RECENT PROGRESSES IN THE CALABI-YAU PROBLEM FOR MINIMAL SURFACES

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ABSTRACT. In the last forty years, interest of many geometers and analysts has concentrated on the global theory of complete minimal surfaces. Because there were no sufficiently complicated examples for exact investigation, this new development proceeded only slowly. However, last few years have seen an important progress on many long-standing problems in global theory of complete minimal surfaces in $\mathbb{R}^3$. One of these has been the Calabi-Yau problem, which dates back to the 1960s. Calabi asked whether or not it is possible for a complete minimal surface in $\mathbb{R}^3$ to be contained in the ball $\mathbb{B} = \{x \in \mathbb{R}^3 \mid \|x\| < 1\}$. Much work has been done on it over the past four decades. The most important result in this line was obtained by N. Nadirashvili in [24] where he constructed a complete minimal surface in $\mathbb{B}$.

After Nadirashvili's work, Calabi-Yau problem continues generating literature. Questions related with embeddedness and properness of completed bounded minimal surfaces has been particularly interesting. In this survey we pretend to review the most important advances in this area.

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1. INTRODUCTION

By the maximum principle for harmonic functions, there are no compact minimal surfaces in $\mathbb{R}^3$. Moreover, a basic observation of the classical examples of complete nonflat minimal surfaces (catenoid, helicoid, Riemann minimal examples, ...) reveals that one cannot bound any coordinate function for these surfaces. Even more, none of these examples is contained in a halfspace. These facts motivated E. Calabi to conjecture in 1965, the following:

Conjecture 1 (Calabi)**.** *There are no complete bounded minimal surfaces in* $\mathbb{R}^3$.

Conjecture 2 (Calabi)**.** *A complete nonflat minimal surface in* $\mathbb{R}^3$ *has an unbounded projection in every straight line. In particular it cannot be contained in a halfspace.*

Both conjectures turned out to be false. The first example of a complete minimal surface with a bounded coordinate function was a disk constructed by L. P. Jorge and F. Xavier in 1980 [7].

Other examples of complete minimal surfaces in a slab were constructed by F. J. López [8] and by H. Rosenberg and E. Toubiana [25]. Rosenberg and Toubiana obtained a complete minimal cylinder in a slab, and López constructed examples of this type, with arbitrary genus.

On the other hand, F. F. Brito [2] developed an alternative method of construction for this kind of surfaces, by using lacunary series. Using this method, Costa and Simoes [5] obtained examples with higher genus.

The analytic arguments introduced by Jorge and Xavier in their construction were quite ingenious, and have been present in almost all the papers devoted to find complete minimal surfaces with some kind of boundedness on their coordinates. In particular, their idea of using a labyrinth of compact sets around the boundary of a simply connected domain, jointly with Runge's theorem in order to get completeness, was afterward used by N. Nadirashvili [24] in a more elaborated way to construct an example of a complete minimal surface in a ball of $\mathbb{R}^3$.

Theorem 1 (Nadirashvili)**.** *There exists a complete minimal immersion* $f : \mathbb{D} \to \mathbb{B}$ *from the open unit disk* $\mathbb{D}$ *into the open unit ball* $\mathbb{B} \subset \mathbb{R}^3$. *Furthermore the immersion can be constructed with negative Gaussian curvature.*

Nadirashvili's idea consists of constructing, in a recursive way, a sequence of minimal disks (with boundary):

$$f_n : \mathbb{D} \longrightarrow \mathbb{B}_{R_n},$$

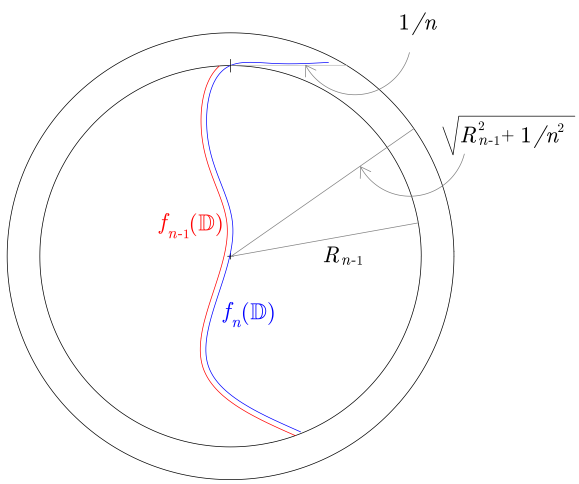
so that:

(1) $\|f_n - f_{n-1}\| < \frac{1}{n^2}$, in $\mathbb{D}_{1-1/n}$;

(2) $\operatorname{dist}_{f_n}(0, \partial\mathbb{D}) \approx \sum_{k=1}^{n} \frac{1}{k}$;

(3) $R_n \approx \sqrt{\sum_{k=1}^{n} \frac{1}{k^2}}$

To get the immersion f_n from the previous one f_{n-1}, we deform the original immersion around its boundary. The deformation acts tangentially to the sphere of radius R_{n-1} in such a way that we increase the intrinsic distance in $1/n$. Hence, it is clear that the new immersion is contained in a bigger ball of radius $R_n = \sqrt{R_{n-1}^2 + 1/n^2}$. Property 1



gives us that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence of harmonics maps, and so it converges $\{f_n\}_{n\in\mathbb{N}} \to f$. Property 2 easily gives the completeness of the limit immersion f, and Property 3 trivially implies that the limit immersion is bounded.

These conjectures were revisited in Yau's 2000 millennium lecture where Yau stated the following questions:

(A) Are there complete embedded minimal surfaces in a ball of $\mathbb{R}^3$?

(B) Are there complete proper minimal immersions $f\colon \mathbb{D} \to \mathbb{B}$?, where by proper we mean that $f^{-1}(C)$ is compact for any $C \subset \mathbb{B}$ compact.

(C) How is the asymptotic behavior ?

(D) How is the spectrum of the Laplacian operator ?

Questions (A) and (B) have been extensively studied in the last few years. As we will see along this survey paper, both problems are intrinsically related.

2. THE CALABI-YAU PROBLEM FOR EMBEDDED MINIMAL SURFACES

Regarding the existence of complete embedded minimal surfaces in a ball, T. Colding and W. Minicozzi [3] have proved that a complete embedded minimal surface with finite topology in $\mathbb{R}^3$ must be properly embedded in $\mathbb{R}^3$. In particular it cannot be contained in a ball.

Theorem 2 (Colding, Minicozzi)**.** *A complete embedded minimal surface with finite topology in $\mathbb{R}^3$ must be proper (in $\mathbb{R}^3$.)*

Very recently, Colding-Minicozzi result has been generalized in two different directions. On one hand W. H. Meeks III, J. Pérez and A. Ros [19] have proved that if M is a complete embedded minimal surface in $\mathbb{R}^3$ with finite genus and a countable number of ends, then M is properly embedded in $\mathbb{R}^3$.

Theorem 3 (Meeks, Pérez, Ros)**.** *If M is a complete embedded minimal surface in $\mathbb{R}^3$ with finite genus and a countable number of ends, then M is proper.*

On the other hand, Meeks and Rosenberg [21] have obtained that if a complete embedded minimal surface M has injectivity radius $I_M > 0$, then M is proper in space. This is a consequence of a more general result that asserts

Theorem 4 (The minimal lamination closure theorem; Meeks, Rosenberg)**.** *Let M be a complete embedded minimal surface in a Riemannian 3-manifold N, such that the injectivity radius of M is positive. Then, $\overline{M}$ has the structure of a $C^{1,\alpha}$-minimal lamination of N.*

As a consequence of the above results, it is natural to conjecture:

Conjecture 3 (Meeks)**.** *If $M \subset \mathbb{R}^3$ is a complete embedded minimal surface with finite genus, then M is proper.*

I would like to mention that the conjecture seems to be false under the assumption of infinite genus, as Meeks, Pérez and Traizet are working in the existence of a nontrivial minimal lamination of $\mathbb{R}^3$ with leaves which are non proper, and with infinite genus. In particular, they would be able to construct a complete embedded minimal surface which is contained in a half space.

3. Complete bounded minimal surfaces with non-trivial topology

In relation with Question (B), López, Martín and Morales have got several results in this line. Summarizing all the information, we have:

Theorem 5 (López, Martín, Morales)**.** *There are examples of complete bounded minimal surfaces with arbitrary finite topology (orientable or not.)*

After Colding-Minicozzi and Meeks-Pérez-Ros theorems, the construction of a complete minimal surface in a ball with infinite genus has become rather interesting, because if an embedded example exists, then it must have infinite genus.

4. Complete proper minimal immersions in bounded regions of $\mathbb{R}^3$

The topological property of being proper has played an important role in the theory of minimal submanifolds and a great number of classical results in the subject assume that the submanifold is proper.

It is trivial to see that a compact submanifold is automatically proper. On the other hand, there is no reason to expect a general immersion (or even embedding) to be proper. However it was long thought that a minimal immersion (or embedding) should be better behaved. This principle was captured by the Calabi-Yau problem. As we mentioned before, the immersed version of this conjecture turned out to be false. In contrast with Nadirashvili's existence result, Colding and Minicozzi have proved that any complete embedded minimal surface with finite topology is proper. From the definition of proper, it is clear that a minimal surface properly immersed in Euclidean space must be unbounded, so Nadirashvili's surfaces are neither embedded nor proper. However, we can ask about the possibility of constructing complete minimal immersions $f : M \to \mathbb{B}$ that were proper in unit ball, in the sense that $f^{-1}(K)$ is compact for any compact $K \subset \mathbb{B}$.

After that, Martín and Morales [12] introduced an additional ingredient into Nadirashvili's machinery in order to produce a complete minimal disk which is properly immersed in a ball of $\mathbb{R}^3$. An example of a complete proper minimal annulus which lies between two parallel planes was constructed earlier by H. Rosenberg and E. Toubiana in [25]. This example is related to the previous construction of Jorge and Xavier.

Recently, Martín and Morales [13] have answered the question to the existence of complete simply connected minimal surfaces which are proper in convex regions of space. To be more precise, their main Theorem establishes that:

Theorem 6 (Martín, Morales)**.** *If $B \subset \mathbb{R}^3$ is a convex domain (not necessarily bounded or smooth), then there exists a complete proper minimal immersion $\psi : \mathbb{D} \to B$.*

Convex domains are a huge and very well known family of domains in Euclidean space. However, one can question whether the convexity hypothesis is necessary or not. A result proved by Nadirashvili shows that convexity cannot be removed from the hypotheses of

Theorem 6. Nadirashvili found a domain of space in which there are no complete proper minimal immersions with finite topology. Nadirashvili have called it the *Magic Cage*.

5. THE TYPE PROBLEM AND UNIVERSAL REGIONS FOR MINIMAL SURFACES

We would also like to mention that, in some sense, Theorem 6 is related with an intrinsic question associated to the underlying complex structure: the so called *type problem* for a minimal surface M, i.e. whether M is hyperbolic or parabolic (as we have already noticed, the elliptic (compact) case is not possible for a minimal surface). Classically, a Riemann surface without boundary is called *hyperbolic* if it carries a nonconstant positive superharmonic function, and *parabolic* if it is neither compact nor hyperbolic. In the case of a Riemann surface with boundary, we say that M is *parabolic* if every bounded harmonic function on M is determined by its boundary values, otherwise M is called *hyperbolic*. It turns out that the parabolicity for Riemann surfaces without boundary is equivalent to the recurrence of Brownian motion on such surfaces. If the boundary of M is nonempty, then M is *parabolic* if, and only if, there exists a point p in the interior of M such that the probability of a Brownian path beginning at p, of hitting the boundary ∂M is 1.

In this setting, given a connected region $W \subset \mathbb{R}^3$ which is either open or the closure of an open set, we say that W is *universal for surfaces* if every complete, connected, properly immersed minimal surface $M \subset W$ is either recurrent ($\partial M = \emptyset$) or a parabolic surface with boundary. The open question of determining which regions of space are universal for surfaces has been proposed by W. H. Meeks and J. Pérez in [16]. Theorem 6 implies that a convex domain of $\mathbb{R}^3$ is not *universal for surfaces.* In contrast with this result, on one hand, Martín, Meeks and Nadirashvili [10] proved the existence of bounded open regions of $\mathbb{R}^3$ which do not admit complete properly immersed minimal surfaces with an annular end. In particular, these domains do not contain a complete properly immersed minimal surface with finite topology.

On the other hand, it is known [4] that the closure of a convex domain is universal for surfaces. This is a consequence of the following theorem by Collin, Kusner, Meeks, and Rosenberg:

Theorem 7 (Collin, Kusner, Meeks, Rosenberg)**.** *Let M be a connected properly immersed minimal surface in $\mathbb{R}^3$, possibly with boundary. Then, every component of the intersection of M with a closed halfspace is a parabolic surface with boundary. In particular, if M has empty boundary and intersects some plane in a compact set, then M is recurrent.*

Until recently, complete minimal surfaces of hyperbolic type played a marginal role in the global theory of minimal surfaces. However, the following recent result by A. Alarcón, L. Ferrer and F. Martín [1] suggests that complete hyperbolic minimal surfaces in $\mathbb{R}^3$ are present in some of the most interesting aspects of minimal surfaces theory.

Theorem 8 (Density theorem; Alarcón, Ferrer, Martín)**.** *Properly immersed, hyperbolic minimal surfaces of finite topology are dense in the space of all properly immersed minimal surfaces in* $\mathbb{R}^3$, *endowed with the topology of smooth convergence on compact sets.*

As a particular case of this theorem, we can obtain the following existence result that improves Theorem 6.

Theorem 9 (Alarcón, Ferrer, Martín)**.** *For any convex domain* D *in* $\mathbb{R}^3$ *(not necessarily bounded or smooth) there exists a complete proper minimal immersion* $\psi : M \to D$*, where* M *is an open Riemann surface with arbitrary finite topology.*

One of the most interesting applications of the Density Theorem is the construction of the first example of a complete minimal surface properly immersed in $\mathbb{R}^3$ with an uncountable number of ends.

Theorem 10 (Alarcón, Ferrer, Martín)**.** *There exists a domain* $\Omega \subset \mathbb{C}$ *and a complete proper minimal immersion* $\psi : \Omega \to \mathbb{R}^3$ *which has uncountably many ends.*

6. THE ASYMPTOTIC BEHAVIOR

To finish, we would like to comment some interesting problems related with the asymptotic behavior of a complete proper minimal immersion. Given a complete minimal disk M in $\mathbb{R}^3$, we define the limit set as $L(M) \overset{\text{def}}{=} \overline{M} \setminus M$. If M is proper in the ball, then it is not difficult to see that $L(M)$ is a closed connected subset of $\mathbb{S}^2$. In general, if we deal with a minimal surface of finite topology, the number of connected components of the limit set is less than or equal to the number of ends of our immersion. Furthermore, it is very easy to check that this limit set has no isolated points. In general, it would be very interesting to know more about the behavior of this limit set.

Recently, Martín and Morales [14] improved their original techniques and were able to show that every bounded domain with $C^{2,\alpha}$-boundary admits a complete properly immersed minimal disk whose limit set is close to a prescribed simple closed curve on the boundary of the domain. In this line of results, Martín and Nadirashvili [15] found Jordan curves in Euclidean space spanning complete minimal surfaces. Moreover they proved that Jordan curves of this kind are dense in the space of Jordan curves with the Hausdorff metric.

Theorem 11 (Martín, Nadirashvili)**.** *There exist complete conformal minimal immersions* $f : \mathbb{D} \to \mathbb{R}^3$ *so that they admit a continuous extension to the closed disk* $F : \overline{\mathbb{D}} \to \mathbb{R}^3$*. The map* $F_{|\mathbb{S}^1}$ *is an embedding and* $F(\mathbb{S}^1)$ *is a non-rectifiable Jordan curve with Hausdorff dimension 1.*

Moreover, for any Jordan curve Γ *in* $\mathbb{R}^3$ *and any* $\epsilon > 0$*, there exists a minimal immersion satisfying the above conditions and such that the Hausdorff distance between* Γ *and* $F(\mathbb{S}^1)$ *is smaller than* ϵ*.*

REFERENCES

[1] A. Alarcón, L. Ferrer and F. Martín, Density theorems for complete minimal surfaces in $\mathbb{R}^3$. Preprint.
[2] Francisco F. de Brito, *Power series with Hadamard gaps and hyperbolic complete minimal surfaces.* Duke Math. J. **68** (1992), no. 2, 297–300.
[3] T. H. Colding and W. P. Minicozzi II, *The Calabi-Yau conjectures for embedded surfaces.* Preprint.
[4] P. Collin, R. Kusner, W. H. Meeks III, and H. Rosenberg, *The topology, geometry and conformal structure of properly embedded minimal surfaces.* J. Differential Geom. **67** (2004), no. 2, 377–393.
[5] C. J. Costa and P. A. Q. Simões, *Complete minimal surfaces of arbitrary genus in a slab of* $\mathbb{R}^3$. Ann. Inst. Fourier (Grenoble) **46** (1996), no. 2, 535–546.
[6] D. Hoffman and W. H. Meeks III, *The strong halfspace theorem for minimal surfaces.* Inventiones Math., Vol. **101** (1990), 373–377.
[7] L. P. M. Jorge and F. Xavier, *A complete minimal surface in* $\mathbb{R}^3$ *between two parallel planes.* Ann. Math., Vol **112** (1980), 203–206.
[8] F. J. López, *A nonorientable complete minimal surface in* $\mathbb{R}^3$ *between two parallel planes.* Proc. Amer. Math. Soc. **103** (1988), no. 3, 913–917.
[9] F. J. López, F. Martín, and S. Morales, *Adding handles to Nadirashvili's surfaces.* J. Differential Geom. **60** (2002), no. 1, 155–175.
[10] F. Martín, W.H. Meeks III and N. Nadirashvili, *Bounded domains which are universal for minimal surfaces*. American J. Math. (to appear).
[11] F. Martín and S. Morales, *A complete bounded minimal cylinder in* $\mathbb{R}^3$. Michigan Math. J. **47** (2000), 499–514.
[12] F. Martín and S. Morales, *On the asymptotic behavior of a complete bounded minimal surface in* $\mathbb{R}^3$. Trans. Amer. Math. Soc. **356** (2004), no. 10, 3985–3994 (electronic).
[13] F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of* $\mathbb{R}^3$. Duke Math. J. **128** (2005), no. 3, 559–593.
[14] F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of* $\mathbb{R}^3$*, II. The behaviour of the limit set.* Comment. Math. Helv., **81** (2006), 699–725.
[15] F. Martín and N. Nadirashvili, *A Jordan curve spanned by a complete minimal surface.* To appear in Arch. Ration. Mech. Anal.
[16] W. H. Meeks III and J. Pérez, *Conformal properties in classical minimal surface theory.* Surveys of Differential Geometry IX - Eigenvalues of Laplacian and other geometric operators, International Press, Somerville, MA 2004, 275–335.
[17] W. H. Meeks III and J. Pérez, *Uniqueness of the helicoid and the classical theory of minimal surfaces.* Preprint.
[18] W. H. Meeks III, J. Pérez, and A. Ros, *The Geometry of minimal surfaces of finite genus I: curvature estimates and quasiperiodicity*. J. Differential Geom. **66** (2004), no. 1, 1–45.
[19] W. H. Meeks III, J. Pérez, and A. Ros, *The Geometry of minimal surfaces of finite genus IV; Calabi-Yau conjectures*. Preprint
[20] W. H. Meeks III and H. Rosenberg, *The uniqueness of the helicoid.* Ann. of Math. (2) **161** (2005), no. 2, 727–758.
[21] W. H. Meeks III and H. Rosenberg, *The minimal lamination closure theorem.* Duke Math. J. **133** (2006), no. 3, 467–497
[22] W. H. Meeks III and S.-T. Yau, *The classical Plateau problem and the topology of three-dimensional manifolds.* Topology **21**(4) (1982), 409–442.

[23] S. Morales, *On the existence of a proper minimal surface in* $\mathbb{R}^3$ *with a conformal type of disk.* Geom. Funct. Anal., **13**(6) (2003), 1281–1301.

[24] N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces.* Invent. Math. **126** (1996), 457–465.

[25] H. Rosenberg and E. Toubiana, *A cylindrical type complete minimal surface in a slab of* $\mathbb{R}^3$. Bull. Sci. Math. (2) **111**(3) (1987), 241–245.

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