Alarcón, *Compact complete minimal immersions in* $\mathbb{R}^3$. Trans. Amer. Math. Soc. (to appear).
2. A. Alarcón, L. Ferrer and F. Martín, *Density theorems for complete minimal surfaces in* $\mathbb{R}^3$. Geom. Funct. Anal. **18** (1), 1–49 (2008).
3. A. Alarcón and N. Nadirashvili, *Limit sets for complete minimal immersions.* Math. Z. **258** (1), 107–113 (2008).
4. T.H. Colding and W.P. Minicozzi, *The Calabi-Yau conjectures for embedded surfaces.* Ann. of Math. **167** (1), 211–243 (2008).
5. J. Douglas, *Solution of the problem of Plateau.* Trans. Amer. Math. Soc. **33** (1), 263–321 (1931).
6. L. Ferrer, F. Martín and W.H. Meeks III, *The existence of proper minimal surfaces of arbitrary topological type.* Preprint (arXiv:0903.4194).
7. F. Martín and W.H. Meeks III. *Calabi-Yau domains in three manifolds.* Preprint (arXiv:0906.4638).
8. F. Martín, W.H. Meeks III and N. Nadirashvili, *Bounded domains which are universal for minimal surfaces.* Amer. J. Math. **129** (2), 455–461 (2007).
9. F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of* $\mathbb{R}^3$. Duke Math. J. **128** (3), 559–593 (2005).
10. F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of* $\mathbb{R}^3$ *(II): The behavior of the limit set.* Comment. Math. Helv. **81** (3), 699–725 (2006).
11. F. Martín and N. Nadirashvili, *A Jordan curve spanned by a complete minimal surface.* Arch. Ration. Mech. Anal. **184** (2), 285–301 (2007).
12. W.H. Meeks III, J. Pérez and A. Ros, *The embedded Calabi-Yau conjectures for finite genus.* Preprint.
13. W.H. Meeks and H. Rosenberg, *The minimal lamination closure theorem.* Duke Math. J. 133 (3), 467–497 (2006).
14. N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces.* Invent. Math. **126** (3), 457–465 (1996).
15. N. Nadirashvili, *An application of potential analysis to minimal surfaces.* Mosc. Math. J. **1** (4), 601–604 (2001).
16. T. Radó, *On Plateau's problem.* Ann. of Math. **31** (3), 457–469 (1930).
17. S.-T. Yau, *Review of geometry and analysis. Mathematics: frontiers and perspectives.* Amer. Math. Soc., Providence, RI, 353–401 (2000).